Fuzzy dominator colouring on fuzzy soft graphs

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Abstract

A fuzzy soft dominator colouring of a fuzzy soft graph G^S(T,V) is an appropriate fuzzy soft colouring such that every single vertex of G^S(T,V) dominate entire vertex of a colour group. In the present work, we initiate fuzzy dominator colouring on fuzzy soft path, fuzzy soft cycle, complete fuzzy soft graph, complete fuzzy soft bipartite graph and its fuzzy soft dominator chromatic number is presented as well as bounds for fuzzy soft dominator chromatic number on fuzzy soft graph is established.

Keywords: Fuzzy Soft Graph, Fuzzy dominator chromatic number, Fuzzy soft path, Fuzzy soft cycle, Strong arcs.
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1 Introduction

Fuzzy soft graphs are a useful mathematical tool for simulating the ambiguity of the actual world in view of parameters. Fuzzy soft graphs, which are often utilised in many different disciplines, including decision-making issues, combine fuzzy soft sets and the graph model. Scheduling problems are just one of many practical issues involving the allocation of scarce resources for which graph colouring is used as a model. Graph colouring also has an important place in discrete mathematics and combinatorial optimization.

To address ambiguous issues in the fields of engineering, social science, economics, medical research, and environment, **?** created the idea of soft set theory. Fuzzy soft sets, a blend of a fuzzy set and a soft set, were first introduced by Maji **?**. Fuzzy soft graphs were independently introduced by **?** as well. **?** presented fuzzy soft graphs and examined their operations as well as several other graph theoretical ideas.

Domination is a fast growing area of graph theory study, and the numerous ways it is used to networks, distributed computers, social networks, and online graphs helps to explain why there is more interest in this subject. The dominator colouring problem in graphs was first described by **?**. An appropriate colouring of a graph G with the extra characteristic that each vertex in the graph dominates an entire class is known as a dominator colouring of G. The smallest number of colour classes in a graph's dominator colouring is known as the dominator chromatic number. We want to reduce the number of colour categories. **?** explored the dominator chromatic number for the hypercube and more broadly for bipartite graphs. He also presented dominator colorings in bipartite graphs.**?** studied dominator colorings in some classes of graphs. The idea of fuzzy dominator colouring in fuzzy graphs was created by **?**. They investigated the fuzzy dominator chromatic number for a number of fuzzy graphs and explored its boundaries. Domination in fuzzy soft graphs was the subject of study done by **?**.
Fuzzy dominator coloring applied to fuzzy soft graph yields fuzzy soft dominator coloring which concentrates on minimizing the number of color classes of FSG. A fuzzy dominator FSG's colouring should be done in such a way that every node of a colour group is dominated by a vertex of G^S(T,V). The proposed fuzzy soft dominator coloring concentrates on strength of connectedness, strong arc and strong neighbour in classes of fuzzy soft graph in view of parameters from the existing method.

In this paper, we introduce fuzzy soft dominator coloring to fuzzy soft graphs and its fuzzy soft dominator chromatic numbers are determined with bounds.

2 Preliminaries

Definition 2.1. *Let V = $\{x_1, x_2, x_3, \cdot\cdot\cdot, x_n\}$ is a non-empty set, R is a parameter set and T $\subseteq R$. Also let*
(i) α: T $\rightarrow F(V)$ (set of all fuzzy subsets in V)
e $\dashv$ α(e) = $\alpha_e(say)$
α_e : V $\rightarrow [0,1]$
(T,α) : Fuzzy Soft Vertex

(ii) β: T $\rightarrow$ F(V$\times$ V) (set of all fuzzy subsets in V$\times$ V)
e $\dashv$ β(e) = β_e (say)
β_e : V$\times$ V $\rightarrow [0,1]$
(T,β) : Fuzzy Soft Edge
Then ((T,α),(T,β)) is called fuzzy soft graph (FSG) iff $\beta_e(x,y) \leq \alpha_e(x) \wedge \alpha_e(y)$ for all e $\in T$ and this fuzzy soft graphs are symbolized by $G^S(T,V)$. Additionally, a FSG is a parametrized family unit of fuzzy graphs.

