Ratio Mathematica Volume 48, 2023

Results on Binary Soft Topological Spaces

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Abstract

The Baire space is used in the proof of results in many areas of analysis and geometry, including some of the fundamental theorems of functional analysis. The concept of Baire spaces has been studied extensively in general topology in [5, 8, 9, 10, 11, 15]. Thangaraj and Anjalmose [35] studied Baire spaces in the context of fuzzy theory. In this paper, we discussed the notions of the binary soft nowhere dense, binary soft dense, binary soft G_δ-set, binary soft first and second category sets, binary soft Baire spaces. Many of their properties are revealed and different characterizations of each are given. In conclusion, we determined some conditions under which the subspace property of a Baire space is preserved.
Keywords: Binary soft set, binary soft nowhere dense set, binary soft dense set, binary soft G_δ-set, binary soft Baire space.
2020 AMS subject classifications: 54A05, 54E52. [1]

1 Introduction

The idea of soft sets was introduced by Molodtsov [14] in 1999. Soft set theory allows researchers to choose the type of parameters they need, which greatly simplifies decision-making and makes the method more productive in the absence of partial data. Later, Shabir et al. [19] started researching on soft topological spaces. Many researchers continued their work on soft topology, including

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[1]Received on February 18, 2023. Accepted on July 20, 2023. Published on August 1, 2023. DOI: 10.23755/rm.v48i0.1136. ISSN: 1592-7415. eISSN: 2282-8214.

Aygunoghu [4], Ahmad [2], Maji [13] and Hussain [12]. In 2016, Acikgöz et al. [1] presented the idea of binary soft set theory on two universal sets and investigated various features. Later, Patil et al. [16] introduced the separation axioms in binary soft topological spaces such as $n - T_0$, $n - T_1$ and $n - T_2$ spaces.

Baire space is a space through which applications of completeness are made in analysis. Also, it is used in well-known theorems as the closed graph theorem, the open mapping theorem and the uniform boundedness theorem. The Baire category theorem (BCT), which specifies sufficient conditions for a topological space to be a Baire space, is a significant result in general topology and functional analysis. Further, BCT is used to prove Hartogs's theorem, a fundamental result in the theory of several complex variables.

The Baire space was initially introduced in Bourbaki's [7] Topologie Generale Chapter lX and was named in honour of Rene Louis Baire. Baire spaces have been extensively studied in classical topology [5, 8, 9, 10, 11, 15]. The first and second category sets were first established by Rene Louis Baire in 1899 [6]. Thangaraj and Anjalmose introduced and studied the concept of Baire spaces in fuzzy topology [20]. Riaz and Fatima [18] studied the concept of soft Baire spaces in soft metric spaces. Ameen and Khalaf investigated the soft Baire invariance using soft semicontinuous, soft somewhat continuous and soft somewhat open functions in [3].

In this article, we introduced a class of binary soft topological space called binary soft Baire space and studied some topological properties that are preserved by proper subspaces such as binary soft nowhere dense, binary soft dense, binary soft first and second category sets and binary soft Baire spaces. We obtained some conditions under which subspace property is preserved. Later, we shall show that binary soft compact, $n - T_2$ space falls in the class of binary soft Baire space.

We have referred the paper [1] for the definition of binary soft (briefly BS) set, binary absolute soft set, binary null soft set, union of two BS sets, intersection of two BS sets and [16, 17] for BS topology, binary soft topological space (briefly BSTS), BS subspace and BS relative topology.

2 Binary Soft Nowhere Dense Set

Definition 2.1. *A BS subset (F, E) of a BSTS (X_1, X_2, τ_b, E) is called a BS dense set (resp. BS co-dense set) if $\overline{\overline{(F, E)}} = \tilde{\tilde{E}}$ (resp. $(F, E)^{\circledcirc} = \tilde{\tilde{\phi}}$).*

Definition 2.2. *A BS subset (F, E) of a BSTS (X_1, X_2, τ_b, E) is called a BS nowhere dense set if $\left(\overline{\overline{(F, E)}}\right)^{\circledcirc} = \tilde{\tilde{\phi}}$.*

Definition 2.3. *A BS subset (F, E) of a BSTS (X_1, X_2, τ_b, E) is called a BS G_δ-set if it is countable BS intersection of BS open sets.*

Example 2.1. *Let $X_1 = R$ and $X_2 = Q$ be two initial universal sets and $E = \{e_1, e_2\}$ be a set of parameters and $\tau_b = \{\tilde{\tilde{E}}, \tilde{\tilde{\phi}}, (F_1, E), (F_2, E), (F_3, E)\}$, where*
$(F_1, E) = \{(e_1, (N, N)), (e_2, (Q, Q))\}$,
$(F_2, E) = \{(e_1, (Q', \phi)), (e_2, (\phi, \phi))\}$,
$(F_3, E) = \{(e_1, (N \cup Q', N)), (e_2, (Q, Q))\}$
then BS closed sets are $\tilde{\tilde{E}}, \tilde{\tilde{\phi}}$
$(F_1, E)'=\{(e_1, (R \setminus N, Q \setminus N)), (e_2, (Q', \phi))\}$,
$(F_2, E)'=\{(e_1, (Q, Q)), (e_2, (R, Q))\}$,
$(F_3, E)'=\{(e_1, (Q \setminus N, Q \setminus N)), (e_2, (Q', \phi))\}$.
Consider the BS sets
$(F_3, E) = \{(e_1, (Q' \cup N, N)), (e_2, (Q, Q))\}$ *then,* $\overline{\overline{(F_3, E)}} = \tilde{\tilde{E}}$. *Therefore,* (F_3, E) *is a BS dense set in* (R, Q, τ_b, E).
$(G_1, E) = \{(e_1, (Q^-, Q^-)), (e_2, (Q', \phi))\}$ *then,* $(G_1, E)^{\circledcirc} = \tilde{\tilde{\phi}}$ *and* $\left(\overline{\overline{(G_1, E)}}\right)^{\circledcirc} = \tilde{\tilde{\phi}}$. *Therefore,* (G_2, E) *BS co-dense and BS nowhere dense set* (R, Q, τ_b, E).
$(G_2, E) = \{(e_1, (Q^+ \setminus N, Q^+ \setminus N)), (e_2, (Q', \phi))\}$ *then,* $(G_2, E)^{\circledcirc} = \tilde{\tilde{\phi}}$. *Therefore* (G_2, E) *is also a BS nowhere dense set* (R, Q, τ_b, E).

