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Restrained star edge coloring of graphs and its application in optimal & safe storage practices

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Abstract

In this paper we introduce the concept of restrained star edge coloring of graphs by restraining the conditions of the star coloring of graphs. The restrained star edge coloring of graphs is a path based graph coloring which is said to be proper if all the bichromatic subgraphs of the graph are in the form of a galaxy. The minimum requirement for this coloring is its restrained star chromatic index, denoted as $\chi'_{rs}(G)$. This paper exclusively explains, the restrained star edge coloring of several families of graphs including path, cycle, wheel, etc., and provides the exact value of its respective restrained star chromatic index, χ'_{rs} with the usage of appropriate illustrations. In addition to this, an application of this coloring in the optimal utilization of storage spaces and in ensuring safe storage practices is also briefly elaborated.
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1 Introduction

A graph G is considered to be properly vertex (edge) colored if two adjoining vertices (edges) are colored uniquely. Star coloring introduced by Grünbaum [1973] is a path-based coloring in which no path of length 3 should be bicolored. Following this, Fertin et al. [2004] observed that in a star vertex coloring any bichromatic subgraph is in the form of galaxy. Where a galaxy is a forest composing only of stars according to Gallian [2022]. This coloring was introduced along with the acyclic coloring of graphs and several results of star coloring in comparison with the acyclic coloring of graphs can be noted in Albertson et al. [2004] and Brause et al. [2022].

The edge version of the same introduced by Deng [2007] states that a star-edge coloring of a graph is a proper edge coloring if no path of length 4 is bicolored. Following its entry several results and theorems have emerged in said topic and can be seen in Bezegová et al. [2016], Casselgren et al. [2021], Evangeline Lydia and Vijaya Xavier Parthipan [2022] and Lei and Shi [2020]. However, it can be noted that the property "every bichromatic subgraph of a properly star colored graph consists of tree composing only of stars" was not observed in the case of star edge coloring of graphs. Having been inspired by the same we define the concept of restrained star edge coloring of graphs, which is a variation of the star edge coloring problem with an additional constraint as follows.

Definition 1.1. *Given a graph G, the proper edge coloring of G in which all its bichromatic subgraphs are in the form of a galaxy is referred to as the restrained star edge coloring of the graph. Where, a galaxy is a graph whose disjoint components are composed only of stars.*

Moreover, this can be achieved only when no path of length 3 in the graph is bicolored.

Definition 1.2. *The smallest integer $k \in \mathbb{N}$ for which a proper coloring of the graph G is obtained is termed as the restrained star chromatic index and is expressed as $\chi'_{rs}(G)$.*

In this paper we are using the abbreviated term 'RSE coloring' to denote the restrained star edge coloring of graphs. The next section is dedicated for some preliminaries and pre-existing results which are required for the understanding of the upcoming sections. In section 3, we present some properties and results of this coloring including the definite value of the χ'_{rs} of some graph families, and in section 4, we discuss one of the numerous possible application of this coloring in ensuring safe storage practices of hazardous chemicals.

2 Preliminaries

This division includes a few fundamentals that are essential in proving the theorems in the upcoming sections. It is to be noted that all the graphs considered in this paper are undirected, connected and finite.Chartrand and Zhang [2019]

A graph is said to be tree if there exists exactly one path between any pair of vertices. A tree where all the vertices except the two endpoints have degree 2 is a path graph . A path graph whose endpoints are adjacent, such that all its vertices have degree 2 is known as a cycle graph. A cycle graph is said to be a wheel if it has a vertex in the middle such that all the other vertices of the cycle are adjacent to this vertex.

A complete bipartite graph is a graph that can be separated into two partitions in such a way that each vertex of the partition-1 is adjacent to every other vertex in partition-2 and no two vertices of the same partition are connected by an edge. A Petersen graph is a non planar graph consisting of 10 vertices and 15 edges whose chromatic index is 4.

