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Compatible mappings and its variants satisfying generalized $(\psi, \phi)-$weak contraction

Kavita [*], Sanjay Kumar[†], Rajesh Kumar [‡], Nikita [§]

Abstract

Banach contraction principle behaves as a mathematical tool to solve various practical problems arising during mathematical formulation of many theoretical problems. In present work, the existence of a unique common fixed point for pairs of minimal commutative mappings is discussed, which satisfy a generalized $(\psi, \phi)-$weak contraction involving cubic terms of distance functions. Examples are given in support of the obtained results and as an application the existence of solution of system of certain functional equations arising in dynamic programming is discussed.
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[*]Department of Mathematics, Deenbandhu Chhotu Ram University of Science and Technology, Murthal, Sonepat-131039, Haryana(India); kvtlather@gmail.com
[†]Department of Mathematics, Deenbandhu Chhotu Ram University of Science and Technology, Murthal, Sonepat-131039, Haryana(India); drsanjaykumar.math@dcrustm.org
[‡]Department of Mathematics, Institute of Higher Learning, BPS Mahila Vishwavidyalaya, Khanpur Kalan, Sonipat-131305, Haryana(India); rkdubaldhania@gmail.com
[§]Department of Mathematics, Deenbandhu Chhotu Ram University of Science and Technology, Murthal, Sonepat-131039, Haryana(India); kadyannikita2@gmail.com
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1 Introduction

Fixed point theory is an important tool in nonlinear analysis. Its importance lies in finding solutions of many problems of applied sciences, engineering and economics. Banach contraction principle [2] is the versatile result of fixed point theory which ensures the existence and uniqueness of a fixed point for every contraction mapping defined on a complete metric space. It is based on iteration process.

Over a hundred years, researchers have been making efforts to extend, generalize and improve the Banach fixed point theorem in various directions. Jungck [15] was the first to prove a common fixed point theorem for a pair of commuting mappings. This theorem opened the door for researchers to generalize the Banach contraction principle for pair/pairs of mappings. Sessa [28] made an attempt to relax commutative condition of mappings to weak commutative condition .

Further, Jungck [16] weakened the notion of commutative/weak commutative mappings to compatible mappings. In 1993, Jungck, Murthy and Cho [17] generalized the notion of compatible mappings to compatible mappings of type (A). The process of generalizing the concept of compatible mappings is still going on. Pathak and Khan [23], Pathak *et al.* [24, 25], Rohen and Singh [27], Singh and Singh [30] and Jha *et al.* [13] weakened this concept of compatible mappings to compatible mappings of type (B), type (P), type (C), type (R), type (E) and type (K) respectively. One can call the variants of compatible mappings as minimal commutative mappings.

In 1971, Ćirić [8] generalized Banach contraction principle for a self map defined on a metric space $(\mathcal{X}, d)$ (say) satisfying the following condition:

$$d(Tu, Tv) \leq k \max\{d(u, v), d(u, Tu), d(v, Tv), \frac{1}{2}[d(u, Tv) + d(v, Tu)]\}, \quad (1)$$

$0 < k < 1$.

In 2005, Singh and Jain [29] proved a fixed point theorem for pairs of compatible mappings along with weakly compatible mappings satisfying the contractive condition of type (1) as follows.

Theorem 1.1. *[29] Let $(\mathcal{X}, d)$ be a complete metric space and let f, g, L, M, S and T be self mappings on $\mathcal{X}$ such that*

(H_1) *$L(\mathcal{X}) \subset ST(\mathcal{X})$, $M(\mathcal{X}) \subset fg(\mathcal{X})$;*

(H_2) *$ST = TS$, $Lg = gL$, $fg = gf$, $MT = TM$;*

(H_3) *either fg or L is continuous;*

(H_4) *the pair (M, ST) is weakly compatible and the pair (L, fg) is compatible;*

(H_5) *for all* $u, v \in \mathcal{X}$ *and for some* k *,* $0 < k < 1$*,*

$$d(Lu, Mv) \leq k \max\{d(Lu, fgv), d(Mv, STv), d(fgu, STv), \frac{1}{2}[d(Lu, STv) + d(Lv, fgu)]\}.$$

Then L, M, S, T, f *and* g *have a unique common fixed point.*

In 1984, Khan *et al.* [20] gave the idea of altering distance/ control function as follows. An altering distance is an increasing and continuous function $\phi : [0, \infty) \to [0, \infty)$ vanishing only at zero.

Many researchers used the notion of control function having various properties to generalize Banach contraction principle. In this direction, Boyd and Wong [7] introduced ϕ contraction of the form $d(Tu, Tv) \leq \phi(d(u, v))$ for all $u, v \in \mathcal{X}$, where T is a self mapping defined on a complete metric space $\mathcal{X}$ and $\phi : [0, \infty) \to [0, \infty)$ is an upper semi continuous function from right such that $0 \leq \phi(t) < t$ for all $t > 0$. In 1997, Alber and Guerre-Delabriere [1] introduced $\phi-$weak contraction to generalize ϕ contraction in Hilbert spaces.Further, Rhoades [26] extended the result of [1] in the setting of complete metric space using the following contraction.

A self mapping T of a complete metric space $(\mathcal{X}, d)$ is said to be a $\phi-$ weak contraction if for each $u, v \in \mathcal{X}$, there exists a continuous non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(t) > 0$, for all $t > 0$ and $\phi(0) = 0$ such that

$$d(Tu, Tv) \leq d(u, v) - \phi(d(u, v)). \tag{2}$$

Dutta and Chaudhary [9] proved fixed point theorem for a self map satisfying $(\psi, \phi)-$weak contractive condition as follows.

Theorem 1.2. *Let* T *be a self mapping of a complete metric space* $\mathcal{X}$ *satisfying*

$$\psi(d(Tu, Tv)) \leq \psi(d(u, v)) - \phi(d(u, v)),$$

for all $u, v \in \mathcal{X}$ *and for some* ψ, ϕ*, where* $\phi, \psi : [0, +\infty) \to [0, +\infty)$ *such that* ϕ *is a lower semi continuous function and* $\phi^{-1}(0) = 0$ *and* ψ *is a non-decreasing continuous function with* $\psi^{-1}(0) = 0$*, then* T *has a unique fixed point in* $\mathcal{X}$*.*

In 2013, Murthy and Prasad [21] proved a fixed point theorem for a map satisfying a weak contraction involving cubic terms of distance functions.

Theorem 1.3. *[21] Let* T *be a self mapping of a complete metric space* $(\mathcal{X}, d)$ *and* $\phi : [0, \infty) \to [0, \infty)$ *be a continuous function with* $\phi(0) = 0$ *and* $\phi(t) > 0$*,* *for each $t > 0$ such that*

$$[1 + pd(u, v)]d^2(Tu, Tv) \leq p \max\Big\{\frac{1}{2}[d^2(u, Tu)d(v, Tv) + d(u, Tu)d^2(v, Tv)], d(u, Tu)d(u, Tv)d(v, Tu), d(u, Tv)d(v, Tu)d(v, Tv)\Big\} + m(u, v) - \phi(m(u, v)), \tag{3}$$

where

$$m(u, v) = \max\Big\{d^2(u, v), d(u, Tu)d(v, Tv), d(u, Tv)d(v, Tu), \frac{1}{2}[d(u, Tu)d(u, Tv) + d(v, Tu)d(v, Tv)]\Big\}, \tag{4}$$

p is a non negative real number. Then, map T has a unique fixed point in $\mathcal{X}$.

In 2022, Kavita and Kumar [19] introduced a generalized $(\psi, \phi)-$weak contraction involving cubic terms of distance functions and generalized the weak contraction (3).

Motivated by the result of Singh and Jain [29], we establish the existence and uniqueness of a fixed point for pairs of compatible mappings and variants of compatible mappings (type (A), type (B), type (C), type (P), type (R), type (K), type (E)) satisfying the generalized $(\psi, \phi)-$weak contraction involving cubic terms of distance functions. These results generalize the results of Jain *et al.*[10, 11, 12], Jung *et al.*[14], Kang *et al.*[18], Murthy and Prasad [21], Pathak *et al.* [24] and various results presented in the literature.

2 Preliminaries

First, we recall some definitions which will be needed in the sequel.

Definition 2.1. *Let $(\mathcal{X}, d)$ be a metric space. A pair (S, T) of self mappings defined on $\mathcal{X}$ is said to be*

(i) compatible [16] if and only if

$$\lim_{n\to\infty} d(STu_n, TSu_n) = 0,$$

(ii) compatible of type (A) [17] if

$$\lim_{n\to\infty} d(SSu_n, TSu_n) = 0 \textit{ and } \lim_{n\to\infty} d(TTu_n, STu_n) = 0,$$

(iii) compatible of type (P) *[23] if*

$$\lim_{n\to\infty} d(SSu_n, TTu_n) = 0,$$

(iv) compatible of type (B) *[23] if*

$$\lim_{n\to\infty} d(STu_n, TTu_n) \le \frac{1}{2}[\lim_{n\to\infty} d(STu_n, Sz) + \lim_{n\to\infty} d(Sz, SSu_n)],$$

and

$$\lim_{n\to\infty} d(TSu_n, SSu_n) \le \frac{1}{2}[\lim_{n\to\infty} d(TSu_n, Tz) + \lim_{n\to\infty} d(Tz, TTu_n)],$$

(v) compatible of type (C) *[25] if*

$$\lim_{n\to\infty} d(STu_n, TTu_n) \le \frac{1}{3}[\lim_{n\to\infty} d(STu_n, Sz) + \lim_{n\to\infty} d(Sz, SSu_n) + \lim_{n\to\infty} d(Sz, TTu_n)]$$

and

$$\lim_{n\to\infty} d(TSu_n, SSu_n) \le \frac{1}{3}[\lim_{n\to\infty} d(TSu_n, Tz) + \lim_{n\to\infty} d(Tz, TTu_n) + \lim_{n\to\infty} d(Tz, SSu_n)],$$

(vi) compatible of type (R) *[27] if*

$$\lim_{n\to\infty} d(STu_n, TSu_n) = 0$$

and

$$\lim_{n\to\infty} d(SSu_n, TTu_n) = 0,$$

(vii) compatible of type (E) *[30] if*

$$\lim_{n\to\infty} SSu_n = \lim_{n\to\infty} STu_n = Tz$$

and

$$\lim_{n\to\infty} TTu_n = \lim_{n\to\infty} TSu_n = Sz,$$

(viii) compatible of type (K) [13] if

$$\lim_{n\to\infty} SSu_n = Tz \text{ and } \lim_{n\to\infty} TTu_n = Sz,$$

whenever $\{u_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in \mathcal{X}$.