Definition 2.2. *A series of distinctive points $x_1, x_2, \cdot\cdot\cdot x_n$ in a FSG is mean to be a Path such that $\forall$ e $\in T$ and β_e (x_{i-1}, x_i) $>$ 0 $\forall$ i = 1,2,3,$\cdot\cdot\cdot n$.*

Definition 2.3. *A FSG $G^S(T,V)$ is called Fuzzy Soft Cycle if it contain above one lowest arc, $\forall$ $e \in T$.*

Definition 2.4. *A fuzzy soft graph G^S(T,V) is complete if $\beta_e(x,y)$ =$\alpha_e(x) \wedge \alpha_e(y)$, for every x,y $\in \alpha^*$, e $\in T$.*

Definition 2.5. *A FSG G^S(T,V) is supposed to be a fuzzy soft bipartite if the node set V can be divided into 2 non-empty sets V_1^e and V_2^e such that V_1^e and V_2^e are fuzzy independent sets. These sets are called fuzzy bipartition of V, thus each efficient arc of FSG has one end in V_1^e and other end in V_2^e, $\forall$ $e \in T$.*

Definition 2.6. *A fuzzy soft bipartite graph is complete if for an individual node V_1^e, each single node of V_2^e is an efficient neighbour, $\forall$ $e \in T$.*

3 Fuzzy Soft Dominator Colouring on Fuzzy Soft Graphs

Definition 3.1. *The strength of connectedness in a FSG G^S(T,V) joining two nodes x,y is define as the greatest strength of the entire paths among x and y and is indicated by $\beta_e^\infty(x,y)$, $\forall$ $e \in T$.*

Definition 3.2. *An arc (x,y) in FSG G^S(T,V) is said to be a strong arc iff $\beta_e(x,y)$ = β_e^{∞} (x,y) e $\in T$ and x, y is said to be strong adjacent vertices (or) efficiently adjoining nodes.*

Definition 3.3. *A k-fuzzy soft vertex colouring of a fuzzy soft graph G^S(T,V) is an allotment of k –colours, commonly mean as 1,2,$\cdots k$ to the vertices of G^S(T,V). A fuzzy soft vertex colouring is stated as appropriate colouring if two strong adjacent vertices receive distinct colours.*

Definition 3.4. *A fuzzy soft dominator coloring (FSDC) of a FSG G^S(T,V) is an appropriate fuzzy soft colouring such that every single vertex of G^S(T,V) dominate entire vertex of a colour group.*

Definition 3.5. *A fuzzy dominator chromatic number (FDCN) of a FSG is the least quantity of colour groups in a fuzzy soft dominator colouring of G^S(T,V) and it is indicated as χ_{fd}^e (G^S(T,V)), e $\in$ T.*

4 Main Results

Proposition 4.1. *If G^S(T,V) is a fuzzy soft graph such that $G^{S^*}(T,V) = (V,E)$ is a soft path of size n $\geq$ 3, in that case $\chi_{fd}^e(G^S(T,V))$= 1 + $\lceil \frac{n}{3} \rceil$ for n=3,4 and 5 and 2+$\lceil \frac{n}{3} \rceil$ for n $\geq$ 6, $\forall\ e \in T$.*

Proof. Consider a fuzzy soft graph G^S(T,V) of whom fundamental graph is a soft path of size n.

$\Longrightarrow$ All interior points of G^S(T,V) dominate themselves and their connecting points. Hence, it is clear that the fuzzy soft dominator chromatic number of G^S(T,V) for n =3,4 and 5 is 1+$\lceil \frac{n}{3} \rceil$.

