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A family of definite integrals involving Legendre's polynomials

The main objective of this article is to provide the analytical solutions (not previously found and not available in the literature) of some problems related with definite integrals integrands of which are the products of the derivatives of Legendre's polynomials of first kind having different order, with the help of some derivatives of Legendre's polynomials of first kind $\mathrm{P}_n(x)$, Rodrigues formula, Leibnitz's generalized rule for successive integration by parts and certain values of successive differential coefficients of $(x^2-1)^r$ at $x=\pm 1$.

Keywords: Legendre polynomials; Rodrigues formula; Leibnitz generalized rule for successive integration by parts; Murphy formula for Legendre polynomial.

1 Motivation and objectives

Legendre polynomials are studied in most science and engineering mathematics courses, mainly in those courses focused on differential equations or special functions. Legendre polynomials, also known as spherical harmonics or zonal harmonics, were first introduced in 1782 by Adrien-Marie Legendre. Legendre polynomials are used in several areas in physics and mathematics. For example, Legendre and Associate Legendre polynomials are widely used in the determination of wave functions of electrons in the orbits of an atom [1,2] and in the determination of potential functions in the spherically symmetric geometry [3]. In 1784, the significant of Legendre polynomials is sensed when the attraction of spheroids and ellipsoids was studying by A. Legendre. They may arise from solutions of Legendre ODE, such as the analog ODEs in spherical polar coordinates and the famous Helmholtz equation.

The main aim of this work is to fill up the gap in the existing literature on definite integrals integrands of which are the product of the derivatives of two families of classical Legendre's polynomials of first kind, by adding certain definite integrals in the incomplete list, as shown in the following possible combinations of definite integrals:

First combination of definite integrals:

$$\text{Already Solved} \int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m(x)\mathrm{d}x \tag{1}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m{}'(x)\mathrm{d}x \tag{2}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m{}''(x)\mathrm{d}x \tag{3}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m{}'''(x)\mathrm{d}x \tag{4}$$

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$$\text{Already Solved} \int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_n(x)\mathrm{d}x \tag{5}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_n{}'(x)\mathrm{d}x \tag{6}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_n{}''(x)\mathrm{d}x \tag{7}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_n{}'''(x)\mathrm{d}x \tag{8}$$

Second combination of definite integrals:

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_m(x)\mathrm{d}x \tag{9}$$

$$\text{Already Solved} \int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_m{}'(x)\mathrm{d}x \tag{10}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_m{}''(x)\mathrm{d}x \tag{11}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_m{}'''(x)\mathrm{d}x \tag{12}$$

$$\text{Repeated with (6)} \int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_n(x)\mathrm{d}x \tag{13}$$

$$\text{Already Solved} \int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_n{}'(x)\mathrm{d}x \tag{14}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_n{}''(x)\mathrm{d}x \tag{15}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_n{}'''(x)\mathrm{d}x \tag{16}$$

Third combination of definite integrals:

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_m(x)\mathrm{d}x \tag{17}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_m{}'(x)\mathrm{d}x \tag{18}$$

$$\text{Already Solved} \int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_m{}''(x)\mathrm{d}x \tag{19}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_m{}'''(x)\mathrm{d}x \tag{20}$$

$$\text{Repeated with (7)} \int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_n(x)\mathrm{d}x \tag{21}$$

$$\text{Repeated with (15)} \int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_n{}'(x)\mathrm{d}x \tag{22}$$

$$\text{Already Solved} \int_{-1}^{+1} \mathrm{P}_n''(x)\mathrm{P}_n''(x)\mathrm{d}x \tag{23}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n''(x)\mathrm{P}_n'''(x)\mathrm{d}x \tag{24}$$

Fourth combination of definite integrals:

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_m(x)\mathrm{d}x \tag{25}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_m'(x)\mathrm{d}x \tag{26}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_m''(x)\mathrm{d}x \tag{27}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_m'''(x)\mathrm{d}x \tag{28}$$

$$\text{Repeated with (8)} \int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_n(x)\mathrm{d}x \tag{29}$$

$$\text{Repeated with (16)} \int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_n'(x)\mathrm{d}x \tag{30}$$

$$\text{Repeated with (24)} \int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_n''(x)\mathrm{d}x \tag{31}$$

$$\text{Unsolved} \int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_n'''(x)\mathrm{d}x \tag{32}$$

Now there are twenty-six non-repeated combinations of the product of derivatives of two Legendre's polynomials. Out of twenty-six integrals only six integrals are solved. Now we have to solve remaining twenty integrals solutions of which are not available in the literature of special functions.

