**Ratio Mathematica** Volume 48, 2023

# Antimagic Labeling of Some Degree Splitting Graphs

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**Abstract**

A graph with $q$ edges is called *antimagic* if its edges can be labeled with $1, 2, 3, ..., q$ without repetition such that the sums of the labels of the edges incident to each vertex are distinct. As Wang et al. [2012], proved that not all graphs are antimagic, it is interesting to investigate antimagic labeling of graph families. In this paper we discussed antimagic labeling of the larger graphs obtained using degree splitting operation on some known antimagic graphs. As discussed in Krishnaa [2016], antimagic labeling has many applications, our results will be used to expand the network on larger graphs.

**Keywords**: Antimagic labeling, Antimagic graph, Graph operation, Degree splitting graph.
**2020 AMS subject classifications**: 05C78, 05C76. [1]

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[1]Received on June 20, 2023. Accepted on August 16, 2023. Published on September 3, 2023. DOI: 10.23755/rm.v48i0.1253. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

# 1 Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$. For all standard terminologies and notations we follow Clark and Holton [1969]. We will give brief summary of definitions which are useful for the present investigations.

**Definition 1.1.** *A* graph labeling *is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or edge labeling).*

For an extensive survey on graph labeling and bibliographic references we refer to Gallian [2022].

Hartsfield and Ringel [1990] have introduced antimagic labeling as follows.

**Definition 1.2.** *A graph with $q$ edges is called* antimagic *if its edges can be labeled with $1, 2, 3, ..., q$ without repetition such that the sums of the labels of the edges incident to each vertex are distinct.*

They have also discussed antimagic labeling for paths, cycles, wheels and complete graph. Wang [2005] has shown that the toroidal grids $C_{n_1} \times C_{n_2} \times \ldots \times C_{n_k}$ are antimagic. Cheng [2008] has proved that all Cartesian products of two or more regular graphs are antimagic.

Vaidya and Vyas [2012, 2013] have discuss antimagic labeling of some path and cycle related graphs as well as graph obtained using switching of a vertex. Alon et al. [2004] have obtained some conditions on degree of a vertices for graph to be antimagic.

Krishnaa [2016] has illustrated some applications of antimagic labeling.

**Definition 1.3.** *Let $G = (V(G), E(G))$ be a graph with $V = S_1 \cup S_2 \cup S_3 \cup ...S_t \cup T$, where each $S_i$ is a set of vertices having same degree (at least two vertices) and $T = V \setminus \cup_{i=1}^{t} S_i$. The* degree splitting graph *of $G$ denoted by $DS(G)$ is obtained from the graph $G$ by adding vertices $w_1, w_2, w_3, ..., w_t$ and joining $w_i$ to each vertex of $S_i$ for $1 \leqslant i \leqslant t$.*

**Definition 1.4.** *The* Cartesian product of graphs *$G_1$ and $G_2$ denoted by $G_1 \square G_2$ is the graph with vertex set $V(G_1) \times V(G_2) = \{(u, v)/u \in V(G_1)$ and $v \in V(G_2)\}$ and $(u, v)$ is adjacent to $(u', v')$ if and only if either $u = u'$ and $vv' \in E(G_2)$ or $v = v'$ and $uu' \in E(G_1)$.*

**Definition 1.5.** *The* ladder graph *$L_n$ is defined as $L_n = P_n \square K_2$.*

**Definition 1.6.** *Let $G_1$ and $G_2$ be two graphs with no vertex in common, We define the* join *of $G_1$ and $G_2$, denoted by $G_1+G_2$, to be the graph with vertex set $V(G_1+G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1+G_2) = E(G_1) \cup E(G_2) \cup J$, where $J = \{x_1x_2 : x_1 \in V(G_1), x_2 \in V(G_2)\}$.*

**Definition 1.7.** *The* fan graph *$F_n$ is defined as $F_n = P_n + K_1$.*

**Definition 1.8.** *Let $G$ and $H$ be two graphs. The* corona product *of $G$ and $H$, denoted by $G \odot H$, is obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$, and by joining each vertex of the $i^{th}$ copies of $H$ to the $i^{th}$ vertex of $G$, for $1, 2, 3, \ldots, |V(G)|$.*

