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On Prime Index of a Graph

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Abstract

In prime labeling, vertices are labeled from 1 to n, with the condition that any two adjacent vertices have relatively prime labels. Coprime labeling maintains the same criterion as prime labeling with adjacent vertices using any set of distinct positive integers. Minimum coprime number $\mathfrak{pr}G$, is the minimum value k for which G has coprime labeling. There are many graphs that do not possess prime labeling, and hence have coprime labeling. The primary purpose of this work is to change a coprime labeled graph into a prime graph by removing the minimum number of edges. Thus, the prime index $\varepsilon(G)$ is the least number of edges to be removed from a coprime graph G to form a prime graph. In this study, the prime index of various graphs is determined. Also, an algorithmic way to determine the prime index of the complete graph is found.

Keywords: Prime labeling, Coprime labeling, Minimum Coprime Number, Prime index

2020 AMS subject classifications: 05C76, 05C78 [1]

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1 Introduction

The fundamental concepts of combinatorics and number theory are used in the majority of graph labeling techniques. In number theory, one of the greatest achievements of mathematicians is the prime number theorem. It provides a way to find the number of prime numbers less than or equal to n and is denoted by $\pi(n)$ [1].
In recent years, graph labeling acts as a significant topic in the field of research. To face the wide range of application challenges, various labeling techniques have been developed. Tout et al. [2] investigated Entringer's concept of prime labeling. Many researchers have studied and introduced various definitions in the context of prime graphs[3]. As a continuation of the above work for edges, a new labeling technique known as relatively prime edge labeling for a simple graph is introduced [4]. A graph G with relatively prime edge labeling is said to be a relatively prime edge labeled graph.
Coprime labeling came from prime labeling where prime labeling uses labels from 1 to n, but in coprime labeling, the labels have no bound [5]. In coprime labeling, if the vertices are labeled from 1 to k, then the least k is the minimum coprime number of G. From the above work, to make a coprime graph into a prime graph the definition of prime index arises. The prime index $\varepsilon(G)$ is the least number of edges removed from the coprime graph G to make it a prime graph.
The next section is discussing on preliminaries of the current work. Section 3 is on defining the parameter prime index of a graph with some results followed by the prime index of the complete graph in section 4. Section 5 is discussed the results of relatively prime index of a graph and lastly, section 6 is the conclusion.

2 Preliminaries

This section briefs some definitions needed for further discussion.

A graph $G = (V, E)$ is a prime graph,
if there exists a bijection $f\colon V \to \{1, 2, 3, ..., \mid V \mid\}$ such that, for every edge $e(= uv) \in E$, $\gcd(f(u), f(v)) = 1$ [6].
Asplund et al. emphasize defining minimal coprime number for the coprime graphs [7]. In particular, the vertices of coprime labeling are labeled with relatively prime labels for the adjacent vertices from the set $\{1, 2, ..., k\}$, for some $k \geq n$, . Also, the minimum coprime number $\mathfrak{pr}G$ is the minimum number k for which G has coprime labeling [8]. The corresponding labeling of G is called minimal coprime labeling of G. If $\mathfrak{pr}G = n$, then a corresponding minimal coprime labeling of G is called prime labeling of G and we call G prime.
The coprime number of complete graph and wheel graph are found to be p_{n-1} and