Theorem 2.1. *Let (F, E) be a BS subset of a BSTS (X_1, X_2, τ_b, E) then the following assertions are interchangeable:*

1. *(F, E) is BS nowhere dense in (X_1, X_2, τ_b, E).*
2. *$\tilde{\tilde{E}} - \overline{\overline{(F, E)}}$ is BS dense in (X_1, X_2, τ_b, E).*
3. *For each non-empty BS open set (G, E) in (X_1, X_2, τ_b, E), there exist a non-empty BS open set (H, E) in (X_1, X_2, τ_b, E) such that $(H, E) \tilde{\tilde{\subseteq}} (G, E)$ and $(H, E) \tilde{\tilde{\cap}} (F, E) = \phi$.*

Proof. (1) $\Rightarrow$ (2), Let (F, E) is BS nowhere dense set in (X_1, X_2, τ_b, E) then $\left(\overline{\overline{(F, E)}}\right)^{\circledcirc} = \tilde{\tilde{\phi}}$. Let (W, E) be a BS open subset of (X_1, X_2, τ_b, E). Since, $\left(\overline{\overline{(F, E)}}\right)^{\circledcirc} = \tilde{\tilde{\phi}}$ therefore (W, E) intersects $\tilde{\tilde{E}} - (\overline{\overline{(F, E)}})$ this holds for all BS open subsets of (X_1, X_2, τ_b, E). Therefore, $\tilde{\tilde{E}} - (\overline{\overline{(F, E)}})$ is BS dense set in (X_1, X_2, τ_b, E).
(2) $\Rightarrow$ (3), Let $\tilde{\tilde{E}} - (\overline{\overline{(F, E)}})$ be a BS dense set in (X_1, X_2, τ_b, E) and (G, E) be non-empty BS open subset of (X_1, X_2, τ_b, E) then, $(G, E) \tilde{\tilde{\cap}} (\tilde{\tilde{E}} - \overline{\overline{(F, E)}}) \neq \phi$. Let $(H, E) = (G, E) \tilde{\tilde{\cap}} (\tilde{\tilde{E}} - \overline{\overline{(F, E)}})$ then clearly $(H, E) \tilde{\tilde{\subseteq}} (G, E)$ and $(H, E) \tilde{\tilde{\cap}} (F, E) = \phi$.
(3) $\Rightarrow$ (1), Suppose (F, E) is not BS nowhere dense set in (X_1, X_2, τ_b, E). That

is $\left(\overline{\overline{(F,E)}}\right)^{\circledcirc} \neq \tilde{\tilde{\phi}}$ then, any non-empty BS open subset of $\left(\overline{\overline{(F,E)}}\right)^{\circledcirc}$ would intersect (F,E), which is a contradiction. Therefore, (F,E) is BS nowhere dense in (X_1, X_2, τ_b, E).

The following theorem gives the relation between BS dense and BS nowhere dense sets.

Theorem 2.2. *If (F,E) is a BS open and dense subset of (X_1, X_2, τ_b, E) then, $(F,E)^c$ is BS nowhere dense set.*

Proof. Let (F,E) is a BS open and dense subset of (X_1, X_2, τ_b, E) then, $\overline{\overline{(F,E)}} = \tilde{\tilde{E}}$, this implies $\tilde{\tilde{E}} - \overline{\overline{(F,E)}} = \tilde{\tilde{\phi}}$, $(\tilde{\tilde{E}} - (F,E))^{\circledcirc} = \tilde{\tilde{\phi}}$, $((F,E)^c)^{\circledcirc} = \tilde{\tilde{\phi}}$. Since, $(F,E)^c$ is BS closed subset. Therefore, $(F,E)^c$ is BS nowhere dense set.

Remark 2.1. *The converse of the preceding theorem is generally untrue. It can be demonstrated using the example below.*

