Ratio Mathematica Volume 50, 2023

Parameter uniform convergence of a finite element method for a singularly perturbed linear reaction diffusion system with discontinuous source terms

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Abstract

A linear system of 'n' second order ordinary differential equations of reaction-diffusion type with discontinuous source terms is considered. On a piecewise uniform Shishkin mesh, a numerical system is built that employs the finite element method. The numerical approximations obtained by this approach are proven to be effectively almost second order convergent.
Keywords: singular perturbation problems, system of differential equations, reaction - diffusion equations, overlapping boundary and interior layers, finite element method, Shishkin mesh, parameter - uniform convergence, discontinuous source terms.
2020 AMS subject classifications: 65L10, 65L11, 65L50, 65L60, 65L70. [1]

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[1]Received on May 12, 2023. Accepted on December 14, 2023. Published on December 31, 2023. doi:10.23755/rm.v50i0.1547. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

Differential equations with small parameters reflect the rapid progress of science and technology and many practical problems, including mathematical boundary layer theory and approximation of solutions are described in [Doolan et al., 1980], [Roos et al., 1996] and [Miller et al., 1996]. Sometimes, solving mathematical problems accurately can be very challenging, if even impossible; especially in these situations, approximations of the solutions are needed. Its derivative often does not converge uniformly at $x = 0$ or $x = 1$ as $\varepsilon \to 0$, whereas the generated asymptotic solution converges uniformly to the solution of reduced difficulty in the prescribed problem throughout the interval $[0, 1]$. Several finite difference methods have been suggested as problems of this type have been taken up for discussion in [Paramasivam et al., 2013, 2010]. In this paper, we consider the type of problem below, but we assume discontinuity of the source term at an interior point of the domain. For second-order singular perturbation problems of reaction diffusion type with discontinuous source term, many authors have studied the finite-difference and finite-element methods; References are included [Paramasivam et al., 2014, Linβ and Madden, 2009]. Motivated by the works of [Miller et al., 1996], in the present paper we discussed a approximate solutions generated by the numerical approach must be globally established at every point throughout the domain of the exact solution to represent a boundary layer with that method. A basic interpolation technique, such as piecewise linear interpolation, takes the numerical solution from a finite-element approach limited to mesh points for the entire domain. Since our method should be extended to complex situations in higher dimensions, we consider only finite-element subspaces via piecewise polynomial basis functions. For small values of parameter ε, the strategy proposed in this work gives better findings and is more suitable.

In the interval $\Omega = \{x : 0 < x < 1\}$, a singularly perturbed linear system of 'n' second order ordinary differential equations of reaction - diffusion type with discontinuous source terms is considered. Assume that the point $d \in \Omega$ occurs as a single discontinuity in the source terms. The jump at d in any function $\vec{\phi}$ is defined by $[\vec{\phi}](d) = \vec{\phi}(d+) - \vec{\phi}(d-)$.

The self-adjoint two-point boundary value problem that corresponds is

$$-E\vec{u}''(x) + A(x)\vec{u}(x) = \vec{f}(x) \text{ on } \Omega^- \cup \Omega^+,\ \vec{u} \text{ given on } \Gamma \text{ and } \vec{f}(d+) \neq \vec{f}(d-) \tag{1}$$

where $\Gamma = \{0, 1\},\ \Omega^- = \{x : 0 < x < d\},\ \Omega^+ = \{x : d < x < 1\}$.

Here $\vec{u}$ is a column $n-$vector, E and $A(x)$ are $n \times n$ matrices, $E = \text{diag}(\vec{\varepsilon})$, $\vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n)$ with $0 < \varepsilon_i \leq 1$ for all $i = 1, \ldots, n$. The parameters are assumed to be distinct and, for convenience, to have the ordering

$$\varepsilon_1 < \cdots < \varepsilon_n.$$

Cases in which any of the parameters are coincident are not considered here for the sake of convenience. The number of layer functions and, as a result, the number of transformation parameters in the Shishkin mesh specified in Section 4 is reduced in these situations. The problem can also be written in the operator form

$$\vec{L}\vec{u} \;=\; \vec{f} \text{ on } \;\Omega^- \cup \Omega^+,\; \vec{u} \text{ given on } \Gamma\;, \text{ and } \vec{f}(d+) \neq \vec{f}(d-),$$

where the operator $\vec{L}$ is defined by

$$\vec{L} \;=\; -ED^2 + A, \quad D^2 = \frac{d^2}{dx^2}.$$

For all $x \in \overline{\Omega}$, it is assumed that the components $a_{ij}(x)$ of $A(x)$ satisfy the inequalities

$$a_{ii}(x) > \sum_{\substack{j\neq i\\ j=1}}^{n} |a_{ij}(x)| \;\text{ for }\; 1 \le i \le n \;\text{ and }\; a_{ij}(x) \le 0 \;\text{ for }\; i \neq j \tag{2}$$

and, for some α,

$$0 < \alpha < \min_{\substack{x\in[0,1]\\ 1\le i\le n}} \Big(\sum_{j=1}^{n} a_{ij}(x)\Big). \tag{3}$$

It is assumed that $a_{ij},\, f_i \;\in\; C^{(2)}(\overline{\Omega})$, for $i,j \;=\; 1,\ldots,n.$ Then (1) has a solution $\vec{u} \;\in\; C(\overline{\Omega}) \cap C^{(1)}(\Omega) \cap C^{(4)}(\Omega^- \cup \Omega^+)$.
It is also assumed that

$$\sqrt{\varepsilon}_n \le \frac{\sqrt{\alpha}}{6}. \tag{4}$$

C is a generalised positive constant that is independent of x as well as all singular perturbation and discretization parameters used in this article. The empirical results are discussed for the continuous problem are presented on the following section. In Section 3, piecewise-uniform Shishkin meshes can be used to solve the boundary and interior layers. The discrete problem is described in Section 4, and the corresponding maximum principle and stability result are defined. Interpolation error bounds is defined in Section 5. The parameter-uniform error estimation is defined in Section 6. The numerical diagrams in Section 8 are included. Discussion and conclusions is described in Section 9.

2 Analysis of the finite element method

Let V be a given Hilbert space with norm $||\cdot||_V$ and scalar product $(\cdot,\cdot)$. V is usually a subspace of the Sobolev space $H^1(\Omega^- \cup \Omega^+) = H^1(\Omega^-) \cup H^1(\Omega^+)$.

Consider the weak formulation, find $\vec{u} \in H_0^1(\Omega^- \cup \Omega^+)^n$ in particular $u_i \in H_0^1(\Omega^- \cup \Omega^+)$ for $i = 1, \dots, n$ such that

$$\beta_i(u_i, v_i) = f_i(v_i) \quad \forall\, v_i \in H_0^1(\Omega^- \cup \Omega^+) \tag{5}$$

$$\beta_i(u_i, v_i) = -\varepsilon_i(u_i^{'}, v_i^{'}) + \left(\sum_{j=1}^{n} (a_{ij} u_j), v_i \right)$$

and

$$f_i(v_i) = (f_i, v_i)$$

where $(u_i, v_i) = \int\limits_0^1 u_i v_i \, dx$. $\beta_i(u_i, v_i)$ is a bilinear form on $H_0^1(\Omega^- \cup \Omega^+)^n$ and $f_i(v_i)$, a given continuous linear functional on $H_0^1(\Omega^- \cup \Omega^+)^n$ and $f_i(v_i(d+)) \neq f_i(v_i(d-))$.