$n+2$, respectively where p_{n-1} denotes the $n-1^{th}$ prime number [7]. Also, the minimum coprime number of the sum of path and cycles are found in [8].
Coxeter et al. introduced the concept of the generalized Petersen graph $P(n,k)$ [9] and it was named by Mark Watkins [10].
The generalized Petersen graphs $P(n,k)$ with $n \geq 3$ and $1 \leq k \leq \frac{n}{2}$ are defined to be a graph with $V(P(n,k)) = \{u_i, v_i \colon 1 \leq i \leq n\}$ and $E(P(n,k)) = \{v_1v_{i+1}, v_iu_i, u_iu_{i+k} \colon 1 \leq i \leq n \text{ , subscript mod } n\}$. For the graph G and H, the sum $G+H$, is the graph obtained by taking disjoint copies of G and H and then adding every edge xy, where $x \in V(G)$ and $y \in V(H)$.
The Wheel graph W_n, $(n \geq 3)$ of $n+1$ vertices is a graph that contains a cycle with vertex set $\{v_1, v_2, ..., v_n\}$ and connects all the vertices of the cycle to the central vertex v. Here v denotes the central vertex and the other vertices $v_1, v_2, ..., v_n$ be on the rim of the wheel graph [12].

3 Prime Index of a Graph

Definition 3.1. *For a coprime graph G, the prime index is the least number of edges removed from G to form a prime graph G^*. And is denoted by, $\varepsilon(G)$. In other words, $\varepsilon(G) = min\{\mid E(H) \mid \colon H \subseteq G \; and \; G - E(H) \; is \; prime\}$*

Every prime graph is a coprime graph, but the converse is not true. The next theorem finds the prime index of the wheel graph W_n, for odd n.

Theorem 3.1. *For odd n, we have $\varepsilon(W_n) = 1$.*

Proof. The vertices of wheel graphs are $\{v_1, v_2, \cdots, v_{n+1}\}$ having v_1 as the central vertex, and the remaining n vertices form a cycle. Now as in Fig 1, label

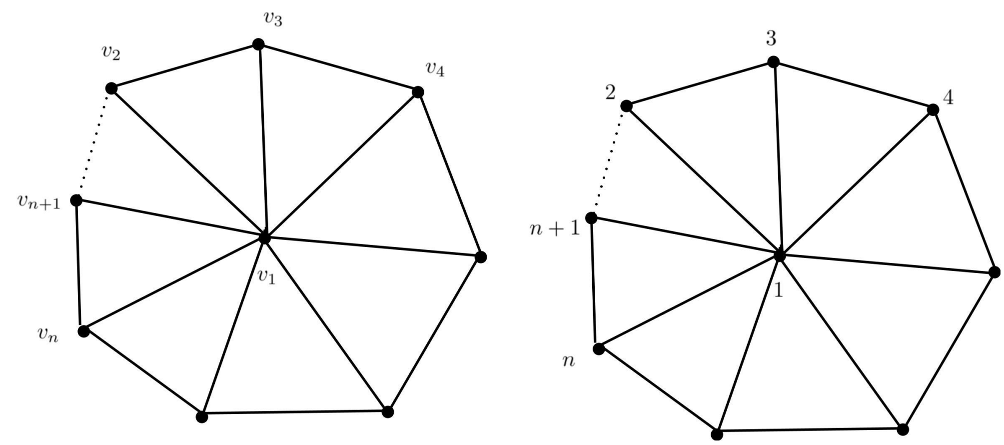

Figure 1: Wheel Graph, $\varepsilon(W_n) = 1$

the vertices v_i as i , for all $i \in \{1, 2, \cdots, n+1\}$, which fails to form prime graph. Since v_2 receives the label 2 and v_{n+1} receives the label $n+1$, which is even. Therefore, for odd n to make w_n a prime graph it is necessary to remove an edge v_2v_{n+1}. Hence $\varepsilon(W_n) = 1$, for odd n.

$P(n,k)$ is a coprime graph , for n odd is found by U. M. Prajapati et.al.[13]. The next theorem finds the prime index of $P(n,2)$, for even n.

Theorem 3.2. *$P(n,k)$ contains d disjoint inner cycles of length $\frac{n}{d}$, only if $gcd(n,k) = d$. [13]*

Theorem 3.3. *For even n , the prime index of $P(n,2)$ is $\varepsilon(P(n,2)) \leq n/2$.*

Proof. We know that for n even, the Generalised Petersen graph $P(n,2)$ is not prime. Hence it is enough to prove that the removal of $n/2$ edges from $P(n,2)$ forms a prime graph. In $P(n,2)$, $v_1, v_2, \cdots, v_n$ represents the outer vertices and $u_1, u_2, \cdots, u_n$ represents the inner vertices as in Fig 2.