Lemma 2.1. *Deng [2007], Wang et al. [2019]. A cycle of order* $n \geq 3$ *has a proper 3-star edge coloring except when* $n \neq 5$ *and 4-star edge coloring when* $n = 5$

Lemma 2.2. *Wang et al. [2019]. The star chromatic index of a wheel graph with order* $n \geq 4$ *is described as*

$$\chi'_s(W_n) = \begin{cases} n+2 \text{ if } n = 5 \\ n+1 \text{ if } n = 4,6,7 \\ n-1 \text{ if } n \geq 8 \end{cases}$$

Lemma 2.3. *Chartrand and Zhang [2019]. The chromatic index of a complete bipartite graph is*

$$\chi'(k_{m,n}) = max\{m, n\}$$

Lemma 2.4. *(Vizing's Theorem) Chartrand and Zhang [2019]. For any graph G with maximum degree* Δ*, satisfies the inequality,*

$$\Delta \leq \chi'(G) \leq \Delta + 1$$

3 Restrained star edge coloring

In this section we provide some results on the RSE coloring and its corresponding restrained star chromatic index, χ'_{rs}. In addition the RSE coloring for certain families of graphs is also discussed.

Remark 3.1. *Any graph G satisfies the inequality, $\chi(G) \leq \chi'_s(G) \leq \chi'_{rs}(G)$ where $\chi(G), \chi'_s(G)$ and $\chi'_{rs}(G)$ represents the chromatic index, star chromatic index and restrained star chromatic index of the graph G respectively.*

Remark 3.2. *For every graph G with order, $n \geq 4$, $\chi'_{rs}(G) \geq 3$*

Example 3.3. *The Petersen's graph exhibits a proper 5-RSE coloring.*

From Remark 3.1 we obtain the inequalities $3 \leq \chi' \leq 4$ and $\chi'(G) = 4 \leq \chi'_{rs}(G)$. The structure of the Petersen's graph consists of K_5 without the outer edges, whose vertices are joined to vertices of an outer cycle C_5 by an edge. Now the steps involved in coloring the Petersen's graph can be separated into three parts namely, coloring the C_5, the inner K_5 without the outer edges and finally coloring the inner edges joining C_5 and K_5. It is evident that 5 distinct colors are required to properly color a 5-cycle. Thus, $5 \leq \chi'_{rs}(G)$. The inner K_5 without its outer edges is equal to a C_5, since the vertices of the outer and inner cycle are separated by edges, they can be colored using the same 5 colors. By making sure that the parallel colors are distinct, the edge joining these two cycles can be colored by the remaining color that is not used by the 4 adjacent edges. As per these coloring no path of length 3 is bicolored, such that the coloring is proper.

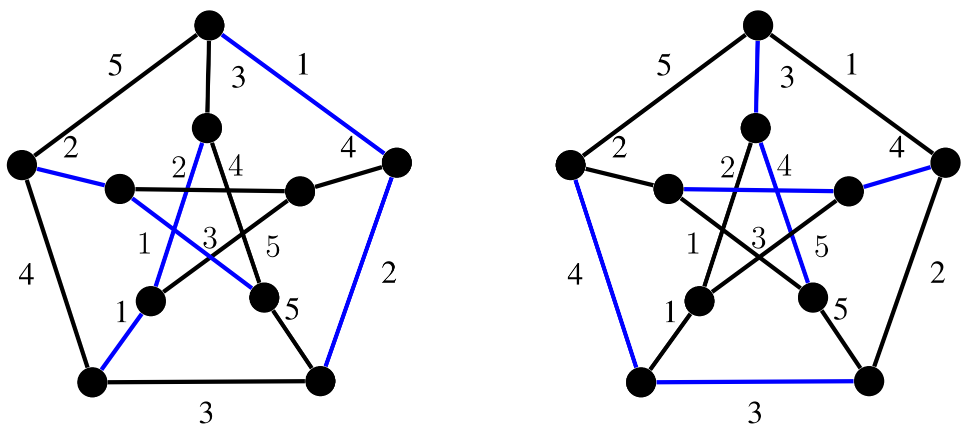

Figure 1: Some of the bichromatic subgraphs of RSE colored Petersen's graph is highlighted

Remark 3.4. *Every bichromatic subgraph of the graph G that is properly RSE colored results in a forest composing only of stars of orders 2 and 3.*

Proof. Now a forest is a graph that contains disjoint union of trees. Assume a graph of order n, the following two cases are possible as per the assignment of the colors.

Case 1. Suppose if a color pair occur in consecutive adjacent edges, then its bichromatic subgraph will be of the form S_3 and since as per our definition of RSE coloring, 2 colors cannot consecutively occur more than once, the order of the star graph cannot be more than 3.

Case 2. Suppose if the pair of colors did not occur in consecutive adjacent edges, then its bichromatic subgraph will consist of disjoint star graphs of order 2, S_2 and since we are considering the edges, the minimum order of the bichromatic subgraph cannot be less than 2.