Example 2.2. *Let* $X_1 = \{a_1, a_2, a_3\}$, $X_2 = \{b_1, b_2, b_3\}$, $E = \{e_1, e_2\}$ *and*
$\tau_b = \{\tilde{\tilde{E}}, \tilde{\tilde{\phi}}, (F_1,E), (F_2,E), (F_3,E), (F_4,E), (F_5,E), (F_6,E), (F_7,E), (F_8,E)\}$
where $(F_1,E) = \{(e_1, (\{a_1\}, \{b_1\})), (e_2, (\{a_2\}, \{b_2\}))\}$,
$(F_2,E) = \{(e_1, (\{a_2\}, \{b_2\})), (e_2, (\{a_3\}, \{b_1\}))\}$,
$(F_3,E) = \{(e_1, (\{a_1, a_2\}, \{b_1, b_2\})), (e_2, (\{a_2, a_3\}, \{b_1, b_2\}))\}$,
$(F_4,E) = \{(e_1, (\{a_1\}, \{b_2\})), (e_2, (\{a_2\}, \{b_1\}))\}$,
$(F_5,E) = \{(e_1, (\{a_1\}, \{b_1, b_2\})), (e_2, (\{a_2\}, \{b_1, b_2\}))\}$,
$(F_6,E) = \{(e_1, (\{a_1\}, \phi)), (e_2, (\{a_2\}, \phi))\}$,
$(F_7,E) = \{(e_1, (\{a_1, a_2\}, \{b_2\})), (e_2, (\{a_2, a_3\}, \{b_1\}))\}$,
$(F_8,E) = \{(e_1, (\phi, \{b_2\})), (e_2, (\phi, \{b_1\}))\}$
then binary soft closed sets are $\tilde{\tilde{E}}, \tilde{\tilde{\phi}}$
$(F_1,E)' = \{(e_1, (\{a_2, a_3\}, \{b_2, b_3\})), (e_2, (\{a_1, a_3\}, \{b_1, b_3\}))\}$,
$(F_2,E)' = \{(e_1, (\{a_1, a_3\}, \{b_1, b_3\})), (e_2, (\{a_1, a_2\}, \{b_2, b_3\}))\}$,
$(F_3,E)' = \{(e_1, (\{a_3\}, \{b_3\})), (e_2, (\{a_1\}, \{b_3\}))\}$,
$(F_4,E)' = \{(e_1, (\{a_2, a_3\}, \{b_1, b_3\})), (e_2, (\{a_1, a_3\}, \{b_2, b_3\}))\}$,
$(F_5,E)' = \{(e_1, (\{a_2, a_3\}, \{b_3\})), (e_2, (\{a_1, a_3\}, \{b_3\}))\}$,
$(F_6,E)' = \{(e_1, (\{a_2, a_3\}, \{b_1, b_2, b_3\})), (e_2, (\{a_1, a_3\}, \{b_1, b_2, b_3\}))\}$,
$(F_7,E)' = \{(e_1, (\{a_3\}, \{b_1, b_3\})), (e_2, (\{a_1\}, \{b_2, b_3\}))\}$,
$(F_8,E)' = \{(e_1, (\{a_1, a_2, a_3\}, \{b_1, b_3\})), (e_2, (\{a_1, a_2, a_3\}, \{b_2, b_3\}))\}$.
Let $(H,E) = \{(e_1, (\{a_3\}, \phi)), (e_2, (\{a_3\}, \{b_3\}))\}$ *be a binary soft subset of* (X_1, X_2, τ_b, E). *Clearly,* $\overline{\overline{(H,E)}} = \{(e_1, (\{a_2, a_3\}, \{b_3\})), (e_2, (\{a_1, a_3\}, \{b_3\}))\}$ *then* $\left(\overline{\overline{(H,E)}}\right)^{\circledcirc} = \tilde{\tilde{\phi}}$. *Therefore,* (H,E) *is BS nowhere dense set in* (X_1, X_2, τ_b, E) *but,* $(H,E)^c = \{(e_1, (\{a_1, a_2\}, \{b_1, b_2, b_3\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2\}))\}$ *is not BS open set.*

Theorem 2.3. *If* (F, E) *is a BS closed and nowhere dense subset of* (X_1, X_2, τ_b, E) *then,* $(F, E)^c$ *is BS dense subset of* (X_1, X_2, τ_b, E).

Proof: Let (F, E) is a BS closed and nowhere dense subset of (X_1, X_2, τ_b, E) then, $\left(\overline{\overline{(F, E)}}\right)^{\circledcirc} = \tilde{\tilde{\phi}}$, this implies $(F, E)^{\circledcirc} = \tilde{\tilde{\phi}}$, $\tilde{\tilde{E}} - (F, E)^{\circledcirc} = \tilde{\tilde{E}}$, $\overline{\overline{(\tilde{\tilde{E}} - (F, E))}} = \tilde{\tilde{E}}$, $\overline{\overline{((F, E)^c)}} = \tilde{\tilde{E}}$. Therefore, $(F, E)^c$ is BS dense set.

Remark 2.2. *The previous theorem's converse is generally not true. The example below can be used to illustrate it.*

Example 2.3. *In the example 2.2 consider,*
$(G, E) = \{(e_1, (\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2, b_3\}))\}$ *be a BS dense subset of* (X_1, X_2, τ_b, E) *but,* $(G, E)^c = \{(e_1, (\phi, \phi)), (e_2, (\{a_3\}, \phi))\}$ *is not BS closed set.*

Theorem 2.4. *Let* (Y, τ_{b_y}, E) *be a BS subspace of* (X_1, X_2, τ_b, E) *and* (F, E) *be a BS subset of* $\tilde{\tilde{Y}}$. *If* (F, E) *is BS nowhere dense set in* $\tilde{\tilde{Y}}$*, then* (F, E) *is BS nowhere dense set in* (X_1, X_2, τ_b, E).

Proof: Suppose (F, E) is a BS nowhere dense subset of $\tilde{\tilde{Y}}$, Let (G, E) be a BS open subset of (X_1, X_2, τ_b, E) then, there exists a non-empty BS open set (H, E) in $\tilde{\tilde{Y}}$ such that $(H, E) \tilde{\tilde{\subseteq}} (G, E) \tilde{\tilde{\cap}} (Y, E)$ and $(H, E) \tilde{\tilde{\cap}} (F, E) = \tilde{\tilde{\phi}}$. Now there exists a BS open set (W, E) in (X_1, X_2, τ_b, E) such that $(H, E) = (W, E) \tilde{\tilde{\cap}} (Y, E)$ thus $(W, E) \tilde{\tilde{\subseteq}} (G, E)$ and $(W, E) \tilde{\tilde{\cap}} (F, E) = \tilde{\tilde{\phi}}$. Therefore, (F, E) is BS nowhere dense set in (X_1, X_2, τ_b, E).