Now we have to prove for the case n $\geq$ 6 as 2+$\lceil \frac{n}{3} \rceil$.

case (i) n$\equiv$ 0 (mod 3)

Consider a fuzzy soft graph G^S(T,V) with points v_1,v_2,...,v_{3_k}. To obtain the fuzzy soft dominator chromatic number of G^S(T,V) we have to use no less than 3k uncommon colours. With no loss of generality, pick the mid point of every 3 nodes, then we have v_2,v_5,...,v_{3_k-1} having k = $\lceil \frac{n}{3} \rceil$ colour groups of G^S(T,V).

Now colour the resting nodes of G^S(T,V). We need two colour groups to colour the resting nodes v_3 and v_4 to obtain appropriate fuzzy soft colouring since v_3 and v_4 are strong arcs. So allot the colour k+1 to v_1,v_2,...,v_{3_k} and assign colour k+2 to v_4,v_7,...,v_{3_k-2}

In the end, we have k+2 =$\lceil \frac{n}{3} \rceil$+2 colour groups in a least fuzzy soft dominator colouring.

case (ii) n$\equiv$ 2 (mod 3)

We should colour an additional node 3k+2 in this case, because it is not strong adjoin to any node in the equal colour group. So put on this node in the colour group v_1,v_2,...,v_{3_k}. Hence we get a least fuzzy soft dominator colouring of G^S(T,V) as k+2 =$\lceil\frac{n}{3}\rceil$+2.

Proposition 4.2. *If G^S(T,V) is a fuzzy soft cycle of length n, then* $\chi^e_{fd}(G^S(T,V))$= $\lceil\frac{n}{3}\rceil$ *for n=4 and* $\lceil\frac{n}{3}\rceil$+2 *for n* $\geq$ *5,*$\forall$ $e \in T$.

Proof. Let C^e_4 = $v_1v_2v_3v_4v_1$ be a fuzzy soft cycle of size n = 4, for every e $\in T$. If the cycle be even, we have two fuzzy independent sets which are v_1,v_3 and v_2,v_4. Then allot colour 1 to the nodes v_1,v_3 and colour 2 to the nodes v_2,v_4. Hence we obtain the least fuzzy soft dominator colouring of C^e_n with $\chi^e_{fd}(G^S(T,V))$= $\lceil\frac{n}{3}\rceil$ = $\lceil\frac{4}{3}\rceil$ = 1.3= 2.
Now we examine the case n$\geq$ 5 of C^e_n.

Consider a sub graph P^e_n of C^e_n, $e \in T$ with nodes v_1,v_2,...,v_n with the equal membership function as in C^e_n. By Proposition 4.1, we have $\chi^e_{fd}(P^e_n)$ = 2+$\lceil\frac{n}{3}\rceil$, $e \in T$ and also there exists a least fuzzy soft dominator colouring such that the points v_1 and v_n are dominated by themselves or by its adjacent nodes v_2 and v_{n-1}.

From the sub graph P^e_n we get the fuzzy soft cycle C^e_n by joining an edge v_1v_n. If the colours of v_1v_n are alike, say colour m (1$\leq m \leq \chi^e_{fd}(P^e_n)$)) $e \in T$ which is one of the repeated colour in fuzzy dominator colouring of P^e_n.

Because $\chi^e_{fd}(G^S(T,V))$ = 2+$\lceil\frac{n}{3}\rceil$, $e \in T$ there is at least one other colour that repeats say n (1$\leq n \neq m \leq \chi^e_{fd}(P^e_n)$) allot colour n to v_1. Now the colour v_2 is a non-consecutive colour since v_1 need to dominate a few colour group in P^e_n and so the colour v_1 and v_2 are distinct. The resting nodes have an appropriate fuzzy soft colouring since it was transmitted from P^e_n. The new fuzzy soft colouring is a fuzzy soft dominator colouring because it was a fuzzy soft dominator colouring of P^e_n and the transform in the additional arc were analyzed.