Lemma 2.1. *Suppose that the bilinear form $\beta_i(\cdot, \cdot)$, $i = 1, \dots, n$, is continuous on $H_0^1(\Omega^- \cup \Omega^+)^n$ is coercive, that*

$$|\beta_i(u_i, v_i)| \leq \gamma ||u_i||\, ||v_i|| \tag{6}$$

$$\beta_i(v_i, v_i) \geq \alpha ||v_i||^2 \tag{7}$$

where α and γ are constants that are indepentent of u_i and v_i. Then for any continuous linear functional $f_i(\cdot)$, the problem (5) *has a unique solution.*

A natural norm on $H_0^1(\Omega^- \cup \Omega^+)^n$ associated with the bilinear form $\beta_i(\cdot, \cdot)$ is the energy norm

$$||v_i||_{\varepsilon_i} = (\varepsilon_i ||v_i||_1^2 + \alpha ||v_i||_0^2)$$

where $||v_i||_1 = (v_i^{'}, v_i^{'})^{1/2}$, $||v_i||_0 = (v_i, v_i)^{1/2}$ on $H_0^1(\Omega^- \cup \Omega^+)^n$.

Lemma 2.2. *A bilinear functional $\beta_i(u_i, v_i)$, $i = 1, \dots, n$, satisfies the coercive property with respect to*

$$||v_i||_{\varepsilon_i}^2 \leq \beta_i(v_i, v_i)$$

Proof. For $i = 1, \dots, n$

$$\begin{aligned} \beta_i(v_i, v_i) &= -\varepsilon_i(v_i^{'}, v_i^{'}) + \left(\sum_{j=1}^{n} (a_{ij} v_j), v_i \right) \\ &= \varepsilon_i ||v_i||_1^2 + \int\limits_0^1 \left(\sum_{j=1}^{n} (a_{ij} v_j) \cdot v_i \right) dx \\ &\geq \varepsilon_i ||v_i||_1^2 + \alpha ||v_i||_0^2 \end{aligned}$$

□

3 The Shishkin mesh

A piecewise uniform Shishkin mesh with N mesh-intervals is now constructed on $\Omega^- \cup \Omega^+$ as follows. Let $\Omega^N = \Omega^{-N} \cup \Omega^{+N}$ where $\Omega^{-N} = \{x_k\}_{k=1}^{\frac{N}{2}-1}, \Omega^{+N} = \{x_k\}_{k=\frac{N}{2}+1}^{N-1}$ $\overline{\Omega}^N = \{x_k\}_{k=0}^{N}$ and $\Gamma^N = \Gamma$. The mesh $\overline{\Omega}^N$ is a piecewise uniform mesh on $[0,1]$ that was generated by dividing $[0,d]$ into $2n+1$ mesh-intervals as follows:

$$[0,\sigma_1] \cup \cdots \cup (\sigma_{n-1},\sigma_n] \cup (\sigma_n, d-\sigma_n] \cup (d-\sigma_n, d-\sigma_{n-1}] \cup \cdots \cup (d-\sigma_1, d].$$

The points separating the uniform meshes are determined by the n parameters σ_r, which are defined by $\sigma_0 = 0$, $\sigma_{n+1} = \frac{1}{2}$,

$$\sigma_n = \min\left\{\frac{d}{4}, 2\frac{\sqrt{\varepsilon_n}}{\sqrt{\alpha}} \ln N\right\} \tag{8}$$

and, for $r = n-1, \ldots 1$,

$$\sigma_r = \min\left\{\frac{r\sigma_{r+1}}{r+1}, 2\frac{\sqrt{\varepsilon_r}}{\sqrt{\alpha}} \ln N\right\}. \tag{9}$$

Clearly

$$0 < \sigma_1 < \ldots < \sigma_n \leq \frac{d}{4}, \qquad \frac{3d}{4} \leq 1-\sigma_n < \ldots < 1-\sigma_1 < d.$$

Then a uniform mesh of $\frac{N}{4}$ mesh-points is placed on the sub-interval $(\sigma_n, d-\sigma_n]$, and a uniform mesh of $\frac{N}{8n}$ mesh-points is placed on each of the sub-intervals $(\sigma_r, \sigma_{r+1}]$ and $(d-\sigma_{r+1}, d-\sigma_r]$, $r = 0, 1, \ldots, n-1$, respectively.
The remaining was generated by dividing $[d,1]$ into $2n+1$ mesh-intervals as follows:

$$[d, d+\tau_1] \cup \cdots \cup (d+\tau_{n-1}, d+\tau_n] \cup (d+\tau_n, 1-\tau_n] \cup (1-\tau_n, 1-\tau_{n-1}] \cup \cdots \cup (1-\tau_1, 1].$$

The points separating the uniform meshes are determined by the n parameters τ_r, which are defined by $\tau_0 = \frac{1}{2}$, $\tau_{n+1} = 1$,

$$\tau_n = \min\left\{\frac{1-d}{4}, 2\frac{\sqrt{\varepsilon_n}}{\sqrt{\alpha}} \ln N\right\} \tag{10}$$

and, for $r = n-1, \ldots 1$,

$$\tau_r = \min\left\{\frac{r\tau_{r+1}}{r+1}, 2\frac{\sqrt{\varepsilon_r}}{\sqrt{\alpha}} \ln N\right\}. \tag{11}$$

Clearly

$$d \;<\; d+\tau_1 \;<\; \dots \;<\; d+\tau_n \;\leq\; \frac{1-d}{4}, \qquad \frac{3(1-d)}{4} \leq 1-\tau_n < \;\dots\; < 1-\tau_1 < 1.$$

Then a uniform mesh of $\frac{N}{4}$ mesh-points is placed on the sub-interval $(d+\tau_n, 1-\tau_n]$, and a uniform mesh of $\frac{N}{8n}$ mesh-points is placed on each of the sub-intervals $(d+\tau_r, d+\tau_{r+1}]$ and $(1-\tau_{r+1}, 1-\tau_r],\;\; r = 0, 1, \dots, n-1,$ respectively.
In practice, it is convenient to take

$$N = 8n\delta, \;\; \delta \geq 3, \tag{12}$$

where n denotes the number of distinct singular perturbation parameters involved in the experiment (1). This produces a class of 2^{n+1} piecewise uniform Shishkin meshes $\overline{\Omega}^N$.
When all of the parameters σ_r and $\tau_r,\;\; r = 1, \dots, n,$ are set to the left, the Shishkin mesh $\overline{\Omega}^N$ becomes a classical uniform mesh with the transformation parameters σ_r, τ_r and a scale N^{-1} from 0 to 1.
The following inequalities hold for the mesh Ω^N , $s = 1, \dots, n-1$

$$\begin{array}{lll}
h_k \leq 2/N & \text{for} & 1 \leq k \leq N \\
h_k \geq 1/N & \text{for} & \frac{N}{8} \leq k \leq \frac{3N}{8} \text{ and } \frac{5N}{8} \leq k \leq \frac{7N}{8} \\
h_k \leq 1/N & \text{for} & 1 \leq k \leq \frac{N}{8} \text{ and } \frac{3N}{8} \leq k \leq \frac{N}{2} \\
h_k \leq 1/N & \text{for} & \frac{N}{2} \leq k \leq \frac{5N}{8} \text{ and } \frac{7N}{8} \leq k \leq N \\
h_k \geq \frac{N}{8s} & \text{for} & \frac{N}{8s} \leq k \leq \frac{N}{8(s+1)} \text{ and } (d - \frac{N}{8(s+1)}) \leq k \leq (d - \frac{N}{8s}) \\
h_k \geq \frac{N}{8s} & \text{for} & d + \frac{N}{8s} \leq k \leq d + \frac{N}{8(s+1)} \text{ and } (1 - \frac{N}{8(s+1)}) \leq k \leq (1 - \frac{N}{8s}) \\
h_k \leq \frac{N}{8s} & \text{for} & 1 \leq k \leq \frac{N}{8(s)} \text{ and } (d - \frac{N}{8(s)}) \leq k \leq \frac{N}{2} \\
h_k \leq \frac{N}{8s} & \text{for} & \frac{N}{2} \leq k \leq d + \frac{N}{8(s)} \text{ and } (1 - \frac{N}{8(s)}) \leq k \leq N
\end{array} \tag{13}$$

4 The discrete problem

In this segment, a numerical method for (5) is constructed using a finite element method with a suitable Shishkin mesh. Let for $i = 1, \dots, n$ and $k = 1, \dots, N-1 \backslash \{\frac{N}{2}\},\; V_{i,k} \subset H_0^1(\Omega^- \cup \Omega^+)^n$ be the space of piecewise linear functionals on $\Omega^- \cup \Omega^+$, that vanish $x = 0, d$, and 1.