For the sake of convenience we shall use the following notations and other results:
Suppose $D^{+1}\{F(x)\} = \frac{d}{dx}\{F(x)\}$, $D^m\{F(x)\} = \frac{d^m}{dx^m}\{F(x)\}$, $D^{-1}\{F(x)\} = \frac{1}{D}\{F(x)\} = \int\{F(x)\}\,dx$,
$D^{-m}\{F(x)\} = \frac{1}{D^m}\{F(x)\} = \underbrace{\int\int\int ...m\text{ - times}... \int\int}\ \{F(x)\}\ \underbrace{\mathrm{d}x\ \mathrm{d}x\ \mathrm{d}x...m\text{ - times }...\mathrm{d}x\ \mathrm{d}x}$.
Some derivatives of Legendre's polynomials of first kind $\mathrm{P}_n(x)$, using Rodrigues formula [4; p.162, Eq.(7)]:

$$\mathrm{P}_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n,$$

$$D\{\mathrm{P}_n(x)\} = \mathrm{P}_n'(x) = \frac{1}{2^n n!}\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n,$$

$$D^2\{\mathrm{P}_n(x)\} = \mathrm{P}_n''(x) = \frac{1}{2^n n!}\frac{d^{n+2}}{dx^{n+2}}(x^2-1)^n,$$

$$D^3\{\mathrm{P}_n(x)\} = \mathrm{P}_n'''(x) = \frac{1}{2^n n!}\frac{d^{n+3}}{dx^{n+3}}(x^2-1)^n, \tag{33}$$

where n is zero and positive integer.

Leibnitz's (also Leibniz) generalized rule for successive integration by parts:

$$I = \int U(x).T(x)dx = \int U.Tdx$$

$$\begin{aligned} I = (-1)^0 \left\{D^0U\right\}\left\{D^{-1}T\right\} + (-1)^1 \left\{D^1U\right\}\left\{D^{-2}T\right\} + \\ +(-1)^2 \left\{D^2U\right\}\left\{D^{-3}T\right\} + (-1)^3 \left\{D^3U\right\}\left\{D^{-4}T\right\} + \\ +\ldots\ldots\ldots\ldots\ldots + (-1)^J \left\{D^JU\right\}\left\{D^{-J-1}T\right\} + \\ +(-1)^{J+1}\int \left\{D^{J+1}U\right\}\left\{D^{-J-1}T\right\} \mathrm{d}x + \text{constant of integration.} \end{aligned} \tag{34}$$

$$\frac{d^{2n}(x^2-1)^n}{dx^{2n}} = D^{2n}(x^2-1)^n = (2n)!$$

$$\{\text{Factorial of any negative integer }\}^{-1} = 0.$$

Our present investigation is motivated by the work collected in beautiful monographs of [5–14]. The article is organized as follows. In Section 2, we present some values of successive differential coefficients of $(x^2-1)^r$ at $x = \pm 1$. In Section 3, we mention six known definite integrals. In Section 4, we establish twenty new definite integrals. In Section 5, we have given the derivation of these new definite integrals.

2 Some successive differential coefficients of $(x^2-1)^r$ *at* $x = \pm 1$

$$\left[D^p(x^2-1)^r\right]_{x=\pm1} = \begin{cases} \text{nonzero}, & \text{if } p \geq r \\ \text{zero}, & \text{if } p < r \end{cases} \tag{35}$$

$$\left[D^r(x^2-1)^r\right]_{x=1} = 2^r r! \tag{36}$$

$$\left[D^r(x^2-1)^r\right]_{x=-1} = 2^r r!(-1)^r \tag{37}$$

$$\left[D^{r+1}(x^2-1)^r\right]_{x=1} = \frac{2^r r! r(r+1)}{2} \tag{38}$$

$$\left[D^{r+1}(x^2-1)^r\right]_{x=-1} = \frac{(-1)2^r r! r(r+1)(-1)^r}{2} \tag{39}$$

$$\left[D^{r+2}(x^2-1)^r\right]_{x=1} = \frac{2^r r!(r+1)(r+2)r(r-1)}{8} \tag{40}$$

