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Products on Maximal Compact Frames

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Abstract

In topological spaces, many topological properties such as separation properties, paracompactness etc. are preserved under the act of taking product of topological spaces. The well known Tychonoff theorem in topological spaces which states that product of compact spaces is compact. Many of these results can be extended to the "generalized topological spaces", known as locales(frames). According to Tychonoff product theorem for locales, locale product(coproduct of frames) of compact locales(frames) is compact. In this paper, we examine whether the coproduct of maximal compact frames is maximal compact. We examine the case for a finite coproduct and for an arbitrary coproduct of maximal compact frames. Every subframe of a compact frame is compact but a sublocale need not be. We provide a characterization for a sublocale of a maximal compact frame to be a maximal compact sublocale.

Keywords: Frame, locale, spatial frame, maximal compact frame, subframe, sublocale.

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1 Introduction

Garrett Birkhoff, in 1936, pointed out the notion of the comparison of two different topologies on the same basic set. He had done this by ordering these topologies as a lattice under set inclusion. A topological space (X, T) with property R is said to be maximal R if T is a maximal element in the set $R(X)$ of all topologies on the set X having property R with the partial ordering of set inclusions. The set of all topologies sharing a given property may not have a greatest element, but it may have maximal elements.

In topological spaces, a closed subspace of a compact space is compact and a compact subspace of a Hausdorff space is closed. Thus in a compact Hausdorff space, closed subspaces coincide with compact subspaces. ”A topological space is maximal compact if and only if its compact subsets are precisely the closed sets”, as proved in [12]. Norman Levine named those spaces in which closed subsets coincide with compact subsets as *C-C Spaces*. A detailed analysis of its properties are discussed in [10].

A locale is a categorical extension of a topological space whereas a frame is an object in its opposite category. In 1972, through the paper [3], J.R.Isbell pointed out the need for a seperate terminology for the dual category of frames. The objects of this dual category are named as “locales” by him and they are actually the “generalized spaces”. A pointfree analogue of the maximal compact space and its associated properties are discussed in [6]. A maximal compact frame is a compact frame which is not properly contained in any other compact frame. A characterization of the maximal compact frame is that a subframe A of any non-compact frame L is maximal compact if and only if the closed sublocales of A are exactly the compact sublocales of A. It is also proved that a compact Hausdorff frame is maximal compact.

We know that many of the topological properties such as separation properties, paracompactness etc. are preserved under the act of taking product of topological spaces. The well known Tychonoff theorem in topological spaces which states that product of compact spaces is compact is equivalent to the axiom of choice by [9]. Coproducts(products) in frames(locales) are the categorical counterpart of product of topological spaces. Frame theory has the advantage that many results in topology requiring *Axiom of Choice* or some of its variants can be proved without its use. The Tychonoff's theorem for locales is constructively proved by [7] without using the axiom of choice.

In this paper, we examine whether the coproduct of maximal compact frames is maximal compact. We examine the case for a finite coproduct and for an arbitrary coproduct of maximal compact frames. Every subframe of a compact frame is compact but a sublocale need not be. We provide a characterization for a sublocale of a maximal compact frame to be a maximal compact sublocale.

2 Preliminaries

The term frame was coined by C.H. Dowker. A remarkable study on frames was done by them in [1]. A *frame* is a complete lattice L in which the infinite distributive law $a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}$ holds for all $a \in L, S \subseteq L$. A map between frames that preserves arbitrary joins and finite meets is called a *frame homomorphism*. Associated with a frame homomorphism $h : M \to L$ is its right adjoint $h_* : L \to M$ given by $h_*(b) = \bigvee \{x \in M : h(x) \leq b\}$. We denote the top element and the bottom element of a frame by 1 and 0 respectively. The category of frames and frame homomorphisms is denoted by **Frm**. The dual category **Frm**^{op} is referred to as the category of locales denoted by **Loc**. The morphisms in **Loc**, called *localic maps*, are given by the right adjoints of frame homomorphisms between two objects. A frame is said to be *spatial*, if it is isomorphic to the topology ΩX of a topological space $(X, \Omega X)$.