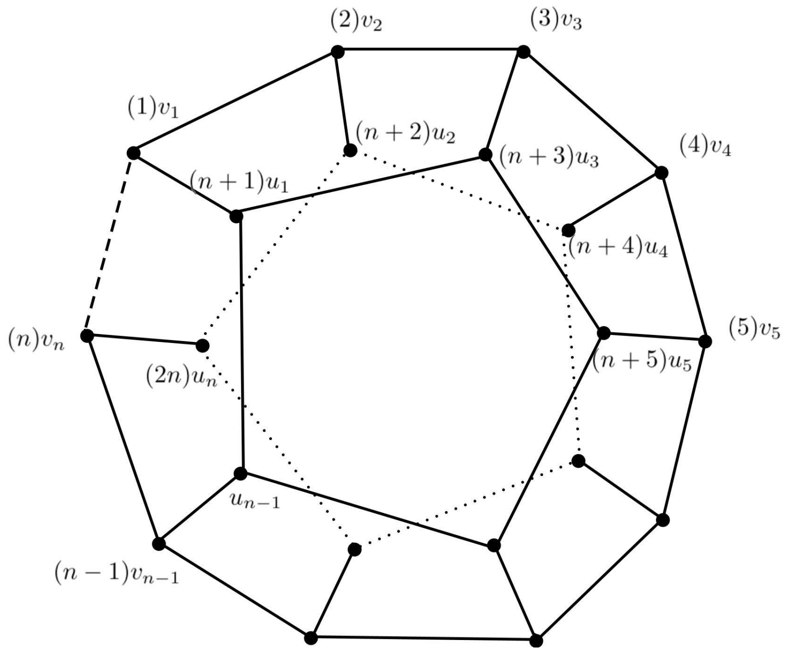

Figure 2: $P(n,2)$

Edges of $P(n,2)$ are $\{v_1v_{i+1}, v_iu_i, u_iu_{i+2} \colon 1 \leq i \leq n, subscript\ mod\ n\}$. From corollary 2, there exist 2 interior cycles of length $n/2$. Now, vertices of $P(n,2)$ are labeled in such a way that $L(v_i) = i\ \&L(u_i) = n+i, 1 \leq i \leq n$. From the definition of the generalized Petersen graph, each u_i is adjacent to u_{i+2}. Thus, there are $n/2$ vertices having the label $n+2, n+4, \cdots, 2n$. Therefore, by removing atleast $\{u_2u_4, u_4u_6, \ldots, u_{n-2}u_n\}$ edges from $P(n, 2)$ results a prime graph. Hence the proof.

It is obvious that, for even m, $P_m + P_2$ is not a prime graph.The removal of at least an edge from $P_m + P_2$ forms a prime graph. Thus, all possible spanning subgraph of $P_m + P_2$ is again a prime graph, for even m.

Theorem 3.4. *For even m, we have $\varepsilon(P_m + P_2) = 1$.*

Proof. The proof is obvious.

4 Prime Index of complete graph

Euler's function, $\phi(n)$ in number theory finds the number of integer k in the range 1 to n for which $gcd(n, k) = 1$. In this section, the prime index of a complete graph with an algorithm is derived.