$$\left[D^{r+2}(x^2-1)^r\right]_{x=-1} = \frac{2^r r!(r+1)(r+2)r(r-1)(-1)^r}{8} \tag{41}$$

$$\left[D^{r+3}(x^2-1)^r\right]_{x=1} = \frac{2^r r!(r+1)(r+2)(r+3)r(r-1)(r-2)}{48} \tag{42}$$

$$\left[D^{r+3}(x^2-1)^r\right]_{x=-1} = \frac{(-1)2^r r!(r+1)(r+2)(r+3)r(r-1)(r-2)(-1)^r}{48} \tag{43}$$

$$\left[D^{r+4}(x^2-1)^r\right]_{x=1} = \frac{2^r r!(r+1)(r+2)(r+3)(r+4)r(r-1)(r-2)(r-3)}{384} \tag{44}$$

$$\left[D^{r+4}(x^2-1)^r\right]_{x=-1} = \frac{2^r r!(r+1)(r+2)(r+3)(r+4)r(r-1)(r-2)(r-3)(-1)^r}{384} \tag{45}$$

$$\left[D^{r+5}(x^2-1)^r\right]_{x=1} = \frac{2^r r!(r+1)(r+2)(r+3)(r+4)(r+5)r(r-1)(r-2)(r-3)(r-4)}{3840} \tag{46}$$

$$\left[D^{r+5}(x^2-1)^r\right]_{x=-1} = \frac{(-1)2^r r!(r+1)(r+2)(r+3)(r+4)(r+5)r(r-1)(r-2)(r-3)(r-4)(-1)^r}{3840} \tag{47}$$

With the help of Rodrigues formula and derivatives of the hypergeometric forms (Murphy formula [4; p.166, Eqs.(2) and (3)]) of Legendre's polynomials $\mathrm{P}_r(x)$, we can derive successive differential coefficients of $(x^2-1)^r$ at $x=\pm 1$.

3 Six known definite integrals

Integral(1).
$$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m(x)\mathrm{d}x = 0, \qquad \text{if } \ m \neq n. \tag{48}$$

Integral(5).
$$\int_{-1}^{+1} \{\mathrm{P}_n(x)\}^2 \,\mathrm{d}x = \frac{2}{2n+1}. \tag{49}$$

The integrals (1) or (48) and (5) or (49) were derived by A. M. Legendre in the years 1784 and 1789 respectively ([15; p.281]; see also[16; p.277, Eqn's (13) and (14)]).
Integral(10). When m and n are positive integers and $m \geq n \geq 1$, then

$$\int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_m{}'(x)\mathrm{d}x = \frac{n(n+1)}{2}\left\{1+(-1)^{m+n}\right\}. \tag{50}$$

Special case of the integral (50)

$$\int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_m{}'(x)\mathrm{d}x = \begin{cases} 0 & \text{if } (m+n) \text{is an odd integer and } \ m \geq n \geq 1, \\ n(n+1) & \text{if } (m+n) \ \text{ is an even integer and } \ m \geq n \geq 1. \end{cases}$$

Integral(14). When m and n are positive integers and $m=n$ in equation (50), then

$$\int_{-1}^{+1} \{\mathrm{P}_n{}'(x)\}^2 \,\mathrm{d}x = n(n+1). \tag{51}$$

The integrals (10) or (50) and (14) or (51) were asked in examination of Clare College London, Cambridge University (1898)[17; p.170, Q.N.11];[18; p.309, e.g.(3)].
Integral(19). When m and n are positive integers such that $m \geq n \geq 2$, then

$$\int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_m{}''(x)\mathrm{d}x = \frac{(n+2)!}{(n-2)!(48)}\left\{1+(-1)^{m+n}\right\}\{3m(m+1)-n(n+1)+6\}. \tag{52}$$

Special case of the integral (52)

$$\int_{-1}^{+1} \mathrm{P}_n''(x)\mathrm{P}_m''(x)\mathrm{d}x = \begin{cases} 0 & \text{if } (m+n) \text{ is an odd integer and } m \geq n \geq 2, \\ \frac{n(n+1)(n+2)(n-1)}{24}\{3m(m+1)-n(n+1)+6\} & \text{if } (m+n) \text{ is an even integer and } m \geq n \geq 2. \end{cases}$$

Integral(23). When m and n are positive integers and $m=n$ in equation (52), then

$$\int_{-1}^{+1} \{\mathrm{P}_n''(x)\}^2 \, \mathrm{d}x = \frac{(n+2)!}{(n-2)!(12)}\{n(n+1)+3\}. \tag{53}$$

The integrals (19) or (52) and (23) or (53) were asked in examination of Mathematical Tripos, Cambridge University (1897) [15; p.308, Q.N.2]; [18; p.309, e.g.(4)]. But the solutions of (52) and (53) are not available in the literature of special functions.