Remark 2.3. *The previous theorem's converse is generally not true. The example below can be used to illustrate it.*

Example 2.4. *In the example 2.2, consider the BS subspace*
$\tilde{\tilde{Y}} = \{(e_1, (\{a_3\}, \{b_3\})), (e_2, (\{a_3\}, \{b_3\}))\}$ *with BS subspace topology*
$\tau_{b_y} = \{\tilde{\tilde{Y}}, \tilde{\tilde{\phi}}, (G_1, E)\}$*, where* $(G_1, E) = \{(e_1, (\phi, \phi)), (e_2, (\{a_3\}, \phi))\}$ *and*
$(G_1, E)' = \{(e_1, (\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2, b_3\}))\}$.
Let $(H, E) = \{(e_1, (\{a_3\}, \phi)), (e_2, (\{a_3\}, \{b_3\}))\}$ *be a binary soft subset of* (X_1, X_2, τ_b, E). *Clearly,* (H, E) *is a BS nowhere dense set in* (X_1, X_2, τ_b, E). *In BS subspace* (Y, τ_{b_y}, E), $\overline{\overline{(H, E)}}^{y} = \tilde{\tilde{Y}}$ *then* $\left(\overline{\overline{(H, E)}}^{y}\right)^{\circledcirc_y} = (\tilde{\tilde{Y}})^{\circledcirc_y} \neq \tilde{\tilde{\phi}}$.
Therefore, (H, E) *is not BS nowhere dense set in* $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$.

Theorem 2.5. *Let* $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ *is BS open or BS dense subspace of* (X_1, X_2, τ_b, E) *and* (F, E) *is BS nowhere dense in* (X_1, X_2, τ_b, E) *then,* (F, E) *is BS nowhere dense in* $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$.

Proof: Let $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ be a BS open subspace of (X_1, X_2, τ_b, E) and (F, E) is BS nowhere dense in (X_1, X_2, τ_b, E). Let (G, E) is BS open subset of $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ then (G, E) is BS open subset of (X_1, X_2, τ_b, E). Therefore, there exists a non-empty BS open subset (H, E) of (G, E) such that $(H, E)\tilde{\tilde{\cap}}(F, E) = \tilde{\tilde{\phi}}$, (H, E) is also BS open subset of $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$. Therefore, (F, E) is also BS nowhere dense in $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$.
The proof is similar in the case of $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ is BS dense subspace of (X_1, X_2, τ_b, E).

Corolary 2.1. *The above theorem need not be true for BS closed subspaces. That is, if $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ is a BS closed subspace of a BS topological space (X_1, X_2, τ_b, E) and (F, E) is BS nowhere dense set in (X_1, X_2, τ_b, E) then, (F, E) need not to be BS nowhere dense set in $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$.*

Example 2.5. *Let $X_1 = \{a_1, a_2, a_3\}$, $X_2 = \{b_1, b_2, b_3\}$, $E = \{e_1, e_2\}$ and*
$\tau_b = \{\tilde{\tilde{E}}, \tilde{\tilde{\phi}}, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E), (F_6, E), (F_7, E), (F_8, E), (F_9, E), (F_{10}, E), (F_{11}, E)\}$
where $(F_1, E) = \{(e_1, (\{a_1, a_2\}, \{b_1, b_2\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2\}))\}$,
$(F_2, E) = \{(e_1, (\{a_3\}, \{b_3\})), (e_2, (\{a_3\}, \{b_3\}))\}$,
$(F_3, E) = \{(e_1, (\{a_2\}, \{b_1\})), (e_2, (\{a_1\}, \{b_2\}))\}$,
$(F_4, E) = \{(e_1, (\{a_2, a_3\}, \{b_1, b_3\})), (e_2, (\{a_1, a_3\}, \{b_2, b_3\}))\}$,
$(F_5, E) = \{(e_1, (\{a_3\}, \{b_1\})), (e_2, (\{a_3\}, \{b_2\}))\}$,
$(F_6, E) = \{(e_1, (\{a_1, a_2, a_3\}, \{b_1, b_2\})), (e_2, (\{a_1, a_2, a_3\}, \{b_1, b_2\}))\}$,
$(F_7, E) = \{(e_1, (\phi, \{b_1\})), (e_2, (\phi, \phi))\}$,
$(F_8, E) = \{(e_1, (\phi, \{b_2\})), (e_2, (\phi, \{b_1\}))\}$,
$(F_9, E) = \{(e_1, (\{a_3\}, \{b_1, b_3\})), (e_2, (\{a_3\}, \{b_2, b_3\}))\}$,
$(F_{10}, E) = \{(e_1, (\{a_2, a_3\}, \{b_1\})), (e_2, (\{a_1, a_3\}, \{b_2\}))\}$,
$(F_{11}, E) = \{(e_1, (\phi, \{b_1\})), (e_2, (\phi, \{b_2\}))\}$
then binary soft closed sets are $\tilde{\tilde{E}}, \tilde{\tilde{\phi}}$
$(F_1, E)' = \{(e_1, (\{a_3\}, \{b_3\})), (e_2, (\{a_3\}, \{b_3\}))\}$,
$(F_2, E)' = \{(e_1, (\{a_1, a_2\}, \{b_1, b_2\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2\}$,
$(F_3, E)' = \{(e_1, (\{a_1, a_3\}, \{b_2, b_3\})), (e_2, (\{a_2, a_3\}, \{b_1, b_3\}))\}$,
$(F_4, E)' = \{(e_1, (\{a_1\}, \{b_2\})), (e_2, (\{a_2\}, \{b_1\}))\}$,
$(F_5, E)' = \{(e_1, (\{a_1, a_2\}, \{b_2, b_3\})), (e_2, (\{a_1, a_2\}, \{b_1, b_3\}))\}$,
$(F_6, E)' = \{(e_1, (\phi, \{b_3\})), (e_2, (\phi, \{b_3\}))\}$,
$(F_7, E)' = \{(e_1, (\{a_1, a_2, a_3\}, \{b_2, b_3\})), (e_2, (\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}))\}$,
$(F_8, E)' = \{(e_1, (\{a_1, a_2\}, \{b_2\})), (e_2, (\{a_1, a_2\}, \{b_1\}))\}$,
$(F_9, E)' = \{(e_1, (\{a_1, a_2\}, \{b_1, b_2, b_3\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2, b_3\}))\}$,
$(F_{10}, E)' = \{(e_1, (\{a_1\}, \{b_2, b_3\})), (e_2, (\{a_2\}, \{b_1, b_3\}))\}$,
$(F_{11}, E)' = \{(e_1, (\{a_1, a_2, a_3\}, \{b_2, b_3\})), (e_2, (\{a_1, a_2, a_3\}, \{b_1, b_3\}))\}$.