Therefore, if some pair of colors occur as adjacent pairs in one position and individually in another position, then its bichromatic subgraph will consist of disjoint star graphs of order 2 and 3 only. □

Proposition 3.5. *For any path graph P_n with $n \geq 4$, $\chi'_{rs}(P_n) = 3$*

Proof. Let us consider a path of order 4, consisting of three edges. Thus, it is obvious that 3 unique colors are required to properly color the graph as per the definition. Suppose the colors used are c_1, c_2, c_3, As the order of the graph increases, it is evident that in each subsequent cases it is possible to color the graph such that each color is repeated only at a distance of three. Since these three colors are distinct, it is possible to color the graph P_n using these colors. □

Theorem 3.6. *For any cycle graph C_n of order $n \geq 3$,*

$$\chi'_{rs}(C_n) = \begin{cases} 3 \text{ if } 3|n \\ 5 \text{ if } n = 5 \\ 4 \; otherwise \end{cases}$$

Proof. We shall examine the possibilities for χ'_{rs} by taking two cases on the order of the cycle, C_n.

Case 1. In the case of a cycle of length 5, let the edges be $\{e_1, e_2, e_3, e_4, e_5\}$. It can be noted from Remark 3.1 that $\chi'_{rs}(C_5) \geq 3$. Thus, $3 \leq \chi'_{rs}(C_5) \leq 5$. Now, $\chi'_{rs}(C_5) = 5$ as the usage of 4 colors will result in at least one bichromatic subgraph becoming a P_4 which is not a star.

Case 2. When $n \neq 5$, we examine the coloring of the graph with the utilization of three cases.

Subcase 2.1. Suppose $n = 3k$, where $k = \{1, 2, 3, \dots\}$. Then for all values of k, the edge set $\{e_1, e_2, e_3, \dots, e_{3k}\}$ can be separated into k paths of distance 3 consisting of edges $\{e_{3i+1}, e_{3i+2}, e_{3i+3}\}$ where $i = \{0, 1, 2, \ldots, (k-1)\}$ and even if the color class$\{c_1, c_2, c_3\}$ is repeated no path of length 3 will become bicolored, thus ensuring all its bichromatic subgraphs form a galaxy. Thus, $\chi'_{rs}(G) = 3$, as per the coloring defined below.

$$c(e_i) = \begin{cases} c_1 \text{ if } i \equiv 1 \ (mod\, 3) \\ c_2 \text{ if } i \equiv 2 \ (mod\, 3) \\ c_3 \text{ if } i \equiv 0 \ (mod\, 3) \end{cases} \quad \forall\, i \in \mathbb{N} \; and \; i \leq n$$

Subcase 2.2. Suppose $n = 3k + 1$, then for any value of $k \in \mathbb{N}$, the edge set will consist of k paths of distance 3 consisting of edges consisting of edges $\{e_{3i+1}, e_{3i+2}, e_{3i+3}\}$ where $i = \{0, 1, 2, \ldots (k-1)\}$ along with 1 additional edge e_k which cannot be colored using the same color class. Thus, we require an additional color c_4 to reach a proper coloring, therefore $\chi'_{rs}(G) = 4$, as per the coloring defined below.

$$c(e_i) = \begin{cases} c_1 \text{ if } i \equiv 1 \ (mod\, 3) \\ c_2 \text{ if } i \equiv 2 \ (mod\, 3) \\ c_3 \text{ if } i \equiv 0 \ (mod\, 3) \end{cases} \quad \forall\, i \in \mathbb{N} \; and \; i \leq n-1, c(e_i) = c_4 \; for \; i = n$$

Subcase 2.3. Suppose $n = 3k + 2$, then for any value of $k \in \mathbb{N}$, the edge set can be divided into k paths of distance 3 along with 2 additional edges. Depending on the nature of n, two possibilities remain.

Subcase 2.3.1. Suppose n is even, then separate the edge set to equal partition of length $n/2$. The edge partition would be $\{e_1, e_2, \ldots, e_l\}$ and $\{e_{l+1}, e_{l+2}, \ldots, e_{2l}\}$ where, $l = 2k$ color the first edge of both the partition $\{e_1, e_{l+1}\}$ using the color c_1. The remaining edges $\{e_1, e_2 \ldots, e_{2l}\}/\{e_1, e_{l+1}\}$ of the partitions can be colored using the rest of colors in the color class $\{c_2, c_3, c_4\}$ by repeating the colors between edges of distance more than 3, thus $\chi'_{rs}(G) = 4$. The coloring can be defined as follows,