The finite element approach is now established for the discrete two-point boundary value problem, $U_{i,k} \in V_{i,k}$

$$\beta_i(U_{i,k}, v_{i,k}) = f(v_{i,k}), \quad \forall\, v_{i,k} \in V_{i,k}, \quad v_{i,\frac{N}{2}} = 0. \tag{14}$$

By $Lax - Migram\, Lemma$ implies that

1. The discrete problem has a unique solution,

2. The discrete problem is stable.

From (3) on A implies that for arbitrary $x \in (\Omega^- \cup \Omega^+)$

$$\xi^T A\, \xi \geq \alpha\, \xi^T \xi \qquad \forall\, \xi \text{ on } \; V_{i,k}^*$$

$V_{i,k}^*$ is dual space for $V_{i,k}$.
Let $\{\phi_{i,k} : k = 1, \cdots, N-1 \backslash \{\frac{N}{2}\}\}$ be a basis for $V_{i,k}$, where $N = N(i,k)$ is the dimension of $V_{i,k}$. Then

$$U_{i,k} = \sum_{k=1}^{\frac{N}{2}-1} C_{i,k}\phi_{i,k} + \sum_{k=\frac{N}{2}+1}^{N-1} C_{i,k}\phi_{i,k}, \qquad \phi_{i,\frac{N}{2}} = 0$$

where the unknowns $C_{i,k}$ satisfiy the linear system

$$AU = B$$

with $A = \beta_i(\phi_{i,k_1}, \phi_{i,k_2})$, $U = C_{i,k}$, $B = f_i(\phi_{i,k})$.
The corresponding difference scheme is

$$\begin{pmatrix} \beta_1(\phi_{1,1},\phi_{1,1}) & \beta_1(\phi_{1,1},\phi_{1,2}) & \cdots & \beta_1(\phi_{1,1},\phi_{n,N-1}) \\ \beta_1(\phi_{1,2},\phi_{1,1}) & \beta_1(\phi_{1,2},\phi_{1,2}) & \cdots & \beta_1(\phi_{1,2},\phi_{n,N-1}) \\ \vdots & \vdots & & \vdots \\ \beta_n(\phi_{n,N-1},\phi_{n,1}) & \beta_n(\phi_{n,N-1},\phi_{n,2}) & \cdots & \beta_n(\phi_{n,N-1},\phi_{n,N-1}) \end{pmatrix} \begin{pmatrix} C_{1,1} \\ C_{1,2} \\ \vdots \\ C_{n,N-1} \end{pmatrix} = \begin{pmatrix} (f_1,\phi_{1,1}) \\ (f_1,\phi_{1,2}) \\ \vdots \\ (f_n,\phi_{n,N-1}) \end{pmatrix}$$

For $k = 1, \ldots, N-1$

$$\phi_{1,k} = \phi_{2,k} = \cdots = \phi_{n,k}$$

$$C_{1,k} = C_{2,k} = \cdots = C_{n,k}$$

The nonzero contribution from a particular element is

$$A_{i,k} = \begin{pmatrix} \int\limits_{x_{k-1}}^{x_k} \phi_{i,k-1}.\phi_{i,k-1}dx & \int\limits_{x_{k-1}}^{x_k} \phi_{i,k-1}.\phi_{i,k}dx \\ \int\limits_{x_k}^{x_{k+1}} \phi_{i,k}.\phi_{i,k}dx & \int\limits_{x_k}^{x_{k+1}} \phi_{i,k}.\phi_{i,k+1}dx \end{pmatrix}$$

Similarly, the local load vector is

$$B_{i,k} = \begin{pmatrix} \int\limits_{x_k}^{x_{k+1}} f_i.\phi_{i,k}dx \\ \int\limits_{x_k}^{x_{k+1}} f_i.\phi_{i,k+1}dx \end{pmatrix}$$

For $k = \frac{N}{2}$, the local load vector $(\int\limits_{x_{k-1}}^{x_k} f_i(\frac{N}{2} - 1) + \int\limits_{x_k}^{x_{k+1}} f_i(\frac{N}{2} + 1))/2$

5 Interpolation error bounds

Lemma 5.1. *Let $u^*_{i,k}$ be the $V_{i,k}$-interpolant of the solution $u_{i,k}$ of* (1) *on the fitted mesh Ω^N. Then*

$$\max_{i=1,\dots,n} \sup_{0<\varepsilon_i\leq 1} ||u^*_{i,k} - u_{i,k}||_{\Omega^N} \leq C(N^{-2}lnN)^2,$$

where C is a constant independent of the parameters ε_i.

Proof. The estimate is obtained separately on each subinterval $\Omega_k = (x_{k-1}, x_k) \in \Omega^- \cup \Omega^+$,
$k = 1, \dots, N-1\backslash\{\frac{N}{2}\}$. Note that for any function $g_{i,k}$ on Ω_k

$$g^*_{i,k} = g_{i,k-1}\phi_{i,k-1} + g_{i,k}\phi_{i,k},$$

and so it is obvious that, on Ω_k,

$$|g^*_{i,k}(x)| \leq \max_{\Omega_k} |g_{i,k}(x)|, \tag{15}$$

and it's easy to see that by using sufficient Taylor expansions

$$|g^*_{i,k}(x) - g_{i,k}(x)| \leq Ch_k^2 \max_{\Omega_k} |g^{''}_{i,k}(x)|. \tag{16}$$

For $i = 1, \dots, n$ from (16) and using Lemma 3 in [Paramasivam et al., 2014], on $\Omega_k \in \Omega^- \cup \Omega^+$,

$$|u^*_{i,k}(x) - u_{i,k}(x)| \leq Ch_k^2 \max_{\Omega_k} |u^{''}_{i,k}(x)|$$

$$\leq C\frac{h_k^2}{\varepsilon_i}. \tag{17}$$

Also, (17) using Lemma 6 and Lemma 7 are in [Paramasivam et al., 2014] on $\Omega_k \in \Omega^-$, for $k = 1, \ldots, \frac{N}{2} - 1$

$$|u_{i,k}^*(x) - u_{i,k}(x)| = |v_{i,k}^*(x) + w_{i,k}^*(x) - v_{i,k}(x) - w_{i,k}(x)|$$

$$\leq |v_{i,k}^*(x) - v_{i,k}(x)| + |w_{i,k}^{*L}(x) - w_{i,k}^L(x)| + |w_{i,k}^{*R}(x) - w_{i,k}^R(x)|$$

$$\leq Ch_k^2 \max_{\Omega_k} |v_{i,k}''(x)| + Ch_k^2 \max_{\Omega_k} |w_{i,k}^{L''}(x)| + Ch_k^2 \max_{\Omega_k} |w_{i,k}^{R''}(x)|$$

$$\leq C(1 + \sum_{q=i}^{n} B_q(x)) + C\sum_{q=i}^{n} \frac{B_q^L(x)}{\varepsilon_q} + C\sum_{q=i}^{n} \frac{B_q^R(x)}{\varepsilon_q} \tag{18}$$

