Ratio Mathematica Volume 48, 2023

Analytical solution of time-fractional N-dimensional Black-Scholes equation using LHPM

S. J. Ghevariya *
C. N. Patel †

Abstract

A famous Black-Scholes differential equation is used for pricing options in financial world which represents financial derivatives more closely. Option is one of the crucial financial derivatives. Sawangtong P., Trachoo K., Sawangtong W. and Wiwattanapataphee B. obtained analytical solution of Black-Scholes equation with two assets in the Liouville-Caputo time-fractional derivative sense using Laplace homotopy perturbation method (LHPM). The aim of this paper is to derive solution of Liouville-Caputo time-fractional Black-Scholes equation with n assets using LHPM. Numerical results shows that our approach gives very accurate results and our formulas are quite close to the plain vanilla options.

Keywords: Financial derivatives, European options, n-dimensional Black-Scholes Equation; Liouville-Caputo Fractional Derivative, Laplace homotopy perturbation method.

2020 AMS subject classifications: Primary: 91G20. Secondary: 35Q91. [1]

*Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar, Gujarat, India; sj_ghevariya@spuvvn.edu

†S. G. M. English Medium College of Commerce and Management, CVM University, Vallabh Vidyanagar, Gujarat, India; chetanbhai.patel@cvmu.edu.in

[1]Received on May 25, 2023. Accepted on December 10, 2023. Published on December 31, 2023. doi: 10.23755/rm.v48i0.1230. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

An agreement between two parties or financial institutes is called financial derivatives. Options, Forward contacts, Futures, etc. are examples of financial derivatives. Financial derivatives are used to manage risk associated with the underlying asset significantly. Actually, financial derivatives are used to transfer risk to another trader or financial institution. Options are without a doubt the main component of a derivative that is frequently utilized in the financial market. Options will help to modified portfolios of the investors. As a result, the concept of option trading has continually evolved. A contract between two parties to buy or sell an underlying asset at a specific future date and price is known as option. The specific future date is called expiry date of option while specific price is called the striking price of option. There are mainly two types of options, namely call and put options. Call (Put) option is an option in which holder of the option has the right to buy (sell) the underlying asset at a specific future date and price. The cost paid by the holder of option is known as option price or premium. Options exist in two styles in the financial market, namely American and European. American option can be exercised before expiry date while European option is exercised at the expiry date only. In 1973, Fisher Black and Myron Scholes derived a model for pricing options for plain vanilla payoffs. This model was recognized worldwide when Black and Scholes awarded a Nobel prize in 1997. Consequently, lots of researcher worked with different payoffs Dedania and Ghevariya [2013a,b], Ghevariya [2020], R. J. Haber, P. J. Schönbucher and Wilmott [1999], Haug [2007], P. Wilmott, S. Howison and Dewynne [1993]. Note that differential equation derived by Black and Scholes known as Black-Scholes differential equation and Black-Scholes model is a solution of that equation with initial and boundary conditions. The solution of the Black-Scholes equation becomes the intrinsic aim of numerous researchers. As a result, the Black-Scholes equation has been solved using a variety of approaches to obtain approximate or closed-form solutions with a variety of payoffs such as projected differential transform method (PDTM) S. O. Edeki , O. O. Ugberbor and Owoloko [2015], S. O. Edeki , R. M. Jena, O. P. Ogundile and Chakraverty [2021], Ghevariya [2021, 2022b], binomial method P. Wilmott, S. Howison and Dewynne [2002], homotopy perturbation method (HPM) Ghevariya [2022a], Mellin transform Fadugba and Nwozo [2016], Ghevariya [2019], Panini and Srivastav [2004], Yoon [2014], etc. The Black-Scholes equation is a second order linear equation having parabolic nature.

It can be seen that fractional calculus plays an important role for finding analytical solution of partial differential equations with initial conditions Miller and Ross [2003], Podlubny [1999]. Fractional calculus has been used in various fields such as finance, physics, engineering, etc. Fractional differential equations and its applications attracted many researchers in the field of Black-Scholes theory during the last four decades S. Kumar, D. Kumar and Singh [2014], Mehrdoust and Najafi [2017], Meng and Wang [2010], Song and Wang [2013], H. Zhang, F. Liu, I. Turner and Yang [2016]. Recently, various methods have been proposed to find solution of fractional differential equations namely, HPM, Adomian decomposition method (ADM), iteration method, Laplace homotopy perturbation method (LHPM), etc. LHPM is the powerful method to obtain explicit solution which will be transform fractional differential equations into algebraic equations and solving them we get solution using inverse Laplace transform. Many problems arise in real world with known physical interpretation. Liouville-Caputo fractional derivative is applicable to such problems because it has initial conditions same as the traditional differential equation. Let f be a real valued n-times differentiable function on $[a, t],\ t > a$ where $a \in \mathbb{R}$. Then for any $\alpha > 0,\ \alpha \notin \{1, 2, ..., n\}$ and $n \in \mathbb{N}$ with $n - 1 < \alpha < n$, the Liouville-Caputo fractional derivative of f is defined as

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\eta)^{n-\alpha-1} f^{(n)}(\eta) d\eta. $$

Many researcher worked with multi-dimensional Black-Scholes equation S. J. Ghevariya, C. N. Patel and Fadugba [2022], Guillaue [2019], J. Kim, T. Kim, J. Jo, Y. Choi, S. Lee, H. Hwang, M. Yoo and Jeong [2016], Prathumwan and Trachoo [2020], P. Sawangtong, K. Trachoo, W. Sawangtong and Wiwattanapataphee [2018], K. Trachoo, W. Sawangtong and Sawangtong [2017]. In this paper, the LHPM has been used to solve n-dimensional Black-Scholes equation with Liouville-Caputo fractional derivative sense. Note that LHPM is a method that combines Laplace transform and homotopy perturbation method. Using a Liouville-Caputo fractional derivative, Sawangtong P., Trachoo K., Sawangtong W., and Wiwattanapataphee B. found an analytical solution to the Black-Scholes equation for two assets P. Sawangtong, K. Trachoo, W. Sawangtong and Wiwattanapataphee [2018]. For European call and put options, we will provide an analytical solution to the time-fractional Black-Scholes equation with n assets. This paper is summarized as follows. Section-2 deals with the n-dimensional Black-Scholes differential equation with Liouville-Caputo fractional derivative. In section-3, we shall discuss about time- fractional LHPM. Using LHPM, we will solve the time-fractional Black-Scholes differential equation analytically in Section 4. We compare these formulas with the solution of known n-dimensional Black-Scholes equation. The conclusion is covered in the final section.

