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Weighted Sharing of Meromorphic Functions Concerning Certain Type of Linear Difference Polynomials

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Abstract

In this research article, with the help of Nevanlinna theory we study the uniqueness problems of transcendental meromorphic functions having finite order in the complex plane $\mathbb{C}$, of the form is given by $\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)$ and $\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)$ where $L(z,\phi) = \sum_{j=1}^{d} a_j\phi(z+c_j)$ which share a non-zero polynomial $p(z)$ with finite weight. By considering the concept of weighted sharing introduced by I. Lahiri (Complex Variables and Elliptic equations,2001,241-253), we investigate difference polynomials for the cases $(0,2),(0,1),(0,0)$. Our new findings extends and generalizes some classical results of Sujoy Majumder[11]. Some examples have been exhibited which are relevant to the content of the paper.

Keywords: Meromorphic functions, Linear difference polynomial, weighted sharing, uniqueness, non-zero polynomial.
2020 AMS subject classifications: 30D35 [1]

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1 Introduction

The Nevanlinna theory is a branch of complex analysis that has seen extensive research work, which mainly deals with the study of distribution of solutions of the equation $f(z) = a$ in a disc $|z| \leq r$, where $f(z)$ is an meromorphic(entire) function in the complex plane $\mathbb{C}$ and $a \in \mathbb{C} \cup \{\infty\}$. In recent years many researchers have been interested in value distribution of meromorphic functions. The proofs in this paper uses the Nevanlinna theory and one can refer(W. K Hayman[4], C. C Yang and H. X. Yi[13]) for the standard notations and definitions.

Let ϕ and ψ be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ and $k \in \mathbb{Z}^+ \cup \infty$, the set $E(a, \phi) = \{z : \phi(z) - a = 0\}$ denotes all those $a-$ points of ϕ, where each $a-$ point of ϕ with multiplicity k is counted k times in the set and the set $\overline{E}(a, \phi) = \{z : \phi(z) - a = 0\}$ denotes all those $a-$ points of ϕ, where the multiplicities are ignored. If $\phi(z) - a$ and $\psi(z) - a$ assumes the same zeros with the same multiplicities, then we say that $\phi(z)$ and $\psi(z)$ share the value a CM (Counting Multiplicity) and we have $E(a, \phi) = E(a, \psi)$. Suppose if $\phi(z) - a$ and $\psi(z) - a$ assumes the same zeros ignoring the multiplicities, then we say that $\phi(z)$ and $\psi(z)$ share the value a IM (Ignoring Multiplicity) and we will have $\overline{E}(a, \phi) = \overline{E}(a, \psi)$.

In general, for a meromorphic function $\phi(z)$, the quantity $m(r, \phi)$ denotes the proximity function of $\phi(z)$, while $N(r, \phi)$ denotes the counting function of poles $\phi(z)$ whose multiplicities are taken into account (respectively $\overline{N}(r, \phi)$ denotes the reduced counting function when multiplicities are ignored). The quantity $N(r, a; \phi)$ denotes the counting function of $a-$ points of $\phi(z)$ whose multiplicities are taken into account(respectively $\overline{N}(r, a; \phi)$ denotes the reduced counting function when multiplicities are ignored). The notation $N(r, a; \phi| = 1)$ denotes the counting function of simple $a-$ points of ϕ and the notation $N(r, a; \phi| \geq 2)$ denotes the counting function of those $a-$ points of ϕ whose multiplicities are atleast 2(respectively $\overline{N}(r, a; \phi| = 1)$ and $\overline{N}(r, a; \phi| \geq 2)$ denotes the reduced counting functions).

Suppose ϕ and ψ share 1 IM z_0 is a zero of $\phi(z) - 1$ of order s and also a zero $\psi(z) - 1$ of order t, then $\overline{N}_L(r, 1; \phi)$ counts those 1-points of $\phi(z)$ and $\psi(z)$ where $s > t$, $\overline{N}_E^{1)}(r, 1; \phi)$ counts those 1-points of $\phi(z)$ and $\psi(z)$ where $s = t = 1$, $\overline{N}_E^{(2}(r, 1; \phi)$ counts those 1-points of $\phi(z)$ and $\psi(z)$ where $s = t \geq 2$, $\overline{N}_{\phi>2}(r, 1; \psi)$ counts those 1-points of $\phi(z)$ and $\psi(z)$ where $s > t = 2$. It is to be noted that each point in these counting functions are counted only once. Similarly $\overline{N}_L(r, 1; \psi)$, $\overline{N}_E^{(2}(r, 1; \psi)$, $\overline{N}_{\psi>2}(r, 1; \phi)$ are defined.

The Nevanlinna characteristic function of a meromorphic function ϕ plays a very important role in the value distribution theory and it is denoted by $T(r, \phi)$. We have $T(r, \phi) = m(r, \phi) + N(r, \phi)$, which clearly shows that $T(r, \phi)$ is nonnegetive. A meromorphic function $\alpha(z)$ is called a small function with respect to $\phi(z)$, if $T(r,\alpha) = S(r,\phi)$, where $S(r,\phi)$ any quantity satisfying $S(r,\phi) = o\{T(r,\phi)\}$ as $r \to \infty$ possibly outside a set of finite linear measure.