**Definition 1.9.** *Let $C_n$ be a cycle of $n$ vertices. The* crown graph *is defined as $C_n \odot K_1$. .*

**Definition 1.10.** *Let $P_n$ be a path of $n$ vertices. The* comb graph *is defined as $P_n \odot K_1$.*

**Definition 1.11.** *Let $P_n$ be a path of $n$ vertices. The* double comb graph *is defined as $P_n \odot 2K_1$.*

In this paper, we have discuss antimagic labeling of degree splitting graphs of path, ladder graph, fan graph, crown graph, comb graph and double comb graph.

# 2 Main Results

**Theorem 2.1.** *The graph $DS(P_n)$ is an antimagic graph.*

*Proof.* Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of $P_n$ with $d(v_1) = d(v_2) = 1$. To construct $DS(P_n)$, add vertices $w_1$ and $w_2$ such that $w_1$ is adjacent to each vertex of degree 1 and $w_2$ is adjacent to each vertex of degree 2 in $P_n$. Then $|V(DS(P_n))| = n+2$ and $|E(DS(P_n))| = 2n-1$ for $n \geqslant 4$.
We define $f : E(DS(P_n)) \to \{1, 2, \ldots, 2n-1\}$, as per following three cases.
**Case - 1** For $n = 2$ and 3.
$DS(P_2)$ and $DS(P_3)$ are also known as cycle $C_3$ and cycle $C_4$ respectively. The antimagic labeling of $C_3$ and $C_4$ have been investigated by Hartsfield and Ringel [1990].
**Case - 2** For $n = 4$.

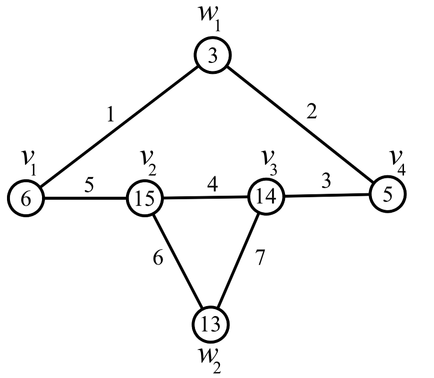


Figure 1: $DS(P_4)$ and its antimagic labeling.

**Case - 3** For $n > 4$.
$f(v_1w_1) = 2(n-1)$,
$f(v_nw_1) = 2n-1$,
$f(v_iv_{i+1}) = i$; For $1 \leqslant i \leqslant n-1$,
$f(v_iw_2) = 2n-i-1$; For $2 \leqslant i \leqslant n-1$.
Above define edge labeling function will generate distinct vertex labels for all the vertices of $DS(P_n)$. Thus, $f$ is an antimagic labeling.
Hence, the graph $DS(P_n)$ is an antimagic graph. □