Remark: We have verified the definite integrals of Legendre's polynomials (48), (49), (50), (51), (52) and (53) numerically by using Mathematica software.

4 Twenty unsolved and new definite integrals

Integral(2). When m and n are positive integers such that $m \geq n$, then

$$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m'(x)\mathrm{d}x = \{1-(-1)^{m+n}\}. \tag{54}$$

Special case of the integral (54)

$$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m'(x)\mathrm{d}x = \begin{cases} 2 & \text{if } (m+n) \text{ is an odd integer and } m \geq n, \\ 0 & \text{if } (m+n) \text{ is an even integer and } m \geq n. \end{cases}$$

Integral(6). When m and n are positive integers such that $m=n$ in equation (54), then

$$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_n'(x)\mathrm{d}x = 0.$$

Integral(3). When m and n are positive integers such that $m \geq n$, then

$$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m''(x)\mathrm{d}x = \frac{\{1+(-1)^{m+n}\}}{2}\{m(m+1)-n(n+1)\}. \tag{55}$$

Special case of the integral (55)

$$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m''(x)\mathrm{d}x = \begin{cases} 0 & \text{if } (m+n) \text{ is an odd integer and } m \geq n, \\ m(m+1)-n(n+1) & \text{if } (m+n) \text{ is an even integer and } m \geq n. \end{cases}$$

Integral(7). When m and n are positive integers such that $m = n$ in equation (55), then

$$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_n{}''(x)\mathrm{d}x = 0.$$

Integral(4). When m and n are positive integers such that $m \geq n$, then

$$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m{}'''(x)\mathrm{d}x = \frac{\{1-(-1)^{m+n}\}}{8}\times$$
$$\times\{m(m+1)(m+2)(m-1) - 2n(n+1)m(m+1) + n(n+1)(n+2)(n-1)\}. \tag{56}$$

Special case of the integral (56)

$$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_m{}'''(x)\mathrm{d}x$$
$$= \begin{cases} \frac{1}{4}\{m(m+1)(m+2)(m-1)- \\ -2n(n+1)m(m+1)+ \\ +n(n+1)(n+2)(n-1)\} & \text{if } (m+n) \text{ is an odd integer and } m \geq n, \\ 0 & \text{if } (m+n) \text{ is an even integer and } m \geq n. \end{cases}$$

Integral(8). When m and n are positive integers such that $m = n$ in equation (56), then

$$\int_{-1}^{+1} \mathrm{P}_n(x)\mathrm{P}_n{}'''(x)\mathrm{d}x = 0.$$

Integral(9). When m and n are positive integers such that $m \geq n$, then

$$\int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_m(x)\mathrm{d}x = 0. \tag{57}$$

Repeated Integral(13). When m and n are positive integers such that $m = n$ in equation (57), then

$$\int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_n(x)\mathrm{d}x = 0.$$

Integral(11). When m and n are positive integers such that $m \geq n$, then

$$\int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_m{}''(x)\mathrm{d}x = \frac{n(n+1)}{8}\left\{1-(-1)^{m+n}\right\}\{2m(m+1)-(n+2)(n-1)\}. \tag{58}$$

Special case of the integral(58)

$$\int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_m{}''(x)\mathrm{d}x$$
$$= \begin{cases} \frac{n(n+1)}{4}\{2m(m+1)-(n+2)(n-1)\} & \text{if } (m+n) \text{ is an odd integer and } m \geq n, \\ 0 & \text{if } (m+n) \text{ is an even integer and } m \geq n. \end{cases}$$

Integral(15). When m and n are positive integers such that $m = n$ in equation (58), then

$$\int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_n{}''(x)\mathrm{d}x = 0.$$

Integral(12). When m and n are positive integers such that $m \geq n$, then

$$\int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_m{}'''(x)\mathrm{d}x = \frac{n(n+1)}{48}\left\{1+(-1)^{m+n}\right\} \times$$

$$\times \left\{3m(m+1)(m+2)(m-1) - 3(n+2)(n-1)m(m+1) + (n+2)(n+3)(n-1)(n-2)\right\}. \quad (59)$$