The discussion now centres on whether $2\sqrt{\varepsilon_n}\, lnN/\sqrt{\alpha} \geq d/4$ or $2\sqrt{\varepsilon_n} lnN/\sqrt{\alpha} \leq d/4$ should be used. In the first case $1/\varepsilon_n \leq C\,(lnN)^2$ and the result follows at once from (13) and (17). In the second case $\sigma_n = 2\sqrt{\varepsilon_n} lnN/\sqrt{\alpha}$. Suppose that k satisfies $N/8 \leq k \leq 3N/8$. Then $h_k = 2(d - 2\sigma_n)/N$ and therefore

$$\frac{h_k}{\varepsilon_n} = 2N^{-1}\frac{d - 2\sigma_n}{\varepsilon_n},$$

$\sigma_n \leq 1 - x_k$, and so

$$e^{-\sqrt{\alpha}(1-x_k)/\sqrt{\varepsilon_n}} \leq e^{-\sqrt{\alpha}\sigma_n/\sqrt{\varepsilon_n}} = e^{-2\,lnN} = N^{-2}. \tag{19}$$

Using (19) and (13) in (18) gives the required result.
On the other hand, if k satisfies $1 \leq k \leq N/8$ and $3N/8 \leq k \leq N/2$ and $r = n - 1, \ldots, 1,$ then the discussion now centres on whether $2\sqrt{\varepsilon_r}\, lnN/\sqrt{\alpha} \geq r\sigma_{r+1}/r + 1$ or $2\sqrt{\varepsilon_r} lnN/\sqrt{\alpha} \leq r\sigma_{r+1}/r + 1$should be used.
In the first case $1/\varepsilon_r \leq C\,(lnN)^2$ and the result follows at once from (13) and (17).
In the second case $\sigma_r = 2\sqrt{\varepsilon_r} lnN/\sqrt{\alpha}$ and $s = 1, \ldots, n - 2$.

1. Suppose that k satisfies $N/8(s + 1) \leq k \leq N/8(s)$ and $d - (N/8(s)) \leq k \leq d - (N/8(s + 1))$. Then $h_k = 8n(\sigma_{r+1} - \sigma_r)/N$ or $8n(\sigma_r - \sigma_{r+1})/N$ and $\sigma_r \leq 1 - x_k$ therefore

$$\frac{h_k}{\varepsilon_r} = 8nN^{-1}\frac{\sigma_{r+1} - \sigma_r}{\varepsilon_r} \text{ or } 8nN^{-1}\frac{\sigma_r - \sigma_{r+1}}{\varepsilon_r}, \tag{20}$$

   Using (20) and (13) in (18) gives the required result.

2. If k satisfies $1 \leq k \leq N/8(s+1)$ and $d-(N/8(s+1)) \leq k \leq N/2$ Then $h_k = 8n(\sigma_{r+1}-\sigma_r)/N$ or $8n(\sigma_{r+1}-\sigma_r)/N$ and therefore

$$\frac{h_k}{\varepsilon_r} = 8nN^{-1}\frac{(\sigma_{r+1}-\sigma_r)}{\varepsilon_r} \text{ or } 8nN^{-1}\frac{(\sigma_r-\sigma_{r+1})}{\varepsilon_r}, \tag{21}$$

Using (21) and (13) in (18) gives the required result.

Also, (17) using Lemma 6 and Lemma 7 are in [Paramasivam et al., 2014] on $\Omega_k \in \Omega^+$, for $k = \frac{N}{2}+1,\ldots,N-1$

$$\begin{aligned}
|u^*_{i,k}(x) - u_{i,k}(x)| &= |v^*_{i,k}(x) + w^*_{i,k}(x) - v_{i,k}(x) - w_{i,k}(x)| \\
\leq |v^*_{i,k}(x) - v_{i,k}(x)| &+ |w^{*L}_{i,k}(x) - w^L_{i,k}(x)| + |w^{*R}_{i,k}(x) - w^R_{i,k}(x)| \\
\leq Ch_k^2 \max_{\Omega_k} |v''_{i,k}(x)| &+ Ch_k^2 \max_{\Omega_k} |w^{L''}_{i,k}(x)| + Ch_k^2 \max_{\Omega_k} |w^{R''}_{i,k}(x)| \\
\leq C(1+\sum_{q=i}^{n} B_q(x)) &+ C\sum_{q=i}^{n}\frac{B^L_q(x)}{\varepsilon_q} + C\sum_{q=i}^{n}\frac{B^R_q(x)}{\varepsilon_q}
\end{aligned} \tag{22}$$

The discussion now centres on whether $2\sqrt{\varepsilon_n}\,lnN/\sqrt{\alpha} \geq (1-d)/4$ or $2\sqrt{\varepsilon_n}lnN/\sqrt{\alpha} \leq (1-d)/4$ should be used. In the first case $1/\varepsilon_n \leq C\,(lnN)^2$ and the result follows at once from (13) and (17). In the second case $\tau_n = 2\sqrt{\varepsilon_n}lnN/\sqrt{\alpha}$. Suppose that k satisfies $5N/8 \leq k \leq 7N/8$. Then $h_k = 2(1-2\tau_n)/N$ and therefore

$$\frac{h_k}{\varepsilon_n} = 2N^{-1}\frac{1-2\tau_n}{\varepsilon_n},$$

$\tau_n \leq 1-x_k$, and so

$$e^{-\sqrt{\alpha}(1-x_k)/\sqrt{\varepsilon_n}} \leq e^{-\sqrt{\alpha}\tau_n/\sqrt{\varepsilon_n}} = e^{-2\,lnN} = N^{-2}. \tag{23}$$

Using (23) and (13) in (22) gives the required result.
On the other hand, if k satisfies $N/2 < k \leq 5N/8$ and $7N/8 \leq k < N$ and $r = n-1,\ldots,1$, then the discussion now centres on whether $2\sqrt{\varepsilon_r}\,lnN/\sqrt{\alpha} \geq r\tau_{r+1}/r+1$ or $2\sqrt{\varepsilon_r}lnN/\sqrt{\alpha} \leq r\tau_{r+1}/r+1$should be used.
In the first case $1/\varepsilon_r \leq C\,(lnN)^2$ and the result follows at once from (13) and (17).
In the second case $\tau_r = 2\sqrt{\varepsilon_r}lnN/\sqrt{\alpha}$ and $s = 1,\ldots,n-2$.

1. Suppose that k satisfies $5N/8(s+1) \leq k \leq 5N/8(s)$ and $1-(N/8(s)) \leq k \leq 1-(N/8(s+1))$. Then $h_k = 8n(\tau_{r+1}-\tau_r)/N$ or $8n(\tau_r-\tau_{r+1})/N$ and $\tau_r \leq 1-x_k$ therefore

$$\frac{h_k}{\varepsilon_r} = 8nN^{-1}\frac{\tau_{r+1}-\tau_r}{\varepsilon_r} \text{ or } 8nN^{-1}\frac{\tau_r-\tau_{r+1}}{\varepsilon_r}, \tag{24}$$

Using (24) and (13) in (22) gives the required result.

2. If k satisfies $N/2 < k \leq N/8(s+1)$ and $1-(N/8(s+1)) \leq k < N$ Then $h_k = 8n(\tau_{r+1}-\tau_r)/N$ or $8n(\tau_{r+1}-\tau_r)/N$ and therefore

$$\frac{h_k}{\varepsilon_r} = 8nN^{-1}\frac{(\tau_{r+1}-\tau_r)}{\varepsilon_r} \text{ or } 8nN^{-1}\frac{(\tau_r-\tau_{r+1})}{\varepsilon_r}, \tag{25}$$

Using (25) and (13) in (22) gives the required result.