**Illustration 2.1.** *The graph $DS(P_7)$ and its antimagic labeling is shown in Figure 2.*

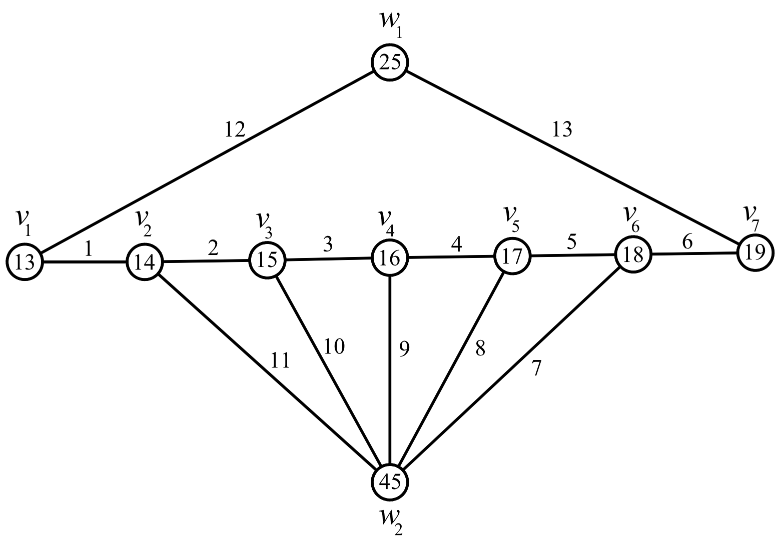


*Figure 2: $DS(P_7)$ and its antimagic labeling.*

**Theorem 2.2.** *The graph $DS(L_n)$ is an antimagic graph.*

*Proof.* Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of first path and $v_{n+1}, v_{n+2}, v_{n+3}, \ldots, v_{2n}$ be the vertices of second path in ladder $L_n$. To construct $DS(L_n)$, add vertices $w_1$ and $w_2$ such that $w_1$ is adjacent to each vertex of degree 2 and $w_2$ is adjacent to each vertex of degree 3 in ladder $L_n$. Then $|V(DS(L_n))| = 2n + 2$ and $|E(DS(L_n))| = 5n - 2$ for $n \geqslant 3$.
We define $f : E(DS(L_n)) \to \{1, 2, \ldots, 5n - 2\}$ as per following two cases.
**Case - 1** For $n = 2$.
$DS(L_2)$ is also known as wheel $W_4$. The antimagic labeling of $W_4$ has been investigated by Hartsfield and Ringel [1990].
**Case - 2** For $n \geqslant 3$.

| | |
|---|---|
| $f(v_iv_{i+1}) = i;$ | For $1 \leqslant i \leqslant n-1,$ |
| $f(v_iv_{n+i}) = 2n - i;$ | For $1 \leqslant i \leqslant n,$ |
| $f(v_{n+i}v_{n+1+i}) = 2n + i - 1;$ | For $1 \leqslant i \leqslant n-1,$ |
| $f(v_iw_2) = 3(n-1) + i;$ | For $2 \leqslant i \leqslant n-1,$ |
| $f(v_{n+i}w_2) = 4n - 5 + i;$ | For $2 \leqslant i \leqslant n-1,$ |
| $f(v_1w_1) = 5(n-1) + 2,$ | |
| $f(v_nw_1) = 5(n-1) + 1,$ | |
| $f(v_{n+1}w_1) = 5(n-1) + 3,$ | |
| $f(v_{2n}w_1) = 5(n-1).$ | |

Above define edge labeling function will generate distinct vertex labels for all the vertices of $DS(L_n)$. Thus, $f$ is an antimagic labeling.
Hence, the graph $DS(L_n)$ is an antimagic graph. □

