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Non-deterministic-M-fuzzy Lattices

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Abstract

Goran Trajakoviski and *Tepavčević* introduced the L-fuzzy lattice in which a bounded lattice is fuzzified using a complete lattice. Using a complete consistent distributive multilattice M, we broaden the concept of L-fuzzy lattice to Nd-M-fuzzy lattice in this study. We also propose the idea of Nd-M-order relation on a set X. Additionally, we demonstrated that the equivalent condition for a Nd-M-fuzzy subset to be a Nd-M-fuzzy sublattice of a bounded lattice L.

Keywords: Nd-M-fuzzy subset, Nd-M-fuzzy relation, Nd-M-fuzzy lattice.
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1 Introduction

Giving suitable fuzzification to crisp concepts is a significant challenge that many academics find intriguing. The membership function, however, provides a unique value in L for each member of its domain in L-fuzzy sets [Birkhoff [1940], Goguen [1967], Klir and Yuan [1995]]. In this case, we employ a membership function to fuzzify a clear idea by providing each member in its domain with a set of values. Consequently, by employing non-deterministic membership functions [Gireesan and Thrivikraman [2015], K.K.Gireesan [2014]], a non-deterministic M-fuzzy set was created, where M is a multilattice [Cordero et al. [2008], Avallone [2002], Johnston [1990], Medina et al. [2006]]. A different concept for a multilattices that is connected to lattices was recently presented by Cordero [Cordero et al. [2008]]. We are looking on how to extend fuzzy-related concepts using these kinds of structures.

Goran Trajakoviski and Tepavc ˇevic ´ developed L-fuzzy lattice , who used a complete lattice to fuzzify a bounded lattice [Tepavčević and Trajkovski [2001]]. Two types of fuzzy lattice were proposed by them. The first form of fuzzy lattice is created by fuzzying up the membership of the elements in a crisp lattice, and the second type is created by fuzzying up the order relation in a crisp lattice. They concluded that these two varieties of fuzzy lattices are identical.

As shown below, the document is organised. Defining the words multilattice, Nd-M-fuzzy subsets, and "L-fuzzy Lattice" in Section 2 as well as outlining some preliminary (theoretical) findings. The concept of Nd-M-fuzzy order relation is explored in more detail in section 3. The concept of Nd-M-fuzzy lattice is introduced and studied in section 4.

2 Preliminary Results

Assume that $F \subseteq M$ and that (M,$\leq$) is a partially ordered set. The minimal element of the set of upper bounds of F is called the multisupremum of F, and is denoted by Multisup(F). Similarly the maximal element of the set of lower bounds of F is called the multiinfimum of F,and is denoted by multiinf(F).

Definition 2.1. *[Cordero et al. [2008]] A partially ordered set (M,$\leq$) is said to be ordered multilattice iff for all i, j, and k with $i \leq k$ and $j \leq k$, there exist $q \in Multisup\,\{i, j\}$ such that $q \leq k$.*

It should be emphasised that our sets will instead contain a set of multisuprema rather than a supremum. Therefore, here we present some ordering, namely the Hoare ordering, the Smyth ordering, and the Egli-Milner ordering, respectively among the subsets of posets.

Definition 2.2. *[Medina et al. [2006]] let $G, H \in 2^M$, then*
$G \sqsubseteq_H H$ iff $\forall\, g \in G \;\exists\, h \in H$ such that $g \leq h$;
$G \sqsubseteq_S H$ iff $\forall\, h \in H \;\exists\, g \in G$ such that $g \leq h$;
$G \sqsubseteq_{EM} H$ iff $G \sqsubseteq_H H$ and $G \sqsubseteq_S H$.

Definition 2.3. *[Johnston [1990]] Let G and H are two subsets of a algebraic multilattice$(M, \wedge, \vee)$ and $m \in M$, then*
$m \wedge G = \cup\, \{(m \wedge g)/g \in G\}$;
$m \vee G = \cup\, \{(m \vee g)/g \in G\}$;
$G \wedge H = \;\cup\, \{(g \wedge h)/g \in G, h \in H\}$;
$G \vee H = \cup\, \{(g \vee h)/g \in G, h \in H\}$.