Let $\tilde{\tilde{Y}} = \{(e_1, (\{a_1, a_2\}, \{b_1, b_2, b_3\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2, b_3\}))\}$ *be a BS closed subset of* (X_1, X_2, τ_b, E) *then the BS subspace topology is*
$\tau_y = \{\tilde{\tilde{Y}}, \tilde{\tilde{\phi}}, (G_1, E), (G_2, E), (G_3, E), (G_4, E), (G_5, E), (G_6, E), (G_7, E)\}$
where $(G_1, E) = \{(e_1, (\{a_1, a_2\}, \{b_1, b_2\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2\}))\}$,
$(G_2, E) = \{(e_1, (\phi, \{b_3\})), (e_2, (\phi, \{b_3\}))\}$,
$(G_3, E) = \{(e_1, (\{a_2\}, \{b_1\})), (e_2, (\{a_1\}, \{b_2\}))\}$,
$(G_4, E) = \{(e_1, (\{a_2\}, \{b_1, b_3\})), (e_2, (\{a_1\}, \{b_2, b_3\}))\}$,
$(G_5, E) = \{(e_1, (\phi, \{b_1\})), (e_2, (\phi, \{b_2\}))\}$,
$(G_6, E) = \{(e_1, (\phi, \{b_1\})), (e_2, (\phi, \phi))\}$,
$(G_7, E) = \{(e_1, (\phi, \{b_1, b_3\})), (e_2, (\phi, \{b_2, b_3\}))\}$
then BS closed sets are $\tilde{\tilde{Y}}, \tilde{\tilde{\phi}}$ $(G_1, E)' = \{(e_1, (\phi, \{b_3\})), (e_2, (\phi, \{b_3\}))\}$,
$(G_2, E)' = \{(e_1, (\{a_1, a_2\}, \{b_1, b_2\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2\}))\}$,
$(G_3, E)' = \{(e_1, (\{a_1\}, \{b_2, b_3\})), (e_2, (\{a_2\}, \{b_1, b_3\}))\}$,
$(G_4, E)' = \{(e_1, (\{a_1\}, \{b_2\})), (e_2, (\{a_2\}, \{b_1\}))\}$,
$(G_5, E)' = \{(e_1, (\{a_1, a_2\}, \{b_2, b_3\})), (e_2, (\{a_1, a_2\}, \{b_1, b_3\}))\}$,
$(G_6, E)' = \{(e_1, (\{a_1, a_2\}, \{b_2, b_3\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2, b_3\}))\}$,
$(G_7, E)' = \{(e_1, (\{a_1, a_2\}, \{b_2\})), (e_2, (\{a_1, a_2\}, \{b_1\}))\}$.
Consider the BS set $\{(e_1, (\phi, \{b_3\})), (e_2, (\phi, \{b_3\}))\}$ *which is BS nowhere dense set in* (X_1, X_2, τ_b, E)*, but it is not BS nowhere dense set in the BS closed subspace* $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$.

3 Binary Soft Baire Spaces

Definition 3.1. *A BS subset* (F, E) *of a BSTS* (X_1, X_2, τ_b, E) *is called BS first category set if it is the BS union of countable family of BS nowhere dense sets.*

Definition 3.2. *A BS subset* (F, E) *of a BSTS* (X_1, X_2, τ_b, E) *is called BS second category set if it is not BS first category set.*

Definition 3.3. *A BS Baire space is a BSTS such that every non-empty BS open subset is BS second category.*

Example 3.1. *Consider the example 2.1, the BS set*
$(H, E) = \{(e_1, (Q \setminus N, Q \setminus N)), (e_2, (Q', \phi))\}$ *is a BS first category set, because* $(H, E) = (G_1, E)\tilde{\tilde{\cup}}(G_2, E)$*. Also,* (R, Q, τ_b, E) *is a BS Baire space.*

Theorem 3.1. *The following assertions are interchangeable for a BSTS* (X_1, X_2, τ_b, E)*:*

1. (X_1, X_2, τ_b, E) *is a BS baire space.*

2. *The BS intersection of any sequence of BS dense open sets is BS dense in* (X_1, X_2, τ_b, E).