Special case of the integral (59)

$$\int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_m{}'''(x)\mathrm{d}x$$

$$= \begin{cases} 0 & \text{if } (m+n) \text{ is an odd integer and } m \geq n, \\ \frac{n(n+1)}{(24)}\left\{3m(m+1)(m+2)(m-1) - \right. & \\ -3(n+2)(n-1)m(m+1)+ & \\ \left. +(n+2)(n+3)(n-1)(n-2)\right\} & \text{if } (m+n) \text{ is an even integer and } m \geq n. \end{cases}$$

Integral(16). When m and n are positive integers such that $m = n$ in equation (59), then

$$\int_{-1}^{+1} \mathrm{P}_n{}'(x)\mathrm{P}_n{}'''(x)\mathrm{d}x = \frac{(n+3)!}{(n-3)!(24)};\ n \geq 3.$$

Integral(17). When m and n are positive integers such that $m \geq n$, then

$$\int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_m(x)\mathrm{d}x = 0. \quad (60)$$

Repeated Integral(21). When m and n are positive integers such that $m = n$ in equation (60), then

$$\int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_n(x)\mathrm{d}x = 0.$$

Integral(18). When m and n are positive integers and $m \geq n \geq 2$, then

$$\int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_m{}'(x)\mathrm{d}x = \frac{(n+2)!}{(n-2)!(8)}\left\{1-(-1)^{m+n}\right\}. \quad (61)$$

Special case of the integral (61)

$$\int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_m{}'(x)\mathrm{d}x = \begin{cases} \frac{(n+2)!}{(n-2)!(4)} & \text{if } (m+n) \text{ is an odd integer and } m \geq n \geq 2, \\ 0 & \text{if } (m+n) \text{ is an even integer and } m \geq n \geq 2. \end{cases}$$

Repeated Integral(22). When m and n are positive integers such that $m = n$ in equation (61), then

$$\int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_n{}'(x)\mathrm{d}x = 0.$$

Integral(20). When m and n are positive integers such that $m \geq n \geq 2$, then

$$\int_{-1}^{+1} \mathrm{P}_n{}''(x)\mathrm{P}_m{}'''(x)\mathrm{d}x = \frac{(n+2)!}{(n-2)!(384)}\left\{1-(-1)^{m+n}\right\} \times$$

$$\times \left\{6m(m+1)(m+2)(m-1) - 4(n+3)(n-2)m(m+1) + (n+3)(n+4)(n-2)(n-3)\right\}. \quad (62)$$

Special case of the integral(62)

$$\int_{-1}^{+1} \mathrm{P}_n''(x)\mathrm{P}_m'''(x)\mathrm{d}x = \begin{cases} \frac{(n+2)!}{(n-2)!(192)} \{6m(m+1)(m+2)(m-1)- \\ -4(n+3)(n-2)m(m+1)+ \\ +(n+3)(n+4)(n-2)(n-3)\} & \text{if } (m+n) \text{ is an odd integer and } m \geq n \geq 2, \\ 0 & \text{if } (m+n) \text{ is an even integer and } m \geq n \geq 2. \end{cases}$$

Integral(24). When m and n are positive integers such that $m = n$ in equation (62), then

$$\int_{-1}^{+1} \mathrm{P}_n''(x)\mathrm{P}_n'''(x)\mathrm{d}x = 0.$$

Integral(25). When m and n are positive integers such that $m \geq n$, then

$$\int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_m(x)\mathrm{d}x = 0. \tag{63}$$

Repeated Integral(29). When m and n are positive integers such that $m = n$ in equation (63), then

$$\int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_n(x)\mathrm{d}x = 0.$$

Integral(26). When m and n are positive integers and $m \geq n \geq 3$, then

$$\int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_m'(x)\mathrm{d}x = \frac{(n+3)!}{(n-3)!(48)} \left\{1 + (-1)^{m+n}\right\}. \tag{64}$$

Special case of the integral (64)

$$\int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_m'(x)\mathrm{d}x = \begin{cases} 0 & \text{if } (m+n) \text{ is an odd integer and } m \geq n \geq 3, \\ \frac{(n+3)!}{(n-3)!(24)} & \text{if } (m+n) \text{ is an even integer and } m \geq n \geq 3. \end{cases}$$