2 N-Dimensional Black-Scholes Equation with Liouville-Caputo Fractional Derivative

The n-dimensional Black-Scholes equation Joonglee and Yongsik [2013] for European call option is given by

$$\frac{\partial \mathcal{C}}{\partial \tau} + \frac{1}{2}\sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 \mathcal{C}}{\partial S_i \partial S_j} + r\sum_{i=1}^{n} S_i \frac{\partial \mathcal{C}}{\partial S_i} - r\mathcal{C} = 0 \tag{1}$$

with initial condition $\mathcal{C}(S_1, S_2, ..., S_n, \tau) = 0$, if at least one $S_i = 0$ for $1 \leq i \leq n$ & boundary conditions $\mathcal{C}(S_1, S_2, ..., S_n, T) = \max\Big(\sum_{i=1}^{n} a_i S_i - K, 0\Big)$ and $\mathcal{C}(S_1, S_2, ..., S_n, \tau) = \sum_{i=1}^{n} a_i S_i - Ke^{-r(T-\tau)}$, if at least one $S_i \to \infty$ for $1 \leq i \leq n$, where $\mathcal{C}(S_1, S_2, ..., S_n, \tau)$ is the price of call option at time τ, S_i is the value of i^{th} underlying asset, σ_i is the volatility, a_i is portion of the i^{th} underlying asset, r is the risk free interest rate, K is the striking price and ρ_{ij} is the correlation of i^{th} and j^{th} underlying assets for $i \leq i, j \leq n$.

Replacing $\mathcal{C}$ by $\mathcal{P}$ in Equation (1), we get Black-Scholes differential equation for European put option with initial condition $\mathcal{P}(S_1, S_2, ..., S_n, \tau) = Ke^{-r(T-\tau)}$, if at least one $S_i = 0$ for $1 \leq i \leq n$ & boundary conditions $\mathcal{P}(S_1, S_2, ..., S_n, T) = \max\Big(K - \sum_{i=1}^{n} a_i S_i, 0\Big)$ and $\mathcal{P}(S_1, S_2, ..., S_n, \tau) = 0$, if at least one $S_i \to \infty$ for $1 \leq i \leq n$, where $\mathcal{P}(S_1, S_2, ..., S_n, \tau)$ is the price of put option at time τ.

Now for each $i = 1, 2, ..., n$, taking $x_i = \ln(S_i) - \big(r - \frac{1}{2}\sigma_i^2\big)\tau, t = T - \tau$ and $\mathcal{C}(x_1, x_2, ..., x_n, \tau) = e^{-r(T-\tau)}\, v(x_1, x_2, ..., x_n, \tau)$, the Equation (1) reduces to

$$\frac{\partial v}{\partial t} - \frac{1}{2}\sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} = 0, \quad (x_1, x_2, ..., x_n, t) \in \Re^n \times [0, T] \tag{2}$$

with initial and boundary conditions for call option transformed as

$$v(x_1, x_2, ..., x_n, 0) = \max\Big(\sum_{i=1}^{n} \tilde{a}_i e^{x_i} - K, 0\Big), \tag{3}$$

$$v(x_1, x_2, ..., x_n, t) = 0, \text{ if at least one } x_i \to -\infty \text{ and} \tag{4}$$

$$v(x_1, x_2, ..., x_n, t) = \sum_{i=1}^{n} \tilde{a}_i e^{x_i + \frac{1}{2}\sigma_i^2 t} - K, \text{ if at least one } x_i \to \infty, \tag{5}$$

where $\tilde{a}_i = a_i e^{(r - \frac{1}{2}\sigma_i^2)T}$. By replacing Liouville-Caputo fractional derivative in Equation (2), we get time-fractional Black-Scholes equation for call option with $\alpha \in (0, 1]$

$$D_t^{\alpha} v = \frac{1}{2}\sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j}, \tag{6}$$

where initial and boundary conditions are given by Equations (3)-(5). Similarly, one can derive the time-fractional Black-Scholes equation for put option.

3 Introduction to Time Fractional LHPM

The HPM is introduced by He He [2003]. The basic idea about Laplace homotopy perturbation method (LHPM) was given by Khan and Wu Khan and Wu [2011]. Consider a general partial differential equation in n variables

$$G(v(x_1,x_2,...,x_n,t)) - g(x_1,x_2,...,x_n,t) = 0,\ t \in \Omega \tag{7}$$

with initial and boundary conditions are given by

$$v(x_1,x_2,...,x_n,0) = h(x_1,x_2,...,x_n) \text{ and } B\Big(v, \frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, ..., \frac{\partial v}{\partial x_n}, \frac{\partial v}{\partial t}\Big) = 0,$$

where $(x_1,x_2,...,x_n) \in \Re^n, \Omega$ is the domain, B is the boundary operator, G is a differential operator, $v(x_1,x_2,...,x_n,t)$ is to be determined and $g(x_1,x_2,...,x_n,t)$ is a known analytic function. Further, G can be split into two parts as given below.

$$G(v(x_1,x_2,...,x_n,t)) = D_t^\alpha v(x_1,x_2,...,x_n,t) + N(v(x_1,x_2,...,x_n,t)),$$

where D_t^α is the Liouville-Caputo fractional derivative with $\alpha \in (0,1]$, N is the remaining part by taking out the simple part with first term. Hence, Equation (7) can be written as

$$D_t^\alpha v(x_1,x_2,...,x_n,t) + N(v(x_1,x_2,...,x_n,t)) = g(x_1,x_2,...,x_n,t). \tag{8}$$

Now, applying the Laplace transform with respect to t on both sides of Equation (8), we get

$$\mathcal{L}[D_t^\alpha v(x_1,x_2,...,x_n,t)] + \mathcal{L}[N(v(x_1,x_2,...,x_n,t))] = \mathcal{L}[g(x_1,x_2,...,x_n,t)].$$