**Illustration 2.2.** *The graph $DS(L_6)$ and its antimagic labeling is shown in Figure 3.*

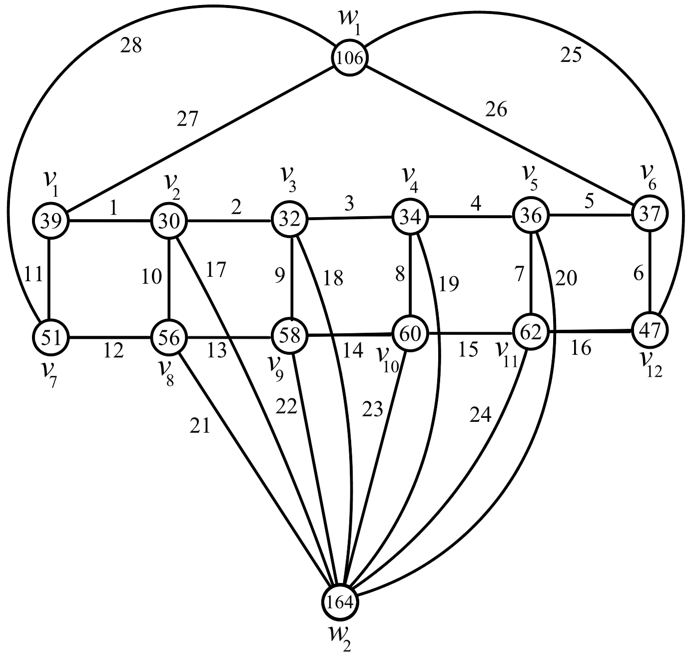


*Figure 3: $DS(L_6)$ and its antimagic labeling.*

**Theorem 2.3.** *The graph $DS(F_n)$ is an antimagic graph.*

*Proof.* Let $v$ be a apex vertex and $v_1, v_2, v_3, ..., v_n$ be the vertices corresponding to path $P_n$ in graph $F_n = P_n + K_1$. To construct $DS(F_n)$, add vertices $w_1$ and $w_2$ such that $w_1$ is adjacent to each vertex of degree 2 and $w_2$ is adjacent to each vertex of degree 3 in fan $F_n$. Then $|V(DS(F_n))| = n + 3$ and $|E(DS(F_n))| = 3n - 1$ for $n \geqslant 4$.

We define $f : E(DS(F_n)) \rightarrow \{1, 2, ..., 3n - 1\}$, as per following three cases.

**<u>Case - 1</u>** For $n = 2$.

$DS(F_2)$ is also known as wheel $W_3$. The antimagic labeling of $W_3$ has been investigated by Hartsfield and Ringel [1990].

**<u>Case - 2</u>** For $n = 3$.

The graph $DS(F_3)$ is an antimagic graph by Alon et al. [2004].

**<u>Case - 3</u>** For $n \geqslant 4$.

$f(v_iv_{i+1}) = i;$ For $1 \leqslant i \leqslant n - 1$,

$f(v_iw_2) = 2n - i - 1;$ For $2 \leqslant i \leqslant n - 1$,

$f(vv_i) = 2n + i - 1;$ For $1 \leqslant i \leqslant n$,

$f(v_1w_1) = 2(n - 1),$

$f(v_nw_1) = 2n - 1.$

Above define edge labeling function will generate distinct vertex labels for all the vertices of $DS(F_n)$. Thus, $f$ is an antimagic labeling.