2 Definitions and Theorems

Definition 2.1. *[6] Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a,\phi)$ the set of all $a-$ points of ϕ where an a- points of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a,\phi) = E_k(a,\psi)$, we say that ϕ, ψ share the value a with weight k.*

Definition 2.2. *[13] Let $a \in \mathbb{C} \cup \{\infty\}$. For a positive integer k, we denote by*
(i) $N_{k)}(r, \frac{1}{\phi-a})$ the counting function of $a-$ points of f with multiplicity $\leq k$.
(ii) $N_{(k}(r, \frac{1}{\phi-a})$ the counting function of $a-$ points of ϕ with multiplicity $\geq k$. Similarly the reduced counting function $\overline{N}_{k)}(r, \frac{1}{\phi-a})$ and $\overline{N}_{(k}(r, \frac{1}{\phi-a})$ are defined.

Definition 2.3. *[6] Let $p \in \mathbb{N} \cup \{\infty\}$. We denote by $N_p(r,a;\phi)$ the counting function of $a-$ points of ϕ, where an a- point of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. Then*
$N_p(r,a;\phi) = \overline{N}(r,a,\phi) + \overline{N}(r,a,\phi| \geq 2) + ... + \overline{N}(r,a,\phi| \geq p)$. Clearly $N_1(r,a;\phi) = \overline{N}(r,a,\phi)$.

Definition 2.4. *[15] Let $\phi(z)$ be a meromorphic function, Linear difference polynomial $L(z,\phi)$ defined as follows.*
$L(z,\phi) = a_1(z)\phi(z+c_1) + a_2(z)\phi(z+c_2) + a_3(z)\phi(z+c_3).......a_d(z)\phi(z+c_d)$ be a linear difference polynomial, where $a_1(z), a_2(z)...a_d(z)$ are non-zero small functions relative to $\phi(z)$, where $c_1, c_2, c_3...c_d$ are complex constants.

In 2011, K. Liu, X. L, Liu and T. B Cao [9] studied the uniqueness of the difference monomials and obtained the following results.

Theorem 2.1. *[9] Let ϕ and ψ be two transcendental meromorphic functions with finite order. Suppose $c \in \mathbb{C}\backslash\{0\}$ and $n \in \mathbb{N}$. If $n \geq 14$, $\phi^n(z)\phi(z+c)$ and $\psi^n(z)\psi(z+c)$ share 1 CM, then $\phi(z) \equiv t\psi(z)$ or $\phi(z)\psi(z) \equiv t$, where $t^{n+1} = 1$.*

Theorem 2.2. *[9] Let ϕ and ψ be two transcendental meromorphic functions with finite order. Suppose $c \in \mathbb{C}\backslash\{0\}$ and $n \in \mathbb{N}$. If $n \geq 26$, $\phi^n(z)\phi(z+c)$ and $\psi^n(z)\psi(z+c)$ share 1 IM, then $\phi(z) \equiv t\psi(z)$ or $\phi(z)\psi(z) \equiv t$, where $t^{n+1} = 1$.*

In 2015, Y. Liu, J. P. Wang and F. H. Liu [10] improved Theorems 2.1 and 2.2 obtained the following results.

Theorem 2.3. *[10] Let $c \in \mathbb{C}\backslash\{0\}$ and let $\phi(z)$ and $\psi(z)$ be two transcendental meromorphic functions with finite order, and $n(\geq 14), k(\geq 3)$ be two positive integers. If $E_k(1, \phi^n(z)\phi(z+c)) = E_k(1, \psi^n(z)\psi(z+c))$, then $\phi(z) \equiv t_1\psi(z)$ or $\phi(z)\psi(z) \equiv t_2$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

Theorem 2.4. *[10] Let $c \in \mathbb{C}\backslash\{0\}$ and let $\phi(z)$ and $\psi(z)$ be two transcendental meromorphic functions with finite order and $n(\geq 16)$ be a positive integer. If $E_2(1, \phi^n(z)\phi(z+c)) = E_2(1, \psi^n(z)\psi(z+c))$, then $\phi(z) \equiv t_1\psi(z)$ or $\phi(z)\psi(z) \equiv t_2$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

Theorem 2.5. *[10] Let $c \in \mathbb{C}\backslash\{0\}$ and let $\phi(z)$ and $\psi(z)$ be two transcendental meromorphic functions with finite order, and $n(\geq 22)$ be a positive integers. If $E_1(1, \phi^n(z)\phi(z+c)) = E_1(1, \psi^n(z)\psi(z+c))$, then $\phi(z) \equiv t_1\psi(z)$ or $\phi(z)\psi(z) \equiv t_2$, for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

In 2017, Sujoy Majumder [11] replaced the sharing value 1 by a nonzero polynomial $p(z)$ in 2.3,2.4 and 2.5 and obtained the following results.

Theorem 2.6. *[11] Let ϕ and ψ be two transcendental meromorphic functions with finite order. Suppose $c \in \mathbb{C}\backslash\{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 14$. Let $p(\not\equiv 0)$ be a polynomial such that $deg(p) < (n-1)/2$. If $\phi^n(z)\phi(z+c) - p(z)$ and $\psi^n(z)\psi(z+c) - p(z)$ share (0,2), then one of the following two cases holds:*

1. *$\phi(z) \equiv t\psi(z)$ for some constant t such that $t^{n+1} = 1$,*
2. *$\phi(z)\psi(z) \equiv t$, where $p(z)$ reduces to a non-zero constant c and t is a constant such that $t^{n+1} = c^2$.*

Theorem 2.7. *[11] Let ϕ and ψ be two transcendental meromorphic functions with finite order. Suppose $c \in \mathbb{C}\backslash\{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 16$. Let $p(\not\equiv 0)$ be a polynomial such that $deg(p) < (n-1)/2$. If $\phi^n(z)\phi(z+c) - p(z)$ and $\psi^n(z)\psi(z+c) - p(z)$ share (0,1). Then the conclusion of Theorem 2.6 holds.*

Theorem 2.8. *[11] Let ϕ and ψ be two transcendental meromorphic functions with finite order. Suppose $c \in \mathbb{C}\backslash\{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 26$. Let $p(\not\equiv 0)$ be a polynomial such that $deg(p) < (n-1)/2$. If $\phi^n(z)\phi(z+c) - p(z)$ and $\psi^n(z)\psi(z+c) - p(z)$ share (0,0). Then the conclusion of Theorem 2.6 holds.*

It is quite natural to ask the following question.