Using Laplace transform of Liouville-Caputo fractional derivative Miller and Ross [2003] in the above Equation, we get

$$\mathcal{L}[v(x_1,x_2,...,x_n,t)] = s^{-\alpha}h(x_1,x_2,...,x_n) - s^{-\alpha}\mathcal{L}[N(v(x_1,x_2,...,x_n,t))] \tag{9}$$
$$+ s^{-\alpha}\mathcal{L}[g(x_1,x_2,...,x_n,t)]. \tag{10}$$

Applying the inverse Laplace transform on both sides of Equation (9), we get

$$v(x_1,x_2,...,x_n,t) = A(x_1,x_2,...,x_n,t) - \mathcal{L}^{-1}[s^{-\alpha}\mathcal{L}[N(v(x_1,x_2,...,x_n,t))]], \tag{11}$$

where $A(x_1, x_2, ..., x_n, t)$ can be find using the initial and boundary conditions given in Equation (7). Consider a homotopy of Equation (11), $u(t,p) : \Omega \times [0,1] \to \mathbb{R}$ satisfying

$$\begin{aligned} H(u,p) =&(1-p)[u-\tilde{u}_0] + p\big[(u-A) \\ &+ \mathcal{L}^{-1}\big[s^{-\alpha}\mathcal{L}\big[N(v(x_1, x_2, ..., x_n, t; p))\big]\big]\big] = 0,\ t \in \Gamma, \end{aligned} \tag{12}$$

where Γ is boundary of the domain Ω, p is an embedding parameter and $\tilde{u_0}$ is the boundary condition given in Equation (7) which can be freely chosen. From Equation (12), we can see that

$$\begin{aligned} &H(u,0) = u - \tilde{u_0} = 0 \ and \\ &H(u,1) = u - A + \mathcal{L}^{-1}\big[s^{-\alpha}\mathcal{L}\big[N(v(x_1, x_2, ..., x_n, t; p))\big]\big] = 0. \end{aligned} \tag{13}$$

The Equation (13) shows that when p changes from 0 to 1, u changes from $\tilde{u_0}$ to v. Hence, the approximate solution of Equation (7) can be assumed to be

$$u(x_1, x_2, ..., x_n, t; p) = \sum_{n=0}^{\infty} p^n u_n(x_1, x_2, ..., x_n, t). \tag{14}$$

From Equations (12) and (14), we get

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n(x_1, x_2, ..., x_n, t) = &\tilde{u}_0(x_1, x_2, ..., x_n, t) \\ &- p\big[\tilde{u}_0(x_1, x_2, ..., x_n, t) - A(x_1, x_2, ..., x_n, t) \\ &+ \mathcal{L}^{-1}\big[s^{-\alpha}\mathcal{L}\big[N\big(\sum_{n=0}^{\infty} p^n u_n(x_1, x_2, ..., x_n, t)\big)\big]\big]\big]. \end{aligned} \tag{15}$$

By equating the coefficients of powers of p on both sides of Equation (15), we get

$$\begin{aligned} u_0(x_1, x_2, ..., x_n, t) &= \tilde{u}_0(x_1, x_2, ..., x_n, t), \\ u_1(x_1, x_2, ..., x_n, t) &= A(x_1, x_2, ..., x_n, t) - \tilde{u}_0(x_1, x_2, ..., x_n, t) \\ &\quad - \mathcal{L}^{-1}\big[s^{-\alpha}\mathcal{L}\big[N(\tilde{u}_0(x_1, x_2, ..., x_n, t))\big]\big] \\ u_m(x_1, x_2, ..., x_n, t) &= -\mathcal{L}^{-1}\big[s^{-\alpha}\mathcal{L}\big[N(\tilde{u}_{m-1}(x_1, x_2, ..., x_n, t))\big]\big], \\ &\qquad \forall\, m \geq 2. \end{aligned} \tag{16}$$

Thus, the approximate solution of Equation (7) will be

$$v(x_1, x_2, ..., x_n, t) = \lim_{p \to 1} \sum_{m=0}^{\infty} u_m(x_1, x_2, ..., x_n, t). \tag{17}$$

The convergence of the above series has been discussed in He [1999].

4 A Solution of Liouville-Caputo Time Fractional Black-Scholes Equation by LHPM

The use of LHPM to solve the Liouville-Caputo time fractional Black-Scholes equation for European call options with n variables is the topic of discussion in this section.

Theorem 4.1. *The solution of n-dimensional Black-Scholes equation discussed in Equation* (6)*, is given by*

$$\begin{aligned}
&\mathcal{C}(S_1, S_2, ..., S_n, \tau) = \\
&e^{-r(T-\tau)}\Bigg[\max\Big(\sum_{i=1}^{n} a_i S_i e^{(r-\frac{1}{2}\sigma_i^2)(T-\tau)} - K, 0\Big) + (T-\tau)^{\alpha} e^{-nr(T-\tau)}\Delta \\
&+\frac{1}{2}(T-\tau)^{\alpha}\sum_{i=1}^{n}\Big[\sigma_i^2 \max\big(a_i S_i e^{(r-\frac{1}{2}\sigma_i^2)(T-\tau)}, 0\big)\ E_{\alpha,\alpha+1}\Big(\frac{1}{2}\sigma_i^2 (T-\tau)^{\alpha}\Big)\Big] \\
&+\frac{1}{2}(T-\tau)^{2\alpha}\ \Gamma(\alpha+1)\ e^{-nr(T-\tau)}\ \Delta\ E_{\alpha,2\alpha+1}(w)\sum_{i,j=1}^{n}\sigma_i\sigma_j\rho_{ij} \\
&-(T-\tau)^{\alpha}\ \Gamma(\alpha+1)\ e^{-nr(T-\tau)}\Delta\ E_{\alpha,\alpha+1}(w)\Bigg],
\end{aligned}$$

where

$$\Delta = \prod_{i=1}^{n} S_i e^{\frac{1}{2}\sigma_i^2(T-\tau)} \ \ \textit{and}\ \ w = \frac{1}{2}(T-\tau)^{\alpha}\sum_{i,j=1}^{n}\sigma_i\sigma_j\rho_{ij}. \tag{18}$$