Hence, the graph $DS(F_n)$ is an antimagic graph. □

**Illustration 2.3.** *The graph $DS(F_7)$ and its antimagic labeling is shown in Figure 4.*

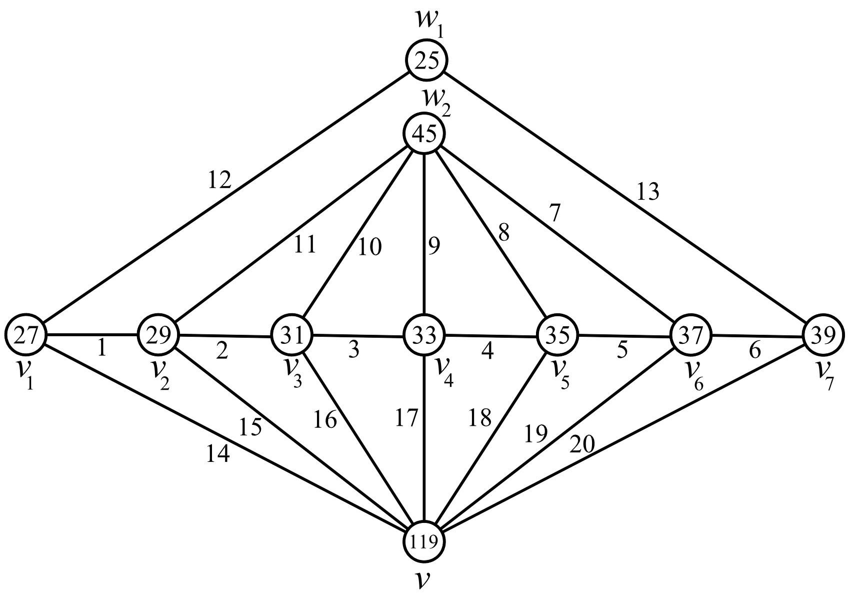


*Figure 4: $DS(F_7)$ and its antimagic labeling*

**Theorem 2.4.** *The graph $DS(C_n \odot K_1)$ is an antimagic graph.*

*Proof.* Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices corresponding to cycle $C_n$ and $v_i'$ be the pendant vertices attached to $v_i$ for each $i = 1, 2, \ldots, n$ respectively in graph $C_n \odot K_1$. To construct $DS(C_n \odot K_1)$, add vertices $w_1$ and $w_2$ such that $w_1$ is adjacent to each vertex of degree 3 and $w_2$ is adjacent to each vertex of degree 1 in $C_n \odot K_1$. Then $|V(DS(C_n \odot K_1))| = 2n + 2$ and $|E(DS(C_n \odot K_1))| = 4n$.
We define $f : E(DS(C_n \odot K_1)) \to \{1, 2, \ldots, 4n\}$ as follows.
$f(v_iv_{i+1}) = 2n - i + 1$; For $1 \leqslant i \leqslant n - 1$,
$f(v_1v_n) = n + 1$,
$f(v_iv_i') = i$; For $1 \leqslant i \leqslant n$,
$f(v_1w_1) = 2n + 1$,
$f(v_iw_1) = 3n - i + 2$; For $2 \leqslant i \leqslant n$,
$f(v_i'w_2) = 3n + i$; For $1 \leqslant i \leqslant n$.
Above define edge labeling function will generate distinct vertex labels for all the vertices of $DS(C_n \odot K_1)$. Thus, $f$ is an antimagic labeling.
Hence, the graph $DS(C_n \odot K_1)$ is an antimagic graph. □

**Illustration 2.4.** *The graph $DS(C_6 \odot K_1)$ and its antimagic labeling is shown in Figure 5.*

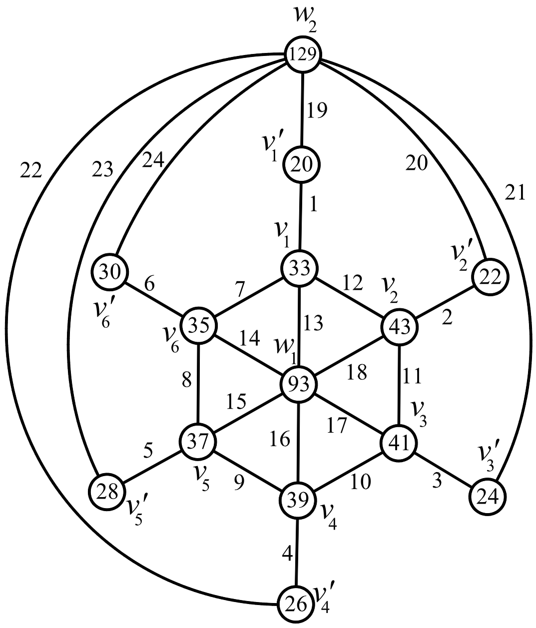


*Figure 5: $DS(C_6 \odot K_1)$ and its antimagic labeling.*

**Theorem 2.5.** *The graph $DS(P_n \odot K_1)$ is an antimagic graph.*

*Proof.* $v_1, v_2, v_3, \ldots, v_n$ be the vertices corresponding to path $P_n$ and $v_i'$ be the pendant vertices attached to $v_i$ for each $i = 1, 2, \ldots, n$ respectively in graph $P_n \odot K_1$. To construct $DS(P_n \odot K_1)$, add vertices $w_1$, $w_2$ and $w_3$ such that $w_1$ is adjacent to each vertex of degree 1, $w_2$ is adjacent to each vertex of degree 2 and $w_3$ is adjacent to each vertex of degree 3 in $P_n \odot K_1$. Then $|V(DS(P_n \odot K_1))| = 2n + 3$ and $|E(DS(P_n \odot K_1))| = 4n - 1$ for $n \geqslant 4$.
We define $f : E(DS(P_n \odot K_1)) \to \{1, 2, \ldots, 4n - 1\}$ as per following two cases.
<u>**Case - 1**</u> For $n = 2$ and 3.