A subset of a frame which is closed under arbitrary joins and finite meets in that frame is called a *subframe*. A *sublocale* M of a locale L can be represented in terms of an onto frame homomorphism $h : L \to M$ in the sense that the image of M under the right adjoint $h_* : M \to L$ will represent that sublocale. For a locale L, denote $\uparrow a = \{x \in L : x \geq a\}$ and $\downarrow b = \{x \in L : x \leq b\}$. Then the sublocale given by the frame homomorphism $j : L \to \uparrow a$ defined by $x \to a \vee x$ for any $a \in L$ is called a *closed sublocale* of L. A *cover* in a frame L is a subset S of L with $\bigvee S = 1_L$. A frame L is said to be *compact* if each cover A of L has a finite subcover.

The construction of coproducts in frames which was first presented in [1]. The construction is provided below through the following definitions.

Definition 2.1. *Let $R \subseteq A \times A$ be an arbitrary binary relation on a frame A. An element $s \in A$ is* R-saturated *if $aRb \Rightarrow (a \wedge c \leq s \Leftrightarrow b \wedge c \leq s)$ forall $a, b, c \in A$*

Let A/R denotes the set of all saturated elements of A. Define $\nu : A \to A/R$ by $\nu(a) = \nu_R(a) = \bigwedge \{s \in A : a \leq s\}$ where s is saturated. Then ν is a surjective frame homomorphism. For a semilattice A, define $\mathcal{D}(A) = \{U \subseteq A : \phi \neq U = \downarrow U\}$. Then $(\mathcal{D}(A), \subseteq)$ is a frame. Define $\lambda_A : A \to \mathcal{D}A$ by $\lambda_A(a) = \downarrow a$ which is a semilattice homomorphism between them. Let $A_i, i \in I$ be frames. Set $\prod'_{i\in I} A_i = \{(a_i)_{i\in I} \in \prod_{i\in I} A_i : a_i = 1 \text{ for all but finitely many i }\} \bigcup \{(0)_{i\in I}\}$ Define $\gamma_j : A_j \to \prod'_{i\in I} A_i$ by setting

$$(\gamma_j(a))_i = \begin{cases} a & \text{if } i = j \\ 1 & \text{otherwise} \end{cases}$$

Consider the frame $\mathcal{D}(A)$ where $A = \prod'_{i\in I} A_i$.
$R = \left\{(\lambda_A \gamma_j(\bigvee_{m\in M} a_m), \bigvee_{m\in M} \lambda_A \gamma_j(a_m)) : j \in I, a_m \in A_j\right\}$ where M is any set is a relation.

Definition 2.2. *The frame* $\bigoplus_{i\in I} A_i = \mathcal{D}(\prod'_{i\in I} A_i)/R$*, containing all R-saturated elements of the frame* $\mathcal{D}(\prod'_{i\in I} A_i)$ *is called the* frame coproduct.

Definition 2.3. *The mapping* $\nu : \mathcal{D}(\prod'_{i\in I} A_i) \to \bigoplus_{i\in I} A_i$ *be as defined above. The maps* $p_j = \nu \circ \lambda \circ \gamma_j : A_j \to \bigoplus_{i\in I} A_i$ *are frame homomorphisms, called* coproduct injections. *Also* $\bigwedge_{i\in I} p_i(a_i) = \oplus(a_i)_{i\in I}$. *Thus the set of all elements of the form* $\oplus(a_i)_{i\in I}$ *is a join basis of* $\oplus(A_i)_{i\in I}$. *The right adjoint of* p_j *denoted by* ${p_j}^*$ *is called the* projection *of the locale* $\bigoplus_{i\in I} A_i$ *to the locale* A_j.

A frame A is called a *Hausdorff frame* if for any $U \in A \oplus A$, the codiagonal $\nabla : A \oplus A \to A$ defined by

$$\nabla(U) = \bigvee \{a \wedge b : (a, b) \in U\}$$

is a closed sublocale.

The frame L is said to be a *regular* frame if $a = \bigvee \{x \in L / x \prec a\}$ for all $a \in L$.

For a detailed reading regarding frames we refer to [11].

3 Maximal Compact Frames and Frame coproducts

An intense research has been done on maximal and minimal topologies so far. The properties of maximal topologies for compactness, countable compactness, and sequential compactness etc. are investigated in [2] and the relations between these spaces are investigated. The paper also presented discussions on some interesting product theorems. Analogously maximal compactness in the pointfree counterpart is studied by [6]. A maximal compact frame is defined here as given below

Definition 3.1. *[6] A frame A is said to be* maximal compact *if,*

1. *A is compact,*
2. *if A is a proper subframe of the frame* L*, then* L *is not compact.*

A characterization for maximal compact frames is obtained in [6].