Proof. We have Liouville-Caputo time fractional Black-Scholes equation for n variables discussed in Equation (6) is given by

$$D_t^{\alpha} v = \frac{1}{2}\sum_{i,j=1}^{n}\sigma_i\sigma_j\rho_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j}, \tag{19}$$

where initial and boundary conditions are given by Equations (3)-(5). Consider

$$N(v(x_1, x_2, ..., x_n, t)) = \frac{1}{2}\sum_{i,j=1}^{n}\sigma_i\sigma_j\rho_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j}.$$

Thus, Equation (19) will be

$$D_t^{\alpha} v(x_1, x_2, ..., x_n, t) = N(v(x_1, x_2, ..., x_n, t)). \tag{20}$$

Now, applying Laplace transform with respect to time variable t on both sides of Equation (20), we get

$$\mathcal{L}\big[v(x_1, x_2, ..., x_n, t)\big] = s^{-\alpha} \max\Big(\sum_{i=1}^{n} \tilde{a}_i e^{x_i} - K, 0\Big) + \frac{1}{2} s^{-\alpha} \mathcal{L}\Big[\sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j}\Big]. \quad (21)$$

The inverse Laplace transform of Equation (21) is obtained as

$$v(x_1, x_2, ..., x_n, t) = \max\Big(\sum_{i=1}^{n} \tilde{a}_i e^{x_i} - K, 0\Big) + \frac{1}{2} \mathcal{L}^{-1}\Big[s^{-\alpha} \mathcal{L}\Big[\sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j}\Big]\Big]. \quad (22)$$

Applying a homotopy discussed in Equation (12) to the Equation (22), we get

$$(1-p)[v(x_1, x_2, ..., x_n, t; p) - \tilde{v}_0(x_1, x_2, ..., x_n, t)] + p\Big[v(x_1, x_2, ..., x_n, t; p) - \max\Big(\sum_{i=1}^{n} \tilde{a}_i e^{x_i} - K, 0\Big) - \frac{1}{2} \mathcal{L}^{-1}\Big[s^{-\alpha} \mathcal{L}\Big[\sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j}\Big]\Big]\Big] = 0, \quad (23)$$

where $p \in [0, 1]$ is an embedding parameter and $\tilde{v}_0(x_1, x_2, ..., x_n, t)$ can be chosen independently. Here we choose

$$\tilde{v}_0(x_1, x_2, ..., x_n, t) = \max\Big(\sum_{i=1}^{n} \tilde{a}_i e^{x_i} - K, 0\Big) + t^\alpha \prod_{i=1}^{n} e^{x_i}. \quad (24)$$

Hence, Equation (23) reduces to

$$v(x_1, x_2, ..., x_n, t; p) = \max\Big(\sum_{i=1}^{n} \tilde{a}_i e^{x_i} - K, 0\Big) + t^\alpha \prod_{i=1}^{n} e^{x_i} - p\Big[t^\alpha \prod_{i=1}^{n} e^{x_i} - \frac{1}{2} \mathcal{L}^{-1}\Big[s^{-\alpha} \mathcal{L}\Big[\sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j}\Big]\Big]\Big]. \quad (25)$$

From HPM, the solution of Equation (19) can be assumed as

$$v(x_1, x_2, ..., x_n, t; p) = \sum_{m=0}^{\infty} p^m u_m(x_1, x_2, ..., x_n, t). \quad (26)$$

Thus, Equations (25) and (26), yields

$$\sum_{m=0}^{\infty} p^m u_m(x_1, x_2, ..., x_n, t) = \max\Big(\sum_{i=1}^{n} \tilde{a}_i e^{x_i} - K, 0\Big) + t^\alpha \prod_{i=1}^{n} e^{x_i} - p\Big[t^\alpha \prod_{i=1}^{n} e^{x_i} - \frac{1}{2}\mathcal{L}^{-1}\Big[s^{-\alpha}\mathcal{L}\Big[\sum_{i,j=1}^{n} \sigma_i\sigma_j\rho_{ij} \sum_{m=0}^{\infty} p^m \frac{\partial^2 u_m}{\partial x_i \partial x_j}\Big]\Big]\Big]. \tag{27}$$

By equating the coefficients of powers of p on both sides of Equation (27), we get

$$u_0(x_1, x_2, ..., x_n, t) = \max\Big(\sum_{i=1}^{n} \tilde{a}_i e^{x_i} - K, 0\Big) + t^\alpha \prod_{i=1}^{n} e^{x_i}$$

$$u_m(x_1, x_2, ..., x_n, t) = \frac{t^{m\alpha}}{2^m \Gamma(m\alpha + 1)} \Big(\sum_{i=1}^{n} \sigma_i^{2m} \max(\tilde{a}_i e^{x_i}, 0)\Big) + \frac{t^{(m+1)\alpha}\Gamma(\alpha+1)}{2^m \Gamma((m+1)\alpha + 1)} \prod_{i=1}^{n} e^{x_i} \Big(\sum_{i,j=1}^{n} \sigma_i\sigma_j\rho_{ij}\Big)^m - \frac{t^{m\alpha}\Gamma(\alpha+1)}{2^{m-1}\Gamma(m\alpha+1)} \prod_{i=1}^{n} e^{x_i} \Big(\sum_{i,j=1}^{n} \sigma_i\sigma_j\rho_{ij}\Big)^{m-1}, \ \forall\, m \geq 1.$$