Suppose, if the colours of v_1 and v_n were uncommon colours in a fuzzy soft dominator colouring of P^e_n, later the colouring would not be optimal. And so, at the most one of the node v_1 and v_n will have an uncommon colour say v_n. Then certainly, v_{n-1} has a common colour in an optimal fuzzy soft dominator colouring of P^e_n. Then joining an edge v_1 v_n will maintain the fuzzy soft dominator colouring of P^e_n of C^e_n. Thus $\chi^e_{fd}(G^S(T,V))$ = 2+$\lceil\frac{n}{3}\rceil$ for n $\geq$ 5, $\forall$ $e \in T$.

Proposition 4.3. *If G^S(T,V) is a complete fuzzy soft graph then* $\chi^e_{fd}(G^S(T,V))$ = *n,* $\forall$ $e \in T$.

Proof. Given that G^S(T,V) is complete, then every vertex of G^S(T,V) dominates itself and its adjacent vertices, since all arcs are strong arcs in complete fuzzy soft graph. Hence every node of G^S(T,V) dominates a colour group. Therefore G^S(T,V) consist of n colour groups v_1,v_2,...,v_n, $\chi^e_{fd}(G^S(T,V))$ = n, $\forall$ $e \in T$.

Proposition 4.4. *If G^S(T,V) is a complete fuzzy soft bipartite graph then* $\chi^e_{fd}(G^S(T,V))$ = 2, $\forall e \in T$.

Proof. Since $\chi^e_{fd}(G^S(T,V))$ = 2 , which means the FSG contains 2 colour groups say V^e_1 and V^e_2. This implies every point of V^e_1 dominates each point ofV^e_2, then we have a bipartition as V^e_1 and V^e_2 respectively, $\forall\ e \in T$.
Therefore G^S(T,V) must be a complete fuzzy soft bipartite graph.

4.1 Bounds on Fuzzy Dominator Chromatic Number of Fuzzy Soft Graphs

Let G^S(T,V) be a connected fuzzy soft graph of size n$\geq$ 2. Then we need no less than two different colours in a fuzzy soft dominator colouring since there are two nodes which are strong adjacent to each other. Futhermore, if each node gains its unique colour, we also obtain a fuzzy soft dominator colouring.Thus

$$2 \leq \chi^e_{fd}(G^S(T,V)) \leq n,\ \forall\ e \in T.$$

and these bounds are sharp for complete fuzzy soft bipartite graph and complete fuzzy soft graph.

Proposition 4.5. *Let G^S(T,V) be a connected fuzzy soft graph. Then $\chi^e_{fd}(G^S(T,V))$ = 2, $\forall\ e \in T$ iff G^S(T,V) is a complete fuzzy soft bipartite graph.*

Proof. Consider a fuzzy soft graph with $\chi^e_{fd}(G^S(T,V))$ = 2 (i.e.) the graph contains 2 colour groups V^e_1 and V^e_2 (say). If $|V^e_1|$ = 1 (or)$|V^e_2|$ = 1, $\forall\ e \in T$.
$\Longrightarrow$ G^S(T,V) is complete fuzzy soft bipartite graph with bipartition $|X^e_1|$=1 and $|X^e_2|$= n-1, since G^S(T,V) is connected. Suppose we have $|V^e_1| \geq 2$ and $|V^e_2| \geq$ 2,$\forall e \in T$. Let v $\in V^e_1$. Since V^e_1 is a fuzzy independent set and $|V^e_1| \geq 2$, it implies that the vertex v is not dominated by any node in V^e_1. This results that the vertex v is efficiently adjoin to every point in V^e_2. Equivalently, for any random point in V^e_2, that point is efficiently adjoin to each point in V^e_1. As a result, G^S(T,V) is a complete fuzzy soft bipartite graph. Converse part follows from the proposition 4.4.