Now, we highlight the relationship among various types of compatible mappings.

Remark 2.1. *[6] Notions of compatible mappings and its variants are independent to each other. Although, all types of compatible mappings are equivalent to each other when one consider continuity of all mappings.*

Remark 2.2. *Notion of compatible mappings of type (R) is a combination of the notion of compatible mappings and compatible mappings of type (P), but it is stronger than compatible mappings and compatible mappings of type (P), see ([27, Example 1-2]).*

Remark 2.3. *If $Sz = Tz$, then compatible of type (E) implies compatible, compatible of type (A), type (B), type (C) and type (P), however the converse may not true (see [31, Example 2.4]).*

Remark 2.4. *If $Sz \neq Tz$, then compatible of type (E) is neither compatible nor compatible of type (A),type (C), type (P), (see [31, Example 2.3]).*

In 1998, Pant [22] introduced the notion of reciprocal continuous mappings as follows.

Definition 2.2. *[22] A pair (S, T) of self mappings of a metric space $(\mathcal{X}, d)$ is said to be reciprocal continuous, if $\lim_{n\to\infty} STu_n = Sz$ and $\lim_{n\to\infty} TSu_n = Tz$, whenever $\{u_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in \mathcal{X}$.*

Remark 2.5. *It is clear that a pair of continuous self mappings is reciprocal continuous, but the converse may not true (see [22]).*

In 2011, Singh and Singh [31] split the concept of compatible mappings of type (E) to the concept of $S-$compatible mappings of type (E) and $T-$ compatible mappings of type (E) and further, split the notion of reciprocal continuous to the notion of $S-$reciprocal continuous and $T-$reciprocal continuous.

Definition 2.3. *[31] Let $(\mathcal{X}, d)$ be a metric space and $S, T : \mathcal{X} \to \mathcal{X}$ be two mappings. The pair (S, T) is said to be*

(i) $S-$compatible of type (E), if $\lim_{n\to\infty} SSu_n = \lim_{n\to\infty} STu_n = Tz$,

(ii) $T-$compatible type (E), if $\lim_{n\to\infty} TTu_n = \lim_{n\to\infty} TSu_n = Sz$,

(iii) $S-$ reciprocal continuous, if $\lim_{n\to\infty} STu_n = Sz$,

(iv) $T-$ reciprocal continuous, if $\lim_{n\to\infty} TSu_n = Tz$,

whenever $\{u_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in \mathcal{X}$,

Remark 2.6. *Compatible of type (E) implies both $S-$compatible of type (E) and $T-$compatible of type (E), however, the converse may not true, see the example given below.*

Example 2.1. *Let $\mathcal{X} = [0, 5]$ and d be a usual metric. Let $S, T : \mathcal{X} \to \mathcal{X}$ be two mappings defined as $Su = 5, Tu = 1,$ for $u \in [0, \frac{5}{2}] - \{\frac{5}{4}\}$, $Su = 0, Tu = 5,$ for $u = \frac{5}{4}$ and $Su = \frac{5-u}{2}, Tu = \frac{u}{2}$, for $u \in (\frac{5}{2}, 5]$. Clearly, S and T are not continuous at $u = \frac{5}{2}, \frac{5}{4}$. Let us assume that $u_n \to \frac{5}{2}$, $u_n > \frac{5}{2}$, for all n. Then, $Su_n = \frac{5-u_n}{2} \to \frac{5}{4} = t$ and $Tu_n = \frac{u_n}{2} \to \frac{5}{4} = t$. Therefore, we have $SSu_n = S(\frac{5-u_n}{2}) = 5 \to 5,$ $STu_n = S(\frac{u_n}{2}) = 5 \to 5, Tt = 5$ and $TTu_n = T(\frac{u_n}{2}) = 1 \to 1,$ $TSu_n = T(\frac{5-u_n}{2}) = 1 \to 1,$ $St = 0$. Thus, the pair (S, T) is $S-$compatible of type (E), but not compatible of type (E).*

Remark 2.7. *The reciprocal continuity of the pair (S, T) implies both $S-$reciprocal continuity and $T-$ reciprocal continuity, however, the converse may not true, see example given below.*

Example 2.2. *Let $\mathcal{X} = [0, 5]$ and d be a usual metric. Let $S, T : \mathcal{X} \to \mathcal{X}$ be two mappings defined as $Su = 5, Tu = 0$ for $u \in [0, \frac{5}{2})$ and $Su = 5 - u, Tu = u,$ for $u \in [\frac{5}{2}, 5]$. Let $\{u_n\}$ be a sequence in $\mathcal{X}$ such that $u_n \to \frac{5}{2}$, $u > \frac{5}{2}$, for all n. Then $Su_n = 5-u_n \to \frac{5}{2}$, $Tu_n = u_n \to \frac{5}{2} = t$, $STu_n = S(u_n) = 5-u_n \to \frac{5}{2}$, $St = \frac{5}{2}$ and $TSu_n = T(5 - u_n) = 0 \to 0$, $Tt = \frac{5}{2}$. It follows that $\lim_{n\to\infty} STu_n = \frac{5}{2} = St$ and $\lim_{n\to\infty} TSu_n = 0 \neq Tt = \frac{5}{2}$. Therefore, the pair (S, T) is $S-$reciprocal continuous, but it is neither $T-$reciprocal continuous nor reciprocal continuous.*

Now, we present some prepositions which are useful for our work.
Let$(\mathcal{X}, d)$ be a metric space and (S, T) be a pair of self mappings of $\mathcal{X}$.

Proposition 2.1. *[17] Suppose the pair (S, T) is compatible of type (A). If either S or T is continuous, then (S, T) is compatible.*

Proposition 2.2. *[16, 27] Suppose the pair (S, T) is compatible or compatible of type (R). If*

(i) $Sz = Tz$, *then* $STz = SSz = TTz = TSz$, *for some* $z \in \mathcal{X}$.

(ii) $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, *for some* $z \in \mathcal{X}$, *then*

- *(a)* $\lim_{n\to\infty} TSu_n = Sz$, *if* S *is continuous at* z,
- *(b)* $\lim_{n\to\infty} STu_n = Tz$, *if* T *is continuous at* z,
- *(c)* $STz = TSz$ *and* $Sz = Tz$, *if* S *and* T *are continuous at* z.

Proposition 2.3. *[23, 24, 25] Suppose the pair (S, T) is compatible of type (B) or type (C) or type (P) on $\mathcal{X}$. If for some z in $\mathcal{X}$,*

(i) $Sz = Tz$, *then* $STz = SSz = TTz = TSz$.

(ii) $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, *then*

- *(a)* $\lim_{n\to\infty} TTu_n = Sz$, *if* S *is continuous at* z;
- *(b)* $\lim_{n\to\infty} SSu_n = Tz$, *if* T *is continuous at* z;
- *(c)* $STz = TSz$ *and* $Sz = Tz$, *if* S *and* T *are continuous at* z.

Proposition 2.4. *[31] Let $\{u_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in \mathcal{X}$. If one of the following conditions is satisfied:*

(i) the pair (S, T) is $S-$compatible of type (E) and $S-$ reciprocally continuous,

(ii) the pair (S, T) is $T-$compatible of type (E) and $T-$ reciprocally continuous,

Then (a) $Sz = Tz$ and (b) if there exists $t \in \mathcal{X}$ such that $St = Tt = z$, then $STt = TSt$.

3 Main Results

In this section, we establish the existence and uniqueness of fixed point for pairs of minimal commutative mappings satisfying generalized $(\psi, \phi)-$weak contraction involving cubic terms of distance functions, where $\psi \in \Psi$ and $\phi \in \Phi$,

Ψ is a collection of all non decreasing, upper semi continuous (in each coordinate variables) functions $\psi : [0, \infty)^4 \to [0, \infty)$ such that

$$\max\{\psi(t, t, 0, 0), \psi(0, 0, 0, t), \psi(0, 0, t, 0), \psi(t, t, t, t)\} \leq t, \text{ for each } t > 0.$$

and Φ is a collection of all continuous functions $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) > 0$ for each $t > 0$ and $\phi(0) = 0$.

Let $(\mathcal{X}, d)$ be a metric space and f, g, L, M, S and T be self mappings defined on $\mathcal{X}$ satisfying the following conditions:

(I) $f(\mathcal{X}) \subset LM(\mathcal{X})$ and $g(\mathcal{X}) \subset ST(\mathcal{X})$,

(II) $LM = ML$, $ST = TS$, $fT = Tf$ and $gM = Mg$,

(III) one of f, g, LM and ST is continuous,

(IV) for all $u, v \in \mathcal{X}$, there exists a function $\psi \in \Psi$, a function $\phi \in \Phi$ and a real number $p \geq 0$ such that

$$\begin{aligned}[1 + pd(STu, LMv)]d^2(fu, gv) \leq p\psi\Big(& d^2(STu, fu)d(LMv, gv),\\ & d(STu, fu)d^2(LMv, gv),\\ & d(STu, fu)d(STu, gv)d(LMv, fu),\\ & d(STu, gv)d(LMv, fu)d(LMv, gv)\Big)\\ & + m(STu, LMv) - \phi(m(STu, LMv)),\end{aligned} \tag{5}$$

where

$$\begin{aligned}m(STu, LMv) = \max\Big\{& d^2(STu, LMv),\\ & d(STu, fu)d(LMv, gv),\\ & d(STu, gv)d(LMv, fu),\\ & \frac{1}{2}[d(STu, fu)d(STu, gv) + d(LMv, fu)d(LMv, gv)]\Big\}.\end{aligned} \tag{6}$$

Let $u_0 \in \mathcal{X}$ be arbitrary point. Using condition (I), one can find $u_1, u_2 \in \mathcal{X}$ such that $fu_0 = LMu_1 = v_0$ and $gu_1 = STu_2 = v_1$. Continuing in this manner, one can construct sequences such that

$$v_{2n} = fu_{2n} = LMu_{2n+1} \quad \text{and} \quad v_{2n+1} = gu_{2n+1} = STu_{2n+2}, \tag{7}$$

for each $n = 0, 1, 2, 3 \ldots$

First, we prove a fixed point theorem for pairs of compatible mappings.