Theorem 4.1. *For a complete graph K_n the prime index $\varepsilon(G)$ of a complete graph is $\varepsilon(K_n) = \binom{n}{2} - \sum_{k=2}^{n} \phi(k)$*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the n vertices of the complete graph K_n having nc_2 edges. Maximal subgraph in K_n containing the closed path is the cycle C_n. Now, label the vertices of K_n as $L(v_i) = i$. As every vertex is adjacent to every other vertices, it is enough to find the pairs for which $gcd(n, k) \neq 1$. Hence, the number of pairs in the range 1 to n for which $gcd(n, k) \neq 1$ equals the difference between the total number of pairs in the range 1 to n and the numbers of pairs in the range 1 to n for which $gcd(n, k) = 1$. That is, $\varepsilon(k_n) = nc_2 - \sum_{k=2}^{n} \phi(k)$.

Super prime graph SP_n is a graph with vertex set $V(SP_n) = \{1, 2, ..., n\}$ and two vertices $u, v \in V(SP_n)$ are adjacent iff $(u, v) = 1$.Also, the super prime graph SP_n is maximal and unique[15]. The next theorem shows the method for a complete graph to be a super-prime graph.

Theorem 4.2. *The removal of $\varepsilon(K_n)$ edges from a complete graph K_n results a super prime graph.*

Proof. The proof is obvious.

4.1 Algorithm to find prime Index

GCD of any two numbers can be found by using the Euclidean algorithm. Here, an algorithmic way to find the prime index of the complete graph is defined.

INPUT: No. of vertices $= n$.
OUTPUT: Prime Index of K_n.

1. Calculate, the number of edges $E = nC_2$, for the given input n.
2. Calculate $\gcd(r, s)$ as, if $r < s$, exchange r and s. Then, divide r by s, with remainder R.
   (a) If $R = 0$, report $s = \gcd(r, s)$
   (b) Else if $R \neq 0$, replace r by s and replace s by R.
   Return to the previous step a.
3. Run a loop for $i = 2$ to n,
   $\phi(n)$ counts the number of pairs having $\gcd(i, n) = 1$.
4. For $n = 2$ to n, $sum(n)$ counts the value $\phi(n)$.
5. Finally, $Prime\, index = E - sum(n)$

4.2 Program to find the prime index of complete graph

C program to find the prime index of complete graph is given below.

```
#include <stdio.h>
int phi(int n);
int gcd(int a, int b);
int sum(int n);
int main( )
{
     int i, n, E, PI;
     printf(" Number of vertices:  ");
     scanf("%d",&n);
     E = (n*(n-1))/2;
     printf("Number of Edges = %d \n",E);
for (i = 1; i <= n; i++)
     printf("phi(%d) = %d \n", i, phi(i));
     PI = E -  sum(n);
     printf("\n Prime index = %d",PI);
     return 0;
  }

// Function finds GCD of r and s
int gcd(int r, int s)
{
  if (r == 0)
      return s;
```

```
  return gcd(s % r, r);
}
// Function to return phi(n)
int phi(int n)
{
      int result = 1;
      for (int i = 2; i < n; i++)
  if (gcd(i, n) == 1)
    result++;
       return result;
}

// Function counts phi(n) from 2 to n
int sum(int n)
{
    int sum;
   for(int k=2; k<=n; k++)
           {
            if (k==2)
            sum =1;
            else
            sum = sum + phi(k);
             }
    return sum;
  }
}
```

```
Enter the number:  7
Edges = 21
phi(1) = 1
phi(2) = 1
phi(3) = 2
phi(4) = 2
phi(5) = 4
phi(6) = 2
phi(7) = 6

 Prime index = 4

...Program finished with exit code 0
Press ENTER to exit console.
```

Figure 3: Output

The necessary and sufficient condition for $K_{m,n}$ to be prime was put forward by Fu and Huang[14]. Also, in [14] by taking, $q(t, v)$ to be the set of all primes x such that $t < x \leq v$, the following proposition exists.

Theorem 4.3. *For the positive integers a, b with $a < b$, $K_{a,b}$ is prime iff $a \leq | q(\frac{a+b}{2}, a+b) | +1$.*

The next theorem finds the prime index of a complete bipartite graph $K_{m,m}$. For that, let V_1 with a set of odd numbers and V_2 with a set of even numbers. Define $\mu(V_1, V_2) = \{u_i v_i \colon u_i \in V_1 \ \& \ v_i \in V_2 \mid (u_i, v_i) \neq 1\}$.