Substituting these values in Equation (26), we obtain

$$v(x_1, x_2, ..., x_n, t; p) = \max\Big(\sum_{i=1}^{n} \tilde{a}_i e^{x_i} - K, 0\Big) + t^\alpha \prod_{i=1}^{n} e^{x_i} + \sum_{m=0}^{\infty} p^{m+1} \Big[\frac{t^{(m+1)\alpha}}{2^{m+1}\Gamma((m+1)\alpha+1)} d + \frac{t^{(m+2)\alpha}\Gamma(\alpha+1)}{2^{m+1}\Gamma((m+2)\alpha+1)} \prod_{i=1}^{n} e^{x_i} \Big(\sum_{i,j=1}^{n} \sigma_i\sigma_j\rho_{ij}\Big)^{m+1} - \frac{t^{(m+1)\alpha}\Gamma(\alpha+1)}{2^m \Gamma((m+1)\alpha+1)} \prod_{i=1}^{n} e^{x_i} \Big(\sum_{i,j=1}^{n} \sigma_i\sigma_j\rho_{ij}\Big)^{m}\Big].$$

where $d = \sum_{i=1}^{n} \sigma_i^{2(m+1)} \max(\tilde{a}_i e^{x_i}, 0)$

Taking $p \to 1$ in above Equation and simplifying, we get

$$\begin{aligned} v(x_1, x_2, , ..., x_n, t) =& \max\Big(\sum_{i=1}^{n} \tilde{a}_i e^{x_i} - K, 0\Big) + t^\alpha \prod_{i=1}^{n} e^{x_i} \\ &+ \frac{t^\alpha}{2} \sum_{i=1}^{n} \left(\sigma_i^2 \ \max(\tilde{a}_i e^{x_i}, 0)\ E_{\alpha,\alpha+1}\Big(\frac{1}{2}\sigma_i^2 t^\alpha\Big)\right) \\ &+ \frac{t^{2\alpha}}{2} \Gamma(\alpha+1) \prod_{i=1}^{n} e^{x_i} E_{\alpha,2\alpha+1}\left(\frac{t^\alpha}{2}\delta\right)\delta \\ &- t^\alpha \Gamma(\alpha+1) \prod_{i=1}^{n} e^{x_i} E_{\alpha,\alpha+1}\left(\frac{t^\alpha}{2}\delta\right), \end{aligned} \tag{28}$$

where $E_{a,b}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(ak+b)}$ is the Mittag-Leffler function, with $a > 0$ and $b \in \mathbb{R}$ and $\delta = \sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij}$. But $x_i = \ln(S_i) - \big(r - \frac{1}{2}\sigma_i^2\big)\tau, \mathcal{C}(x_1, x_2, ..., x_n, \tau) = e^{-r(T-\tau)} v(x_1, x_2, ..., x_n, \tau)$ and $t = T - \tau$, Equation (28) can be written as

$$\begin{aligned} &\mathcal{C}(S_1, S_2, ..., S_n, \tau) = \\ &e^{-r(T-\tau)} \Bigg[\max\Big(\sum_{i=1}^{n} a_i S_i e^{(r-\frac{1}{2}\sigma_i^2)(T-\tau)} - K, 0\Big) + (T-\tau)^\alpha e^{-nr(T-\tau)} \Delta \\ &+ \frac{1}{2}(T-\tau)^\alpha \sum_{i=1}^{n} \Big[\sigma_i^2 \max\big(a_i S_i e^{(r-\frac{1}{2}\sigma_i^2)(T-\tau)}, 0\big)\ E_{\alpha,\alpha+1}\Big(\frac{1}{2}\sigma_i^2 (T-\tau)^\alpha\Big)\Big] \\ &+ \frac{1}{2}(T-\tau)^{2\alpha}\ \Gamma(\alpha+1)\ e^{-nr(T-\tau)}\ \Delta\ E_{\alpha,2\alpha+1}(w) \sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij} \\ &- (T-\tau)^\alpha\ \Gamma(\alpha+1)\ e^{-nr(T-\tau)} \Delta\ E_{\alpha,\alpha+1}(w) \Bigg], \end{aligned}$$

where Δ and w are given in Equation (18). □

Similarly, one can derive solution of Liouville-Caputo time fractional Black-Scholes equation for n variables of put option using LHPM. This formula is stated without proof.

Theorem 4.2. *The solution of n-dimensional Black-Scholes equation for European put option discussed in Section 2, is given by*

$$\begin{aligned}
&\mathcal{P}(S_1, S_2, ..., S_n, \tau) = \\
&e^{-r(T-\tau)}\Bigg[\max\Big(K - \sum_{i=1}^{n} a_i S_i e^{(r-\frac{1}{2}\sigma_i^2)(T-\tau)}, 0\Big) + (T-\tau)^{\alpha} e^{-nr(T-\tau)}\Delta \\
&+\frac{1}{2}(T-\tau)^{\alpha}\sum_{i=1}^{n}\Big[\sigma_i^2 \max\big(-a_i S_i e^{(r-\frac{1}{2}\sigma_i^2)(T-\tau)}, 0\big)\ E_{\alpha,\alpha+1}\Big(\frac{1}{2}\sigma_i^2 (T-\tau)^{\alpha}\Big)\Big] \\
&+\frac{1}{2}(T-\tau)^{2\alpha}\ \Gamma(\alpha+1)\ e^{-nr(T-\tau)}\ \Delta\ E_{\alpha,2\alpha+1}(w)\sum_{i,j=1}^{n}\sigma_i\sigma_j\rho_{ij} \\
&-(T-\tau)^{\alpha}\ \Gamma(\alpha+1)\ e^{-nr(T-\tau)}\Delta\ E_{\alpha,\alpha+1}(w)\Bigg],
\end{aligned}$$

where Δ and w are given in Equation (18)*.*

Theorem 4.3. *S. J. Ghevariya, C. N. Patel and Fadugba [2022] Using the Laplace transform homotopy perturbation method (LTHPM), the n-dimensional Black-Scholes model for European options is given by*