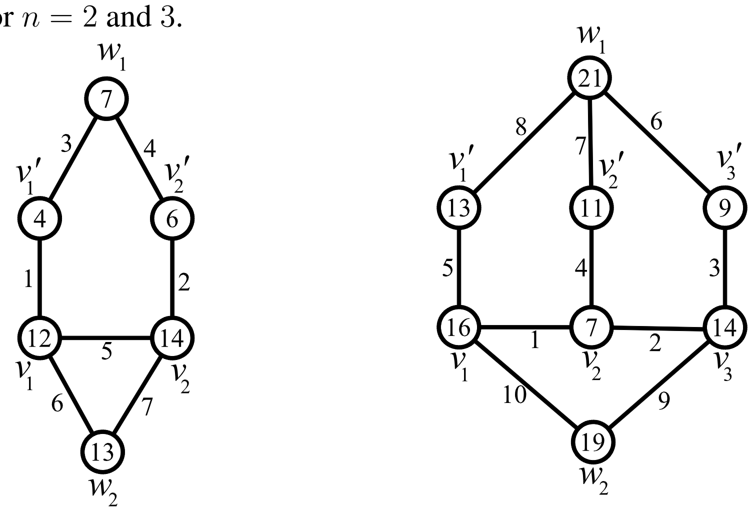


Figure 6: $DS(P_2 \odot K_1)$ and $DS(P_3 \odot K_1)$ and their antimagic labeling.

**Case - 2** For $n \geqslant 4$.

$f(v_i v_i') = i;$ For $1 \leqslant i \leqslant n$,
$f(v_i v_{i+1}) = 2n - i;$ For $1 \leqslant i \leqslant n-1$,
$f(v_i w_3) = 4n - i - 1;$ For $2 \leqslant i \leqslant n-1$,
$f(v_i' w_1) = 2n + i - 1;$ For $1 \leqslant i \leqslant n$,
$f(v_1 w_2) = 2(2n-1)$,
$f(v_n w_2) = 4n - 1$.

Above define edge labeling function will generate distinct vertex labels for all the vertices of $DS(P_n \odot K_1)$. Thus, $f$ is an antimagic labeling.
Hence, the graph $DS(P_n \odot K_1)$ is an antimagic graph. □