Proposition 4.6. *Let G^S(T,V) be a connected fuzzy soft graph. Then $\chi^e_{fd}(G^S(T,V))$ = n, $\forall\ e \in T$ if and only if G^S(T,V) is a fuzzy soft graph whose fundamental graph is complete and the arcs are strong arcs.*

Proof. Consider a connected fuzzy soft graph G^S(T,V) with fuzzy soft dominator chromatic number n. SupposeG^S(T,V) is not a fuzzy soft graph with complete graph as an underlying crisp graph with strong arcs. Then there exists at the minimum of one couple of points u,v such that u and v are not efficiently adjoin. Represent fuzzy soft colouring on G^S(T,V) in which u and v taking the same colour and the rest of the nodes meet with unique colour. Therefore,G^S(T,V) receives a fuzzy soft dominator colouring with $\chi^e_{fd}(G^S(T,V))$ = n-1 which is a contradictory statement for $\chi^e_{fd}(G^S(T,V))$ = n, $\forall\ e \in T$. Converse part follows from proposition 4.3.

Proposition 4.7. *A fuzzy soft graph G^S(T,V) have a point of efficiently adjoin degree n-1 then $\chi^e_{fd}(G^S(T,V))$ = $\chi^e_{f}(G^S(T,V)), \forall\ e \in T$.*

Proof. Let G^S(T,V) be a connected fuzzy soft graph with n points. Suppose that G^S(T,V) having a node of strong adjacent degree n-1. Let v_1, v_2 ... v_n be the points of G^S(T,V). With no loss of generality, let v_i for some i (i=1,2,... n) be the point of degree n-1 which is strong. Then v_i is efficiently adjoin to every point of G^S(T,V). If c_1, c_2, c_3 ... c_k be the colour groups of G^S(T,V) such that $\chi^e_f(G^S(T,V))$ = k, $\forall e \in T$. Clearly the node v_i is in one of the colour group say c_i. This is a fuzzy soft colouring which is appropriate and each node of G^S(T,V) dominates at least one colour group (c_i.) Hence we have a fuzzy soft dominator colouring of k colour groups. Therefore $\chi^e_{fd}(G^S(T,V))$ = k = $\chi^e_f(G^S(T,V))$, $\forall$ $e \in T$.

Corolary 4.1. *If the fuzzy soft domination number of G^S(T,V) is equal to 1, then $\chi^e_{fd}(G^S(T,V))$ = $\chi^e_f(G^S(T,V))$, $\forall\ e \in T$.*

Proposition 4.8. *Let G^S(T,V) be a connected fuzzy soft graph. Then,*
max ($\chi^e_f(G^S(T,V))$,$\gamma^e(G^S(T,V))$) $\leq \chi^e_{fd}(G^S(T,V)) \leq \chi^e_f(G^S(T,V))$ + $\gamma^e(G^S(T,V))$, $\forall\ e \in T$.

Proof. Let G^S(T,V) be a connected fuzzy soft graph, since every fuzzy soft dominator colouring must be an appropriate fuzzy soft colouring, we have $\chi^e_f(G^S(T,V)) \leq \chi^e_{fd}(G^S(T,V))$. In addition, let c be a least fuzzy soft dominator colouring ofG^S(T,V). For every colour group ofG^S(T,V) let u_i be a point in the group i with 1$\leq\ i\ \leq \chi^e_{fd}(G^S(T,V))$. Then S = u_i:1$\leq$ i $\leq \chi^e_{fd}(G^S(T,V))$ is a dominating set which gives $\gamma^e(G^S(T,V)) \leq |S|$ = c =$\chi^e_{fd}(G^S(T,V))$. Therefore max ($\chi^e_f(G^S(T,V))$,$\gamma^e(G^S(T,V))$) $\leq \chi^e_{fd}(G^S(T,V))$, $\forall\ e \in T$.