For $k = \frac{N}{2}$, the source terms is assumed by

$$(\int_{x_{k-1}}^{x_k} f_i(\frac{N}{2}-1) + \int_{x_k}^{x_{k+1}} f_i(\frac{N}{2}+1))/2$$

$h_k = (h_{k-1}+h_{k+1})/2,\ h_{k-1} = (x_{k-2}-x_{k-1})$ and $h_{k+1} = (x_{k+1}-x_{k+2})$, $h_{k-1} = 8n(\sigma_{n-1}-\sigma_n)/N,\quad h_{k+1} = 8n(\tau_1-\tau_2)/N$

$$\frac{h_k}{\varepsilon_i} = \frac{h_{k+1}+h_{k-1}}{2\varepsilon_i} = \frac{4nN^{-1}((\sigma_{n-1}-\sigma_n)+(\tau_1-\tau_2))}{\varepsilon_i} \tag{26}$$

Using (26) and (13) in (22) gives the required result. $\square$

Lemma 5.2. *Let $u^*_{i,k}$ be the $V_{i,k}$-interpolant of the solution $u_{i,k}$ of* (1) *on the fitted mesh Ω^N. Then*

$$\max_{i=1,\dots,n}\sup_{0<\varepsilon_i\leq 1}||u^*_{i,k}-u_{i,k}||_{\varepsilon_i} \leq C(N^{-1}lnN)^2,$$

where C is a constant independent of ε_i.

Proof. For $i = 1,\dots,n$ from the definition of the energy norm

$$||u^*_{i,k}-u_{i,k}||_{\varepsilon_i} = \varepsilon_i((u^*_{i,k}-u_{i,k})', (u^*_{i,k}-u_{i,k})') + \alpha(u^*_{i,k}-u_{i,k}, u^*_{i,k}-u_{i,k}). \tag{27}$$

Each term on the right is now treated separately. It is easy to see that the second term satisfies

$$(u^*_{i,k}-u_{i,k}, u^*_{i,k}-u_{i,k}) \leq ||u^*_{i,k}-u_{i,k}||^2. \tag{28}$$

Using integration by parts and noting that $(u^*_{i,k}-u_{i,k})(x_k) = 0$, for each k, the first term can be bounded as follows

$$\varepsilon_i((u^*_{i,k}-u_{i,k})', (u^*_{i,k}-u_{i,k})') = \varepsilon_i \sum_{k=1,k\neq\frac{N}{2}}^{N-1} \int_{x_{i-1}}^{x_i} ({u^*_{i,k}}'(s)-{u_{i,k}}'(s))^2 ds$$

$$= -\varepsilon_i \sum_{k=1,k\neq\frac{N}{2}}^{N-1} \int_{x_{i-1}}^{x_i} ({u^*_{i,k}}''(s)-{u_{i,k}}''(s))(u^*_{i,k}(s)-u_{i,k}(s)) ds$$

$$= \varepsilon_i \sum_{k=1, k\neq \frac{N}{2}}^{N-1} \int\limits_{x_{i-1}}^{x_i} {u_{i,k}}''(s)(u^*_{i,k}(s) - u_{i,k}(s))ds$$
$$= (\varepsilon_i {u_{i,k}}'', u^*_{i,k} - u_{i,k}),$$

where the fact that ${u^*_{i,k}}'' = 0$ on each Ω_k has been used.
The estimate for the second derivative of the components of $u_{i,k}$ are contains in [Paramasivam et al., 2014], using lemma 6 and lemma 7 in[Paramasivam et al., 2014] then gives

$$|(\varepsilon_i u''_{i,k}, u^*_{i,k} - u_{i,k})| \leq ||u^*_{i,k} - u_{i,k}|| \int\limits_0^d \varepsilon_i |u''_{i,k}| ds + \int\limits_d^1 \varepsilon_i |u''_{i,k}| ds$$

$$|(\varepsilon_i u''_{i,k}, u^*_{i,k} - u_{i,k})| \leq ||u^*_{i,k} - u_{i,k}|| (\int\limits_0^d + \int\limits_d^1)(\varepsilon_i |v''_{i,k}| + \varepsilon_i |w^{L''}_{i,k}| + \varepsilon_i |w^{R''}_{i,k}|) ds$$

$$\leq C||u^*_{i,k} - u_{i,k}|| (\int\limits_0^d + \int\limits_d^1)(1 + \sum_{q=i}^{n} B_q(s)) + C\sum_{q=i}^{n} \frac{B^L_q(s)}{\varepsilon_q} + C\sum_{q=i}^{n} \frac{B^R_q(s)}{\varepsilon_q} ds$$
$$\leq C||u^*_{i,k} - u_{i,k}||,$$

and so

$$\varepsilon_i((u^*_{i,k} - u_{i,k})', (u^*_{i,k} - u_{i,k})') \leq C||u^*_{i,k} - u_{i,k}||. \tag{29}$$

Combining (27)-(29) leads to

$$||u^*_{i,k} - u_{i,k}||_{\varepsilon_i} \leq C||u^*_{i,k} - u_{i,k}||(1 + \alpha||u^*_{i,k} - u_{i,k}||),$$

and the proof is completed using the estimate of $||u^*_{i,k} - u_{i,k}||$ from Lemma 5.1. □

Lemma 5.3. *Let $u^*_{i,k}$ be the $V_{i,k}$ -interpolant of the solution $u_{i,k}$ of* (1) *on the fitted mesh Ω^N. Then*

$$\max_{i=1,\dots,n} \sup_{0<\varepsilon_i\leq 1} ||u^*_{i,k} - u_{i,k}||_{\varepsilon_i, \overline{\Omega}^N} \leq C(N^{-1} lnN)^2.$$

Proof. Since $u^*_{i,k}(x_k) - u_{i,k}(x_k) = 0$, it follows from the definitions of the norms that

$$||u^*_{i,k} - u_{i,k}||^2_{\varepsilon_i, \vec{\Omega}^N}|| = \varepsilon_i((u^*_{i,k} - u_{i,k})', (u^*_{i,k} - u_{i,k})') \leq ||u^*_{i,k} - u_{i,k}||^2_{\varepsilon_i}.$$

Using the estimate in Lemma 5.2 completes the proof. □

6 Interpolation error estimate

Lemma 6.1. *Let $u_{i,k}$ be the solution of* (1) *and $U_{i,k}$ the solution of* (14)*. Suppose that $V_{i,k}$. Then*

$$\max_{i=1,\dots,n} |\beta_i(U_{i,k} - u_{i,k}, v_i)| \leq C(N^{-1} lnN)^2 ||v_{i,k}||_{l^2(\overline{\Omega}^N)},$$

where the constant C is independent of ε_i.

Proof. Since v_i is in $V_{i,k}$, it can be written in the form

$$v_i = \sum_{k=1, k\neq \frac{N}{2}}^{N-1} v_{i,k}\phi_{i,k},$$

and so

$$\beta_i(U_{i,k} - u_{i,k}, v_i) = \sum_{k=1, k\neq \frac{N}{2}}^{N-1} v_{i,k}\beta_i(U_{i,k} - u_{i,k}, \phi_{i,k}) \tag{30}$$

Then, for each k, $1 \leq k \leq N-1 \backslash \{\frac{N}{2}\}$, using (1), (14) and the fact that $(1, \phi_{i,k})_{\Omega^N} = (1, \phi_{i,k}) = \frac{h_k + h_{k+1}}{2}$,

$$\begin{aligned}\beta_i(U_{i,k} - u_{i,k}, \phi_{i,k}) &= \sum_{j=1}^{n}(a_{ij}U_{i,k}, \phi_{i,k}) - \sum_{j=1}^{n}(a_{ij}u_{i,k}, \phi_{i,k}) \\ &= \sum_{j=1}^{n}(a_{ij}u_{j,k}(x_k), \phi_{i,k}) - \sum_{j=1}^{n}(a_{ij}u_{j,k}, \phi_{i,k}) \\ &= \sum_{j=1}^{n}(a_{ij}(u_{j,k}(x_k) - u_{j,k}), \phi_{i,k})\end{aligned}$$

Since

$$|u_{j,k}(x_k) - u_{j,k}| = |\int_{x}^{x_k} u'_{j,k}(s)ds| \leq I_k,$$

where

$$I_k = \int_{x_{k-1}}^{x_{k+1}} |u'_{j,k}(s)|ds,$$

it follows from (13) that

$$|\beta_i(U_{i,k} - u_{i,k}, \phi_{i,k})| \leq C\frac{(h_k + h_{k+1})}{2}(I_k + N^{-1}). \tag{31}$$