Theorem 3.1. *[6] A frame is maximal compact if and only if its closed sublocales are exactly the compact sublocales.*

For a topological space X, the frame of open sets is denoted by ΩX. A relation between a C-C space and the associated frame of open sets is established in the following result.

Theorem 3.2. *[6] The topological space $(X, \Omega X)$ is a C-C space if and only if ΩX is a maximal compact frame.*

A compact Hausdorff topological space is regular and the same is true locales too.

Theorem 3.3. *[11] A compact Hausdorff locale is regular.*

The property of being a C-C spaces is productive if the component spaces are Hausdorff. The product Hausdorff C-C spaces are C-C and vice-versa by the following result.

Theorem 3.4. *[10] Let (X, τ) be a topological space and let $(X \times X, \tau^*)$ be the cartesian product of (X, τ) with itself. Then $(X \times X, \tau^*)$ is C-C if and only if (X, τ) is C-C and Hausdorff.*

The following three results regarding productive properties of compact frame as well as regular frames are proved in [8]. They are as expected as one learned in topological spaces.

Theorem 3.5. *A compact regular frame(locale) is spatial.*

Theorem 3.6. *The coproduct(product) of compact frames(locales) is compact.*

Theorem 3.7. *The coproduct(product) of regular frames(locales) is regular.*

The frame coproduct of component spaces is isomorphic to frame of open sets of the product topology of these spaces.

Theorem 3.8. *[11] Let $X_i, i \in I$ be family of spaces. Then $\oplus_i(\Omega X_i)$ is isomorphic to $\Omega(\oplus X_i)$ if and only if it is a spatial frame.*

3.1 Coproducts on Maximal Compact Frames

Most of the topological properties are preserved under the act of taking product. The productive properties of *C-C Spaces* are discussed and a characterization is obtained in [10]. The paper also provides some results associated with the productive properties of *C-C Spaces*. The previous section contains characterization for a maximal compact frame and some results associated with it. In this section, we are conducting a study on productive properties of maximal compact frames.

The inverse image of a compact sublocale of a compact frame under the localic projection map in the coproduct a given nonempty family of nonempty frames is again compact by the following result.

Theorem 3.9. *Let $\{A_i : i \in I\}$ be a nonempty family of nonempty compact frames and let A be the frame coproduct. If C_j is a compact sublocale of A_j, then ${p_j}^{*-1}(C_j)$ is compact in A.*

Proof. Take $A_j = C_j$ in $\bigoplus_{i\in I} A_i$, then we have ${p_j}^*(\bigoplus_{i\in I} A_i) = C_j$. Thus ${p_j}^{*-1}(C_j) = \bigoplus_{i\in I} A_i$ where $A_j = C_j$. Since C_j is a compact sublocale and all the other $A_i's$ are compact, by Tychonoff theorem for locales, $\bigoplus_{i\in I} A_i$ is compact. Hence ${p_j}_*^{-1}(C_j)$ is compact in A. □

Theorem 3.10. *A locale L is compact if and only if the product projection $p^* : L \oplus M \to M$(the coproduct injection $p : M \to L \oplus M$ in frame language) is closed for every locale M*

The above result is known as the Kuratowski-Mrowka Theorem for locales[11] and is used for proving the next result..

Theorem 3.11. *Let $\{A_i : i \in I\}$ be a non empty family of non empty compact frames and let A be the frame coproduct. If A is a maximal compact frame, then each A_i is a maximal compact frame.*

Proof. If C_j is closed in A_j, then it is compact as a closed sublocale of a compact frame is compact. Now suppose that C_j is a compact sublocale of A_j. Then by *Theorem* 3.9, $C = {p_j}^{*-1}(C_j)$ is compact in A and hence it is closed in A as A is a CCE Frame. Then ${p_j}^*(C) = C_j$ and since A is compact by Tychonoff theorem for locales C_j is closed as the projections of A to A_i being closed maps by*Theorem* 3.10. Hence each A_i is a maximal compact frame. □

The converse of the above result need not be true. We prove this through the next theorem which tells that the coproduct of a compact frame A with itself is a maximal compact frame if and only if A is a maximal compact frame that is Hausdorff. Hence, if the condition Hausdorffness is dropped, then the coproduct may not be a maximal compact frame.