Theorem 3.1. *Let f, g, L, M, S and T be self mappings defined on a complete metric space $\mathcal{X}$ satisfying the conditions (I)-(IV). If pairs (f, ST) and (g, LM) are compatible, then f, g, L, M, S and T have a unique common fixed point in $\mathcal{X}$.*

Proof. In the view of [19, Theorem 2.3], the sequence $\{v_n\}$, defined by (7), is a Cauchy sequence in $\mathcal{X}$. Since $\mathcal{X}$ is a complete metric space, so the sequence $\{v_n\}$ converges to a point, say, $z \in \mathcal{X}$, as $n \to \infty$. Consequently, the sub sequences $\{fu_{2n}\}$, $\{STu_{2n}\}$, $\{gu_{2n+1}\}$, and $\{LMu_{2n+1}\}$ also converges to the same point $z \in \mathcal{X}$.

Case (i) If f is continuous, then $\{ffu_{2n}\}$ and $\{f(ST)u_{2n}\}$ converges to fz as $n \to \infty$. Since the pairs (f, ST) is compatible on $\mathcal{X}$, it follows from the Proposition 2.2(ii) that $\{(ST)fu_{2n}\}$ converges to fz as $n \to \infty$.

Step 1. We claim that $z = fz$.Taking $u = fu_{2n}$ and $v = u_{2n+1}$ in (5) and (6) and letting $n \to \infty$, we have

$$[1 + pd(fz, z)]d^2(fz, z) \leq p\psi(0, 0, 0, 0) + m(fz, z) - \phi(m(fz, z)),$$

where

$$\begin{aligned} m(fz, z) &= \max\Big\{d^2(fz, z), d(fz, z)d(z, z), d(fz, z)d(z, fz), \\ &\qquad \frac{1}{2}[d(fz, fz)d(fz, z) + d(z, fz)d(z, z)]\Big\} \\ &= d^2(fz,z). \end{aligned}$$

On simplifying, we have $d^2(fz, z) = 0$, i.e., $fz = z$. Since $f(\mathcal{X}) \subset LM(\mathcal{X})$, there exists a point $w \in \mathcal{X}$ such that $z = fz = LMw$.

Step 2. Taking $u = fu_{2n}$ and $v = w$ in (5) and (6) and letting $n \to \infty$, we get

$$\begin{aligned} [1 + pd(z, z)]d^2(z, gw) \leq p\psi\Bigg(& d^2(z, z)d(z, gw), d(z, z)d^2(z, gw), \\ & d(z, z)d(z, gw)d(z, z), d(z, gw)d(z, z)d(z, gw)\Bigg) \\ & + m(z, z) - \phi(m(z, z)), \end{aligned}$$

where

$$\begin{aligned} m(z, z) = \max\Bigg\{& d^2(z, z), d(z, z)d(z, gw), d(z, gw)d(z, z), \\ & \frac{1}{2}\big[d(z, z)d(z, gw) + d(z, z)d(z, gw)\big]\Bigg\} = 0. \end{aligned}$$

Using the property of ϕ, ψ in the above inequality, we have $d^2(z, gw) = 0$, i.e., $gw = z$. Since the pair (g, LM) is compatible on $\mathcal{X}$ and $LMw = gw = z$, therefore, by Proposition 2.2(i), we have $(LM)gw = g(LM)w$ and hence, $LMz = (LM)gw = g(LM)w = gz$.

Step 3. Taking $u = u_{2n}$ and $v = z$ in (5) and (6) and letting $n \to \infty$, we get

$$[1 + pd(z, gz)]d^2(z, gz) \leq p\psi(0, 0, 0, 0) + m(z, gz) - \phi(m(z, gz)),$$

where

$$m(z, gz) = \max\Big\{d^2(z, gz), d(z, z)d(LMz, gz), d(z, gz)d(gz, z), \frac{1}{2}[d(z, z)d(z, gz) + d(LMz, z)d(gz, gz)]\Big\} = d^2(z, gz).$$

On solving, we get $z = gz$. Since $g(\mathcal{X}) \subset ST(\mathcal{X})$, therefore, for this z there exists a point $x \in \mathcal{X}$ such that $z = gz = STx$.

Step 4. We claim that $z = fx$. For this, putting $u = x$ and $v = z$ in (5) and (6), we get

$$\begin{aligned}[1 + pd(STx, LMz)]d^2(fx, gz) \leq p\psi\Big(& d^2(STx, fx)d(LMz, gz), \\ & d(STx, fx)d^2(LMz, gz), \\ & d(STx, fx)d(STx, gz)d(LMz, fx), \\ & d(STx, gz)d(LMz, fx)d(LMz, gz)\Big) \\ & + m(STx, LMz) - \phi\big(m(STx, LMz)\big),\end{aligned}$$

where

$$\begin{aligned}m(STx, LMz) = \max\Big\{& d^2(STx, LMz), d(STx, fx)d(LMz, gz), \\ & d(STx, gz)d(LMz, fx), \frac{1}{2}\big[d(STx, fx)d(STx, gz) \\ & + d(LMz, fx)d(LMz, gz)\big]\Big\} = 0.\end{aligned}$$

After simplification, we conclude that $d^2(fx, z) = 0$. This gives that $fx = z$.

Since (f, ST) is compatible on $\mathcal{X}$ and $fx = STx = z$, therefore, by Proposition 2.2(i), we have $(ST)fx = f(ST)x$, i.e.,$STz = (ST)fx = f(ST)x = fz$.

Step 5. We claim that $Tz = z$. Substituting $u = Tz$, $v = u_{2n+1}$ in (5) and (6) and letting $n \to \infty$, we have

$$[1 + pd((ST)Tz, z)]d^2(fTz, z) \leq p\psi(0,0,0,0) + m((ST)Tz, z) - \phi(m((ST)Tz, z)),$$

where

$$m((ST)Tz, z) = \max\Big\{d^2((ST)Tz, z), d((ST)Tz, fTz)d(z, z), d((ST)Tz, z)d(z, fTz), \frac{1}{2}[d((ST)Tz, fTz)d((ST)Tz, z) + d(z, fTz)d(z, z)]\Big\} = d^2((ST)Tz, z).$$

Since $ST = TS$, $STz = z$, $fT = Tf$, $fz = z$, so $(ST)Tz = (TS)Tz = T(STz) = Tz$ and $fTz = Tfz = Tz$.

On Simplifying, the above inequality reduces to

$$pd^3(Tz, z) + \phi(d^2(Tz, z)) \leq 0,$$

which holds only for $d(Tz, z) = 0$, i.e., $Tz = z$. Also, $STz = z$ implies that $Sz = z$.

Step 6. Next we show that $Mz = z$. Substituting $u = u_{2n}$ $v = Mz$ in (5) and (6) and letting $n \to \infty$, we have

$$[1 + pd(z, (LM)Mz)]d^2(z, gMz) \leq p\psi(0,0,0,0) + m(z, (LM)Mz) - \phi(m(z, (LM)Mz),$$

where

$$m(z, (LM)Mz) = \max\Big\{d^2(z, (LM)Mz), d(z, z)d((LM)Mz, gMz), d(z, gMz)d((LM)Mz, z), \frac{1}{2}[d(z, z)d(z, gMz) + d((LM)Mz, z)d((LM)Mz, gMz)]\Big\} = d^2(z, (LM)Mz).$$

Since $LM = ML$, $LMz = z$, $gM = Mg$, $gz = z$, so $(LM)Mz = (ML)Mz = M(LMz) = Mz$ and $gMz = Mgz = Mz$.

On simplifying, we get

$$pd^3(z, Mz) + \phi(d^2(z, Mz)) \leq 0,$$

which holds only for $d(z, Mz) = 0$, i.e., $Mz = z$. Also, $LMz = z$ implies that $Lz = z$. Thus, z is common fixed point of f, g, S, T, L and M.

Case (ii) If g is continuous, one can complete the proof on the similar lines of case (i).

Case(iii) If ST is continuous. Then $\{(ST)(ST)u_{2n}\}$ and $\{(ST)fu_{2n}\}$ converges to STz as $n \to \infty$. Since the mappings ST and f are compatible on $\mathcal{X}$, it follows from the Proposition 2.2(ii) that $\{f(ST)u_{2n}\}$ converges to STz as $n \to \infty$.

Next, we claim that $STz = z$. For this, substituting $u = STu_{2n}$ and $v = u_{2n+1}$ in equation (5) and inequality (6) and letting $n \to \infty$ with the property of ϕ and ψ, we get

$$[1 + pd(STz, z)]d^2(STz, z) \leq p\psi(0, 0, 0, 0) + m(STz, z) - \phi(m(STz, z)),$$

where

$$\begin{aligned} m(STz, z) = \max\Big\{&d^2(STz, z), d(STz, STz)d(z, z), d(STz, z)d(z, STz), \\ &\frac{1}{2}[d(STz, STz)d(STz, z) + d(z, STz)d(z, z)]\Big\} \\ = {}& d^2(STz, z). \end{aligned}$$

Using the value of $m(STz, z)$ along with the property of ϕ and ψ, the above inequality reduces to $d^2(STz, z) = 0$, i.e., $STz = z$.

Next, we show that $fz = z$. Taking $u = z$ and $v = u_{2n+1}$ in (5) and (6) and letting $n \to \infty$, we get

$$\begin{aligned} [1 + pd(STz, z)]d^2(fz, z) \leq p\psi\Big(&d^2(STz, fz)d(z, z), d(STz, fz)d^2(z, z), \\ &d(STz, Sz)d(STz, z)d(z, fz), \\ &d(STz, z)d(z, fz)d(z, z)\Big) \\ &+ m(STz, z) - \phi\big(m(STz, z)\big), \end{aligned}$$

where

$$\begin{aligned} m(STz, z) = \max\Bigg\{&d^2(STz, z), d(STz, fz)d(z, z), d(STz, z)d(z, fz), \\ &\frac{1}{2}\big[d(STz, fz)d(STz, z) + d(z, fz)d(z, z)\big]\Bigg\} = 0. \end{aligned}$$

After simplification, one gets $d^2(fz, z) = 0$. This implies that $fz = z$. Therefore, $STz = z = fz$. Since $f(\mathcal{X}) \subset LM(\mathcal{X})$, there exists a point $w \in \mathcal{X}$ such that $z = fz = LMw$.