Assume for the moment that

$$I_k \leq C(N^{-1}lnN)^2. \tag{32}$$

Then (30)-(32) and the Cauchy-Schwarz inequality give

$$|\beta_i(U_{i,k}-u_{i,k},v_i)| \leq C(N^{-1}lnN)^2 \sum_{k=1,k\neq\frac{N}{2}}^{N-1} \frac{(h_k+h_{k+1})}{2}^{1/2}|v_{i,k}|\frac{(h_k+h_{k+1})}{2}^{1/2}$$

$$\leq C(N^{-1}lnN)^2||v_{i,k}||_{l^2(\overline{\Omega}^N)},$$

as required.
It remains therefore to verify (32). From the estimate are contain Lemma 3 in [Paramasivam et al., 2014], for the first derivative of the solution, it is clear that

$$I_k \leq C\int_{x_{k-1}}^{x_{k+1}} \varepsilon_i^{-1}(||\vec{u}||_\Gamma + ||\vec{f}||_\Omega)dx$$

It follows that

$$I_k \leq C(h_k+h_{k+1})^2/\varepsilon_i, \tag{33}$$

and that

$$I_k \leq C\frac{(h_k+h_{k+1})^2}{\varepsilon_i} + e^{-\sqrt{\alpha}(1-x_{k+1})/\sqrt{\varepsilon_n}} \tag{34}$$

For $i=1,\dots,n,\ k=1,\dots,\frac{N}{2}-1$, then the discussion now centres on whether $2\sqrt{\varepsilon_n}\,lnN/\sqrt{\alpha}\ \geq d/4$ or $2\sqrt{\varepsilon_n}lnN/\sqrt{\alpha} \leq d/4$ should be used. In the first case $1/\varepsilon_n \leq C\,(lnN)^2$ and the result follows at once from (13) and (34). In the second case $\sigma_n = 2\sqrt{\varepsilon_n}lnN/\sqrt{\alpha}$. Suppose that k satisfies $N/8 \leq k \leq 3N/8$. Then $h_k = 2(d-2\sigma_n)/N$ and therefore

$$\frac{h_k}{\varepsilon_n} = 2N^{-1}\frac{d-2\sigma_n}{\varepsilon_n},$$

$\sigma_n \leq 1-x_k$, and so

$$e^{-\sqrt{\alpha}(1-x_k)/\sqrt{\varepsilon_n}} \leq e^{-\sqrt{\alpha}\sigma_n/\sqrt{\varepsilon_n}} = e^{-2\,lnN} = N^{-2}. \tag{35}$$

Using (35) and (13) in (34) gives the required result.
On the other hand, if k satisfies $1 \leq k \leq N/8$ and $3N/8 \leq k \leq N/2$ and $r=n-1,\dots,1$, then the discussion now centres on whether $2\sqrt{\varepsilon_r}\,lnN/\sqrt{\alpha} \geq r\sigma_{r+1}/r+1$ or $2\sqrt{\varepsilon_r}lnN/\sqrt{\alpha} \leq r\sigma_{r+1}/r+1$should be used.
In the first case $1/\varepsilon_r \leq C\,(lnN)^2$ and the result follows at once from (13) and (34).
In the second case $\sigma_r = 2\sqrt{\varepsilon_r}lnN/\sqrt{\alpha}$ and $s=1,\dots,n-2$.

1. Suppose that k satisfies $N/8(s+1) \leq k \leq N/8(s)$ and $d-(N/8(s)) \leq k \leq d-(N/8(s+1))$. Then $h_k = 8n(\sigma_{r+1}-\sigma_r)/N$ or $8n(\sigma_r-\sigma_{r+1})/N$ and $\sigma_r \leq 1-x_k$ therefore

$$\frac{h_k}{\varepsilon_r} = 8nN^{-1}\frac{\sigma_{r+1}-\sigma_r}{\varepsilon_r} \text{ or } 8nN^{-1}\frac{\sigma_r-\sigma_{r+1}}{\varepsilon_r}, \tag{36}$$

Using (36) and (13) in (34) gives the required result.

2. If k satisfies $1 \leq k \leq N/8(s+1)$ and $d-(N/8(s+1)) \leq k \leq N/2$ Then $h_k = 8n(\sigma_{r+1}-\sigma_r)/N$ or $8n(\sigma_{r+1}-\sigma_r)/N$ and therefore

$$\frac{h_k}{\varepsilon_r} = 8nN^{-1}\frac{(\sigma_{r+1}-\sigma_r)}{\varepsilon_r} \text{ or } 8nN^{-1}\frac{(\sigma_r-\sigma_{r+1})}{\varepsilon_r}, \tag{37}$$

Using (37) and (13) in (34) gives the required result.

3. Finally, suppose that $k = \{N/8(s), d-(N/8(s)), N/8n, d-(N/8n)\}$. Then

$$I_k \leq \Big(\int_{k-1}^{k} + \int_{k}^{k+1}\Big)|u'_{i,k}|dx < I_{k-1} + I_{k+1}$$

$$\leq C(N^{-1}lnN)^2$$

For $i = 1,\ldots,n,\ k = \frac{N}{2}+1,\ldots,N-1$, then the discussion now centres on whether $2\sqrt{\varepsilon_n}\,lnN/\sqrt{\alpha} \geq (1-d)/4$ or $2\sqrt{\varepsilon_n}lnN/\sqrt{\alpha} \leq (1-d)/4$ should be used. In the first case $1/\varepsilon_n \leq C\,(lnN)^2$ and the result follows at once from (13) and (34). In the second case $\tau_n = 2\sqrt{\varepsilon_n}lnN/\sqrt{\alpha}$. Suppose that k satisfies $5N/8 \leq k \leq 7N/8$. Then $h_k = 2(1-2\tau_n)/N$ and therefore

$$\frac{h_k}{\varepsilon_n} = 2N^{-1}\frac{1-2\tau_n}{\varepsilon_n},$$

$\tau_n \leq 1-x_k$, and so

$$e^{-\sqrt{\alpha}(1-x_k)/\sqrt{\varepsilon_n}} \leq e^{-\sqrt{\alpha}\tau_n/\sqrt{\varepsilon_n}} = e^{-2\,lnN} = N^{-2}. \tag{38}$$

Using (38) and (13) in (34) gives the required result.
On the other hand, if k satisfies $N/2 < k \leq 5N/8$ and $7N/8 \leq k < N$ and $r = n-1,\ldots,1$, then the discussion now centres on whether $2\sqrt{\varepsilon_r}\,lnN/\sqrt{\alpha} \geq r\tau_{r+1}/r+1$ or $2\sqrt{\varepsilon_r}lnN/\sqrt{\alpha} \leq r\tau_{r+1}/r+1$should be used.
In the first case $1/\varepsilon_r \leq C\,(lnN)^2$ and the result follows at once from (13) and (34).
In the second case $\tau_r = 2\sqrt{\varepsilon_r}lnN/\sqrt{\alpha}$ and $s = 1,\ldots,n-2$.

1. Suppose that k satisfies $5N/8(s+1) \leq k \leq 5N/8(s)$ and $1-(N/8(s)) \leq k \leq 1-(N/8(s+1))$. Then $h_k = 8n(\tau_{r+1}-\tau_r)/N$ or $8n(\tau_r-\tau_{r+1})/N$ and $\tau_r \leq 1-x_k$ therefore

$$\frac{h_k}{\varepsilon_r} = 8nN^{-1}\frac{\tau_{r+1}-\tau_r}{\varepsilon_r} \text{ or } 8nN^{-1}\frac{\tau_r-\tau_{r+1}}{\varepsilon_r}, \tag{39}$$

Using (39) and (13) in (34) gives the required result.