Theorem 4.4. *The prime index of the complete bipartite graph $K_{m,m}$ with $m > | P(m, 2m) | +1$ is $\varepsilon(K_{m,m}) = | \mu(V_1, V_2) |$.*

Proof. Let $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$ be the two partite of $K_{m,m}$. Label the vertices of V_1 and V_2 with odd and even numbers respectively. Then, the resultant graph is not a prime. Thus to make $K_{m,m}$ to be a prime graph, its necessary to remove $\mu(V_1, V_2)$ edges from $K_{m,m}$. Hence the prime index $\varepsilon(K_{m,m}) = | \mu(V_1, V_2) |$.

As an application part of the prime index of a graph, consider there are certain workers having common skill sets. Represent them using a graph, if the vertex represents the workers and if they have a common skill set an edge can be drawn between them. The label represents the shift hours of each worker. Prime labeling can be used to find the optimal solution so that it covers the maximum skill set in the particular shift. Suppose every worker has some common skill set with every other worker, then the prime index helps to find the common skill that cannot be carried out to get the optimal solution.

5 On Relatively Prime Index of a Graph

The concept of relatively prime edge labeling is just an extension of prime labeling by considering edges into account. In prime labeling, adjacent vertices have relatively prime labels, whereas, in relatively prime edge labeling, adjacent edges have relatively prime labels[4]. The idea to make a coprime edge labeled graph into a relatively prime edge labeled graph results in a definition of a relatively prime index[4].

Definition 5.1. *[4] Let G be a coprime edge labeled graph. A relatively prime index $\varepsilon_r(G)$ is defined to be the minimum number of edges removed from G to form a relatively prime edge labeled graph G^*. In other words, $\varepsilon_r(G) = min\{| E(H) | \colon H \subseteq G \ \& \ G - E(H) \ is \ relatively \ prime \ edge \ labeled \ graph\}$*

The next theorem finds the relatively prime index of the corona product of K_n and K_1.

Theorem 5.1. *For a graph $K_n \odot K_1$,*

$$\varepsilon_r(K_n \odot K_1) = \begin{cases} \frac{n(n-3)}{2} & \text{if } 2n+1 \equiv 0 (mod\, 3) \\ \frac{n(n-3)}{2} - 1 & \text{if } 2n+1 \not\equiv 0 (mod\, 3) \end{cases} \quad (1)$$