Now, we claim that $gw = z$. Substituting $u = z$ and $v = w$ in (5) and (6), we have

$$\begin{aligned}[1 + pd(STz, LMw)]d^2(fz, gw) \leq p\psi\Big(& d^2(STz, fz)d(LMw, gw),\\ & d(STz, fz)d^2(LMw, gw),\\ & d(STz, fz)d(STz, gw)d(LMw, fz),\\ & d(STz, gw)d(LMw, fz)d(LMw, gw)\Big)\\ & +m(STz, LMw) - \phi\big(m(STz, LMw)\big),\end{aligned}$$

where

$$\begin{aligned}m(STz, LMw) = \max\Bigg\{& d^2(STz, LMw), d(STz, fz)d(LMw, gw),\\ & d(STz, gw)d(LMw, fz), \frac{1}{2}\big[d(STz, fz)d(STz, gw)\\ & + d(LMw, fz)d(LMw, gw)\big]\Bigg\}\\ = \max\Bigg\{& d^2(z, z), d(z, z)d(z, gw), d(z, gw)d(z, z),\\ & \frac{1}{2}\big[d(z, z)d(z, gw) + d(z, z)d(z, gw)\big]\Bigg\} = 0.\end{aligned}$$

After simplification, we conclude that $gw = z$.

Since (g, LM) is a pair of compatible mappings in $\mathcal{X}$ and $LMw = z = gw$, therefore, by Proposition 2.2(i), we have $(LM)gw = g(LM)w$ and hence, $LMz = (LM)gw = g(LM)w = gz$.

Now, we claim that z is a fixed point of LM and g. For this, putting $u = v = z$ in (5) and (6), we have

$$\begin{aligned}[1 + pd(STz, LMz)]d^2(fz, gz) \leq p\psi\Big(& d^2(STz, fz)d(LMz, gz),\\ & d(STz, fz)d^2(LMz, gz),\\ & d(STz, Sz)d(STz, gz)d(LMz, fz),\\ & d(STz, gz)d(LMz, fz)d(LMz, gz)\Big)\\ & +m(STz, LMz) - \phi\big(m(STz, LMz)\big),\end{aligned}$$

where

$$m(STz, LMz) = \max\Big\{d^2(STz, LMz), d(STz, fz)d(LMz, gz), \\ d(STz, gz)d(LMz, fz), \\ \frac{1}{2}\big[d(STz, fz)d(STz, gz) \\ + d(LMz, fz)d(LMz, gz)\big]\Big\} = d^2(z, LMz).$$

On solving, the above inequality reduces to

$$[1 + pd(z, LMz)]d^2(z, LMz) \leq p\psi(0, 0, 0, 0) + d^2(z, LMz) - \phi(d^2(z, LMz)),$$

which gives $d^2(z, LMz) = 0$ and hence, $z = LMz = gz$.

Following steps 5 and 6 of case (i), we have $Tz = z$, $Mz = z$. Also, $STz = z$, $LMz = z$ imply that $Sz = z$ and $Lz = z$.

Case(iv) Suppose mapping LM is continuous. Following case (iii), one may obtain the desired result.

Uniqueness: Suppose $w_1 \neq w_2$ be two common fixed point of f, g, S, T, L and M. Putting $u = w_1$ and $v = w_2$ in (5) and (6) and solving, we get

$$[1 + pd(w_1, w_2)]d^2(w_1, w_2) \leq p\psi(0, 0, 0, 0) + m(w_1, w_2) - \phi\big(m(w_1, w_2)\big),$$

where

$$m(w_1, w_2) == \max\Big\{d^2(w_1, w_2), d(w_1, w_1)d(w_2, w_2), d(w_1, w_2)d(w_2, w_1), \\ \frac{1}{2}\big[d(w_1, w_1)d(w_1, w_2) + d(w_2, w_1)d(w_2, w_2)\big]\Big\} \\ = d^2(w_1, w_2).$$

Simplifying the above inequality, we get $d(w_1, w_2) = 0$, i.e., $w_1 = w_2$. Hence, mappings f, g, S, T, L and M have a unique common fixed point in $\mathcal{X}$. □

Now, we establish a common fixed point theorem for pairs of compatible mappings of type (A).

Theorem 3.2. *Let f, g,L,M, S and T be self mappings defined on a complete metric space $\mathcal{X}$ satisfying the conditions (I)-(IV). If pairs (f, ST) and (g, LM) are compatible of type (A), then f, g, L, M, S and T have a unique common fixed point in $\mathcal{X}$.*

Proof. Suppose f is continuous on $\mathcal{X}$ and pairs (f, ST) is compatible of type (A). By Proposition 2.1, (f, ST) is compatible. Then result follows easily from Theorem 3.1.

If g is assumed to be continuous, then by Proposition 2.1, (g, LM) being compatible of type (A) is compatible also. Then result follows from Theorem 3.1

Similarly, assuming ST or LM continuous, one can get the required result. □

Next, we establish a common fixed point theorem for pairs of compatible mappings of type (B).

Theorem 3.3. *Let f, g, L, M, S and T be self mappings defined on a complete metric space $\mathcal{X}$ satisfying the conditions (I)-(IV). If pairs (f, ST) and (g, LM) are compatible of type (B), then f, g, L, M, S and T have a unique common fixed point in $\mathcal{X}$.*

Proof. Following [19, Theorem 2.3], the sequence $\{v_n\}$, defined by (7), is a Cauchy sequence in $\mathcal{X}$. $\mathcal{X}$ being a complete metric space, the sequence $\{v_n\}$ converges to a point, say, $z \in \mathcal{X}$, as $n \to \infty$. Consequently, the sub sequences $\{fu_{2n}\}$, $\{STu_{2n}\}$, $\{gu_{2n+1}\}$, and $\{LMu_{2n+1}\}$ also converges to the same point $z \in \mathcal{X}$.

Suppose g is continuous. Then $\{ggu_{2n+1}\}$ and $\{g(LM)u_{2n+1}\}$ converges to gz as $n \to \infty$. Since the pair (g, LM) is compatible of type (B), it follows from the Proposition 2.3(ii) that $\{(LM)(LM)u_{2n+1}\}$ converges to gz as $n \to \infty$.

Step 1. Now, we prove that $gz = z$. For this, taking $u = u_{2n}$ and $y = LMu_{2n+1}$ in (5) and (6) and letting $n \to \infty$, we get

$$[1 + pd(z, gz)]d^2(z, gz) \leq p\psi(0, 0, 0, 0) + m(z, gz) - \phi(m(z, gz)),$$

where

$$m(z, gz) = \max\Big\{d^2(z, gz), d(z, z)d(gz, gz), d(z, gz)d(gz, z), \frac{1}{2}[d(z, z)d(z, gz) + d(gz, z)d(gz, gz)]\Big\} = d^2(z, gz).$$

Solving the above inequality, we get $p\, d^3(z, gz) + \phi(d^2(z, gz)) \leq 0$, which is possible only if $d(gz, z) = 0$. This implies that $gz = z$. Since $g(\mathcal{X}) \subset ST(\mathcal{X})$, there exists a point $w \in \mathcal{X}$ such that $z = gz = STw$.

Step 2. We claim that $fw = w$. For this, substituting $u = w$ and $v = u_{2n+1}$ in (5) and (6) and letting $n \to \infty$, we get

$$[1 + pd(z, z)]d^2(fw, z) \leq p\,\psi(0, 0, 0, 0) + m(z, z) - \phi\big(m(z, z)\big),$$

where

$$m(z, z) = \max\left\{d^2(z, z), d(z, fw)d(z, z), d(z, z)d(z, fw), \frac{1}{2}\big[d(z, fw)d(z, z) + d(z, fw)d(z, z)\big]\right\} = 0.$$

Simplifying the above inequality, we get $d(fw, z) = 0$ i.e., $fw = z$. Since (f, ST) is compatible of type (B) and $fw = STw$, therefore, by Proposition 2.3(i), $f(ST)w = (ST)fw$. Hence, $STz = (ST)fw = f(ST)w = fz$.

Step 3. Taking $u = z$ and $v = u_{2n+1}$ in (5) and (6) and letting $n \to \infty$, we get

$$[1 + pd(fz, z)]d^2(fz, z) \le p\,\psi(0, 0, 0, 0) + m(fz, z) - \phi\big(m(fz, z)\big),$$

where

$$m(fz, z) = \max\Big\{d^2(fz, z), d(fz, fz)d(z, z), d(fz, z)d(z, fz), \frac{1}{2}\big[d(fz, fz)d(fz, z) + d(z, fz)d(fz, z)\big]\Big\} = d^2(fz, z).$$

After simplification, we get $p\,d^3(fz, z) + \phi(d^2(fz, z)) \le 0$, which is possible only if $d(fz, z) = 0$, i.e., $fz = z$. Since $f(\mathcal{X}) \subset LM(\mathcal{X})$, there exists a point $x \in \mathcal{X}$ such that $z = fz = LMx$.

Step 4. We claim that $gx = z$. For this, putting $u = u_{2n}$ and $v = x$ in (5) and (6) and letting $n \to \infty$, we get

$$[1 + pd(z, z)]d^2(z, gx) \le p\,\psi(0, 0, 0, 0) + m(z, z) - \phi\big(m(z, z)\big),$$

where

$$m(z, z) = \max\left\{d^2(z, z), d(z, z)d(z, gx), d(z, gx)d(z, z), \frac{1}{2}\big[d(z, z)d(z, gx) + d(z, z)d(z, gx)\big]\right\} = 0.$$

Solving the above inequality, we conclude that $d(z, gx) = 0$, i.e., $gx = z$. Since the pair (g, LM) is compatible of type (B) in $\mathcal{X}$ and $gx = LMx$, therefore, by Proposition 2.3(i), we have $(LM)gx = g(LM)x$ and hence, $LMz = (LM)gx = g(LM)x = gz = z$.