For the upper bound, let c be an appropriate fuzzy soft colouring of G^S(T,V) with $\chi^e_f(G^S(T,V))$ colours. Now assign colours $\chi^e_f(G^S(T,V))$ + 1, $\chi^e_f(G^S(T,V))$ +2,... $\chi^e_f(G^S(T,V))$ + $\gamma^e(G^S(T,V))$ to the points of least dominating set of G^S(T,V) departing the remaining points coloured already. Clearly this is a fuzzy soft dominator colouring of G^S(T,V). Hence $\chi^e_{fd}(G^S(T,V)) \leq \chi^e_f(G^S(T,V))$ + $\gamma^e(G^S(T,V))$, $\forall\ e \in T$.

Corolary 4.2. *If G^S(T,V) be complete fuzzy soft bipartite graph, then the lower bound max ($\chi^e_f(G^S(T,V))$),γ^e(G^S(T,V)) $\leq \chi^e_{fd}(G^S(T,V))$,$\forall e \in T$ is sharp.*

Proof. Let G^S(T,V) be complete fuzzy soft bipartite graph, then max ($\chi_f^e(G^S(T,V))$,$\gamma^e(G^S(T,V))$) = max(2,2) = 2 $\leq \chi_{fd}^e(G^S(T,V))$= 2, $\forall\ e \in T$.

Corolary 4.3. *If G^S(T,V) be a fuzzy soft path of length n≥6, then the upper bound $\chi_{fd}^e(G^S(T,V)) \leq \chi_f^e(G^S(T,V))$ + $\gamma^e(G^S(T,V))$, $\forall\ e \in T$ is sharp.*

Proof. Let G^S(T,V) be a fuzzy soft path of length n≥ 6, then $\chi_{fd}^e(G^S(T,V))$=2+$\lceil \frac{n}{3} \rceil \leq \chi_f^e(G^S(T,V))$ + $\gamma^e(G^S(T,V))$=2 + $\lceil \frac{n}{3} \rceil$, $\forall\ e \in T$.

5 Conclusion

In this work, we defined fuzzy soft dominator colouring for classes of fuzzy soft graphs and its fuzzy soft dominator chromatic numbers through bounds were studied. In this study, we analysed the bounds for the connected fuzzy soft graph of size n $\geq$ 2 is 2 $\leq \chi_{fd}^e(G^S(T,V)) \leq$ n, $\forall\ e \in T$ which is sharp for complete fuzzy soft bipartite graph and complete fuzzy soft graph. And also, the lower bound for complete fuzzy soft bipartite and the upper bound for fuzzy soft path is sharp in terms of domination number. We further extend this study on some more types of fuzzy soft graphs.

References

Chellali and Maffary. Dominator colorings in some classes of graphs. *Graphs Combin*, 28:97–107, 2012.

D.Molodstov. Soft set theory – first results. *Computers and Mathematics with Applications*, 37:19–31, 1999.

P. M. et.al. Fuzzy soft sets. *Journal of Fuzzy Mathematics*, 9(3):589–602, 2001.

R. Gera. On the dominator colorings in bipartite graphs. *Proceedings of the 4th International Conference on Information Technology:New Generations*, pages 947–952, 2007a.

R. Gera. On the dominator colorings in graph. *Graph Theory Notes of New York LII*, pages 25–30, 2007b.

R. J. Hussain and K. K. Fathima. On fuzzy dominator colouring in fuzzy graphs. *Applied Mathematical Sciences*, 9:1131–1137, 2015.

R. J. Hussain and S. Hussain. Domination in fuzzy soft graphs. *International Journal of Fuzzy Mathematical Archieve*, 14:243–252, 2017.

M.Akram and S.Nawaz. On fuzzy soft graphs. *Italian journal of pure and applied mathematics*, 34:497–514, 2015.

Mohinta and Samanta. An introduction to fuzzy soft graph. *Mathematica Moravica*, 19(2):35–48, 2015.