$$\begin{aligned}
\mathcal{C}_{\text{LTHPM}} &= \max\Big(\sum_{i=1}^{n} a_i S_i - K, 0\Big) \\
&+ \sum_{\substack{i,j=1 \\ i<j}}^{n} \rho_{ij} S_i^2\Big((\sigma_i^2 + r)E_{2,\sigma_i^2+r}(T-\tau) - E_{1,\sigma_i^2+r}(T-\tau) + (T-\tau)\Big) \\
&+ \sum_{\substack{i,j=1 \\ i<j}}^{n} \rho_{ij} S_j^2\Big((\sigma_j^2 + r)E_{2,\sigma_j^2+r}(T-\tau) - E_{1,\sigma_j^2+r}(T-\tau) + (T-\tau)\Big) \\
&- rE_{1,-r}(T-\tau)\Big(\sum_{i=1}^{n}\max(a_i, 0)S_i + \max\Big(\sum_{i=1}^{n} a_i S_i - K, 0\Big)\Big),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{P}_{\text{LTHPM}} &= \max\Big(K-\sum_{i=1}^{n}a_iS_i,0\Big)\\
&+ \sum_{\substack{i,j=1\\ i<j}}^{n}\rho_{ij}S_i^2\Big((\sigma_i^2+r)E_{2,\sigma_i^2+r}(T-\tau)-E_{1,\sigma_i^2+r}(T-\tau)+(T-\tau)\Big)\\
&+ \sum_{\substack{i,j=1\\ i<j}}^{n}\rho_{ij}S_j^2\Big((\sigma_j^2+r)E_{2,\sigma_j^2+r}(T-\tau)-E_{1,\sigma_j^2+r}(T-\tau)+(T-\tau)\Big)\\
&- rE_{1,-r}(T-\tau)\Big(\sum_{i=1}^{n}\max(-a_i,0)S_i-\max\Big(K-\sum_{i=1}^{n}a_iS_i,0\Big)\Big),
\end{aligned}$$

where $E_{a,b}(t)=t^a\sum_{k=0}^{\infty}\frac{(bt)^k}{\Gamma(a+k+1)}$ *is the Mellin-Ross function with* $a,b>0$.

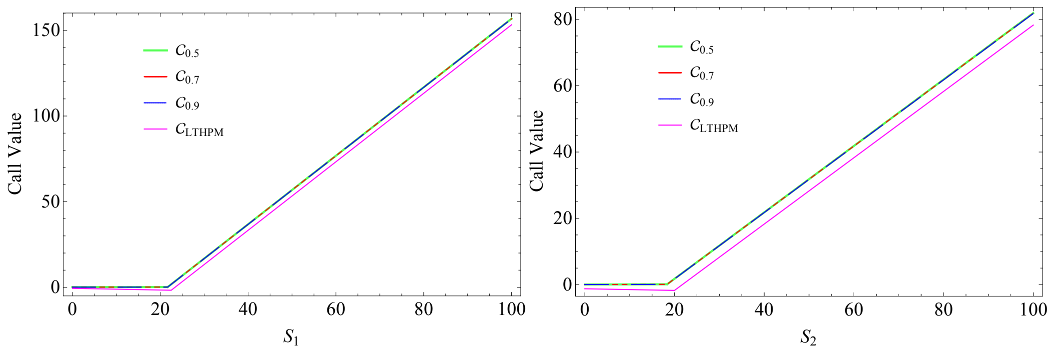
Theorem 4.4. *[Haug, 2007, P.2] The Black-Scholes model for plain vanilla payoffs are given by*

$$\begin{aligned}
\mathcal{C}_{\text{Vanilla}} &= SN(d_1)-Ke^{-r(T-t)}N(d_2)\\
\mathcal{P}_{\text{Vanilla}} &= Ke^{-r(T-t)}N(d_2)-SN(-d_1)
\end{aligned}$$

where $N(\cdot)$ *is cumulative distribution function of standard normal random variable and* $d_1=\frac{\ln(S/K)-(r-\frac{1}{2}\sigma^2)}{\sigma\sqrt{T-t}}$, $d_2=d_1-\sigma\sqrt{T-t}$.

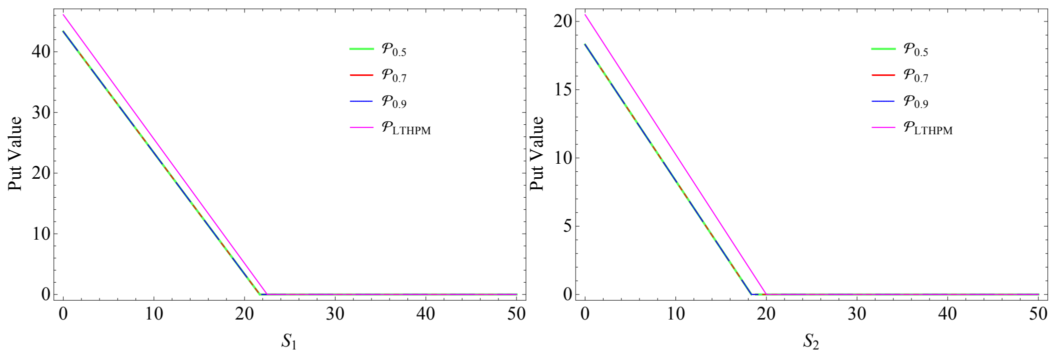
5 An Illustrative Example

In this section, we consider Black-Scholes models using LTHPM and Liouville-Caputo time fractional derivative sense for $n=2$. Further, the comparisons of these models for $n=1$ with the Black-Scholes model for plain vanilla payoffs have been discussed. We examine call and put values against different values of α $(0.5, 0.7, 0.9)$ and asset prices. For that, take the striking price $K=70$, the risk free interest rate, $r=0.05$, $a_1=2, a_2=1, \sigma_1=0.05, \sigma_2=0.1, t=0$ and time to expiration, $T=0.5$. In Figures 1 & 2, we consider the correlation coefficient, $\rho=0.25$. Figure 1(a) (Figure 2(a)) represents values of call (put) option against values of S_1 by taking $S_2=25$, while Figure 1(b) (Figure 2(b)) represents values of call (put) option against values of S_2 by taking $S_1=25$. Figures 3(a) & 3(b) represents values of call and put option against values of S_1 by taking $a_1=1$. Further, it can be seen from Figures 1 & 2 that the variations of call and put values for different values of α with the values from LTHPM are not far away. Also, from Figure 3, we observe that the model using Liouville-Caputo time fractional derivative sense for $\alpha = 0.9$ coincide with the model for plain vanilla payoffs. This suggest that our model is more accurate than the model using LTHPM.