Repeated Integral(30). When m and n are positive integers such that $m = n$ in equation (64), then

$$\int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_n'(x)\mathrm{d}x = \frac{(n+3)!}{(n-3)!(24)};\ n \geq 3.$$

Integral(27). When m and n are positive integers and $m \geq n \geq 3$, then

$$\int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_m''(x)\mathrm{d}x = \frac{(n+3)!}{(n-3)!(384)} \left\{1 - (-1)^{m+n}\right\} [4m(m+1) - (n+4)(n-3)]. \tag{65}$$

Special case of the integral (65)

$$\int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_m''(x)\mathrm{d}x = \begin{cases} \frac{(n+3)!}{(n-3)!(192)} \times \\ \times \{4m(m+1) - (n+4)(n-3)\} & \text{if } (m+n) \text{ is an odd integer and } m \geq n \geq 3, \\ 0 & \text{if } (m+n) \text{ is an even integer and } m \geq n \geq 3. \end{cases}$$

Repeated Integral(31). When m and n are positive integers such that $m = n$ in equation (65), then

$$\int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_n''(x)\mathrm{d}x = 0.$$

Integral(28). When m and n are positive integers such that $m \geq n \geq 3$, then

$$\int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_m'''(x)\mathrm{d}x = \frac{(n+3)!}{(n-3)!(3840)} \left\{1 + (-1)^{m+n}\right\} \times$$

$$\times \left\{10m(m+1)(m+2)(m-1) - 5(n+4)(n-3)m(m+1) + (n+4)(n+5)(n-3)(n-4)\right\}. \quad (66)$$

Special case of the integral(66)

$$\int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_m'''(x)\mathrm{d}x$$

$$= \begin{cases} 0 & \text{if } (m+n) \text{ is an odd integer and } m \geq n \geq 3, \\ \frac{(n+3)!}{(n-3)!(1920)} \times \{10m(m+1)(m+2)(m-1)- \\ -5(n+4)(n-3)m(m+1)+ \\ +(n+4)(n+5)(n-3)(n-4)\} & \text{if } (m+n) \text{ is an even integer and } m \geq n \geq 3. \end{cases}$$

Integral(32). When m and n are positive integers such that $m = n$ in equation (66), then

$$\int_{-1}^{+1} \{\mathrm{P}_n'''(x)\}^2\mathrm{d}x = \frac{(n+3)!}{(n-3)!(960)}\{3n^4 + 6n^3 + 7n^2 + 4n + 120\};\ n \geq 3. \quad (67)$$

Remark: We have verified the definite integrals from (54) to (67) of Legendre's polynomials numerically by using Mathematica software.

5 Derivation of new definite integrals of section 4

Here, in this section we shall provide the detailed and systematic derivation of any one integral.
Derivation of integral (66):
Consider the integral when $m \geq n \geq 3$

$$I = \left\{2^{m+n}(m!)(n!)\right\} \int_{-1}^{+1} \mathrm{P}_n'''(x)\mathrm{P}_m'''(x)\mathrm{d}x. \quad (68)$$

Using the equation (33), we have

$$I = \int_{-1}^{+1} D^{n+3}(x^2-1)^n D^{m+3}(x^2-1)^m \mathrm{d}x.$$

Taking $U = D^{n+3}(x^2-1)^n$, $T = D^{m+3}(x^2-1)^m$ and using the Leibnitz integration formula (34) (with suitable value of $J = n-4$), we get

$$\begin{aligned} I = \Big[& (-1)^0 \left\{D^{n+3}(x^2-1)^n\right\} \left\{D^{m+2}(x^2-1)^m\right\} + \\ & +(-1)^1 \left\{D^{n+4}(x^2-1)^n\right\} \left\{D^{m+1}(x^2-1)^m\right\} + \\ & +(-1)^2 \left\{D^{n+5}(x^2-1)^n\right\} \left\{D^{m}(x^2-1)^m\right\} + \\ & +(-1)^3 \left\{D^{n+6}(x^2-1)^n\right\} \left\{D^{m-1}(x^2-1)^m\right\} + \\ & +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\ & +(-1)^{n-4} \left\{D^{2n-1}(x^2-1)^n\right\} \left\{D^{m-n+6}(x^2-1)^m\right\} \Big]_{-1}^{+1} + \\ & +(-1)^{n-3} \int_{-1}^{+1} \left\{D^{2n}(x^2-1)^n\right\} \left\{D^{m-n+6}(x^2-1)^m\right\} \mathrm{d}x, \end{aligned}$$