Definition 2.4. *[Gireesan and Thrivikraman [2015], K.K.Gireesan [2014]] Let $A \neq \emptyset$ and M be a complete distributive multi lattice. Then a function from A to 2^M is called Non deterministic M-Fuzzy subset of A (Nd-M-fuzzy subset) and Non-deterministic-M-fuzzy space is the set of all Nd-M-fuzzy subsets of A and is denoted by $(2^M)^A$.*

Definition 2.5. *If a completely distributive multilattice M has an order reversing involution $' : M \to M$, then M is said to be F multilattice and for any $P \in (2^M)^A$, $P'(a) = [P(a)]' = \{p' \mid p \in P(a)\}$*

Definition 2.6. *[Gireesan and Thrivikraman [2015]] Set relations rules on $(2^M)^X$.*
Suppose H and K belongs to $(2^M)^A$. Then

1. *$H = K$, if $H(i) = K(i)$, for every $i \in A$;*
2. *$H \leq K$, if $H(i) \sqsubseteq_{EM} K(i)$;*
3. *$L = H \vee K$, if $L(i)$ = multisup $\{(H(i), K(i)) \mid \forall\, i \in A\} = \cup\{(h \vee k) \mid h \in H(i), k \in K(i)\}$, $\forall\, i \in A$;*
4. *$I = H \wedge K$, if $I(i)$ = multiinf $\{(H(i), K(i)) \mid \forall\, i \in A\} = \cup\{(h \wedge k) \mid h \in H(i), k \in K(i)\}$, $\forall\, i \in A$.*

Definition 2.7. *[Ajmal and Thomas [1994], Tepavčević and Trajkovski [2001]] Suppose $(A, \vee, \wedge)$ be a lattice and $(L, \vee_L, \wedge_L)$ is a complete lattice with 0_L and 1_L and $\Omega \in L^A$. Then the set $\Omega_\beta = \{t \in A \mid \Omega(t) \geq \beta\}$ is called the β cut of Ω . Also Ω is called a fuzzy sub lattice of L, if*

$$\Omega(s \wedge t) \wedge_L \Omega(s \vee t) \geq \min(\Omega(s), \Omega(t)) \;\; s, t \in A.$$

That is,

$$\Omega(s \wedge t) \wedge_L \Omega(s \vee t) \geq \Omega(s) \wedge_L \Omega(t).$$

Remark 2.1. *$\Omega \in L^A$ is an L-fuzzy sub lattice of A iff Ω_β is a sublattice of A for each $\beta \in L$.*

3 Non-deterministic-M-fuzzy relation

Definition 3.1. *Suppose $(M, \wedge_M, \vee_M))$ be a complete and consistent multilattice with greatest element 1_M and least element 0_M, where $\vee_M$ and $\wedge_M$ are multi-supremum and multiinfimum of elements of M and A be a set,$A \neq \phi$. Then a Non-deterministic M-fuzzy relation on A is a mapping $\Phi : A \times A \to A$.*

Definition 3.2. *Assume $\beta \in 2^M$, then the set $\Phi_\beta = \{(a, b) : \beta \sqsubseteq_{EM} \Phi(a, b)\}$ is called the β-level of Φ.*

Definition 3.3. *An Nd-M-fuzzy relation Φ is called*

1. *Nd-M- reflexive, if $\Phi(i, j) = \{1\}$,$\forall\, i, j \in A$;*
2. *Weakly Nd-M- reflexive, if*

$$\Phi(i, j) \sqsubseteq_{EM} \Phi(i, i) \ \ and\ \ \Phi(j, i) \sqsubseteq_{EM} \Phi(i, i)\ \forall\ \ , i, j \in A;$$

3. *Nd-M- anti-symmetric, if*

$$\Phi(i, j) \wedge_M \Phi(j, i) = \{0\}\ \forall\ i, j \in A\ \ with\ \ i \neq j;$$

4. *Nd-M- transitive, if*

$$\Phi(i, j) \wedge \Phi(j, i) \sqsubseteq_{EM} \Phi(i, k)\ \ ,\forall\ i, j, k \in A.$$

If Φ is Nd-M-reflexive, Nd-M- antisymmetric and Nd-M-transitive, then Φ is called Nd-M-fuzzy ordering relation on A.