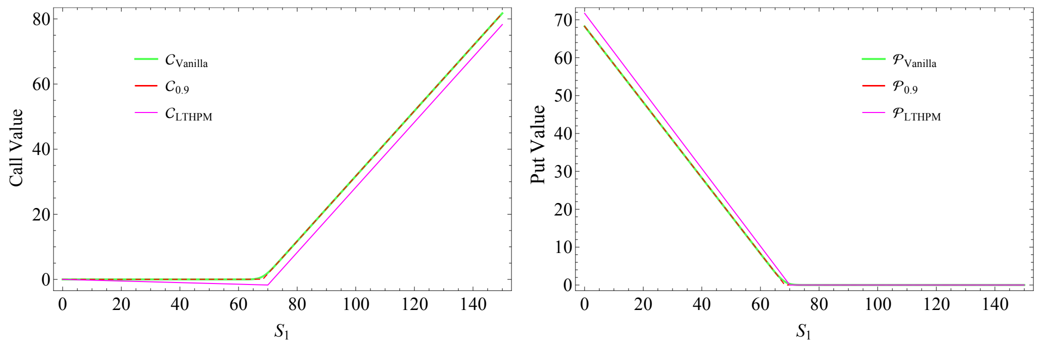
(a) $S_2 = 25$ (b) $S_1 = 25$

Figure 1: Call value with $n = 2$



(a) $S_2 = 25$ (b) $S_1 = 25$

Figure 2: Put value with $n = 2$



(a) Call (b) Put

Figure 3: $\mathcal{C}_{\text{Vanilla}}$, $\mathcal{C}_0.9$ and $\mathcal{C}_{\text{LTHPM}}$ with $n = 1$

Table 1 shows the relative errors of call option prices for $n = 2$ from Theorem 4.1 of different values of α with values from Theorem 4.3. For that take $K = 70, r = 0.05, \sigma_1 = 0.05, \sigma_2 = 0.1, \rho_{12} = 0.25, a_1 = 2, a_2 = 1, S_1 = 100, S_2 = 25, T = 0.5, t = 0$. From Theorem 4.3, we get $\mathcal{C}_{\text{LTHPM}} = 153.272$.

α	$\mathcal{C}_\alpha$	Error (%)
0.1	156.909	2.318
0.2	156.897	2.310
0.3	156.881	2.300
0.4	156.861	2.288
0.5	156.840	2.275
0.6	156.818	2.261
0.7	156.795	2.247
0.8	156.772	2.233
0.9	156.750	2.219
1	156.728	2.205

Table 1: Relative error $\mathcal{C}_\alpha$ and $\mathcal{C}_{\text{LTHPM}}$ with $n = 2$

Further, for put option, consider $S_1 = 10, S_2 = 12$. From Theorem 4.2 and Theorem 4.3, we get $\mathcal{P}_{\text{LTHPM}} = 38.938$ and $\mathcal{P}_\alpha = 36.314$. Hence the relative error for put option is 7.226%. Table 2 shows the the relative errors of call option prices for $n = 1$ from Theorems 4.1 & 4.4 of different values of α with values from Theorem 4.3. For that take $K = 70, r = 0.05, \sigma = 0.05, a_1 = 1, S = 150, T = 0.5, t = 0$. From Theorem 4.3 & 4.4, we get $\mathcal{C}_{\text{LTHPM}} = 153.272$ and $\mathcal{C}_{\text{Vanilla}} = 81.728$.

α	C_α	Error (%)	
		$\mathcal{C}_\alpha$ & $\mathcal{C}_{\text{Vanilla}}$	$\mathcal{C}_\alpha$ & $\mathcal{C}_{\text{LTHPM}}$
0.1	81.819	0.110	4.335
0.2	81.813	0.103	4.328
0.3	81.804	0.093	4.318
0.4	81.795	0.081	4.307
0.5	81.784	0.068	4.295
0.6	81.773	0.055	4.282
0.7	81.762	0.041	4.268
0.8	81.750	0.027	4.255
0.9	81.739	0.013	4.242
1	81.728	0.000	4.229

Table 2: Relative error $\mathcal{C}_\alpha$, $\mathcal{C}_{\text{LTHPM}}$ and $\mathcal{C}_{\text{Vanilla}}$ with $n = 1$

Further, for put option, consider $S = 10$. From Theorems 4.2, 4.3 and 4.4, we get $\mathcal{P}_{\text{LTHPM}} = 61.481, \mathcal{P}_\alpha = 58.278$ and $\mathcal{P}_{\text{Vanilla}} = 58.272$. Hence the relative error for $\mathcal{P}_{\text{Vanilla}}$ and $\mathcal{P}_\alpha$ is 0.011%, while relative error for $\mathcal{P}_\alpha$ and $\mathcal{P}_{\text{LTHPM}}$ is 5.497%.

6 Discussion

The famous Black-Scholes model is used for valuing option price with single asset as given in Theorem 4.4 derived by Fischer Black and Myron Scholes in 1973. Many authors have tried to improve it and in 2017, K. Trachoo,W. Sawangtong and P. Sawangtong presented a formula for valuing options based on two assets using LTHPM K. Trachoo, W. Sawangtong and Sawangtong [2017]. In 2022, S. J. Ghevariya, C. N. Patel and S. E. Fadugba had generalized it for n assets S. J. Ghevariya, C. N. Patel and Fadugba [2022]. The researchers have also presented a formula for valuing option price with two assets using Liouville-Caputo fractional derivative Prathumwan and Trachoo [2020], P. Sawangtong, K. Trachoo, W. Sawangtong and Wiwattanapataphee [2018]. In this paper we derived the formula for valuing option price based on n assets using Liouville-Caputo fractional derivative.

For $n = 2$, examples were presented to analyze the option prices derived from Liouville-Caputo time fractional Black-Scholes equation and LTHPM given in Theorem 4.3 by considering different values of α, while these option prices compared with plain vanilla options given in Theorem 4.4 by considering $n = 1$. Figures 1 and 2 shows values of call and put options with different values of α against the underlying asset prices, in the context of fixed values of the parameters: K, r, a_1, a_2, σ_1, σ_2, T, ρ. Figure 3 indicates the value of option prices from Liouville-Caputo time fractional Black-Scholes equation, LTHPM and plain vanilla. It is observed from all Figures that the values of call option increase and the value of put option decrease linearly with values of asset prices.

7 Conclusion

Tables 1 & 2 show the performance, accuracy and validation of solution derived in the context of Liouville-Caputo fractional derivative sense with respect to LTHPM and plain vanilla options. Figures 1 & 2 indicate how close the call and put values derived by authors and LTHPM for different values of α. Also Figure 3 shows that our solution is quite close to the well known model for plain vanilla options for single asset.