Theorem 3.12. *Let A be any compact frame and let $A \oplus A$ be the coproduct of A with itself. Then $A \oplus A$ is a maximal compact frame if and only if A is maximal compact and Hausdorff.*

Proof. Suppose that $A \oplus A$ is a maximal compact Frame. Then A is a maximal compact frame by *Theorem* 3.11. Let $\Delta(U) = \bigvee \{a \wedge b : (a, b) \in U\}$ where $U \in A \oplus A$. Then $\Delta : A \oplus A \to A$ called the codiagonal is a surjective frame homomorphism and hence its right adjoint Δ^* is a sublocale map by which A is a sublocale of $A \oplus A$. Since A is a maximal compact frame, it is compact. Hence A is a compact sublocale of $A \oplus A$. Since $A \oplus A$ is a maximal compact frame, A is closed in $A \oplus A$, by *Theorem* 3.1. Thus A is Hausdorff as the diagonal Δ embeds A as a closed sublocale of $A \oplus A$, by definition of a Hausdorff frame.

Assume that A is a maximal compact frame and Hausdorff. A compact Hausdorff frame is regular by *Theorem* 3.3. Since a compact regular frame is spatial by *Theorem* 3.5, A is spatial. Then the topological space corresponding to A is a C-C space by *Corollary* 3.2. Also the space corresponding to a regular frame is regular and hence Hausdorff. Thus A is a C-C space that is Hausdorff and hence by *Theorem* 3.4, the product topological space is C-C. Then by *Corollary* 3.2, the product topology is a maximal compact Frame. Since A is compact and regular, by *Theorem* 3.6 and *Theorem* 3.7, $A \oplus A$ is compact and regular. Then, by *Theorem* 3.5, $A \oplus A$ is spatial. Now, by *Theorem* 3.8, $A \oplus A$ is isomorphic to the product topology. Hence $A \oplus A$ is a maximal compact frame. □

We know that every subframe of a compact frame is compact. But every sublocale of a compact locale need not be compact. It happens when the sublocale becomes a closed sublocale. We know that a sublocale is different from a subframe. A sublocale is a quotient frame and hence it cannot be regarded as a subframe of a frame. Hence a maximal compact frame can have a sublocale which in its own respect may become a maximal compact frame. In the next theorem, we prove that the above situation occurs when the sublocale is closed.

Theorem 3.13. *Let A be a maximal compact frame. A sublocale K of A is maximal compact if and only if K is closed in A.*

Proof. Suppose that the sublocale K of A is maximal compact. Then it is a compact sublocale of A and hence closed, as A is a maximal compact frame.

Conversely, suppose that K is a closed sublocale of A. Then K is a compact sublocale of A as it is maximal compact. Thus any closed sublocale of K is compact. Now assume that K_1 is a compact sublocale of K. Then it is a compact sublocale of A. Since A is a maximal compact Frame K_1 is closed in A. Therefore $K_1 = \uparrow_A a$ where $a \in A$. Since K_1 is a sublocale of K, we have $\uparrow_A a = \uparrow_K a$. Hence K_1 is closed in K. Thus K is a maximal compact locale. □

4 Conclusions

As discussed above, the study of productive properties in topological spaces were extensively studied in [2]. But the extension of these results to the "generalized spaces" were not done to that extent. At first we characterized the maximal compact frames in [6] and consequently took the first step to examine the productive property of maximal compact frames. We conclude that if an arbitrary coproduct of compact frames is maximal compact then each component frame is maximal compact and the converse need not hold. We also examined the same result in finite case and observed that Hausdorffness of the component frames is necessary for the two way existence of the result.

A frame is said to be *reversible*[5], if every order preserving self bijection is a frame isomorphism. A characterization for reversible frames is given in [5]. It is also proved that a frame that is maximal or minimal with respect to some frame isomorphic property is reversible. Hence the characterization for maximal compact frames can be used as a method to identify reversible frames . Thereby one can get some information about the automorphism groups possible on finite frames that are maximal compact[4]. The possible automorphism groups on infinite frames is still an open problem.

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