2. If k satisfies $N/2 < k \leq N/8(s+1)$ and $1-(N/8(s+1)) \leq k < N$ Then $h_k = 8n(\tau_{r+1}-\tau_r)/N$ or $8n(\tau_{r+1}-\tau_r)/N$ and therefore

$$\frac{h_k}{\varepsilon_r} = 8nN^{-1}\frac{(\tau_{r+1}-\tau_r)}{\varepsilon_r} \text{ or } 8nN^{-1}\frac{(\tau_r-\tau_{r+1})}{\varepsilon_r}, \tag{40}$$

Using (40) and (13) in (34) gives the required result.

3. Finally, suppose that $k = \{d+N/8(s), 1-(N/8(s)), d+N/8n, 1-(N/8n)\}$. Then

$$I_k \leq (\int\limits_{k-1}^{k} + \int\limits_{k}^{k+1})|u'_{i,k}|dx < I_{k-1} + I_{k+1}$$

$$\leq C(N^{-1}lnN)^2$$

For $k = \frac{N}{2}$, the source terms is assumed by

$$(\int\limits_{x_{k-1}}^{x_k} f_i(\frac{N}{2}-1) + \int\limits_{x_k}^{x_{k+1}} f_i(\frac{N}{2}+1))/2$$

$h_k = (h_{k-1}+h_{k+1})/2,\ h_{k-1} = (x_{k-2}-x_{k-1})$ and $h_{k+1} = (x_{k+1}-x_{k+2})$, $h_{k-1} = 8n(\sigma_{n-1}-\sigma_n)/N,\ \ h_{k+1} = 8n(\tau_1-\tau_2)/N$

$$\frac{h_k}{\varepsilon_i} = \frac{h_{k+1}+h_{k-1}}{2\varepsilon_i} = \frac{4nN^{-1}((\sigma_{n-1}-\sigma_n)+(\tau_1-\tau_2))}{\varepsilon_i} \tag{41}$$

Using (41) and (13) in (34) gives the required result.

□

7 Discretization error

Lemma 7.1. *Let $u^*_{i,k}$ be the $V_{i,k}$-interpolant of the solution $u_{i,k}$ of* (1) *and $U_{i,k}$ the solution of* (14). *Then*

$$\max_{i=1,\dots,n} ||U_{i,k} - u^*_{i,k}|_{\varepsilon_i,\overline{\Omega}^N} \leq C(N^{-1}lnN)^2,$$

where the constant C is independent of the parameters ε_i.

Proof. From the coercivity of $\beta_i(.,.)$ in Lemma 2.1 and since $U_{i,k} - u^*_{i.k} \in V_{i,k}$,

$$||U_{i,k} - u^*_{i,k}||^2_{\varepsilon_i,\overline{\Omega}^N} \leq C\beta_i(U_{i,k} - u^*_{i,k}, U_{i,k} - u^*_{i,k})$$

$$\leq C[\beta_i(U_{i,k} - u_{i,k}, U_{i,k} - u^*_{i,k}) + \beta_i(u_{i,k} - u^*_{i,k}, U_{i,k} - u^*_{i,k})].$$

Using Lemma 6.1, with $v_i = U_{i,k} - u^*_{i,k}$, then gives

$$||U_{i,j} - u^*_{i,k}||^2_{\varepsilon_i,\overline{\Omega}^N} \leq C(N^{-1}lnN)^2 ||U_{i,k} - u^*_{i,k}||_{\varepsilon_i,\overline{\Omega}^N},$$

Cancelling the common factor gives

$$||U_{i,k} - u^*_{i,k}||_{\varepsilon_i,\overline{\Omega}^N} \leq C(N^{-1}lnN)^2,$$

as required. □

Theorem 7.1. *Let $u_{i,k}$ be the solution of* (1) *and $U_{i,k}$ the solution of* (14). *Then*

$$\max_{i=1,\dots,n} ||U_{i,k} - u_{i,k}||_{\varepsilon_i,\overline{\Omega}^N} \leq C(N^{-1}lnN)^2,$$

where the constant C is independent of the parameters ε_i.

Proof. Since

$$||U_{i,k} - u_{i,k}||_{\varepsilon_i,\overline{\Omega}^N} \leq ||U_{i,k} - u^*_{i.k}||_{\varepsilon_i,\overline{\Omega}^N} + ||u^*_{i,k} - u_{i,k}||_{\varepsilon_i,\overline{\Omega}^N},$$

the result follows by combining Lemma 5.1 and Lemma 7.1. □

Theorem 7.2. *Let $u_{i,k}$ be the solution of* (1) *and $U_{i,k}$ the solution of* (14). *Then the following parameter uniform error estimate holds*

$$\max_{i=1,\dots,n} \sup_{0<\varepsilon_i\leq 1} ||U_{i,k} - u_{i,k}||_{\varepsilon_i,\overline{\Omega}^N} \leq C(N^{-1}lnN)^2,$$

where the constant C is independent of the parameters ε_i.

Proof. Since $\sigma_r \leq 2\sqrt{\varepsilon_r} lnN/\sqrt{\alpha}$, $r = n, \dots, 1$, consider k satisfies, $1 \leq k \leq N/8s$ and $(3N/8s) \leq k \leq \frac{N}{2}$, $s = 1, \dots, n-1$ on a neighbourhood of the boundary layers.
Using the Cauchy Schwarz inequality and Theorem 7.1,

$$|(U_{i,k} - u_{i,k})(x_k)| = |\int_{\Omega_k} (U_{i,k} - u_{i,k})(s) ds|$$

$$\leq (\frac{1}{\varepsilon_r} \int_{\Omega_k} 1^2 ds)^{1/2} (\varepsilon_r \int_{\Omega_k} |(U_{i,k} - u_{i,k})'(s)|^2 ds)^{1/2}$$

$$\leq \frac{\sigma_r}{\varepsilon_r} ||U_{i,k} - u_{i,k}||_{\varepsilon_r, \overline{\Omega}^N}$$

$$\leq C(N^{-1} lnN)^2. \quad (42)$$

On the other hand, Suppose that k satisfies $N/8 \leq k \leq 3N/8$, outside the boundary layers, $h_k \geq 1/N$ and so

$$|(U_{i,k} - u_{i,k})(x_k)|^2 \leq N h_k |(U_{i,k} - u_{i,k})(x_k)|^2$$

$$\leq N \sum_{k=N/8}^{\frac{3N}{8}} h_k |(U_{i,k} - u_{i,k})(x_k)|^2$$

$$\leq N ||U_{i,k} - u_{i,k}||^2_{l^2(\Omega^N)}.$$

Using Theorem 7.1 then leads to

$$|(U_{i,k} - u_{i,k})(x_k)| \leq ||U_{i,k} - u_{i,k}||_{l^2(\Omega^N)})$$

$$\leq C(N^{-1} lnN)^2. \quad (43)$$

Combining (42) and (43) as required results
Since $\tau_r \leq 2\sqrt{\varepsilon_r} lnN/\sqrt{\alpha}$, $r = n, \dots, 1$, consider k satisfies, $\frac{N}{2} < k \leq 5N/8s$ and $1 - (7N/8s) \leq k \leq N-1$, $s = 1, \dots, n-1$ on a neighbourhood of the boundary layers.
Using the Cauchy Schwarz inequality and Theorem 7.1,

$$|(U_{i,k} - u_{i,k})(x_k)| = |\int_{\Omega_k} (U_{i,k} - u_{i,k})(s) ds|$$

$$\leq (\frac{1}{\varepsilon_r} \int_{\Omega_k} 1^2 ds)^{1/2} (\varepsilon_r \int_{\Omega_k} |(U_{i,k} - u_{i,k})'(s)|^2 ds)^{1/2}$$

$$\leq \frac{\sigma_r}{\varepsilon_r}||U_{i,k} - u_{i,k}||_{\varepsilon_r,\overline{\Omega}^N}$$
$$\leq C(N^{-1}(lnN)^2. \tag{44}$$