Question 1. Suppose $L(z, \phi) = \sum_{j=1}^{d} a_j\phi(z + c_j)$ is a linear difference polynomial of finite order of meromorphic function ϕ with $a_1(z), a_2(z)...a_d(z)$ are nonzero small functions relative to $\phi(z)$, where $c_1, c_2, c_3...c_d$ are complex constants, then what can we say about relation between two finite order non-constant meromorphic functions ϕ and ψ, if their linear difference polynomials $\phi^n(z)\sum_{j=1}^{d} a_j \phi(z+c_j)$ and $\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)$ share $p(z)$?

In this paper, we have attempted to answer the above question successfully.

Following are the main results of our paper.

3 Main Results

Theorem 3.1. *Let ϕ and ψ be two transcendental meromorphic functions of finite order and n be a positive integer such that $n > 4d+5$. Suppose that $c_j \in \mathbb{C}\backslash\{0\}$ for $j = 1, 2, 3...d$. Let $\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j) - p(z)$ and $\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j) - p(z)$ share $(0, 2)$ where p be a nonzero polynomial such that $deg(p) < (n-d)/2$, then one of the following two cases holds:*

1. *$\phi(z) \equiv t\psi(z)$ for some constant t such that $t^{n+d} = 1$,*
2. *$\phi(z)\psi(z) \equiv t$, where $p(z)$ reduces to a nonzero constant c and t is a constant such that $t^{n+d} = c^2$.*

Theorem 3.2. *Let ϕ and ψ be two transcendental meromorphic functions of finite order and n be a positive integer such that $n > \frac{9d+11}{2}$. Suppose that $c_j \in \mathbb{C}\backslash\{0\}$ for $j = 1, 2, 3...d$. Let $\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j) - p(z)$ and $\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j) - p(z)$ share $(0, 1)$ where p be a nonzero polynomial such that $deg(p) < (n-d)/2$, then the conclusion of Theorem 3.1 holds.*

Theorem 3.3. *Let ϕ and ψ be two transcendental meromorphic functions of finite order and n be a positive integer such that $n > 6d+8$. Suppose that $c_j \in \mathbb{C}\backslash\{0\}$ for $j = 1, 2, 3...d$. Let $\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j) - p(z)$ and $\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j) - p(z)$ share $(0, 0)$ where p be a nonzero polynomial such that $deg(p) < (n-d)/2$, then the conclusion of Theorem 3.1 holds.*

Remark 3.1. *Theorems 3.1 - 3.3 extends and generalizes of Theorems 2.6 - 2.8 respectively.*

Example 1. Let $\phi(z) = sinz$, $\psi(z) = cosz$. Choose $c = 2\pi, a_j = 1$ then $\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)$ and $\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)$ share $p(z)$. Clearly we get $\phi \equiv t\psi$ for a constant t.

Example 2. Let $\phi(z) = (7z-3)^2(z+1)e^{(z-1)^3}$, $\psi(z) = (7z-3)^2(z+1)e^{-(z-1)^3}$. Choose $c_j = 1, a_j = 1$ then $\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)$ and $\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)$ share $p(z)$. Clearly we get $\phi \not\equiv t\psi$ for a constant t.

4 Auxiliary Lemmas

In this section, we present some necessary Lemmas.
Denote H by the following function.

$$H = \Big(\frac{\Phi''}{\Phi'} - \frac{2\Phi'}{\Phi - 1}\Big) - \Big(\frac{\Psi''}{\Psi'} - \frac{2\Psi'}{\Psi - 1}\Big). \tag{1}$$

Lemma 4.1. *[12] Let ϕ be a non-constant meromorphic function and let $a_1, a_2, ...a_n$ be finite complex numbers, $a_n \not\equiv 0$, $a_{n-1}, ..., a_0$ be meromorphic functions such that $T(r, a_i) = S(r, f)$ for $i = 0, 1, 2...n$ Then*

$$T(r, a_n\phi^n + a_{n-1}\phi^{n-1} + ... + a_1\phi + a_0) = nT(r, \phi) + S(r, \phi).$$

Lemma 4.2. *[3] Let ϕ be a non-constant meromorphic function of finite order ρ and let $c \in \mathbb{C}\backslash\{0\}$ be fixed. Then for each $\epsilon > 0$, we have*

$$m\left(r, \frac{\phi(z+c)}{\phi(z)}\right) + m\left(r, \frac{\phi(z)}{\phi(z+c)}\right) = O\left(r^{\rho-1+\epsilon}\right) = S(r, \phi).$$

The following lemma is a slight modification of the original version.