The formulas derived in this paper are limited to assets paying no dividend. Some extensions of the methodology can be explored for further research instead of linear payoffs and assets paying no dividend.

References

H. V. Dedania and S. J. Ghevariya. Option Pricing Formula for Modified Log-payoff Function. *Inter. Journal of Mathematics and Soft Computing*, 3(2):129–140, 2013a.

H. V. Dedania and S. J. Ghevariya. Option Pricing Formulas for Fractional Polynomial Payoff Function. *Inter. Jr. of Pure and Applied Mathematical Sciences*, 6(1):43–48, 2013b.

S. E. Fadugba and C. R. Nwozo. Valuation of European Call Options via the Fast Fourier Transform and the Improved Mellin Transforml. *Jr. of Math. Finance*, 6:338–359, 2016.

S. J. Ghevariya. An improved Mellin transform approach to BSM formula of ML-payoff function. *Journal of Interdisciplinary Mathematics*, 22(6):863–871, 2019.

S. J. Ghevariya. BSM Model for ML-Payoff Function through PDTM. *Asian-European Jr. of Mathematics*, 13(1):2050024 (6 pages), 2020.

S. J. Ghevariya. PDTM Approach to Solve Black Scholes Equation for Powered MLPayoff Function. *Computational Methods for Differential Equations*, 10(2): 320–326, 2021.

S. J. Ghevariya. Homotopy Perturbation Method to Solve Black Scholes Differential Equation for ML-Payoff Function. *Journal of Interdisciplinary Mathematics*, 25(2):553–561, 2022a.

Sanjay. J. Ghevariya. PDTM approach to solve Black Scholes equation for powered ML-Payoff function. *Computational Methods for Differential Equation*, 10(2):320–326, 2022b.

T. Guillaue. On the multidimensionl Black-Scholes partial differential equation. *Annals of Operations Research*, 281:229–251, 2019.

H. Zhang, F. Liu, I. Turner and Q. Yang. Numerical solution of the time fractional Black-Scholes model governing European options. *Computers and Mathematics with Applications*, 71(9):1772–1783, 2016.

E. G. Haug. *The Complete Guide to Option Pricing Formulas*. McGraw-Hill, 2007.

J. H. He. Homotopy perturbation technique. *Computer Methods in Applied Mechanics and Engineering*, 178(3-4):257–262, 1999.

J. H. He. Homotopy perturbation method: a new nonlinear analytical technique. *Applied Mathematics and Computation*, 135(1):73–79, 2003.

J. Kim, T. Kim, J. Jo, Y. Choi, S. Lee, H. Hwang, M. Yoo and D. Jeong. A practical finite difference method for the three-dimensional Black-Scholes equation. *European Journal of Operational Research*, 252(1):183–190, 2016.

J. Joonglee and K. Yongsik. Comparision of numerical schemes on multi-dimensional Black-Scholes equations. *Bull. Korean Math. Soc.*, 6:2035–2051, 2013.

K. Trachoo, W. Sawangtong and P. Sawangtong. Laplace transform homotopy perturbation method for the two dimensional Black Scholes model with European call option. *Mathematical and computational applicatios*, 22(1):1–11, 2017.

Y. Khan and Q. Wu. Homotopy perturbation transform method for nonlinear equations using He's polynomials. *Comput. Math.Appl.*, 61(8):1963–1967, 2011.

F. Mehrdoust and A. R. Najafi. Pricing european options under fractional black-scholes model with a weak payoff function. *Computational Economics*, 52: 685–706, 2017.

L. Meng and W. Wang. Comparison of Black-Scholes formula with fractional Black-Scholes formula in the foreign exchange option market with changing volatility. *Asia-Pacific Financial Markets*, 17(2):99–111, 2010.

K. S. Miller and B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Jhon Willey & Sons, USA, 2003.

P. Sawangtong, K. Trachoo, W. Sawangtong and B. Wiwattanapataphee. The Analytic Solution for the Black-Scholes Equation with Two Assets in the Liouville-Caputo Fractional Derivative Sense. *Mathematics*, 6(8):129, 2018.

P. Wilmott, S. Howison and J. Dewynne. *Option Pricing: Mathematical Models and Computation*. Oxford Financial Press, England, 1993.

P. Wilmott, S. Howison and J. Dewynne. *The Mathematics of Financial Derivatives*. Cambridge University Press, 2002.

R. Panini and R. P. Srivastav. Option pricing with Mellin Transform. *Mathematical and Computer Modelling*, 40:43–56, 2004.

I. Podlubny. *Fractional Differential Equations*. Academic press, San Diego, 1999.

D. Prathumwan and K. Trachoo. On the solution of two-dimensional fractional Black-Scholes equation for European put option. *Advances in Difference Equations*, 146:9, 2020.

R. J. Haber, P. J. Schönbucher and P. Wilmott. Pricing Parisan Options. *Journal of Derivatives*, 6(3):71–79, 1999.

S. J. Ghevariya, C. N. Patel and S. E. Fadugba. Laplace Transform Homotopy Perturbation Method for Black-Scholes Multi-Assets Model. *Eurasian Journal of Mathematical and Computer Applications*, 10(4):62–72, 2022.

S. Kumar, D. Kumar and J. Singh. Numerical computation of fractional Black-Scholes equation arising in financial market. *Egyptian Journal of Basic and Applied Sciences*, 1(3-4):177–183, 2014.

S. O. Edeki , O. O. Ugberbor and E. A. Owoloko. Analytical Solutions of the Black-Scholes Pricing Model for European Option Valuation via a Projected Differential Transform Method. *Entropy*, 17:7510–7521, 2015.

S. O. Edeki , R. M. Jena, O. P. Ogundile and S. Chakraverty. PDTM for the solution of a time-fractional barrier option Black-Scholes model. *Journal of Physics, Conference Series*, page 1734, 2021.

L. Song and W. Wang. Solution of the fractional Black Scholes option pricing model by finite difference method. *Abstract and Applied Analysis*, page 10, 2013.

J. H. Yoon. Mellin Transform method for European option pricing with Hull-White stochastic interest rate. *Journal of Applied Mathematics*, page 7, 2014.