Definition 3.4. *Suppose $(L, \vee_L, \wedge_L, 0, 1)$ is a bounded lattice and $(M, \vee_M, \wedge_M)$ be a non-trivial complete and consistent multilattice. Let $\Omega \in (2^M)^L$. Then the $\Omega_\beta = \{t \in L \mid \beta \sqsubseteq_{EM} \Omega(t)\}$ is the β -level of Ω, where $\beta \in 2^M$.*

4 Non-deterministic-M-fuzzy lattice

Definition 4.1. *If a Nd-M fuzzy subset Ω belongs to $(2^M)^L$ and if $\beta \sqsubseteq_{EM} \Omega(s) \wedge_M \mu(t)$, $\forall\, s, t \in \Omega_\beta$ and Ω_β is a sub lattice of L for each $\beta \in 2^M$, then Ω is called a non-deterministic-M-fuzzy sub lattice (Nd-M-fuzzy sub lattice) of L.*

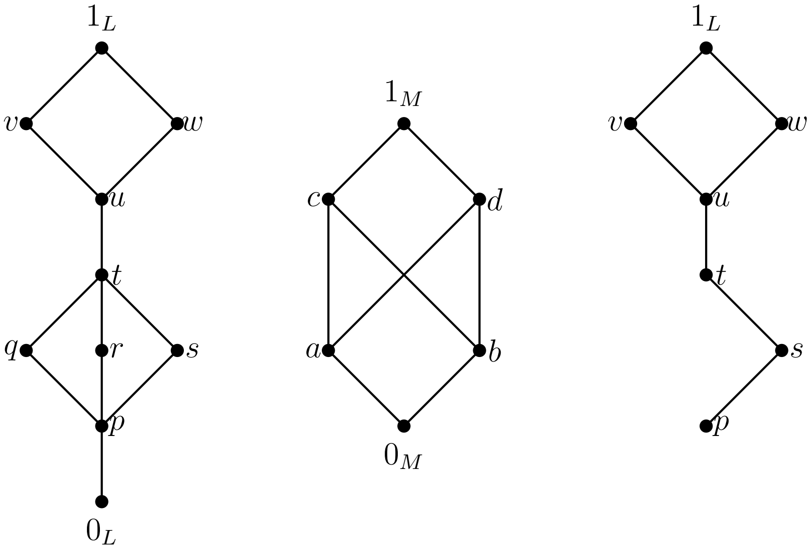

Figure 1: The lattice L, the multilattice M and the β-level Ω_β where $\beta = \{a\}$ in Example 4.1

Example 4.1. *1. Any L-fuzzy lattice is a Nd-M-fuzzy lattice.*

2. *Choose $\beta \in 2^M$ such that $\beta \wedge \beta = \beta$. Let $\Omega : L \to 2^M$ defined by $\Omega(t) = \beta$,$\forall\ t \in L$, Ω is a Nd-M- fuzzy lattice.*

3. *If $(M, \wedge_M, \vee_M)$ be a multilattice with $0_M$and 1_M and $(L, \vee_L, \wedge_L)$ be a lattice where $M = \{0_M, a, b, c, d, 1_M\}$ and $L = \{0_L, p, q, r, s, t, u, v, w, 1_L\}$, then*

$$\Omega = \begin{pmatrix} 0_L & p & q & r & s & t & u & v & w & 1_L \\ \{b\} & \{a\} & \{b\} & \{a,b\} & \{a\} & \{d\} & \{d\} & \{c,d\} & \{d\} & 1_M \end{pmatrix}$$

is an $Nd - M-$ fuzzy lattice (see Fig.1).
If $\beta = \{a\}$, then $\Omega_\beta = \{p, s, t, u, v, w, 1_L\}$.