Lemma 4.3. *[5] Let ϕ be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then*

$$\begin{aligned} N(r, \infty; \phi(z+c)) &\leq N(r, \infty; \phi) + S(r, \phi) \\ N(r, 0; \phi(z+c)) &\leq N(r, 0; \phi) + S(r, \phi) \\ \overline{N}(r, \infty; \phi(z+c)) &\leq \overline{N}(r, \infty; \phi) + S(r, \phi) \\ \overline{N}(r, 0; \phi(z+c)) &\leq \overline{N}(r, 0; \phi) + S(r, \phi). \end{aligned}$$

Lemma 4.4. *[3] Let ϕ be a transcendental meromorphic function of finite order, $c \in \mathbb{C}\backslash\{0\}$ be fixed then*

$$T(r, \phi(z+c) = T(r, \phi) + S(r, \phi).$$

Lemma 4.5. *[14] Let ϕ and ψ be two non-constant meromorphic functions. Then*

$$N(r, \infty; \frac{\phi}{\psi}) - N(r, \infty; \frac{\psi}{\phi}) = N(r, \infty; \phi) + N(r, 0; \psi) - N(r, \infty; \psi) - N(r, 0; \phi).$$

Lemma 4.6. *Let ϕ be a transcendental meromorphic function of finite order and let $\Phi = \phi^n(z)\sum_{j=1}^{d} a_j\phi(z + c_j)$, where n is positive integer. Then*

$$(n - d)T(r, \phi) \leq T(r, \Phi) + S(r, \phi).$$

Proof. By Lemmas 4.1,4.2 and First Fundamental Theorem, we obtain

$$\begin{aligned}
(n+d)T(r,\phi) &= T(r,\phi^{n+d}) + S(r,\phi),\\
&\leq T\left(r, \frac{\phi^d(z)\Phi}{\sum_{j=1}^{d} a_j\phi(z+c_j)}\right) + S(r,\phi),\\
&\leq T(r,\Phi) + T\left(r, \frac{\phi^d(z)}{\sum_{j=1}^{d} a_j\phi(z+c_j)}\right) + S(r,\phi),\\
&\leq T(r,\Phi) + N\left(r, \frac{\sum_{j=1}^{d} a_j\phi(z+c_j)}{\phi^d(z)}\right) + S(r,\phi),\\
&\leq T(r,\Phi) + 2dT(r,\phi) + S(r,\phi),\\
(n-d)T(r,\phi) &\leq T(r,\Phi) + S(r,\phi).
\end{aligned}$$

This completes the proof of Lemma 4.6.

Lemma 4.7. *Let ϕ and ψ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}\backslash\{0\}$ and $n \in \mathbb{N}$ such that $n > d$. Let $p(z)$ be a nonzero polynomial such that $deg(p) < (n-d)/2$. Then*

1. *if $deg(p) \geq 1$, then $\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j).\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j) \not\equiv p^2(z)$;*
2. *if $p(z) = c \in \mathbb{C}\backslash\{0\}$, then the relation $\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)\psi^n(z)$ $\sum_{j=1}^{d} a_j\psi(z+c_j) \equiv p^2(z)$, always implies $\phi.\psi = t$, where t is a constant such that $t^{n+d} = c^2$.*

Proof. Suppose,

$$\left(\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)\right)\left(\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)\right) \equiv p^2(z). \tag{2}$$

Let $h_1 = \phi\psi$ then by equation 2, we have

$$h_1^n(z) = \frac{p^2(z)}{\sum_{j=1}^{d} h_1 a_j(z+c_j)} \tag{3}$$

We now consider the following two cases,
Case 1. *Suppose h_1 is a transcendental meromorphic function. Now by Lemmas*

4.1, 4.2, 4.3 we get

$$\begin{aligned} nT(r,h_1) = T(r,h_1^n) + S(r,h_1) &= T\left(r, \frac{p^2(z)}{\sum_{j=1}^{d} h_1 a_j(z+c_j)}\right) + S(r,h_1), \\ &\leq N\left(r,0;\sum_{j=1}^{d} a_j\phi(z+c_j)\right) + N\left(r,0;\sum_{j=1}^{d} a_j\psi(z+c_j)\right) \\ &\quad + S(r,h_1), \\ &\leq d(T(r,\phi)+T(r,\psi)) + S(r,h_1). \end{aligned}$$

which is a contradiction.
Case 2. *Suppose h_1 is a rational function . Let*

$$h_1 = \frac{h_2}{h_3}, \tag{4}$$

where h_2 and h_3 are two non-zero relatively prime polynomials. By equation 4, we have

$$T(r,h_1) = max\{deg(h_2), deg(h_3)\}logr + O(1). \tag{5}$$

Now by equations 3-5, we have

$$\begin{aligned} n.\, max\{deg(h_2), deg(h_3)\}logr = T(r,h_1^n) + O(1) &\leq d(T(r,\phi)+T(r,\psi)) \\ &+ 2T(r,p) + O(1). \end{aligned} \tag{6}$$

We see that $max\{deg(h_2), deg(h_3)\} \geq 1$. Now by equation 6, we deduce that $(n-d)/2 \leq deg(p)$, which contradicts our assumption that $deg(p) < (n-d)/2$. Hence h_1 must be a non-zero constant. Let

$$h_1 = t \in \mathbb{C}\backslash\{0\}. \tag{7}$$

Now when $deg(p) \geq 1$, by equations 3 and 7, we arrive at a contradiction.Then

$$\left(\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)\right)\left(\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)\right) \not\equiv p^2(z).$$

Suppose $p(z) = c \in \mathbb{C}\backslash\{0\}$. So by equation 3 we see that $h_1^{n+d} \equiv c^2$. By equation 7 we get $t^{n+d} \equiv c^2$.
This completes the proof of Lemma 4.7.