$$\begin{aligned} I = \Big[& (-1)^0 \left\{D^{n+3}(x^2-1)^n\right\} \left\{D^{m+2}(x^2-1)^m\right\} + \\ & +(-1)^1 \left\{D^{n+4}(x^2-1)^n\right\} \left\{D^{m+1}(x^2-1)^m\right\} + \\ & +(-1)^2 \left\{D^{n+5}(x^2-1)^n\right\} \left\{D^{m}(x^2-1)^m\right\} + \\ & +(-1)^3 \left\{D^{n+6}(x^2-1)^n\right\} \left\{D^{m-1}(x^2-1)^m\right\} + \\ & +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\ & +(-1)^{n-4} \left\{D^{2n-1}(x^2-1)^n\right\} \left\{D^{m-n+6}(x^2-1)^m\right\} \Big]_{-1}^{+1} + \\ & +(-1)^{n-3}(2n)! \int_{-1}^{+1} \left\{D^{m-n+6}(x^2-1)^m\right\} \mathrm{d}x, \end{aligned}$$

$$\begin{aligned} I = \Big[& (-1)^0 \left\{D^{n+3}(x^2-1)^n\right\} \left\{D^{m+2}(x^2-1)^m\right\} + \\ & +(-1)^1 \left\{D^{n+4}(x^2-1)^n\right\} \left\{D^{m+1}(x^2-1)^m\right\} + \\ & +(-1)^2 \left\{D^{n+5}(x^2-1)^n\right\} \left\{D^{m}(x^2-1)^m\right\} + \\ & +(-1)^3 \left\{D^{n+6}(x^2-1)^n\right\} \left\{D^{m-1}(x^2-1)^m\right\} + \\ & +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\ & +(-1)^{n-4} \left\{D^{2n-1}(x^2-1)^n\right\} \left\{D^{m-n+6}(x^2-1)^m\right\} + \\ & +(-1)^{n-3}(2n)! \left\{D^{m-n+5}(x^2-1)^m\right\} \Big]_{-1}^{+1}, \end{aligned} \tag{69}$$

$$\begin{aligned} &= [\text{upper limit of right-hand side expression of (69) at } x=1] - \\ &\quad -[\text{lower limit of right-hand side expression of (69) at } x=-1]. \end{aligned}$$

The values of fourth line to last line of right-hand side expression of (69) will be zero at $x = \pm 1$ because $(m-1)$, $(m-2)$,....,$\{m-(n-5)\}$ are less than m, in view of the equation (35).

$$I = \Big[(-1)^0 \left\{D^{n+3}(x^2-1)^n\right\} \left\{D^{m+2}(x^2-1)^m\right\} + (-1)^1 \left\{D^{n+4}(x^2-1)^n\right\} \left\{D^{m+1}(x^2-1)^m\right\} +$$

$$+(-1)^2\left\{D^{n+5}(x^2-1)^n\right\}\left\{D^m(x^2-1)^m\right\}\Bigg]_{-1}^{+1}. \tag{70}$$

Now using the results [see equations (36), (37), (38), (39), (40), (41), (42), (43), (44), (45), (46) and (47)] (with suitable values of p and r) in the equation (70), we get

$$I=\Bigg[\left\{\frac{(10)2^n n!n(n+1)(n+2)(n+3)(n-1)(n-2)}{480}\right\}\left\{\frac{2^m m!m(m+1)(m+2)(m-1)}{8}\right\}-$$

$$-\left\{\frac{(5)2^n n!n(n+1)(n+2)(n+3)(n+4)(n-1)(n-2)(n-3)}{1920}\right\}\left\{\frac{2^m m!m(m+1)}{2}\right\}+$$

$$+\left\{\frac{2^n n!n(n+1)(n+2)(n+3)(n+4)(n+5)(n-1)(n-2)(n-3)(n-4)}{3840}\right\}\{2^m m!\}\Bigg]-$$

$$-\Bigg[\left\{\frac{(10)(-1)2^n n!n(n+1)(n+2)(n+3)(n-1)(n-2)(-1)^n}{480}\right\}\times$$