3. *The BS complement of any BS first category set in* (X_1, X_2, τ_b, E) *is BS dense in* (X_1, X_2, τ_b, E).

4. *Every countable BS union of BS closed sets with empty BS interior in* (X_1, X_2, τ_b, E) *has empty BS interior in* (X_1, X_2, τ_b, E).

Proof:$(1) \Rightarrow (2)$, Suppose (X_1, X_2, τ_b, E) is a BS Baire space. Let $\{(G_i, E)\}$ be a sequence of BS dense open sets in (X_1, X_2, τ_b, E). Let us assume that their BS intersection $\tilde{\tilde{\cap}}(G_i, E)$ is not BS dense in (X_1, X_2, τ_b, E), then there exists a BS open set (H, E) such that it doesnot intersect $\tilde{\tilde{\cap}}(G_i, E)$. That is $(H, E) = \tilde{\tilde{E}} - \tilde{\tilde{\cap}}(G_i, E)$, $(H, E) = \tilde{\tilde{\cup}}(\tilde{\tilde{E}} - (G_i, E))$ where each $\tilde{\tilde{E}} - (G_i, E))$ is BS nowhere dense set in (X_1, X_2, τ_b, E). Therefore, (H, E) is of BS first category set. Which is contradiction to (X_1, X_2, τ_b, E) is a BS baire space. Therefore, $\tilde{\tilde{\cap}}(G_i, E)$ is a BS dense set in (X_1, X_2, τ_b, E).
$(2) \Rightarrow (3)$, Let $\{(F_i, E)\}$ be a sequence of BS closed nowhere dense subsets of (X_1, X_2, τ_b, E), then their BS union $\tilde{\tilde{\cup}}(F_i, E)$ is a BS first category set in (X_1, X_2, τ_b, E), we have to prove $\tilde{\tilde{E}} - \tilde{\tilde{\cup}}(F_i, E)$ is BS dense in (X_1, X_2, τ_b, E). Let us assume that it is not dense in (X_1, X_2, τ_b, E). Consider a BS open set (G, E) in (X_1, X_2, τ_b, E) such that it does not intersect with $\tilde{\tilde{E}} - \tilde{\tilde{\cup}}(F_i, E) = \tilde{\tilde{\cap}}(\tilde{\tilde{E}} - (F_i, E))$, where each $(\tilde{\tilde{E}} - (F_i, E))$ is BS dense openset in (X_1, X_2, τ_b, E) then their binary soft intersection is not BS dense in (X_1, X_2, τ_b, E), which is contradiction. Therefore, $\tilde{\tilde{E}} - \tilde{\tilde{\cup}}(F_i, E)$ is BS dense in (X_1, X_2, τ_b, E).
$(3) \Rightarrow (4)$, Let $\{(F_i, E)\}$ be a sequence of BS closed sets with empty interiors. Suppose $\tilde{\tilde{\cup}}(F_i, E)$ does not has empty BS interior, then $\tilde{\tilde{E}} - \tilde{\tilde{\cup}}(F_i, E)$ would not be BS dense in (X_1, X_2, τ_b, E), which is a contradiction. Therefore, $\tilde{\tilde{\cup}}(F_i, E)$ has empty BS interior.
$(4) \Rightarrow (1)$, Let $\{(F_i, E)\}$ be a sequence of BS nowhere dense subsets in (X_1, X_2, τ_b, E). Suppose $\tilde{\tilde{\cup}}(F_i, E)$ is a BS open set, then each $cl(F_i, E)$ has no BS interior point, that is $\left(\overline{\overline{(F_i, E)}}\right)^{\circledcirc} = \tilde{\tilde{E}}$, then $\tilde{\tilde{\cup}}\overline{\overline{(F_i, E)}} = \overline{\overline{(\tilde{\tilde{\cup}}(F_i, E))}}$, since $\tilde{\tilde{\cup}}(F_i, E)$ is BS open, this implies it contains all its interior points, therefore $\tilde{\tilde{\cup}}\overline{\overline{(F_i, E)}}$ has BS interior points, which is contradiction to (4), therefore $\tilde{\tilde{\cup}}(F_i, E)$ is not BS openset. Therefore, BS opensets cannot be written as BS union of BS nowhere dense sets. Therefore, (X_1, X_2, τ_b, E) is a BS Baire space.

Theorem 3.2. *the following assertions are interchangeable for a BSTS* (X_1, X_2, τ_b, E).

1. (X_1, X_2, τ_b, E) *is a BS baire space.*
2. $(G, E)^{\circledcirc} = \tilde{\tilde{\phi}}$ *for every BS first category set* (G, E) *in* (X_1, X_2, τ_b, E).
3. $\overline{\overline{(H, E)}} = \tilde{\tilde{E}}$ *for every BS residual set* (H, E) *in* (X_1, X_2, τ_b, E).