Lemma 4.8. *Let ϕ and ψ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}\backslash\{0\}$ be finite complex constant such that for $(j=1,2...d)$ and n be* *a positive integer such that* $n \geq 4d+3$. *Let* $\Phi(z) = \left(\frac{\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)}{p(z)}\right)$ *and* $\Psi(z) = \left(\frac{\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)}{p(z)}\right)$, *where* $p(z)$ *is non-zero polynomial and* $H \equiv 0$, *then one of the following conclusions occur.*

(i) $\left(\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)\right)\left(\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)\right) \equiv p^2(z)$, *where* $\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j) - p(z)$ *and* $\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j) - p(z)$ *share 0 CM,*

(ii) $\phi(z) = t\psi(z)$ *for a constant t with* t^{n+d}=*1.*

Proof. By equation 1, $H \equiv 0$ *on integration we get*

$$\frac{1}{\Phi - 1} = \frac{B\Psi + A - B}{\Psi - 1}, \tag{8}$$

where A,B are constants and $A \neq 0$. *From equation 8 it is clear that* Φ *and* Ψ *share* $(1,\infty)$. *We now consider following cases.*
Case 1. *Let* $B \neq 0$ *and* $A \neq B$. *If* $B = -1$, *then from equation 8 we have*

$$\Phi = \frac{-A}{\Psi - A - 1}.$$

Therefore $\overline{N}(r, A+1; \Psi) = \overline{N}(r, \Phi) \leq N(r, 0; p) \leq T(r, p) = S(r, \psi)$. *So in view of Lemma 4.6 and Second Fundamental Theorem, we get*

$$\begin{aligned}
(n-d)T(r,\psi) &\leq T\left(r, \frac{\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)}{p(z)}\right) + S(r.\psi) \\
&\leq T(r,\Psi) + S(r.\psi), \\
&\leq \overline{N}(r,\infty;\Psi) + \overline{N}(r,0;\Psi) + \overline{N}(r,A+1;\Psi) + S(r,\psi), \\
&\leq 2(d+1)T(r,\psi) + S(r,\psi).
\end{aligned}$$

which contradicts $n > 3d+2$.
If $B \neq -1$, *from equation 8 we obtain that*

$$\Phi - \left(1 + \frac{1}{B}\right) = \frac{-A}{B^2\left(\Psi + \frac{A-B}{B}\right)}$$

$$\overline{N}\left(r, \frac{B-A}{B}; \Psi\right) = S(r,\psi).$$

Using the Lemma 4.6 and the same argument as used in the case when $B = -1$ we can get a contradiction.
Case 2. *Let $B \neq 0$ and $A = B$. If $B = -1$, then from equation 8, we have*

$$\left(\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)\right)\left(\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)\right) \equiv p^2(z).$$

when $\left(\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)\right) - p(z)$ and $\left(\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)\right) - p(z)$ share 0 CM. If $B \neq -1$, from equation 8 we have

$$\frac{1}{\Phi} = \frac{B\Psi}{(1+B)\Psi - 1}.$$

$$\overline{N}\left(r, \frac{1}{1+B}; \Psi\right) = \overline{N}(r, 0; \Phi) + S(r.\phi).$$

So in view of Lemmas 4.2, 4.6 and the Second Fundamental Theorem, we get

$$\begin{aligned}(n-d)T(r,\psi) &\leq T(r,\Psi) + S(r,\psi),\\ &\leq \overline{N}(r,\infty;\Psi) + \overline{N}(r,0;\Psi) + \overline{N}(r,(1+B);\Psi) + S(r,\psi),\\ &\leq 2(d+1)T(r,\psi) + (1+d)T(r,\phi) + S(r,\phi) + S(r,\psi),\\ (n-d-3d-3)T(r,\psi) &\leq S(r,\psi).\end{aligned}$$

which is contradiction, since $n > 4d+3$.
Case 3. *Let $B = 0$. From equation 8 we obtain*

$$\Phi = \frac{\Psi + A - 1}{A} \tag{9}$$

If $A \neq 1$, then from equation 9, we obtain $\overline{N}(r, 1-A; \Psi) = \overline{N}(r, 0; \Phi)$.
We can similarly deduce a contradiction as in Case 2. Therefore $A = 1$ and from equation 9, we obtain $\Phi(z) = \Psi(z)$. That is

$$\left(\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)\right) \equiv \left(\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)\right). \tag{10}$$

Let $h = \frac{\phi}{\psi}$, and then substituting $\phi = \psi h$ in equation 10 we deduce

$$h^{n+1} = \frac{\phi}{\psi\sum_{j=1}^{d} ha_j(z+c_j)}, \tag{11}$$

If h is not a constant , then we have

$$(n+1)T(r,h) \leq T\left(r, \frac{\phi}{\sum_{j=1}^{d} a_j\phi(z+c_j)}\right) + T\left(r, \frac{\sum_{j=1}^{d} a_j\psi(z+c_j)}{\psi}\right) + S(r,\phi) + S(r,\psi),$$

where $T(r,h)$=$T(r,\frac{\phi}{\psi}) = T(r,\phi) + T(r,\psi) + S(r,\phi) + S(r,\psi)$. *we obtain* $(n-d)[T(r,\phi) + T(r,\psi)] \leq S(r,\phi) + S(r,\psi)$, *which is impossible.*
Therefore h is a constant , then substituting $\phi = \psi h$ in equation 10, we have $h^{n+d} \equiv 1$. *Therefore $\phi = \psi t$, where t is a constant with $t^{n+d} = 1$.*

This completes the proof of Lemma 4.8

Lemma 4.9. *[7] If $N(r,0;\phi^{(k)}|\phi \neq 0)$ denotes the counting function of those zeros of $\phi^{(k)}(z)$ is counted according to its multiplicity, then*
$N(r,0;\phi^{(k)}|\phi \neq 0) \leq k\overline{N}(r,\infty;\phi) + N(r,0;\phi| < k) + k\overline{N}(r,0;\phi| \geq k) + S(r,\phi)$.