$$c(e_i) = \begin{cases} c_2 \text{ if } i \equiv 2 \ (mod\, 3) \\ c_3 \text{ if } i \equiv 0 \ (mod\, 3) \\ c_4 \text{ if } i \equiv 1 \ (mod\, 3) \end{cases} \quad \forall\, i = \{2, 3 \ldots, l, l+2 \ldots, 2l\}$$

and $c(e_i) = c_1 \; for \; i = 1, l+1$

Subcase 2.3.2. Suppose n is odd, then separate the edge set into two subsequent paths of length 4 consisting of edges $\{e_{4i+1}, e_{4i+2}, e_{4i+3}, e_{4i+4}\}$ for $i = \{0, 1\}$ and $k-2$ subsequent paths of length 3 consisting of edges $\{e_{3i}, e_{3i+1}, e_{3i+2}\}$ where $i =$

$\{3, 4, 5, \ldots k\}$. The paths of length 4 are colored using the colors $\{c_1, c_2, c_3, c_4\}$ and the paths of length 3 are colored using the colors $\{c_1, c_2, c_3\}$ by ensuring that the colors are repeated only at a distance more than 3, thus $\chi'_{rs}(G) = 4$ □

Theorem 3.7. *For any Wheel graph W_n of order $n \geq 4$ and $k \in \mathbb{N}$,*

$$\chi'_{rs}(W_n) = \begin{cases} n+4 & \textit{if } n = 6 \\ n+2 & \textit{if } n = 3k+1 \\ n+3 & otherwise \end{cases}$$

Proof. The process of coloring a wheel can be divided to two parts namely, coloring the edges of the outer cycle and coloring the edges connecting the inner vertex with the cycle. Now, since the a wheel graph of n vertices consist of a cycle of length $n-1$ and $n-1$ edges incident from the center vertex with the inner vertex being adjacent to all other vertices, the colors required to color the edges connecting the inner vertex and the outer cycle cannot be from the same color class.

Case 1. Let $n = 6$, it is trivial that 5 distinct colors are needed to color the edges incident with the center vertex and 5 other distinct colors are necessary to color the outer cycle as each edge is at a length of at most 3, then $\chi'_{rs}(G) = n + 4$.

Case 2. Let $n \neq 6$, then we observe the following cases.

Subcase 2.1 Suppose $n = 3k, k \in \mathbb{N}$, then the graph contains $3k - 1$ edges connecting the outer cycle with center vertex that can be properly colored only with the usage of $3k - 1$ colors, say $\{c_i : where\ i = 1, 2 \ldots, 3k - 1\}$. Now the outer cycle can be colored using 4 additional colors as defined in theorem 3.6 using the colors $\{c_j : where\ j = 3k \ldots, 3k+3\}$. Thus, $3k+3$ colors are required in other words $n + 3$ colors.

Subcase 2.2 Suppose $n = 3k + 1, k \in \mathbb{N}$, then the graph contains $3k$ edges connecting the outer cycle with center vertex that can be properly colored only with the usage of $3k$ colors, say $\{c_i : where\ i = 1, 2 \ldots, 3k\}$. Now the outer cycle can be colored using 3 additional colors as defined in theorem 3.6 using the colors $\{c_j : where\ j = 3k + 1, \ldots, 3k + 3\}$. Thus, $3k + 3$ colors are required in other words $\chi'_{rs}(G) = n + 2$.

Subcase 2.3 Suppose $n = 3k + 2, k \in \mathbb{N}$, then the graph contains $3k + 1$ edges connecting the outer cycle with center vertex that can be properly colored only with the usage of $3k + 1$ colors, say $\{c_i : where\ i = 1, 2 \ldots, 3k + 1\}$. Now the outer cycle can be colored using 4 additional colors as defined in 3.6 using the colors $\{c_j : where\ j = 3k+2, \ldots, 3k+5\}$. Thus, $3k+5$ colors are required, thus $\chi'_{rs}(G) = n+3$. □

Proposition 3.8. *A complete bipartite graph $k_{m,n}$ with $m, n \geq 2$ has a proper mn-RSE coloring.*

Proof. Consider a complete bipartite graph consisting of $m+n$ vertices and mn edges such that $m > n$. A complete bipartite graph expresses a chromatic index of $max\{m, n\}$, which is in turn equal to its maximum degree Δ. Then, according to our hypothesis $\chi'(G) = m$. Therefore, with the usage of Remark 3.1 we acquire the inequality $m \leq \chi'_{rs}(k_{m,n}) \leq mn$.