Step 5. Following steps 5 and 6 of case (i) of Theorem 3.1, we have $Lz = Mz = Tz = Sz = z$. Uniqueness follows easily. Thus, z is a unique common fixed point of S, T, L, M, f and g.

Similarly, one can complete the proof taking the mapping f or ST or LM to be continuous. □

Now, we discuss the existence and uniqueness of a common fixed point for pairs of compatible mappings of type (C).

Theorem 3.4. *Let f, g, L, M, S and T be self mappings defined on a complete metric space $\mathcal{X}$ satisfying the conditions (I)-(IV). If pairs (f, ST) and (g, LM) are compatible of type (C), then f, g, L, M, S and T have a unique common fixed point in $\mathcal{X}$.*

Proof. Following [19, Theorem 2.3], the sequence $\{v_n\}$, defined by (7), is a Cauchy sequence in $\mathcal{X}$ and $(\mathcal{X}, d)$ being a complete metric space the sequence $\{v_n\}$ converges to a point, say, $z \in \mathcal{X}$, as $n \to \infty$. Consequently, the sub sequences $\{fu_{2n}\}$, $\{STu_{2n}\}$, $\{gu_{2n+1}\}$, and $\{LMu_{2n+1}\}$ also converges to the same point z.

Assume that g is continuous. Then sequences $\{ggu_{2n+1}\}$ and $\{g(LM)u_{2n+1}\}$ converges to gz, as $n \to \infty$. Since the pair (g, LM) is compatible of type (C), it follows from the Proposition 2.3(ii) that $\{(LM)(LM)u_{2n+1}\}$ converges to gz, as $n \to \infty$.

Following steps 1-5 of Theorem 3.3, z is a unique common fixed point of S, T, L, M, f and g. Similarly, one can complete the proof, assuming T or ST or LM to be continuous. □

Now, we investigate the existence of a fixed point theorem for pairs of compatible mappings of type(P) as follows.

Theorem 3.5. *Let f, g, S, T, L and M be self mappings of a complete metric space $(\mathcal{X}, d)$. If (f, ST) and (g, LM) are the pairs of compatible mappings of type (P) satisfying conditions (I)-(IV), then all the mappings f, g, S, T, L and M have a unique common fixed point in $\mathcal{X}$.*

Proof. In the view of [19, Theorem 2.3], the sequence $\{v_n\}$, defined by (7), is a Cauchy sequence in $\mathcal{X}$ and $(\mathcal{X}, d)$ being a complete metric space sequence $\{v_n\}$ converges to a point, say, $z \in \mathcal{X}$, as $n \to \infty$. Consequently, the sub sequences $\{fu_{2n}\}$, $\{STu_{2n}\}$, $\{gu_{2n+1}\}$, and $\{LMu_{2n+1}\}$ also converges to the same point z.

Assume that g is continuous. Then sequences $\{ggu_{2n+1}\}$ and $\{g(LM)u_{2n+1}\}$ converges to gz, as $n \to \infty$. Since the pair (g, LM) is compatible of type (P), it follows from the Proposition 2.3(ii) that $\{(LM)(LM)u_{2n+1}\}$ converges to gz, as $n \to \infty$.

Rest of the proof follows on steps 1-5 of Theorem 3.3 □

Now, we study the existence of a unique fixed point for pairs of compatible mappings of type(R) as follows.

Theorem 3.6. *Let f, g, S, T, L and M be self mappings of a complete metric space $(\mathcal{X}, d)$. If (f, ST) and (g, LM) are the pairs of compatible mappings of type (R) satisfying conditions (I)-(IV), then all the mappings f, g, S, T, L and M have a unique common fixed point in $\mathcal{X}$.*

Proof. Following [19, Theorem 2.3], the sequence $\{v_n\}$, defined by (7), is a Cauchy sequence in $\mathcal{X}$ and $(\mathcal{X}, d)$ is a complete metric space, so the sequence $\{v_n\}$ converges to a point, say, $z \in \mathcal{X}$, as $n \to \infty$. Consequently, the sub sequences $\{fu_{2n}\}$, $\{STu_{2n}\}$, $\{gu_{2n+1}\}$, and $\{LMu_{2n+1}\}$ also converges to the same point $z \in \mathcal{X}$.

Assume that f is continuous. Then sequences $\{ffu_{2n}\}$ and $\{f(ST)u_{2n}\}$ converges to fz, as $n \to \infty$. Since the pair (f, ST) is compatible of type (R), it follows from the Proposition 2.2(ii) that $\{(ST)fu_{2n}\}$ converges to fz, as $n \to \infty$.

In the view of steps 1-6 of case (i) of Theorem 3.1, we conclude that z is a unique common fixed point of S, T, L, M, f and g. Assuming either of g, ST, LM to be continuous, the proof follows from cases (ii)-(iv) of Theorem 3.1 □

Now, we prove fixed point theorem for pairs of compatible mappings of type (K) as well as reciprocal continuous mappings.

Theorem 3.7. *Let f, g, L, M, S and T be self mappings of a complete metric space $(\mathcal{X}, d)$ satisfying the conditions (I), (II) and (IV). Then all the mappings f, g, L, M, S and T have a unique common fixed point in $\mathcal{X}$, provided that (f, ST) and (g, LM) are the pairs of reciprocal continuous mappings and compatible mappings of type (K).*

Proof. Following proof of [19, Theorem 2.3], the sequence $\{v_n\}$, defined by (7), is a Cauchy sequence in $\mathcal{X}$. Since $(\mathcal{X}, d)$ is a complete metric space, so the sequence $\{v_n\} \to w \in \mathcal{X}$, as $n \to \infty$. Consequently, the subsequences $\{fu_{2n}\}$, $\{STu_{2n}\}$, $\{gu_{2n+1}\}$, and $\{LMu_{2n+1}\}$ also converges to the same point $w \in \mathcal{X}$.

Since the pair (f, ST) is compatible of type (K), $(ST)(ST)u_{2n} \to fw$, $ffu_{2n} \to STw$ as $n \to \infty$. Also reciprocal continuity of the pair (f, ST) implies that $(ST)fu_{2n} \to STw$ and $f(ST)u_{2n} \to fw$ as $n \to \infty$.

Also, the pair (g, LM) is compatible of type (K) and reciprocal continuous, therefore, $(LM)(LM)u_{2n+1} \to gw$, $ggu_{2n+1} \to LMw$, $(LM)gu_{2n+1} \to LMw$ and $g(LM)u_{2n+1} \to gw$ as $n \to \infty$.

Now, we claim that $LMw = STw$. Taking $u = fu_{2n}$, $v = gu_{2n+1}$ in (5) and (6) and letting $n \to \infty$, we get

$$[1 + pd(STw, LMw)]d^2(STw, LMw) \leq p\psi(0, 0, 0, 0) + m(STw, LMw) \\ - \phi(m(STw, LMw)),$$

where

$$\begin{aligned} m(STw, LMw) = \max\Big\{ & d^2(STw, LMw), d(STw, STw)d(LMw, LMw), \\ & d(STw, LMw)d(LMw, STw), \\ & \frac{1}{2}[d(STw, STw)d(STw, LMw) \\ & + d(LMw, STw)d(LMw, LMw)]\Big\} \\ = & d^2(LMw, STw). \end{aligned}$$

Solving the above inequality, we get $d(LMw, STw) = 0$, which implies that $LMw = STw$. Next, we prove that $LMw = fw$. Letting $u = w$ and $v = gu_{2n+1}$ in (5) and (6) and taking the limit as $n \to \infty$, we get

$$\begin{aligned} [1 + pd(STw, LMw)]d^2(fw, LMw) \leq p\psi\big(& d^2(STw, fw)d(LMw, LMw), \\ & d(STw, fw)d^2(LMw, LMw), \\ & d(STw, fw)d(STw, LMw)d(LMw, fw), \\ & d(STw, LMw)d(LMw, fw)d(LMw, LMw)\big) \\ & + m(STw, LMw) - \phi\big(m(STw, LMw)\big), \end{aligned}$$

where

$$\begin{aligned} m(STw, LMw) = \max\Big\{ & d^2(STw, LMw), d(STw, fw)d(LMw, LMw), \\ & d(STw, LMw)d(LMw, fw), \\ & \frac{1}{2}\big[d(STw, fw)d(STw, LMw) \\ & + d(LMw, fw)d(LMw, LMw)\big]\Big\} = 0. \end{aligned}$$

Simplifying the above inequality, we get $d(fw, LMw) = 0$, i.e., $fw = LMw$. So $STw = LMw = fw$. Next, we claim that $fw = gw$. Putting $u = v = w$ in (5) and (6), we have

$$\begin{aligned} [1 + pd(STw, LMw)]d^2(fw, gw) \leq p\psi\Big(& d^2(STw, fw)d(LMw, gw), \\ & d(STw, fw)d^2(LMw, gw), \\ & d(STw, fw)d(STw, gw)d(LMw, fw), \\ & d(STw, gw)d(LMw, fw)d(LMw, gw)\Big) \\ & + m(STw, LMw) - \phi\big(m(STw, LMw)\big), \end{aligned}$$

where

$$m(STw, LMw) = \max\Big\{d^2(STw, LMw), d(STw, fw)d(LMw, gw), \\ d(STw, gw)d(LMw, fw), \\ \frac{1}{2}\big[d(STw, fw)d(STw, gw) \\ + d(LMw, fw)d(LMw, gw)\big]\Big\} = 0.$$

On solving, we get $d^2(fw, gw) \leq 0$, which is true for $fw = gw$.
Hence, $LMw = gw = STw = fw$, i.e., w is a coincidence point of f, g, ST and LM. Taking $u = u_{2n}$ and $v = w$ in (5) and (6) and letting $n \to \infty$, we get

$$[1 + pd(w, LMw)]d^2(w, gw) \leq p\psi(0, 0, 0, 0) + m(w, LMw) - \phi(m(w, LMw)),$$

where
$m(w, LMw) = \max\{d^2(w, LMw), 0, d(w, gw)d(LMw, w), 0\} = d^2(w, LMw)$.