Lemma 4.10. *[1] If ϕ and ψ be two non-constant meromorphic functions such that they share $(1,1)$. Then*
$2\overline{N}_L(r,1;\phi) + 2\overline{N}_L(r,1;\psi) + \overline{N}_E^{(2}(r,1;\phi) - \overline{N}_{f>2}(r,1;\psi) \leq N(r,1;\psi) - \overline{N}(r,1;\psi)$.

Lemma 4.11. *[2] Let ϕ and ψ share $(1,1)$. Then*

$$\overline{N}_{f>2}(r,1;\psi) \leq \frac{1}{2}\overline{N}(r,0;\phi) + \frac{1}{2}\overline{N}(r,\infty;\phi) - \frac{1}{2}N_0(r,0;\phi') + S(r,\phi),$$

,
where $N_0(r,0;\phi')$ is the counting function of those zeros of ϕ' which are not the zeros of $\phi(\phi-1)$.

Lemma 4.12. *[2] If ϕ and ψ share $(1,0)$. Then*

1. $\overline{N}_{\phi>1}(r,1;\psi) \leq \overline{N}(r,0;\phi) + \overline{N}(r,\infty;\phi) - N_0(r,0;\phi') + S(r,\phi)$;
2. $\overline{N}_{\psi>1}(r,1;\phi) \leq \overline{N}(r,0;\psi) + \overline{N}(r,\infty;\psi) - N_0(r,0;\psi') + S(r,\phi)$.

Lemma 4.13. *[2] Let ϕ and ψ share $(1,0)$. Then*

$$\overline{N}_L(r,1;\phi) \leq \overline{N}(r,0;\phi) + \overline{N}(r,\infty;\phi) + S(r,\phi).$$

Lemma 4.14. *[2] Let ϕ and ψ be two non-constant meromorphic functions sharing $(1,0)$. Then*

$$\overline{N}_L(r,1;\phi) + 2\overline{N}_L(r,1;\psi) + \overline{N}_E^{(2}(r,1;\phi) - \overline{N}_{\phi>1}(r,1;\psi) - \overline{N}_{\psi>1}(r,1;\phi) \leq N(r,1;\psi) - \overline{N}(r,1;\psi).$$

5 Proof of Main results

Proof of Theorem 3.1.
Let $\Phi(z) = \left(\frac{\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)}{p(z)}\right)$ and $\Psi(z) = \left(\frac{\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)}{p(z)}\right)$, it follows that Φ and Ψ share $(1,2)$ except the zeros of $p(z)$.
Case 1. Let $H \not\equiv 0$ from 1, we obtain

$$N(r,\infty;H) \leq \overline{N}_*(r,1;\Phi,\Psi) + \overline{N}(r,0;\Phi| \geq 2) + N(r,0;\Psi| \geq 2) + \overline{N}_0(r,0;\Phi^{'}) + \overline{N}_0(r,0;\Psi^{'}). \tag{12}$$

where $\overline{N}_0(r,0;\Phi^{'})$ is the reduced counting function of those zeros of $\Phi^{'}$ which are not zeros of $\Phi(\Phi-1)$ and $\overline{N}_0(r,0;\Psi^{'})$ is similarly defined. Let z_0 be a simple zero of $\Phi-1$ such that $p(z_0)\neq 0$. Then z_0 is a simple zero of $\Psi-1$ and a zero of H. So

$$N(r,1;\Phi| = 1) \leq N(r,0;H) \leq N(r,\infty;H) + S(r,\phi) + S(r,\psi). \tag{13}$$

From equations 12 and 13 we get

$$\overline{N}(r,1;\Phi) \leq \overline{N}(r,0;\Phi| \geq 2) + \overline{N}(r,0;\Psi| \geq 2) + \overline{N}_*(r,1;\Phi,\Psi) + \overline{N}(r,1;\Phi| \geq 2) + \overline{N}_0(r,0;\Phi^{'}) + \overline{N}_0(r,0;\Psi^{'}) + S(r,\phi) + S(r,\psi). \tag{14}$$

Now in view of Lemma 4.9 we get

$$\overline{N}_0(r,0;\Psi^{'}) + \overline{N}(r,1;\Phi| \geq 2) + \overline{N}_*(r,1;\Phi,\Psi) \leq \overline{N}_0(r,0;\Psi^{'}) + \overline{N}(r,1;\Phi| \geq 2) + \overline{N}(r,1;\Phi| \geq 3) \leq N(r,0;\Psi^{'}|\Psi \neq 0) \leq \overline{N}(r,0;\Psi) + S(r,\psi). \tag{15}$$

Hence using equations 14, 15 and Lemmas 4.2, 4.6 we get from Second fundamental theorem that

$$\begin{aligned}(n-d)T(r,\phi) &\leq T(r,\Phi) + S(r,\phi)\\ &\leq \overline{N}(r,\infty;\Phi) + \overline{N}(r,0;\Phi) + \overline{N}(r,1;\Phi) - N_0(r,0;\Phi^{'}) + S(r,\phi),\\ &\leq \overline{N}(r,\infty;\Phi) + N_2(r,0;\Phi) + N_2(r,0;\Psi) + S(r,\phi) + S(r,\psi),\\ (n-d)T(r,\phi) &\leq (3+2d)T(r,\phi) + (2+d)T(r,\psi) + S(r,\phi) + S(r,\psi). \end{aligned} \tag{16}$$

Similarly, we can obtain

$$(n-d)T(r,\psi) \leq (3+2d)T(r,\psi) + (2+d)T(r,\phi) + S(r,\phi) + S(r,\psi). \tag{17}$$

Combining equations 16 and 17 we see that

$$(n-d-5-3d) \leq S(r,\phi) + S(r,\psi) \tag{18}$$

Since $n > 4d + 5$ equation 18 leads to a contadiction.
Case 2. Let $H \equiv 0$. Then the theorem follows from Lemmas 4.7 and 4.8.
This completes the proof of Theorem 3.1.