Suppose $\chi'_{rs}(G) < mn$, then it implies it is possible to star color the graph properly with $mn-1$ colors. As per the structure of a complete bipartite graph considered, there exist at least one path of P_3 between every two vertices. This would result in at least one path of length 3 becoming bicolored.

Thus, it is not possible to find a proper coloring using $mn-1$ colors, resulting in a contradiction. Which in turn would imply that $\chi'_{rs}(G) = mn$, and the coloring using mn distinct colors is the remaining possibility for a proper restrained star coloring of the graph. □

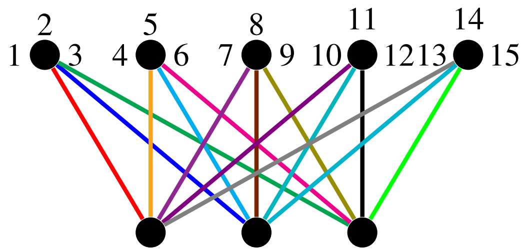

Figure 2: The RSE coloring of $k_{5,3}$

4 Application of Restrained star edge coloring

In this section, we explain one of the many possible applications of RSE coloring of graphs in the proper utilization of storage spaces. Out of all kinds of goods that are stored, the storage of chemicals must be handled with extreme caution as possible misplacing could end up in severe damages. Certain incompatible chemicals should not be stored together as their neutralization reactions might result in production of heat and gas, which in turn might even cause fire to erupt. Such cases have also been reported in areas where the precautionary methods were not followed. And some of these fires are hard to put off due to its chemical reaction with the atmosphere. Thus, the importance of ensuring safe storage of such chemicals is extremely significant.

Generally, for the safety of the environment and our well being, it is absolutely necessary that these chemicals that are incompatible ought to be stored in separate divisions (rooms) if possible but such kind of storage spaces is not available in all laboratories that is low on space such as the chemistry laboratories in schools and colleges. In such cases, it is said that the chemicals should be placed at least at a distance of 3 meters apart. This is where the concept of RSE coloring can be used to ensure that such incompatible chemicals are placed at a distance of at least 3 meters as according to its definition, every bichromatic subgraph is form a galaxy. Therefore, by converting the area available to a graph and by finding its χ'_{rs} we can find the optimal placement of such chemicals, thus minimizing the risk of fire.

Firstly, we assign color labels to the chemicals such that chemicals that cannot be stored together are labelled using the same color. This is applicable as per the basic definition of graph coloring which would ensure that a particular color cannot be positioned together with the same color. Thus, the chemicals labelled by the color say c_1 cannot be placed together with another chemical with the same labelling. After this we compare the storage area available to square grid of length 1.5 meters and construct the graph by plotting its vertices and edges. Then by finding the RSE-coloring of the graph thus constructed we can find the optimal and safe way to store these chemicals.

Algorithm:

Step 1: Compare the area to a grid and figure out the area available for storage

Step 2: Select the area satisfying the requirement and ignore the remaining areas

Step 3: Plot the vertices of the graph by assigning a vertex to every adjacent corner in such a way that each square contains 2 vertices.

Step 4: Join the vertices to get the edges of the graph.

Step 5: Find the RSE coloring of the graph obtained from the previous steps.

Step 6: Convert the coloring obtained by assigning the said color to the squares of the grid.

Step 7: This colored squares would provide the optimal way to store the chemicals in a safe manner.

Example:

Consider the area map as given in the Figure 3, firstly we compare the area with grids of length 1.5 meters and discard the squares whose area is less than that of 2.25 metres. Then two squares will make a distance of 3 meters and two squares diagonally will have a distance of 4 metres between them.

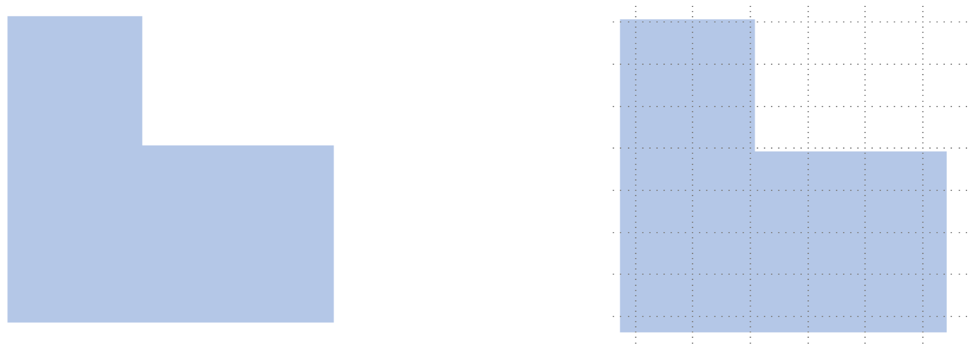

Figure 3: The area map and its comparison with a grid of length 1.5m

Plot the vertices on the corner of each grid in a way that each grid would have two diagonal vertices as illustrated in Figure 3. Then by joining these two vertices of each square we get the edges. And by joining all the vertices we get a connected graph.