After simplification, we get $d(w, gw) = 0$, i.e., $w = gw$. In the view of steps 5 and 6 of case (i) of Theorem 3.1, we have $Tw = w$, $Mw = w$. Using $STw = LMw = w$, we have $Sw = w$ and $Lw = w$. Therefore, w is a common fixed point of mappings f, g, S, T, L and M. The uniqueness follows easily. □

Now, we discuss the existence of a unique common fixed point of compatible mappings of type (E) .

Theorem 3.8. *Self mappings f, g, S, T, L and M of a complete metric space $(\mathcal{X}, d)$ satisfying conditions (I), (II) and (IV) have a unique common fixed point in $\mathcal{X}$, if the pairs (f, ST) and (g, LM) satisfy either of the following conditions:*

(a) *(f, ST) is ST-compatible of type (E) and ST-reciprocal continuous, (g, LM) is LM-compatible of type (E) and $LM-$reciprocal continuous.*

(b) *(f, ST) is f-compatible of type (E) and f-reciprocal continuous, (g, LM) is g-compatible of type (E) and g-reciprocal continuous.*

Proof. Following proof of [19, Theorem 2.3], the sequence $\{v_n\}$ defined by (7), is a Cauchy sequence in $\mathcal{X}$. Since $(\mathcal{X}, d)$ is a complete metric space, therefore, $\{v_n\}$ converges to a point $w \in \mathcal{X}$, as $n \to \infty$. Consequently, the subsequences $\{fu_{2n}\}$, $\{STu_{2n}\}$, $\{gu_{2n+1}\}$, and $\{LMu_{2n+1}\}$ also converges to the same point $w \in \mathcal{X}$.

Suppose that pair (f, ST) is $ST-$compatible of type (E) and $ST-$reciprocal continuous, therefore, by Proposition 2.4, $STw = fw$.

We claim that w is a fixed point of ST, i.e., $STw = w$. Letting $u = w$ and $v = u_{2n+1}$ in (5) and (6) and letting $n \to \infty$, we have

$$[1 + pd(STw, w)]d^2(fw, w) \leq p\psi(0, 0, 0, 0) + m(STw, w) - \phi(m(STw, w)),$$

where

$$m(STw, w) = \max\Big\{d^2(STw, w), d(STw, fw)d(w, w), d(STw, w)d(w, fw), \frac{1}{2}[d(STw, fw)d(STw, w) + d(w, fw)d(w, w)]\Big\} = d^2(STw, w).$$

Solving the above inequality, we get $pd^3(STw, w) + \phi(d^2(STw, w)) \leq 0$. This is true only if $d(STw, w) = 0$, which implies that $STw = w$. Therefore, we have $w = STw = fw$. Since $f(\mathcal{X}) \subset LM(\mathcal{X})$, there exists a point $u^* \in \mathcal{X}$ such that $fw = LMu^*$. Now, we claim that $gu^* = LMu^*$. Taking $u = w$, $v = u^*$ in (5) and (6), we get

$$[1 + pd(STw, LMu^*)]d^2(fw, gu^*) \leq p\psi(0, 0, 0, 0) + m(STw, LMu^*) - \phi(m(STw, LMu^*)),$$

where

$$m(STw, LMu^*) = \max\Big\{d^2(STw, LMu^*), d(STw, fw)d(LMu^*, gu^*), d(STw, gu^*)d(LMu^*, fw), \frac{1}{2}[d(STw, fw)d(STw, gu^*) + d(LMu^*, fw)d(LMu^*, gw)]\Big\} = 0.$$

Using the value of $m(STw, LMu^*)$ along with the property of ϕ and ψ, the above inequality reduces to $d^2(fw, gu^*) \leq 0$.This is true only if $d(fw, gu^*) = 0$, i.e., $gu^* = fw$. Hence, $gu^* = LMu^* = fw = STw = w$. Since the pair (g, LM) is $LM-$compatible of type (E) and $LM-$reciprocal continuous and $LMu^* = gu^*$, by Proposition 2.4, $LMw = (LM)gu^* = g(LM)u^* = gw$.
Now, we prove that w is a fixed point of LM. Putting $u = v = w$ in (5) and (6), we get

$$[1 + pd(w, LMw)]d^2(w, gw) \leq p\psi(0, 0, 0, 0) + m(w, LMw) - \phi(m(w, LMw)),$$

where

$$m(w, LMw) = \max\Big\{d^2(w, LMw), d(w, w)d(LMw, gw), d(w, gw)d(LMw, w), \frac{1}{2}[d(w, w)d(w, gw) + d(LMw, w)d(LMw, gw)]\Big\} = d^2(w, LMw).$$

On Simplifying, we get $d(w, LMw) = 0$, which implies that $w = LMw$. Thus, $w = STw = fw = LMw = gw$. In the view of steps 5 and 6 of case (i) of Theorem 3.1, we have $Tw = w$, $Mw = w$. Therefore, $STw = w$, $LMw = w$ imply that $Sw = w$ and $Lw = w$. Hence, w is common fixed point of mappings f, g, S, T, L and M. Uniqueness follows easily.

Similarly, one can complete the proof when the pairs (f, ST) and (g, LM) satisfy the condition (b). □

4 Consequences and Examples

Taking suitable mappings f, g, S, T, L and M one can derive corollaries involving two, three as well as four self mappings. As an example, one can deduce the following corollaries for four self mappings by taking $T = M = I_{\mathcal{X}}$(identity mappings of $\mathcal{X}$) in Theorems 3.1- 3.6, we have the following result for four mappings.

Corollary 4.1. *Let S, L, f and g be self mappings of a complete metric space $(\mathcal{X}, d)$ satisfying*

(i) $f(\mathcal{X}) \subset L(\mathcal{X})$ and $g(\mathcal{X}) \subset S(\mathcal{X})$,

(ii) one of S, L, f, g is continuous,

(iii) for all $u, v \in \mathcal{X}$, there exists a function $\psi \in \Psi$, a function $\phi \in \Phi$ and a real number $p \geq 0$ such that

$$[1 + pd(Su, Lv)]d^2(fu, gv) \leq p\psi\Big(d^2(Su, fu)d(Lv, gv), d(Su, fu)d^2(Lv, gv), d(Su, fu)d(Su, gv)d(Lv, fu), d(Su, gv)d(Lv, fu)d(Lv, gv)\Big) + m(Su, Lv) - \phi(m(Su, Lv)),$$

where

$$m(Su, Lv) = \max\Big\{d^2(Su, Lv), d(Su, fu)d(Lv, gv), d(Su, gv)d(Lv, fu), \frac{1}{2}[d(Su, fu)d(Su, gv) + d(Lv, fu)d(Lv, gv)]\Big\}.$$

If (S, f) and (L, g) are either of the followings *(a)* *compatible*

(b) *compatible of type* (A)

(c) *compatible of type* (B)

(d) *compatible of type* (C)

(e) *compatible of type* (P)

(f) *compatible of type* (R).

Then S, L, f *and* g *have a unique common fixed point in* $\mathcal{X}$.

Taking $T = M = I_{\mathcal{X}}$(identity mappings of $\mathcal{X}$) in the Theorems 3.7-3.8, we have the following results.

Corollary 4.2. *Let* S, L, f *and* g *be self mappings of a complete metric space* $(\mathcal{X}, d)$ *satisfying conditions (i) and (iii) of Corollary 4.1. If* (S, f) *and* (L, g) *are compatible of type* (K) *and reciprocal continuous, then* S, L, f *and* g *have a unique common fixed point in* $\mathcal{X}$.

Corollary 4.3. *Self mappings* f, g, S *and* L *of a complete metric space* $(\mathcal{X}, d)$ *satisfying conditions (i) and (iii) of Corollary 4.1 have a unique common fixed point in* $\mathcal{X}$*, if the pairs* (S, f) *and* (L, g) *satisfy either of the following conditions:*

(a) (S, f) *is* S*-compatible of type* (E) *and* S*-reciprocal continuous,* (L, g) *is* L*-compatible of type* (E) *and* $L-$*reciprocal continuous.*

(b) (S, f) *is* f*-compatible of type* (E) *and* f*-reciprocal continuous,* (L, g) *is* g*-compatible of type* (E) *and* g*-reciprocal continuous.*

Taking $T = M = I_{\mathcal{X}}$(identity mappings of $\mathcal{X}$), $g = f$, $L = S$ in the Theorem 3.1- 3.6, we have the following result for two self mappings.

Corollary 4.4. *Let* $(\mathcal{X}, d)$ *be a complete metric space and* $S, f : \mathcal{X} \to \mathcal{X}$ *be two mappings satisfying the following conditions*

(C_1) $f(\mathcal{X}) \subset S(\mathcal{X})$,

(C_2) *for all* $u, v \in \mathcal{X}$*, there exists a real number* $p \geq 0$*, a function* $\psi \in \Psi$*, a function* $\phi \in \Phi$*, such that*

$$\begin{aligned}[1 + pd(Su, Sv)]d^2(fu, fv) \leq p\,\psi\Big(& d^2(Su, fu)d(Sv, fv),\\ & d(Su, fu)d^2(Sv, fv),\\ & d(Su, fu)d(Su, fv)d(Sv, fu),\\ & d(Su, fv)d(Sv, fu)d(Sv, fv)\Big)\\ & + m(Su, Sv) - \phi(m(Su, Sv)),\end{aligned}$$

where

$$m(Su,Sv)=\max\Big\{d^2(Su,Sv),d(Su,fu)d(Sv,fv),d(Su,fv)d(Sv,fu),\\ \frac{1}{2}[d(Su,fu)d(Su,fv)+d(Sv,fu)d(Sv,fv)]\Big\},$$

(C_3) *either S or f is continuous.*

If S and f are compatible mappings or variants of compatible mappings (type (A) or type (B) or type (C) or type (P) or type (R)), then S and f have a unique common fixed point in $\mathcal{X}$.

Taking $T=M=I_{\mathcal{X}}$(identity mappings of $\mathcal{X}$), $g=f$, $L=S$ in the Theorems 3.7-3.8, we have the following results for two self mappings.