On the other hand, Suppose that k satisfies $5N/8 \leq k \leq 7N/8$, outside the boundary layers, $h_k \geq 1/N$ and so

$$|(U_{i,k} - u_{i,k})(x_k)|^2 \leq Nh_k|(U_{i,k} - u_{i,k})(x_k)|^2$$
$$\leq N \sum_{k=N/4}^{\frac{3N}{4}} h_k|(U_{i,k} - u_{i,k})(x_k)|^2 \leq N||U_{i,k} - u_{i,k}||^2_{l^2(\Omega^N)}.$$

Using Theorem 7.1 then leads to

$$|(U_{i,k} - u_{i,k})(x_k)| \leq ||U_{i,k} - u_{i,k}||_{l^2(\Omega^N)})$$
$$\leq C(N^{-1}lnN)^2. \tag{45}$$

For $k = \frac{N}{2}$, $h_{\frac{N}{2}} = (h_{\frac{N}{2}-1} + h_{\frac{N}{2}+1})/2$, $h_{\frac{N}{2}-1} = (x_{\frac{N}{2}-2} - x_{\frac{N}{2}-1})$ and $h_{\frac{N}{2}+1} = (x_{\frac{N}{2}+1} - x_{\frac{N}{2}+2})$,

$$|(U_{i,\frac{N}{2}} - u_{i,\frac{N}{2}})(x_{\frac{N}{2}})|^2 \leq Nh_{\frac{N}{2}}|(U_{i,\frac{N}{2}} - u_{i,\frac{N}{2}})(x_{\frac{N}{2}})|^2$$
$$\leq N\frac{h_{\frac{N}{2}-1} + h_{\frac{N}{2}+1}}{2}|(U_{i,\frac{N}{2}} - u_{i,\frac{N}{2}})(x_{\frac{N}{2}})|^2 \leq N||U_{i,\frac{N}{2}} - u_{i,\frac{N}{2}}||^2_{l^2(\Omega^N)}.$$

Using Theorem 7.1 then leads to

$$|(U_{i,\frac{N}{2}} - u_{i,\frac{N}{2}})(x_{\frac{N}{2}})| \leq ||U_{i,\frac{N}{2}} - u_{i,\frac{N}{2}}||_{l^2(\Omega^N)})$$
$$\leq C(N^{-1}lnN)^2. \tag{46}$$

Combining (44) and (46) completes the proof. □

8 Numerical Illustrations

Example 8.1. *Consider the BVP*

$$-E\vec{u}''(x) + A(x)\vec{u}(x) = \vec{f}(x),\ \textit{for}\ x \in (0,1),\ \vec{u}(0) = \vec{0},\ \vec{u}(1) = \vec{0}$$

where $E = \text{diag}(\varepsilon_1,\ \varepsilon_2)$, $A = \begin{pmatrix} 5 & -1 \\ -1 & 5(x+1) \end{pmatrix}$, $\vec{f_1} = (1 + x^2,\ 2)^T$ $\vec{f_2} = (4,\ x^2)^T$. For various values of $\varepsilon_1,\ \varepsilon_2$ $N = 8k,\ k = 2^r,\ r = 3, \cdots, 8,\ d = 0.3$, and $\alpha = 1.9$,

Using the general methods from [Miller et al., 1996] , the ε uniform order of convergence and the ε uniform error constant are computed by applying fitted mesh method to the Example 8.1 shown in the Figure 1. In the following Table 8 outlines the conclusions.

η	Number of mesh points N				
	64	128	256	512	1024
2^{0}	0.7544E-03	0.1717E-03	0.6677E-04	0.2797E-04	0.1303E-04
2^{-2}	0.1786E-02	0.2975E-03	0.1115E-03	0.4510E-04	0.2050E-04
2^{-4}	0.3974E-02	0.7429E-03	0.1842E-03	0.7169E-04	0.3064-04
2^{-6}	0.8120E-02	0.1769E-02	0.3029E-03	0.1139E-03	0.4607E-04
2^{-8}	0.1492E-01	0.3948E-02	0.7378E-03	0.1837E-03	0.7132E-04
2^{-10}	0.2426E-01	0.8082E-02	0.1761E-02	0.3010E-02	0.1129E-03
2^{-12}	0.2426E-01	0.8082E-02	0.1761E-02	0.3010E-02	0.1129E-03
2^{-14}	0.2426E-01	0.8082E-02	0.1761E-02	0.3010E-02	0.1129E-03
D^N	0.2426E-01	0.8082E-02	0.1761E-02	0.3010E-02	0.1129E-03
p^N	0.1319E+01	0.1389E+01	0.1453E+01	0.1473E+01	
C_p^N	0.9233E+00	0.9053E+00	0.7898E+00	0.5031E+00	0.5032E+00
Computed order of $\vec{\varepsilon}$-uniform convergence, $p^* = 1.319$					
Computed $\vec{\varepsilon}$-uniform error constant, $C_{p^*}^N = 0.9233$					

Table 1: Values of $D_\varepsilon^N,\ D^N,\ p^N,\ p^*$ and $C_{p^*}^N$ for $\varepsilon_1 = \dfrac{\eta}{64},\ \varepsilon_2 = \dfrac{\eta}{16}$.

9 Conclusions

The research work presented in this article is based on the fundamental concept developed by [Miller et al., 1996]. They considered convection diffusion problems in one dimension. In this paper, second order parameter uniform convergence has been established for system of n second order differential equations of reaction diffusion type with discontinuous source terms. The proposed method can be extended to higher dimensional problems.

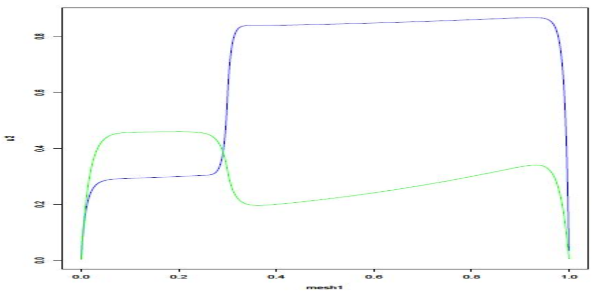

Figure 1: Graphical representation of solution for $\varepsilon = 2^{-4}$ and $N = 512$ of Example 8.1.

References

E. P. Doolan, J. J. H. Miller, and W. H. A. Schilders. *Uniform numerical methods for problems with initial and boundary layers*. Boole Press, Dublin, Ireland, 1980.

T. Linβ and N. Madden. Layer-adapted meshes for a linear system of coupled singularly perturbed reaction-diffusion problems. *IMA J. Num. Anal.*, 29:109–125, 2009.

J. J. H. Miller, E. O. Riordan, and G. I. Shishkin. *Fitted numerical methods for singular perturbation problems. Error estimates in the maximum norm for linear problems in one and two dimensions*. World scientific publishing CO.Pvt.Ltd., Singapore, 1996.

M. J. Paramasivam, S. Valarmathi, and J. J. H. Miller. Second order parameter uniform convergence for a finite difference method for a singularly perturbed linear reaction diffusion system. *Mathematical Communications*, 15:587–612, 2010.

M. J. Paramasivam, S. Valarmathi, and J. J. H. Miller. Second order parameter-uniform numerical method for a partially singularly perturbed linear system of reaction-diusion type. *Mathematical Communications*, 18:271–295, 2013.

M. J. Paramasivam, S. Valarmathi, and J. J. H. Miller. Parameter-uniform convergence for a finite difference method for a singularly perturbed linear reaction-diffusion system with discontinuous source terms. *Int. J. Numer. Anal. Model*, 11:385–399, 2014.

H. G. Roos, M. Stynes, and L. Tobiska. *Numerical methods for singularly perturbed differential equations*. Springer Verlag, 1996.