Proof of Theorem 3.2.
Let $\Phi(z) = \left(\frac{\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)}{p(z)}\right)$ and $\Psi(z) = \left(\frac{\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)}{p(z)}\right)$, it follows that Φ and Ψ share $(1,1)$ except the zeros of $p(z)$. We now consider the following two cases.
Case 1. $H \not\equiv 0$.
Using Lemmas 4.6 , 4.8 , 4.10 and equations 12 and 13 we get

$$\begin{aligned}
\overline{N}(r,1;\Phi) &\le N(r,1;\Phi|=1) + \overline{N}_L(r,1;\Phi) + \overline{N}_L(r,1;\Psi) + \overline{N}_E^{(2}(r,1;\Phi) \\
&\le \overline{N}(r,0;\Phi|\ge 2) + \overline{N}(r,0;\Psi|\ge 2) + \overline{N}_*(r,1;\Phi,\Psi) + \overline{N}_L(r,1;\Phi) \\
&\quad + \overline{N}_L(r,1;\Psi) + \overline{N}_E^{(2}(r,1;\Phi) + \overline{N}_0(r,0;\Phi^{'}) + \overline{N}_0(r,0;\Psi^{'}) \\
&\quad + S(r,\phi) + S(r,\psi), \\
&\le \overline{N}(r,0;\Phi|\ge 2) + \overline{N}(r,0;\Psi|\ge 2) + 2\overline{N}_L(r,1;\Phi) + 2\overline{N}_L(r,1;\Psi) \\
&\quad + \overline{N}_E^{(2}(r,1;\Phi) + \overline{N}_0(r,0;\Phi^{'}) + \overline{N}_0(r,0;\Psi^{'}) + S(r,\phi) + S(r,\psi) \\
&\le \overline{N}(r,0;\Phi|\ge 2) + \overline{N}(r,0;\Psi|\ge 2) + \overline{N}_{\Phi>2}(r,1;\Psi) + N(r,1;\Psi) \\
&\quad - \overline{N}(r,1;\Psi) + \overline{N}_0(r,0;\Phi^{'}) + \overline{N}_0(r,0;\Psi^{'}) + S(r,\phi) + S(r,\psi) \\
&\le \overline{N}(r,0;\Phi|\ge 2) + \frac{1}{2}\overline{N}(r,0;\Phi) + \overline{N}(r,0;\Psi|\ge 2) + \\
&\quad + N(r,0;\Psi^{'}|\Psi\ne 0) + \overline{N}_0(r,0;\Phi^{'}) + S(r,\phi) + S(r,\psi) \\
\overline{N}(r,1;\Phi) &\le \overline{N}(r,0;\Phi|\ge 2) + \frac{1}{2}\overline{N}(r,0;\Phi) + N_2(r,0;\Psi) + \overline{N}_0(r,0;\Phi^{'}) \quad (19) \\
&+ S(r,\phi) + S(r,\psi).
\end{aligned}$$

Hence by using equation 19, Lemmas 4.2 and 4.6 we get from Second Fundamental Theorem that

$$\begin{aligned}
(n-d)T(r,\phi) &\le T(r,\Phi) + S(r,\phi), \\
&\le \overline{N}(r,\infty;\Phi) + \overline{N}(r,0;\Phi) + \overline{N}(r,1;\Phi) - N_0(r,0;\Phi^{'}) + S(r,\phi), \\
&\le \overline{N}(r,\infty;\Phi) + \frac{1}{2}\overline{N}(r,0;\Phi) + N_2(r,0;\Phi) + N_2(r,0;\Psi) \\
&\quad + S(r,\phi) + S(r,\psi), \\
(n-d)T(r,\phi) &\le \frac{5d+7}{2}T(r,\phi) + (2+d)T(r,\psi) + S(r,\phi) + S(r,\psi). \quad (20)
\end{aligned}$$

Similarly,

$$(n-d)T(r,\psi) \leq \frac{5d+7}{2}T(r,\psi) + (2+d)T(r,\phi) + S(r,\phi) + S(r,\psi). \quad (21)$$

Combining equations 20 and 21 we get

$$\left(n - \frac{9d+11}{2}\right)[T(r,\phi) + T(r,\psi)] \leq S(r,\phi) + S(r,\psi) \quad (22)$$

Since $n > \left(\frac{9d+11}{2}\right)$ equation 22 leads to a contradiction.
Case 2. Let $H \equiv 0$. Then the theorem follows from Lemmas 4.7 and 4.8.
This completes the proof of Theorem 3.2.
Proof of Theorem 3.3.
Let $\Phi(z) = \left(\frac{\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)}{p(z)}\right)$ and $\Psi(z) = \left(\frac{\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)}{p(z)}\right)$, it follows that Φ and Ψ share $(1,0)$ except the zeros of $p(z)$.
Here equation 13 changes to

$$N_E^{1)}(r,1;\Phi) \leq N(r,0;H) \leq N(r,\infty;H) + S(r,\Phi) + S(r,\Psi) \quad (23)$$