**Illustration 2.5.** *The graph $DS(P_8 \odot K_1)$ and its antimagic labeling is shown in Figure 7.*

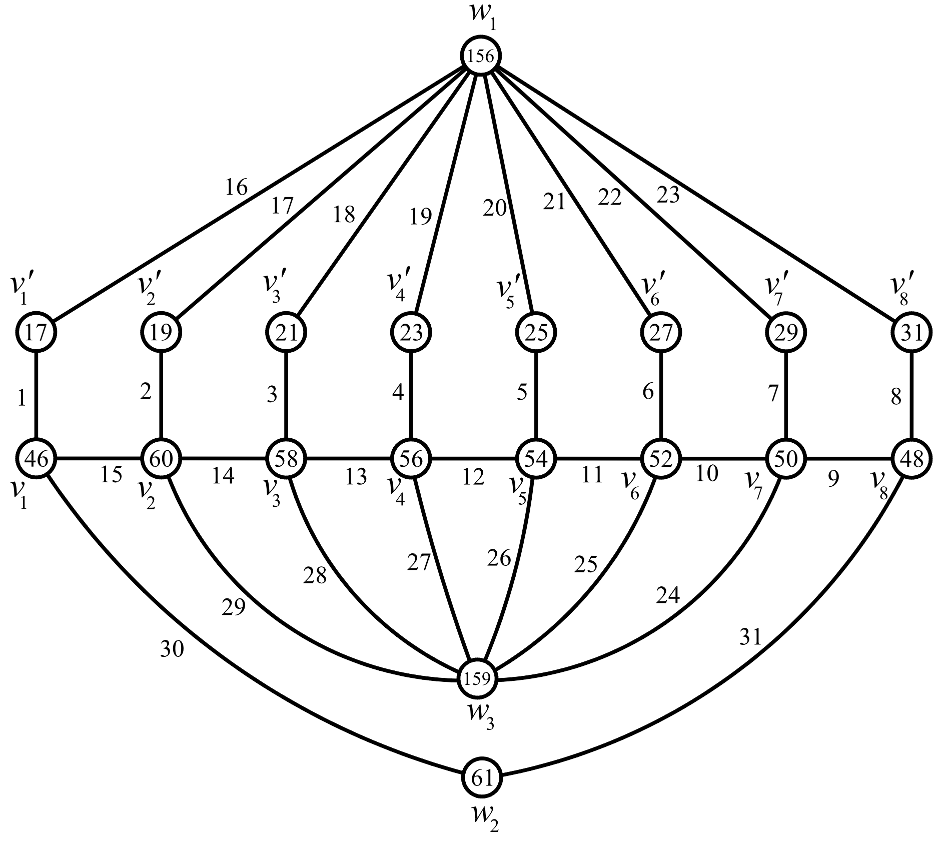


*Figure 7: $DS(P_8 \odot K_1)$ and its antimagic labeling.*

**Theorem 2.6.** *The graph $DS(P_n \odot 2K_1)$ is an antimagic graph.*

*Proof.* $v_1, v_2, v_3, \ldots, v_n$ be the vertices corresponding to path $P_n$ and $v_i'$ and $v_i''$ be the pendant vertices attached to $v_i$ for each $i = 1, 2, \ldots, n$ respectively in graph

$P_n \odot 2K_1$. To construct $DS(P_n \odot 2K_1)$, add vertices $w_1$, $w_2$ and $w_3$ such that $w_1$ is adjacent to each vertex of degree 1, $w_2$ is adjacent to each vertex of degree 3 and $w_3$ is adjacent to each vertex of degree 4 in $P_n \odot 2K_1$. Then $|V(DS(P_n \odot 2K_1))| = 3n + 3$ and $|E(DS(P_n \odot 2K_1))| = 6n - 1$ for $n \geqslant 4$.
We define $f : E(DS(P_n \odot 2K_1)) \to \{1, 2, \ldots, 6n - 1\}$ as per following two cases.
**Case - 1** For $n = 2$ and 3.

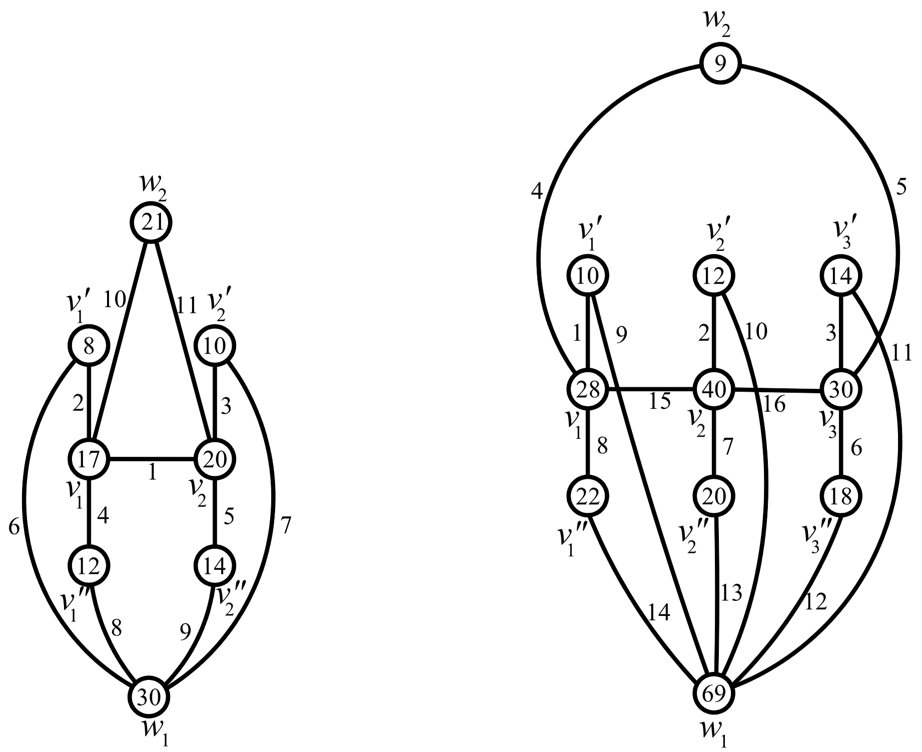

Figure 8: $DS(P_2 \odot 2K_1)$ and $DS(P_3 \odot 2K_1)$ and their antimagic labeling.