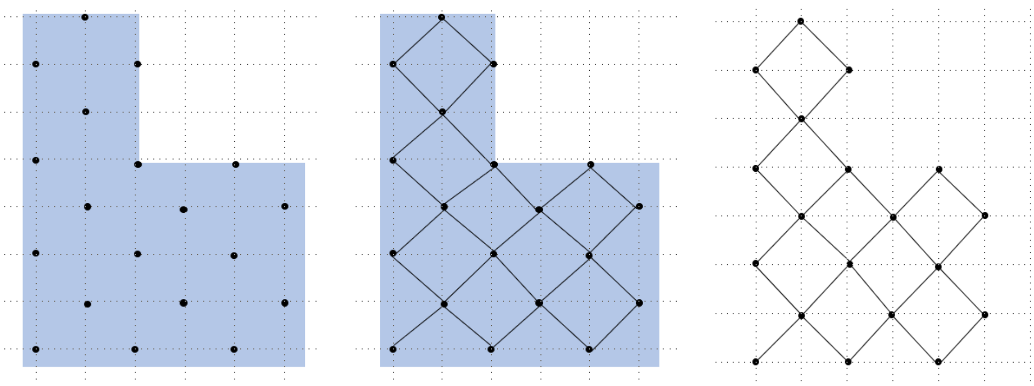

Figure 4: The process of plotting the graph

Then by finding the RSE coloring of this graph, we can find the optimal way to store the chemicals in such a way that every incompatible chemicals are kept at a minimum distance of 3 meters.

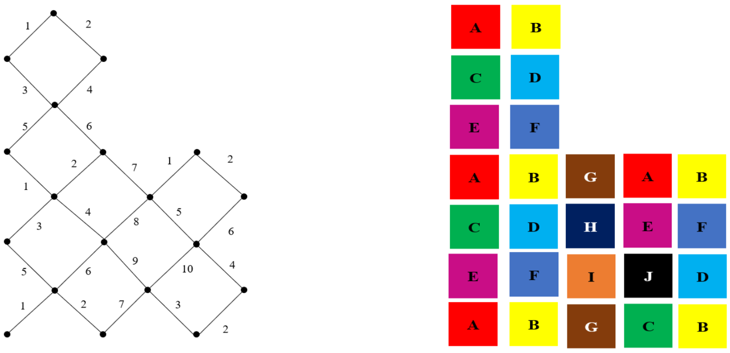

Figure 5: The RSE coloring and the optimal storage as per the area available

It can be noted that each chemical denoted by a particular color say (red) is incompatible to a chemical of the same color (in this case, red). The color that is distinct from that particular color (red) is used to denote either a chemical that is compatible with the chemical of that particular color or empty spaces if no other such chemicals are available. In this we have used the RSE coloring and its respective χ'_{rs} to find the optimal way of storage of these hazardous chemicals. However it can be noted that in the case of smaller scale the same results can be achieved with the help of restrained star vertex coloring and its respective restrained star chromatic number as well.

5 Results and Discussions

The RSE coloring of graphs is found to be an alluring topic with boundless prospects. It is to be noted that it is a challenging task to find the exact value of χ'_{rs} when the graph under consideration is complex, especially if the said graphs is obtained as a result of some graph operations. In such cases this paper would serve as a basis for understanding the said coloring of the basics of such complex graphs and would help in further generalizations of the RSE coloring.

6 Conclusions

Thus, in this paper the RSE coloring of few families of graphs including path, cycle, wheel, complete bipartite graph is defined and the definite values of its respective χ'_{rs} is provided. In addition to this, a possible application of RSE coloring in ensuring safe storage of chemicals is discussed in section 4. It was observed that for any graph G its star chromatic index is less than or equal to restrained star chromatic index, in other words $\chi'_s(G) \leq \chi'_{rs}(G)$.

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