Proof. Let $G = K_n \odot K_1$ be the graph with $2n$ vertices and $\frac{n(n+1)}{2}$ edges and let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of $K_n \cdots K_1$.
Case 1: For $2n+1 \equiv 0(mod\, 3)$.
We now prove that, the removal of $\frac{n(n-3)}{2}$ edges results in a relatively prime edge-labeled graph. Suppose the removal of $\frac{n(n-3)}{2} - 1$ edges in$K_n \odot K_1$ results in a relatively prime edge labeled graph. That is, remaining $\frac{n(n+1)}{2} - \frac{n(n-3)}{2} + 1 = 2n+1$ edges of K_n can be labeled from 1 to $2n+1$. Hence by removing $\frac{n(n-3)}{2} - 1$ interior edges of $K_n \odot K_1$, the resultant graph will be of the form C_n with n edges, n pendent vertices $(u_1, u_2, \ldots, u_n)$ connecting to the each vertex of C_n and an edge connecting any two non-adjacent vertices of C_n. Each vertex $v_1, v_2, \ldots, v_n$ of C_n is of degree 3. By labeling the edges of C_n with $1, 3, 5, \ldots, 2n-1$, each edge incident on the pendant vertices is labeled with $2, 4, 6, \ldots, 2n$ and an edge connecting any two non-adjacent vertices of C_n is labeled with $2n+1$. As $2n+1 \equiv 0(mod\, 3)$, that is $2n+1 = 3m$, the label incident on the vertices of the edge with label $2n+1$ fails to be relatively prime.
Case 2: For $2n+1 \not\equiv 0(mod\, 3)$.
We now prove that the removal of $\frac{n(n-3)}{2} - 1$ edges results in a relatively prime edge labeled graph. Suppose the removal of $\frac{n(n-3)}{2} - 2$ edges in $K_n \odot K_1$ results in a relatively prime edge labeled graph. That is, remaining $\frac{n(n+1)}{2} - \frac{n(n-3)}{2} + 2 = 2n+2$ edges of K_n can be labeled from 1 to $2n+2$. Hence by removing $\frac{n(n-3)}{2} - 2$ interior edges of $K_n \odot K_1$, the resultant graph will be of the form C_n with n edges, n pendent vertices $(u_1, u_2, \ldots, u_n)$ connecting to the each vertex of C_n and two edges connecting any two non-adjacent vertices of C_n. Each vertex $v_1, v_2, \ldots, v_n$ of C_n is of degree 3. By labeling the edges of C_n with $1, 3, 5, \ldots 2n-1$, each edge incident on the pendant vertices is labeled with $2, 4, 6, \ldots, 2n$ and the edge connecting any two non-adjacent vertices of C_n is labeled with $2n+1, 2n+2$. As $2n+1 \not\equiv 0(mod\, 3)$, the label incident on the vertices of the edge with label $2n+2$ fails to be relatively prime.

A set is an edge-independent set of G, if any two edges in the set are not incident one another. The maximum cardinality of all edge-independent sets is the edge independence number of G, $\alpha'(G)$[16].

Theorem 5.2. *For a relatively prime edge labeled graph G having p vertices and q edges, then edge independence number $\alpha'(G) \geq \left\lfloor \frac{q}{2} \right\rfloor$*

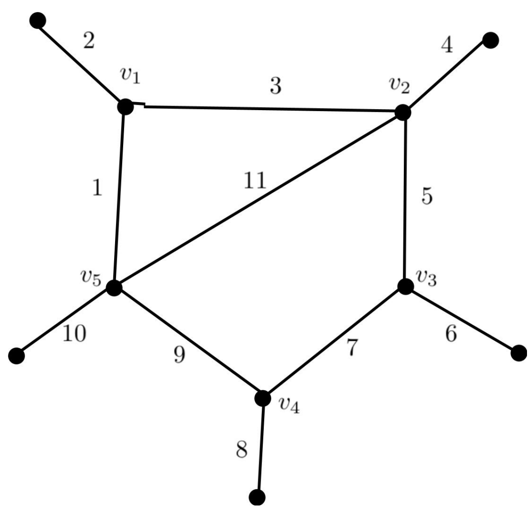

Figure 4: Relatively prime index of $K_5 \odot K_1$ = 4

Proof. Let G be a relatively prime edge labeled graph. The set of edges that are labeled with even numbers is an edge-independent set and its cardinality will be less than or equal to $\left\lfloor \frac{q}{2} \right\rfloor$. Hence the proof.

Theorem 5.3. *Let $K(G)$ be the clique graph of a graph G with q edges and if $\mid E(K(G)) \mid < \left\lfloor \frac{q}{2} \right\rfloor$, then G is not a relatively prime edge labeled graph.*

Proof. From the definition of clique graph, no two edges in the clique graph $K(G)$ can be independent. That is, $\alpha'(G) < \mid E(K(G) \mid < \left\lfloor \frac{q}{2} \right\rfloor$. Also by Theorem - 8, for a relatively prime edge labeled graph $\alpha'(G) \geq \left\lfloor \frac{q}{2} \right\rfloor$, which is a contradiction. Hence G is not a relatively prime edge labeled graph.