Proof: $(1) \Rightarrow (2)$*, Let* (X_1, X_2, τ_b, E) *be a BS baire space, let* (G, E) *be a BS first category set then* $(G, E) = \tilde{\tilde{\cup}}(G_i, E)$ *where* (G_i, E)*'s are BS nowhere dense sets then* $(G, E)^{\circledcirc} = \left(\tilde{\tilde{\cup}}(G_i, E)\right)^{\circledcirc} = \tilde{\tilde{\phi}}$*. Therefore,* $(G, E)^{\circledcirc} = \tilde{\tilde{\phi}}$
$(2) \Rightarrow (3)$*, Let* (H, E) *be a BS residual set then* $(H, E)'$ *is BS first category set i.e.* $\tilde{\tilde{E}} - (H, E)$ *is BS first category set, then* $(\tilde{\tilde{E}} - (H, E))^{\circledcirc} = \tilde{\tilde{\phi}}$*, then* $\tilde{\tilde{E}} - \overline{\overline{(H, E)}} = \tilde{\tilde{\phi}}$*. Therefore,* $\overline{\overline{(H, E)}} = \tilde{\tilde{E}}$*.*
$(3) \Rightarrow (1)$*, Let* (F, E) *be a BS first category set then* $\tilde{\tilde{E}} - (F, E)$ *is BS residual set then* $\overline{\overline{(\tilde{\tilde{E}} - (F, E))}} = \tilde{\tilde{E}}$*, then* $\tilde{\tilde{E}} - (F, E)^{\circledcirc} = \tilde{\tilde{E}}$*,* $(F, E)^{\circledcirc} = \tilde{\tilde{\phi}}$ *therefore* $\left(\tilde{\tilde{\cup}}(F_i, E)\right)^{\circledcirc} = \tilde{\tilde{\phi}}$ *where* (F_i, E)*'s are BS nowhere dense sets. Therefore,* (X_1, X_2, τ_b, E) *is a BS Baire space.*

Theorem 3.3. *Every BS open or BS dense subspace of a BS Baire space is a BS Baire space.*

Proof: Let (X_1, X_2, τ_b, E) be a BS baire space, let $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ be a BS open subspace of (X_1, X_2, τ_b, E), consider (F_i, E) be a sequence of BS nowhere dense sets in $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$, by theorem 2.4 (F_i, E) is also a sequence of BS nowhere dense sets in (X_1, X_2, τ_b, E). Since (X_1, X_2, τ_b, E) is a BS baire space then $\tilde{\tilde{\cup}}(F_i, E)$ is also BS nowhere dense set in (X_1, X_2, τ_b, E). Let us assume that $\tilde{\tilde{\cup}}(F_i, E)$ is not BS nowhere dense set in $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$, then $\left(\overline{\overline{(\tilde{\tilde{\cup}}(F_i, E))}}^{y}\right)^{\circledcirc_y} \neq \tilde{\tilde{\phi}}$ then there exists a BS open set (G, E) in $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ such that $(G, E) \tilde{\tilde{\cap}} (\tilde{\tilde{\cup}}(F_i, E)) \neq \tilde{\tilde{\phi}}$, since $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ is a BS open subspace of (X_1, X_2, τ_b, E). Thus (G, E) is a BS open set in (X_1, X_2, τ_b, E) then $(G, E) \tilde{\tilde{\cap}} (\tilde{\tilde{\cup}}(F_i, E)) \neq \tilde{\tilde{\phi}}$, which is contradiction that (X_1, X_2, τ_b, E) is a binary soft baire space, therefore $\left(\overline{\overline{(\tilde{\tilde{\cup}}(F_i, E))}}^{y}\right)^{\circledcirc_y} = \tilde{\tilde{\phi}}$, thus $\tilde{\tilde{\cup}}(F_i, E)$ is BS nowhere dense set in $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$. Therefore $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ is also a BS Baire space.
The proof is similar for a BS dense subspace.

Example 3.2. *Consider the example 2.5,*
let $\tilde{\tilde{Y}} = \{(e_1, (\{a_1, a_2\}, \{b_1, b_2\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2\}))\}$ *be a binary soft open*

subset of a BS Baire space (X_1, X_2, τ_b, E) then the BS subspace topology is
$\tau_{b_y} = \{\tilde{\tilde{Y}}, \tilde{\tilde{\phi}}, (G_1, E), (G_2, E), (G_3, E)\}$
where $(G_1, E) = \{(e_1, (\{a_2\}, \{b_1\})), (e_2, (\{a_1\}, \{b_2\}))\}$,
$(G_2, E) = \{(e_1, (\phi, \{b_1\})), (e_2, (\phi, \{b_2\}))\}$,
$(G_3, E) = \{(e_1, (\phi, \{b_1\})), (e_2, (\phi, \phi))\}$
then BS closed sets are $\tilde{\tilde{Y}}, \tilde{\tilde{\phi}}$, $(G_1, E)' = \{(e_1, (\{a_1\}, \{b_2\})), (e_2, (\{a_2\}, \{b_1\}))\}$,
$(G_2, E)' = \{(e_1, (\{a_1, a_2\}, \{b_2\})), (e_2, (\{a_1, a_2\}, \{b_1\}))\}$,
$(G_3, E)' = \{(e_1, (\{a_1, a_2\}, \{b_1\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2\}))\}$.
Since, (X_1, X_2, τ_b, E) is a BS Baire space and $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ is a BS open subspace of (X_1, X_2, τ_b, E), then $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ is also BS Baire space.