Corollary 4.5. *Let $(\mathcal{X},d)$ be a complete metric space. Suppose $f,S:\mathcal{X}\to\mathcal{X}$ are two mappings satisfying the conditions (C_1) and (C_2) of Corollary 4.4. If (f,S) is compatible of type (K) as well as reciprocal continuous, then f and S have a unique common fixed point in $\mathcal{X}$.*

Corollary 4.6. *Let f and S be self mappings of a complete metric space $(\mathcal{X},d)$ satisfying the conditions (C_1) and (C_2) of Corollary 4.4. If pair (f,S) satisfies either of the following conditions:*

(a) *(f,S) is $f-$compatible of type (E) and $f-$reciprocal continuous;*

(b) *(f,S) is $S-$compatible of type (E) and $S-$reciprocal continuous.*

Then f and S have a unique common fixed point in $\mathcal{X}$.

Now, we present examples in support of Corollaries 4.1 and 4.3.

Example 4.1. *Let $\mathcal{X}=[2,20]$ and d be a usual metric. Let f,g,S,L be self mappings of $\mathcal{X}$ defined by $Su=2$, $2\le u\le 5$, $Su=u-3$, $5<u\le 20$, $fu=6$, $2<u<5$, $fu=2$, $u=2$ or $5\le u\le 20$, $Lu=2$, $gu=2$, $u=2$, $Lu=6$, $gu=3$, $2<u\le 20$. Let $\phi:[0,\infty)\to[0,\infty)$ be a function defined by $\phi(t)=2t$ and $\psi:[0,\infty)^4\to[0,\infty)$ be a function defined by $\psi(w_1,w_2,w_3,w_4)=\max\{w_1,w_2,w_3,w_4\}$, $w_i\ge 0, i=1,2,3,4$. Consider a sequence $\{u_n\}$ with $u_n=2$, for each n. Clearly $\phi\in\Phi$ and $\psi\in\Psi$. Also, One can easily verified that all the conditions of the Corollary 4.1 (a) are satisfied and 2 is the unique common fixed point of f,g,S and L.*

Example 4.2. *Let $\mathcal{X} = [2, 20]$ and d be a usual metric. Let f, g, L and S be self mappings of $\mathcal{X}$ defined as $Su = 2,\ 2 \le u \le 10,\ Su = u - 8,\ 10 < u \le 20$, $fu = 2,\ 2 \le u \le 20$, $Lu = 2, gu = 2, u = 2,\ Lu = 6, gu = 3,\ 2 < u \le 20$. Here $g(\mathcal{X}) = \{2, 3\} \subset [2, 12] = S(\mathcal{X})$ and $f(\mathcal{X}) = \{2\} \subset \{2, 6\} = L(\mathcal{X})$. For the sequence $\{u_n\}$, where $u_n = 2$, for each n, pairs (S, f) and (L, g) are compatible of type (R). If we define a function $\phi : [0, \infty) \to [0, \infty)$ as $\phi(t) = 2t$, for each $t \ge 0$ and define a function $\psi : [0, \infty)^4 \to [0, \infty)$ as $\psi(w_1, w_2, w_3, w_4) = \max\{w_1,\ w_2,\ w_3,\ w_4\}$, $w_i \ge 0, i = 1, 2, 3, 4$ and take a real number $p \ge \frac{3}{2}$, then all the conditions of Corollary 4.1(f) are satisfied and 2 is the unique common fixed point of f, g, S and L.*

Example 4.3. *Let $\mathcal{X} = [0, 5]$ and d be a usual metric. Define $f, g, S, L : \mathcal{X} \to \mathcal{X}$ as $gu = fu = \frac{5+u}{2}$, $Lu = Su = \frac{5}{2} + u$, $0 \le u < \frac{5}{2}$, $gu = fu = \frac{5}{2}$, $\frac{5}{2} \le u \le 5$, $Su = Lu = \frac{5}{2}$, $u = \frac{5}{2}$ and $Su = Lu = \frac{24}{5}$, $\frac{5}{2} < u \le 5$. Clearly, $f(\mathcal{X}) = [\frac{5}{2}, \frac{15}{4}) = g(\mathcal{X})$ and $S(\mathcal{X}) = L(\mathcal{X}) = [\frac{5}{2}, 5)$. The mappings are not continuous at $u = \frac{5}{2}$. Let $\{u_n\}$ be a sequence in $\mathcal{X}$ such that $u_n \to 0$, $u_n > 0$, for all n. Then fu_n, $Su_n \to \frac{5}{2} = t$ and $ffu_n = S(\frac{5+u_n}{2}) \to \frac{5}{2}$, $fSu_n = f(\frac{5}{2} + u_n) \to \frac{5}{2}$, $SSu_n = S(\frac{5}{2} + u_n) \to \frac{24}{5}$ and $Sfu_n = S(\frac{5+u_n}{2}) \to \frac{24}{5}$. Also, we have $St = \frac{5}{2} = ft$. Thus $ffu_n, fSu_n \to \frac{5}{2} = St = S(\frac{5}{2})$ and $fSu_n \to \frac{5}{2} = ft = f(\frac{1}{2})$. Therefore, the pair (S, f) is $f-$compatible of type (E) and $f-$reciprocal continuous and the pair (L, g) is $g-$compatible of type (E) and $g-$reciprocal continuous. In particular, if we take $\psi(t_1, t_2, t_3, t_4) = \max\{t_1, t_2, t_3, t_4\}$, where $t_i \ge 0, i = 1, 2, 3, 4$, $\phi(t) = 2t, t \ge 0$ and $p = \frac{3}{2}$, then all the conditions of Corollary 4.3 are satisfied and $\frac{5}{2}$ is a unique common fixed point of all the mappings f, g, S and L.*

5 Application

Let U, V denote Banach spaces, $\hat{S} \subset U$, $D \subset V$ are state space and decision space respectively. Let $\mathbb{R}$ denotes the set of all real numbers and $B(\hat{S}) = \{h : \hat{S} \to \mathbb{R}, h \text{ is bounded }\}$. Let $d(h, k) = \sup\{|h(u) - k(u)| : u \in \hat{S}\}$ for any $h, k \in B(\hat{S})$. Obviously, $(B(\hat{S}), d)$ is a complete metric space.

Bellman and Lee [5] gave the basic form of functional equation as follows:

$$g(u) = \operatorname*{opt}_{v} G(u, v, g(\tau(u, v))),$$

where $u \in \hat{S}$, $v \in D$, τ is the transformation process, $g(u)$ is the optimal return with initial state u and the opt denotes $\max$ or $\min$.

Now, we discuss the application of our result in finding a common solution of the following functional equations that are arising in dynamic programming (see

[3, 4, 5])

$$f_i(u) = \sup_{v \in D} \mathcal{F}_i(u, v, f_i(\tau(u, v))), u \in \hat{S} \tag{8}$$

$$g_i(u) = \sup_{v \in D} \mathcal{G}_i(u, v, g_i(\tau(u, v))), u \in \hat{S}, \tag{9}$$

where $\tau : \hat{S} \times D \to \hat{S}$ and $\mathcal{F}_i, \mathcal{G}_i : \hat{S} \times D \times \mathbb{R} \to \mathbb{R}, i = 1, 2$.

Theorem 5.1. *Let $\mathcal{F}_i, \mathcal{G}_i : \hat{S} \times D \times \mathbb{R} \to \mathbb{R}$, be bounded for $i \in \{1, 2\}$. Define the mappings $\mathcal{P}_i, \mathcal{Q}_i : B(\hat{S}) \to B(\hat{S})$ as follows*

$$\begin{aligned} \mathcal{P}_i h(u) &= \sup_{v \in D} \mathcal{F}_i(u, v, h(\tau(u, v))), \\ \mathcal{Q}_i k(u) &= \sup_{v \in D} \mathcal{G}_i(u, v, k(\tau(u, v))), \end{aligned} \tag{10}$$

for all $u \in \hat{S}$, $h, k \in B(\hat{S})$, $i = 1, 2$. Suppose that the following conditions hold:

(a) for all $u, t \in \hat{S}, v \in D$, $h, k \in B(\hat{S})$,

$$\begin{aligned} \big|\mathcal{F}_1(u, v, h(t)) - \mathcal{F}_2(u, v, k(t))\big|^2 \leq \mathcal{M}^{-1}\Big(p\,\psi\big(&d^2(\mathcal{Q}_1 h, \mathcal{P}_1 h) d(\mathcal{Q}_2 k, \mathcal{P}_2 k), \\ &d(\mathcal{Q}_1 h, \mathcal{P}_1 h) d^2(\mathcal{Q}_2 k, \mathcal{P}_2 k), \\ &d(\mathcal{Q}_1 h, \mathcal{P}_1 h) d(\mathcal{Q}_1 h, \mathcal{P}_2 k) d(\mathcal{Q}_2 k, \mathcal{P}_1 h), \\ &d(\mathcal{Q}_1 h, \mathcal{P}_2 k) d(\mathcal{Q}_2 k, \mathcal{P}_1 h) d(\mathcal{Q}_2 k, \mathcal{P}_2 k)\big) \\ &+ m(\mathcal{Q}_1 h, \mathcal{Q}_2 k) - \phi(m(\mathcal{Q}_1 h, \mathcal{Q}_2 k))\Big), \end{aligned}$$

where

$$\begin{aligned} m(\mathcal{Q}_1 h, \mathcal{Q}_2 k) = \max\Big\{&d^2(\mathcal{Q}_1 h, \mathcal{Q}_2 k), d(\mathcal{Q}_1 h, \mathcal{P}_1 h) d(\mathcal{Q}_2 k, \mathcal{P}_2 k), \\ &d(\mathcal{Q}_1 h, \mathcal{P}_2 k) d(\mathcal{Q}_2 k, \mathcal{P}_1 h) \\ &\frac{1}{2}[d(\mathcal{Q}_1 h, \mathcal{P}_1 h) d(\mathcal{Q}_1 h, \mathcal{P}_2 k) \\ &+ d(\mathcal{Q}_2 k, \mathcal{P}_1 h) d(\mathcal{Q}_2 k, \mathcal{P}_2 k)]\Big\}, \end{aligned}$$

$\mathcal{M} = 1 + pd(\mathcal{Q}_1 h, \mathcal{Q}_2 k)$, $\phi \in \Phi$, $\psi \in \Psi$, p is a positive real number,