Taking Lemmas 4.9, 4.12, 4.13, 4.14 and equations 13 and 23 we get

$$\begin{aligned}
\overline{N}(r,1;\Phi) &\leq N_E^{1)}(r,1;\Phi) + \overline{N}_L(r,1;\Phi) + \overline{N}_L(r,1;\Psi) + N_E^{(2}(r,1;\Phi) \\
&\leq \overline{N}(r,0;\Phi| \geq 2) + \overline{N}(r,0;\Psi| \geq 2) + \overline{N}_*(r,1;\Phi,\Psi) + \overline{N}_L(r,1;\Phi) \\
&\quad + \overline{N}_L(r,1;\Psi) + N_E^{(2}(r,1;\Phi) + \overline{N}_0(r,0;\Phi^{'}) + \overline{N}_0(r,0;\Psi^{'}) \\
&\quad + S(r,\phi) + S(r,\psi) \\
&\leq \overline{N}(r,0;\Phi| \geq 2) + \overline{N}(r,0;\Psi| \geq 2) + 2\overline{N}_L(r,1;\Phi) + 2\overline{N}_L(r,1;\Psi) \\
&\quad + \overline{N}_E^{(2}(r,1;\Phi) + \overline{N}_0(r,0;\Phi^{'}) + \overline{N}_0(r,0;\Psi^{'}) + S(r,\phi) + S(r,\psi) \\
&\leq \overline{N}(r,0;\Phi| \geq 2) + \overline{N}(r,0;\Psi| \geq 2) + \overline{N}_{\Phi>1}(r,1;\Psi) + \overline{N}_{\Psi>1}(r,1;\Phi) \\
&\quad + \overline{N}_L(r,1;\Phi) + N(r,1;\Psi) - \overline{N}(r,1;\Psi) + \overline{N}_0(r,0;\Phi^{'}) + \overline{N}_0(r,0;\Psi^{'}) \\
&\quad + S(r,\phi) + S(r,\psi) \\
&\leq N_2(r,0;\Phi) + \overline{N}(r,0;\Phi) + N_2(r,0;\Psi) + N(r,0;\Psi'|\Psi \neq 0) + \overline{N}_0(r,0;\Phi^{'}) \\
&\quad + S(r,\phi) + S(r,\psi) \\
\overline{N}(r,1;\Phi) &\leq N_2(r,0;\Phi) + \overline{N}(r,0;\Phi) + N_2(r,0;\Psi) + \overline{N}(r,0;\Psi) + \overline{N}_0(r,0;\Phi^{'}) \quad (24) \\
&\quad + S(r,\phi) + S(r,\psi).
\end{aligned}$$

Hence taking equation 24, Lemmas 4.2 and 4.6, we get from Second Fundamental

Theorem that

$$
\begin{aligned}
(n-d)T(r,\phi) &\le T(r,\Phi)+S(r,\phi),\\
&\le \overline{N}(r,\infty;\Phi)+\overline{N}(r,0;\Phi)+\overline{N}(r,1;\Phi)-N_0(r,0;\Phi^{'})+S(r,\phi),\\
&\le \overline{N}(r,\infty;\Phi)+2N_2(r,0;\Phi)+N_2(r,0;\Psi)+\overline{N}(r,0;\Psi)\\
&\quad +S(r,\phi)+S(r,\psi),\\
(n-d)T(r,\phi) &\le (3d+5)T(r,\phi)+(2d+3)T(r,\psi)+S(r,\phi)+S(r,\psi). \quad (25)
\end{aligned}
$$

Similarly, we obtain

$$(n-d)T(r,\psi) \le (3d+5)T(r,\psi)+(2d+3)T(r,\phi)+S(r,\phi)+S(r,\psi). \quad (26)$$

Combining equations 25 and 26, we get

$$(n-d-5d-8)[T(r,\phi)+T(r,\psi)] \le S(r,\phi)+S(r,\psi). \quad (27)$$

Since $n > 6d+8$ equation 27 leads to contradiction.
Case 2. Let $H \equiv 0$. Then the theorem follows from Lemmas 4.7 and 4.8.
This completes the proof of Theorem 3.3.

6 Discussion and Conclusions

Nevanlinna theory is a powerful tool in complex analysis, with wide range of applications in diverse areas of mathematics. Its insights into the behavior of meromorphic functions, the distribution of values and the properties of solutions of differential equations have profound implications for various fields of study. By utilizing Nevanlinna's theory, mathematicians can gain a deeper understanding of complex functions and their behavior, leading to advancements in mathematical theory like in signal processing, communication networks, design of filters and controllers for systems and some practical applications.

In this article using the Nevanlinna theory, we investigate the value distribution and uniqueness of linear difference polynomials of the type $\phi^n \sum_{j=1}^{d} a_j\phi(z+c_j)$ and $\psi^n \sum_{j=1}^{d} a_j\psi(z+c_j)$ transcendental meromorphic functions of finite order. Also, by using the concept of weighted sharing introduced by Indrajit Lahiri, we have studied uniqueness problem of linear difference polynomials sharing a non-zero polynomial $p(z)$ with finite weight. Our findings extends and generalizes some previous results of Theorems 2.6, 2.7 and 2.8 respectively.

Continuing further research, we can pose the following open questions:
Open Problems.
1. Is the condition for n sharp in Theorems 3.1,3.2 and 3.3 ?

2. Can $\sum_{j=1}^{d} a_j\phi(z+c_j)$ is replaced by $\Delta_c^n\phi$ in Theorems 3.1,3.2 and 3.3, where $\Delta_c^n\phi = \sum_{r=0}^{n-1}(-1)^r\binom{n}{r}\phi(z+(n-r)c)$?

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