Corolary 3.1. *A BS compact and BS $n - T_2$ space is BS regular.*

Theorem 3.4. *If (X_1, X_2, τ_b, E) is a BS compact and BS $n - T_2$ space then (X_1, X_2, τ_b, E) is a BS baire space.*

Proof: Let (X_1, X_2, τ_b, E) is a BS compact and BS $n - T_2$ space, let $\{(F_i, E)\}$ be a countable collection of closed sets of (X_1, X_2, τ_b, E) having empty BS interiors, to prove $\tilde{\tilde{\cup}}(F_i, E)$ has empty BS interior, that is $\tilde{\tilde{\cup}}(F_i, E)$ does not contain any BS open set. Consider a BS openset (G_1, E) of (X_1, X_2, τ_b, E), then we must find a pair of points $(x_0, y_0) \in (G_1, E)$ but does not lie in any of the sets (F_i, E). Let us first consider the set (F_1, E), by hypothesis (F_1, E) does not contain (G_1, E), that is there exist a point $(x_1, y_1) \in (G_1, E)$ but $(x_1, y_1) \notin^s (F_1, E)$, since (X_1, X_2, τ_b, E) is a BS compact and BS $n - T_2$ space then by corollary 3.1 it is a BS n-regular space, thus for any $(x_1, y_1) \in (G_1, E)$ there exist a BS open set (H_1, E) such that $(x_1, y_1) \in (H_1, E) \tilde{\tilde{\subseteq}} \overline{\overline{(H_1, E)}} \tilde{\tilde{\subseteq}} (G_1, E)$ also for any $(x_1, y_1) \in X_1 \times X_2$ and $(x_1, y_1) \notin^s (F_1, E)$ then, there is a BS open set (H_1, E) such that $(x_1, y_1) \in (H_1, E)$ and $\overline{\overline{(H_1, E)}} \tilde{\tilde{\cap}} (F_1, E) = \tilde{\tilde{\phi}}$ [16], $\overline{\overline{(H_1, E)}} \tilde{\tilde{\subseteq}} (G_1, E)$. There fore given any nonempty binary soft open set G_n, E, we choose a point $(x_n, y_n) \in (G_n, E)$ and it does not lie in the BS closed set (F_i, E) for all i, then we choose a BS open set (H_n, E) such that $\overline{\overline{(H_n, E)}} \tilde{\tilde{\cap}} (F_n, E) = \tilde{\tilde{\phi}}$ and $\overline{\overline{(H_n, E)}} \tilde{\tilde{\subseteq}} (G_n, E)$. Since (X_1, X_2, τ_b, E) is a BS compact space then consider $\{\overline{\overline{(H_n, E)}}\}$ be a family of BS closed sets, by finite intersection property of BS closed sets [16], we have $\tilde{\tilde{\cap}} \overline{\overline{(H_i, E)}} \neq \tilde{\tilde{\phi}}$, let $(x_0, y_0) \in \tilde{\tilde{\cap}} \overline{\overline{(H_i, E)}}$, that is $(x_0, y_0) \in \overline{\overline{(H_i, E)}}$ for all i, this implies $(x_0, y_0) \in (G_i, E)$ also $(x_0, y_0) \notin^s (F_i, E)$ for all i, thus $(x_0, y_0) \notin^s \tilde{\tilde{\cup}}(F_i, E)$, that is $\tilde{\tilde{\cup}}(F_i, E)$ is also has empty BS interior, There fore (X_1, X_2, τ_b, E) is a BS Baire space.

Remark 3.1. *The converse of the preceding theorem is generally untrue. It can be demonstrated using the example below.*

Example 3.3. *Let* $X_1 = \{a_1, a_2\}$, $X_2 = \{b_1, b_2\}$, $E = \{i, ii\}$ *and*
$\tau_b = \{\tilde{\tilde{E}}, \tilde{\tilde{\phi}}, (G_1, E), (G_2, E), (G_3, E), (G_4, E), (G_5, E), (G_6, E), (G_7, E)\}$
where $(G_1, E) = \{(i, (\{a_1, a_2\}, \{b_1\})), (ii, (\{a_1, a_2\}, \{b_1\}))\}$,
$(G_2, E) = \{(i, (\{a_2\}, \{b_1, b_2\})), (ii, (\{a_2\}, \{b_1, b_2\}))\}$,
$(G_3, E) = \{(i, (\{a_1, a_2\}, \{b_2\})), (ii, (\{a_1, a_2\}, \{b_2\}))\}$,
$(G_4, E) = \{(i, (\{a_2\}, \{b_1\})), (ii, (\{a_2\}, \{b_1\}))\}$,
$(G_5, E) = \{(i, (\{a_2\}, \{b_2\})), (ii, (\{a_2\}, \{b_2\}))\}$,
$(G_6, E) = \{(i, (\{a_1, a_2\}, \phi)), (ii, (\{a_1, a_2\}, \phi))\}$,
$(G_7, E) = \{(i, (\{a_2\}, \phi)), (ii, (\{a_2\}, \phi))\}$.
Then, (X_1, X_2, τ_b, E) *is a BS Baire space and BS compact space as it is a finite space. But, it is not* $n - T_2$ *space because,* $(a_1, b_1) \neq (a_2, b_2) \in X_1 \times X_2$ *and there does not exist BS disjoint open sets* (G, E) *and* (F, E) *such that* $(a_1, b_1) \in (G, E)$ *and* $(a_2, b_2) \in (F, E)$.

4 Discussion and Conclusion

This paper contributes to the area of Baire spaces in the BS topological spaces. We defined BS nowhere dense, BS dense, BS first, second category sets and obtained their properties. Also, we have introduced BS Baire space and obtained their characterizations. We determined some conditions under which the subspace property of a Baire space is preserved and established a relation between BS compact and BS Baire space. To support the obtained result and relations we have provided examples.

In the future, we intend to investigate the further properties of BS topological spaces with Baire property.

Acknowledgment

The second author is grateful to Karnataka Science and Technology Promotion Society(KSTePS), Karnataka, India for the financial support to research work. Further, the authors thank the anonymous referees for the helpful remarks and suggestions for the improvement of this paper.

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