(b) either there exists $\mathcal{P}_i \in \{\mathcal{P}_1, \mathcal{P}_2\}$ such that for any sequence $\{k_n\}$ of $B(\hat{S})$ and $k \in B(\hat{S})$

$$\lim_{n \to \infty} \sup_{u \in \hat{S}} |k_n(u) - k(u)| = 0, \textit{ implies } \lim_{n \to \infty} \sup_{u \in \hat{S}} |\mathcal{P}_i k_n(u) - \mathcal{P}_i k(u)| = 0$$

or there exists $\mathcal{Q}_i \in \{\mathcal{Q}_1, \mathcal{Q}_2\}$ *such that for any sequence* $\{k_n\}$ *of* $B(\hat{S})$ *and* $k \in B(\hat{S})$

$$\lim_{n\to\infty} \sup_{u\in\hat{S}} |k_n(u) - k(u)| = 0, \text{ implies } \lim_{n\to\infty} \sup_{u\in\hat{S}} |\mathcal{Q}_i k_n(u) - \mathcal{Q}_i k(u)| = 0;$$

(c) for any $h \in B(\hat{S})$*, there exists* $k_1, k_2 \in B(\hat{S})$ *such that*

$$\mathcal{P}_1 h(u) = \mathcal{Q}_2 k_1(u), \mathcal{P}_2 h(u) = \mathcal{Q}_1 k_2(u), u \in \hat{S};$$

(d) for any $i \in \{1, 2\}$*,* $\lim_{n\to\infty} \sup_{u\in\hat{S}} |\mathcal{Q}_i\mathcal{P}_i k_n(u) - \mathcal{P}_i\mathcal{Q}_i k_n(u)| = 0$*, whenever* $\{k_n\}$ *is a sequence of* $B(\hat{S})$ *such that* $\lim_{n\to\infty} \mathcal{Q}_i k_n = \lim_{n\to\infty} \mathcal{P}_i k_n = h$*, for some* $h \in B(\hat{S})$*.*

Then the system of functional equations (8) and (9) have a unique common solution in $B(\hat{S})$*.*

Proof. From conditions (b) -(d), $\mathcal{P}_i, \mathcal{Q}_i$ are self mappings of $B(\hat{S})$. One of $\mathcal{P}_i, \mathcal{Q}_i$ is continuous for $i \in \{1, 2\}$, $\mathcal{P}_1(B(\hat{S})) \subset \mathcal{Q}_2(B(\hat{S}))$ and $\mathcal{P}_2(B(\hat{S})) \subset \mathcal{Q}_1(B(\hat{S}))$ and the pairs of mappings $(\mathcal{P}_1, \mathcal{Q}_1)$ and $(\mathcal{P}_2, \mathcal{Q}_2)$ are compatible.

For $\eta > 0$, $u \in \hat{S}$ and $k_1, k_2 \in B(\hat{S})$, there exists $v_1, v_2 \in D$ such that

$$\mathcal{P}_i k_i(u) < \mathcal{F}_i(u, v_i, k_i(u_i)) + \eta, \tag{11}$$

where $u_i = \tau(u, v_i), i = 1, 2$. Also, we have

$$\mathcal{P}_1 k_1(u) \geq \mathcal{F}_1(u, v_2, k_1(u_2)), \tag{12}$$
$$\mathcal{P}_2 k_2(u) \geq \mathcal{F}_2(u, v_1, k_2(u_1)). \tag{13}$$

From (11),(13) and condition (a), we have

$$\begin{aligned}(\mathcal{P}_1 k_1(u) - \mathcal{P}_2 k_2(u))^2 &< (\mathcal{F}_1(u, v_1, k_1(u_1)) - \mathcal{F}_2(u, v_1, k_2(u_1)) + \eta)^2 \\ &= (\mathcal{F}_1(u, v_1, k_1(u_1)) - \mathcal{F}_2(u, v_1, k_2(u_1)))^2 + \xi, \\ &\leq \mathcal{M}^{-1}\Big(p\,\psi\big(d^2(\mathcal{Q}_1k_1, \mathcal{P}_1k_1)d(\mathcal{Q}_2k_2, \mathcal{P}_2k_2), \\ &\quad d(\mathcal{Q}_1k_1, \mathcal{P}_1k_1)d^2(\mathcal{Q}_2k_2, \mathcal{P}_2k_2), \\ &\quad d(\mathcal{Q}_1k_1, \mathcal{P}_1k_1)d(\mathcal{Q}_1k_1, \mathcal{P}_2k_2)d(\mathcal{Q}_2k_2, \mathcal{P}_1k_1), \\ &\quad d(\mathcal{Q}_1k_1, \mathcal{P}_2k_2)d(\mathcal{Q}_2k_2, \mathcal{P}_1k_1)d(\mathcal{Q}_2k_2, \mathcal{P}_2k_2)\big) + \\ &\quad m(\mathcal{Q}_1k_1, \mathcal{Q}_2k_2) - \phi(m(\mathcal{Q}_1k_1, \mathcal{Q}_2k_2))\Big) + \xi,\end{aligned} \tag{14}$$

where $\xi = \eta^2 + 2\eta(\mathcal{F}_1 - \mathcal{F}_2)$.

From (11), (12) and condition (a), we have

$$\begin{aligned}(\mathcal{P}_1k_1(u)-\mathcal{P}_2k_2(u))^2 >& (\mathcal{F}_1(u,v_2,k_1(u_2))-\mathcal{F}_2(u,v_2,k_2(u_2))-\eta)^2\\ =&(\mathcal{F}_1(u,v_1,k_1(u_2))-\mathcal{F}_2(u,v_1,k_2(u_2)))^2+\xi_1,\\ &\geq -\mathcal{M}^{-1}\Big(p\,\psi\big(d^2(\mathcal{Q}_1k_1,\mathcal{P}_1k_1)d(\mathcal{Q}_2k_2,\mathcal{P}_2k_2),\\ &d(\mathcal{Q}_1k_1,\mathcal{P}_1k_1)d^2(\mathcal{Q}_2k_2,\mathcal{P}_2k_2),\\ &d(\mathcal{Q}_1k_1,\mathcal{P}_1k_1)d(\mathcal{Q}_1k_1,\mathcal{P}_2k_2)d(\mathcal{Q}_2k_2,\mathcal{P}_1k_1),\\ &d(\mathcal{Q}_1k_1,\mathcal{P}_2k_2)d(\mathcal{Q}_2k_2,\mathcal{P}_1k_1)d(\mathcal{Q}_2k_2,\mathcal{P}_2k_2)\big)+\\ &m(\mathcal{Q}_1k_1,\mathcal{Q}_2k_2)-\phi(m(\mathcal{Q}_1k_1,\mathcal{Q}_2k_2))\Big)-\xi,\end{aligned} \tag{15}$$

where $\xi_1=\eta^2-2\eta(\mathcal{F}_1-\mathcal{F}2)<\xi$.

From (14) and (15), we obtain

$$\begin{aligned}|\mathcal{P}_1k_1(u)-\mathcal{P}_2k_2(u)|^2 \leq& \mathcal{M}^{-1}\Big(p\,\psi\big(d^2(\mathcal{Q}_1k_1,\mathcal{P}_1k_1)d(\mathcal{Q}_2k_2,\mathcal{P}_2k_2),\\ &d(\mathcal{Q}_1k_1,\mathcal{P}_1k_1)d^2(\mathcal{Q}_2k_2,\mathcal{P}_2k_2),\\ &d(\mathcal{Q}_1k_1,\mathcal{P}_1k_1)d(\mathcal{Q}_1k_1,\mathcal{P}_2k_2)d(\mathcal{Q}_2k_2,\mathcal{P}_1k_1),\\ &d(\mathcal{Q}_1k_1,\mathcal{P}_2k_2)d(\mathcal{Q}_2k_2,\mathcal{P}_1k_1)d(\mathcal{Q}_2k_2,\mathcal{P}_2k_2)\big)\\ &+m(\mathcal{Q}_1k_1,\mathcal{Q}_2k_2)-\phi(m(\mathcal{Q}_1k_1,\mathcal{Q}_2k_2))\Big)+\xi,\end{aligned} \tag{16}$$

As $\eta>0$ is arbitrary, so ξ is negligible and (16) is true for all $u\in\hat{S}$, taking supremum, we get

$$\begin{aligned}[1+pd(\mathcal{Q}_1k_1,\mathcal{Q}_2k_2)]d^2(\mathcal{P}_1k_1,\mathcal{P}_2k_2)\leq&\ p\,\psi\big(d^2(\mathcal{Q}_1k_1,\mathcal{P}_1k_1)d(\mathcal{Q}_2k_2,\mathcal{P}_2k_2),\\ &d(\mathcal{Q}_1k_1,\mathcal{P}_1k_1)d^2(\mathcal{Q}_2k_2,\mathcal{P}_2k_2),\\ &d(\mathcal{Q}_1k_1,\mathcal{P}_1k_1)d(\mathcal{Q}_1k_1,\mathcal{P}_2k_2)d(\mathcal{Q}_2k_2,\mathcal{P}_1k_1),\\ &d(\mathcal{Q}_1k_1,\mathcal{P}_2k_2)d(\mathcal{Q}_2k_2,\mathcal{P}_1k_1)d(\mathcal{Q}_2k_2,\mathcal{P}_2k_2)\big)\\ &+m(\mathcal{Q}_1k_1,\mathcal{Q}_2k_2)-\phi(m(\mathcal{Q}_1k_1,\mathcal{Q}_2k_2)).\end{aligned}$$

All the hypotheses of Corollary 4.1 are satisfied and $\mathcal{P}_1,\mathcal{P}_2,\mathcal{Q}_1$ and $\mathcal{Q}_2$ corresponds to mappings f,g,S and L respectively. So, $\mathcal{P}_1,\mathcal{P}_2,\mathcal{Q}_1$ and $\mathcal{Q}_2$ have a unique common fixed point $k^*\in B(\hat{S})$, i.e., $k^*(u)$ is a unique common solution of the system of functional equations (8) and (9). □

6 Conclusion

In this paper, we have established common fixed point theorems for pairs of minimal commutative mappings. These results generalize the results of Jain *et* *al.*[10, 11, 12], Jung *et al.*[14], Kang *et al.*[18], Murthy and Prasad [21], Pathak *et al.* [24] and various results presented in the literature. Our result is useful in studying the existence and uniqueness problems of certain functional equations arising in dynamic programming. This result can be extended for multivalued mappings and family of mappings.

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