6 Conclusions

Prime labeling is a versatile technique that finds applications in various fields such as graph theory and optimization. One of its primary applications is in the domain of scheduling and resource allocation. For example, it can be used to determine the optimal way to assign workers to shifts to maximize the utilization of their skills. In network flow problems, prime labeling can help identify critical edges that should be included in any feasible flow solution to optimize the flow of a commodity through the network. In bioinformatics, prime labeling can be used to identify conserved regions in DNA sequences. By representing DNA sequences as graphs and labeling the vertices based on prime numbers, it can help identify regions of high conservation between different sequences.

In summary, prime labeling is a powerful technique that has wide-ranging applications in areas such as graph theory, optimization, and several other fields in science and engineering.
As a next, to employ the prime index of a graph in a practical way, we can think of a scenario where there are several workers who share certain skills. We can create a graph to represent this scenario, with each worker being a vertex and an edge connecting two vertices if the corresponding workers share a common skill. The label on each vertex indicates the shift hours for that worker. By using prime labeling, we can determine the best solution that covers the maximum number of skills required during a particular shift. If every worker shares at least one skill with every other worker, the prime index can help identify the shared skills that must be present to achieve the optimal solution.
The work presented here finds some new results in the idea of prime labeling of graphs. The main goal of this work is to convert a coprime-labeled graph into a prime graph by removing the fewest edges possible. Finally, a method for computing the prime index of the complete graph using an algorithm has been found. In the future, the prime index of a certain class of graphs can be found. Also, the importance of prime index will be discussed briefly.

References

[1] Jameson, G.J.O. The Prime number theorem. Cambridge University Press, Cambridge. 2003.

[2] Tout, R., Dabboucy, A.N., Howalla, K. Prime labeling of graphs. National Academy Science Letters-India. 5(11), 365–368.1982.

[3] Fu, H.L., Huang, K.C. On prime labellings. Discrete Mathematics. 127,181–186. 1994.

[4] Janani, R., Ramachandran, T. On Relatively Prime Edge Labeling of Graphs. Engineering Letters. 30(2), 659–665. 2022.

[5] Berliner, A. H. et al. Coprime and prime labelings of graphs. Journal of Integer Sequences,19(5), 1–14.2016

[6] Gallian, J.A. A dynamic survey of graph labeling. In The Electronic Journal of Combinatorics. 2002

[7] Asplund,J and Bradley Fox.N. Minimum Coprime Labelings for Operations on Graphs. Integers, 1–19. 2017.

[8] Lee, C . Minimum coprime graph labelings. arXiv preprint arXiv:1907.12670. 2019.

[9] Coxeter, H. et al. Self-dual configurations and regular graphs. Bull. Amer. Math. Soc 56. 413–455. 1950.

[10] Watkins, M. E. A theorem on Tait colorings with an application to the generalized Petersen graphs. J. Comb. Theory 6(2), 152–164. 1969.

[11] Bondy, J.A. Murty, U.S.R. Graph theory with applications 290, London: Macmillan, London. 1976.

[12] Harary,F. Graph Theory. Narosa Publishing House, New Delhi. 1998.

[13] Prajapati, U. M., S. J. Gajjar. Prime labeling of generalized Petersen graph. Internat. J. Math. and Soft Comput 5, 65-71.2015.

[14] Fu, Hung-Lin, Kuo-Ching Huang. On prime labellings. Discrete Mathematics 127,181-186. 1994

[15] Nagoji Rao, Suryaprakash. Graceful and Prime Labelings–Algorithms, Embeddings and Conjectures. arXiv e-prints (2020): arXiv-2006.

[16] Ying-Ying Tan, Yi-Zheng Fan. The vertex (edge) independence number, vertex (edge) cover number and the least eigenvalue of a graph . Linear Algebra and its Applications ,433(4), 790-795. 2010.