Theorem 4.1. *Assume Ω_β,where $\beta \in 2^M$ and $\Omega \in (2^M)^L$, satisfies $\beta \sqsubseteq_{EM} \Omega(s) \wedge_M \Omega(t), \forall s, t \in \Omega_\beta$, then Ω is a Nd-M-fuzzy lattice of L iff for every $s, t \in L$,*

$$multiinf\{\Omega(s), \Omega(t)\} \sqsubseteq_{EM} multiinf\{\Omega(s \wedge_L t), \Omega(s \vee_L t)\}.$$

That is,

$$\Omega(s) \wedge_M \Omega(t) \sqsubseteq_{EM} \Omega(s \wedge_L t) \wedge_M \Omega(s \vee_L t).$$

Proof. Assume $\Omega_\beta = \{t \in L : \beta \sqsubseteq_{EM} \Omega(t)\}$ satisfies $\beta \sqsubseteq_{EM} \Omega(s) \wedge_M \Omega(t), \forall$ $s, t \in \Omega_\beta$. Consider $P = \Omega(s) \wedge_M \Omega(t)$, then $P \sqsubseteq_{EM} \Omega(s)$ and $P \sqsubseteq_{EM} \Omega(t)$. That is, $s, t \in \Omega_P$. But by our hypothesis, Ω_P is a sub lattice of L, then $s \wedge_L t$ and $s \vee_L t$ belongs to Ω_P and so $P \sqsubseteq_{EM} \Omega(s \vee_L t)$ and $P \sqsubseteq_{EM} \Omega(s \wedge_M t$. Since $s \wedge_L t$ and $s \vee_L t$ belongs to Ω_P, $P \sqsubseteq_{EM} \Omega(s \vee_L t) \wedge_M \Omega(s \wedge_L t)$. Therefore, $P = \Omega(s) \wedge_M \Omega(t) \sqsubseteq_{EM} \Omega(s \vee_L t) \wedge_M \Omega(s \wedge_L t)$.

Conversely, assume that Ω satisfies $\Omega(s) \wedge_M \Omega(t) \sqsubseteq_{EM} \Omega(s \vee_L t) \wedge_M \Omega(s \wedge_L t)$. Let P belongs to 2^M. Then For every $s, t \in \Omega_P$, Then $P \sqsubseteq_{EM} \Omega(s)$ and $P \sqsubseteq_{EM} \Omega(t)$. Hence $P \sqsubseteq_{EM} \Omega(s) \wedge_M \Omega(t)$. But our assumption, we have $P \sqsubseteq_{EM} \Omega(s) \wedge_M \Omega(t) \sqsubseteq_{EM} \Omega(s \wedge_L t) \wedge_M \Omega(s \vee_L t)$. Hence $P \sqsubseteq_{EM} \Omega(s \wedge_L t)$ and $P \sqsubseteq_{EM} \Omega(s \vee_L t)$. Since $s \vee_L t \in \Omega_P$ and $s \wedge_L t \in \Omega_P$, we obtain Ω_P is a sublattice of L, and thus Ω is an Nd-M fuzzy lattice.

□

Theorem 4.2. *Suppose $(L, \wedge_L, \vee_L)$ be a lattice $(M, \wedge_M, \vee_M)$ be a bounded multilattice. Then $\Omega : L \to 2^M$ is an Nd-M- fuzzy lattice, the following relations eqivalent.*

1. $\Omega(s) \wedge_M \Omega(t) \sqsubseteq_{EM} \Omega(s \wedge_L t)$;
2. $\Omega(s) \wedge_M \Omega(t) \sqsubseteq_{EM} \Omega(s \vee_L t)$.

The steps for creating a Nd-M- fuzzy lattice with a family of lattices as its family of level sets are shown below.

Consider two posets S_1 and S_2 that have disjoint underlying sets. The posets $(S_1 \cup S_2, \leq)$ is the disjoint union of posets S_1 and S_2, where $\leq$ is defined by $s \leq$ iff $s, t \in S_1$ and $s \leq t$ in S_1 or $s, t \in S_2$ and $s \leq$ in S_2 or $s \in S_1$ and $t \in S_2$. The sum of Posets S_1 and S_2 is called linear sum and is denoted by $(S_1 \oplus S_2, \leq)$.

Theorem 4.3. *Let κ be the disjoint family of lattices. Then there is an Nd-M-fuzzy lattice whose non-trivial β-levels are precisely the lattices from κ.*

Proof. Suppose κ i a collection of lattices $(L_i, \wedge_i, \vee_i)$ with smallest element 0_i and greatest element 1_i. lowest element 0_i and greatest element 1_i, $(i \in I)$.Then create a Nd-M- fuzzy lattice using the assortment of lattices in the manner shown below.