**Case - 2** For $n \geqslant 4$.

| | |
|---|---|
| $f(v_iv_{i+1}) = i;$ | For $1 \leqslant i \leqslant n - 1,$ |
| $f(v_iv_i') = n - 1 + i;$ | For $1 \leqslant i \leqslant n,$ |
| $f(v_iv_i'') = 2n - 1 + i;$ | For $1 \leqslant i \leqslant n,$ |
| $f(v_iw_3) = 6n - 1 - i;$ | For $2 \leqslant i \leqslant n - 1,$ |
| $f(v_i'w_1) = 3n - 1 + i;$ | For $1 \leqslant i \leqslant n,$ |
| $f(v_i''w_1) = 4n - 1 + i;$ | For $1 \leqslant i \leqslant n,$ |
| $f(v_iw_2) = 2(3n - 1),$ | |
| $f(v_nw_2) = 6n - 1.$ | |

Above define edge labeling function will generate distinct vertex labels for all the vertices of $DS(P_n \odot 2K_1)$. Thus, $f$ is an antimagic labeling.
Hence, the graph $DS(P_n \odot 2K_1)$ is an antimagic graph. □

**Illustration 2.6.** *The graph $DS(P_5 \odot 2K_1)$ and its antimagic labeling is shown in Figure 9.*

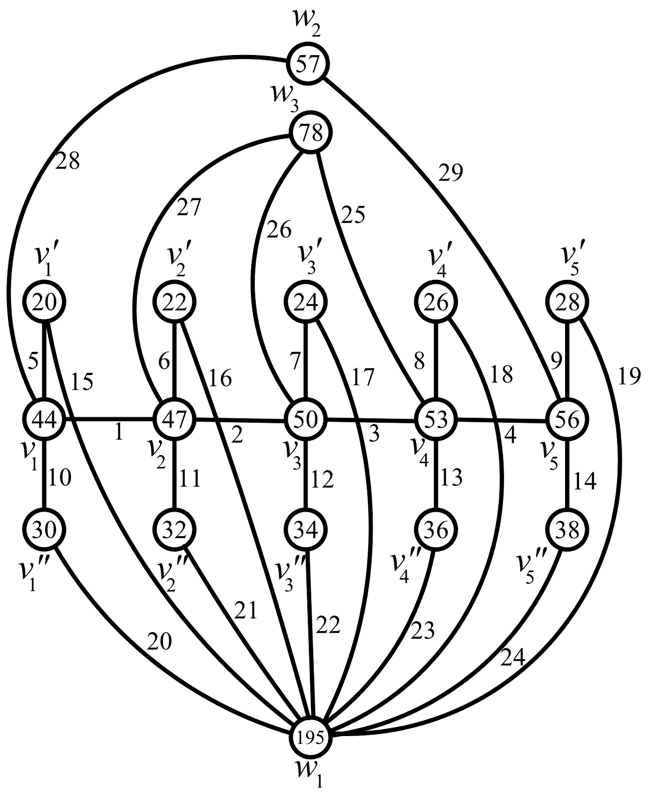


*Figure 9: $DS(P_5 \odot 2K_1)$ and its antimagic labeling.*

# 3 Conclusions

Hartsfield and Ringel [1990] has proved that path is an antimagic graph. Cheng [2007] has derived the antimagic labeling of ladder graph. While Sridharan and Umarani [2012] have discussed antimagic labeling of fan graph and crown graph. In this paper, we have discussed antimagic labeling of larger graphs obtained from Path, ladder graph, fan graph and crown graph using the degree splitting operation. Also antimagic labeling of degree splitting graphs of comb graph and double comb graph have been obtained. To investigated antimagic labeling for different graph families as well as in the context of various graph operations is an open area of research.

## Acknowledgements

The authors are highly thankful to anonymous referees for kind comments and constructive suggestions.

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