Assume $M = \{0_M, 1_M\} \cup \{m_i, n_i : i \in I\}$ be the multilattice, where the order M is defined by $\forall\ i\ m_i \geq n_i, \forall i$ and $m_1 \geq n_2$ and $m_2 \geq n_1$, where n_i,'s are atoms and m_i's are coatoms. Define the linear sum $\oplus$ of these lattices is the set $L = 0_L \oplus \bigcup\limits_{i \in I} L_i \oplus 1_L$, where 0_L and 1_L are the single element lattices. Now L is a lattice. Then define $\Omega : L \to 2^M$ by

$$\Omega\{s\} = \{m_i, n_i\} \text{ iff } s \in L_i, i \in I \text{ and } \Omega\{0_L\} = \{0_M\}, \Omega\{1_L\} = \{1_M\}.$$

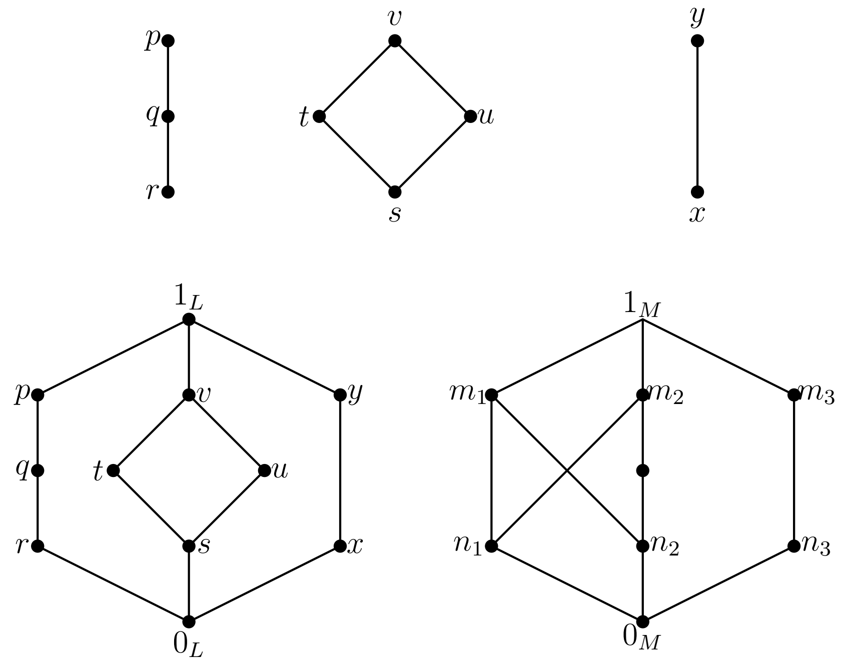

Figure 2: The lattices and the multilattice in Example 4.2

It is then evident that for every $\beta \in 2^M$, all of the non-trivial - β levels of Ω are precisely the lattices in κ. $\square$

Example 4.2. *Let κ has three lattices L_1, L_2 and L_3 (see figure 2). Next, create L and M using the prior theorem.Following the mapping Ω provides the necessary Nd-M- fuzzy lattices.*

$$\bar{L} = \begin{pmatrix} p & q & r & s & t & u \\ \{m_1, n_1\} & \{m_1, n_1\} & \{m_1, n_3\} & \{m_2, n_2\} & \{m_2, n_3\} & \{m_2, n_3\} \\ v & x & y & 0_L & 1_L & \\ \{m_2, n_2\} & \{m_3, n_3\} & \{m_3, n_3\} & 0_M & 1_M & \end{pmatrix}.$$

5 Conclusion

The terms L-fuzzy relation and L-fuzzy lattice were generalised to Nd-M-fuzzy relation and Nd-M-fuzzy lattice, respectively. Furthermore, we proved the necessary and sufficient condition for a Nd-M-fuzzy subset to be a Nd-M-fuzzy sublattice of L. Finally, we noticed that when the membership value in an L-fuzzy lattice is replaced by a set of values in a multilattice in M, the expansion of L-fuzzy lattices to Nd-M-fuzzy lattices is true with regard to the Egli- Milner ordering.

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