$$\times\left\{\frac{2^m m!m(m+1)(m+2)(m-1)(-1)^m}{8}\right\}-$$

$$-\left\{\frac{(5)2^n n!n(n+1)(n+2)(n+3)(n+4)(n-1)(n-2)(n-3)(-1)^n}{1920}\right\}\times$$

$$\times\left\{\frac{(-1)2^m m!m(m+1)(-1)^m}{2}\right\}+$$

$$+\left\{\frac{(-1)2^n n!n(n+1)(n+2)(n+3)(n+4)(n+5)(n-1)(n-2)(n-3)(n-4)(-1)^n}{3840}\right\}\times$$

$$\times\{2^m m!(-1)^m\}\Bigg],$$

$$I=\left\{\frac{2^{m+n}m!n!n(n+1)(n+2)(n+3)(n-1)(n-2)}{3840}\right\}\times$$

$$\times[10m(m+1)(m+2)(m-1)+10m(m+1)(m+2)(m-1)(-1)^{m+n}-5(n+4)(n-3)m(m+1)-$$
$$-5(n+4)(n-3)m(m+1)(-1)^{m+n}+(n+4)(n+5)(n-3)(n-4)+(n+4)(n+5)(n-3)(n-4)(-1)^{m+n}],$$

$$I=\left\{\frac{2^{m+n}m!n!n(n+1)(n+2)(n+3)(n-1)(n-2)}{3840}\right\}\times$$

$$\times[10m(m+1)(m+2)(m-1)\{1+(-1)^{m+n}\}-5(n+4)(n-3)m(m+1)\{1+(-1)^{m+n}\}+$$
$$+(n+4)(n+5)(n-3)(n-4)\{1+(-1)^{m+n}\}],$$

$$I=\{2^{m+n}m!n!\}\left\{\frac{(n+3)(n+2)(n+1)n(n-1)(n-2)(n-3)!}{3840\ (n-3)!}\right\}\{1+(-1)^{m+n}\}\times$$

$$\times[10m(m+1)(m+2)(m-1)-5(n+4)(n-3)m(m+1)+(n+4)(n+5)(n-3)(n-4)]. \tag{71}$$

Finally, cancelling the factor $\{2^{m+n}m!n!\}$ in the equations (68) and (71), we obtain the integral (66). Similarly, we can derive the remaining integrals.

Conclusion

Here in this paper, we obtain some definite integrals related with the product of the derivatives of Legendre's polynomials of first kind of different order, by using the derivatives of Legendre's polynomials of first kind $\mathrm{P}_n(x)$, Rodrigues formula, Leibnitz's generalized rule for successive integration by parts and some values of successive differential coefficients of $(x^2-1)^r$ at $x=\pm 1$.

We conclude our present investigation by observing that, we can evaluate the following integral $\int_{-1}^{+1}\left\{\frac{d^r}{dx^r}\mathrm{P}_m(x)\right\}\left\{\frac{d^s}{dx^s}\mathrm{P}_n(x)\right\}dx$ by taking positive integral values of r and s, in analogous manner. The classical Legendre polynomials $\mathrm{P}_n(x)$ form a sequence of orthogonal polynomials with many historical applications. Their use continues in recent times in applications such as beam theory [19], phone segmentation [20], neural networks [21] and signal processing [22]; see also the recent works [23–25] dealing extensively with the methodology and techniques based on Legendre polynomials.

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Құрамында Лежандр көпмүшелері бар анықталған интегралдар үйірі

Мақаланың негізгі мақсаты–анықталған интегралдармен байланысты кейбір есептерді (бұрын табылмаған және әдебиетте жарияланбаған) аналитикалық жолмен шешу. Анықталған интегралдың интеграл астындағы өрнегі әр түрлі ретті бірінші текті Лежандр полиномдарының туындыларының көбейтіндісі болып табылады, бұл ретте $\mathrm{P}_n(x)$ бірінші текті Лежандр полиномдарының кейбір туындылары, Родригес формулалары, Лейбництің тізбектелген дифференциалдық $(x^2-1)^r$, $x=\pm1$ коэффициенттерінің бөліктері мен кейбір мәндері бойынша тізбектеп интегралдауға арналған жалпыланған ережесі қолданылады.

Кілт сөздер: Лежандр көпмүшелері, Родригес формуласы, біртіндеп бөліктеп интегралдау үшін Лейбництің жалпыланған ережесі, Лежандр көпмүшесі үшін Мерфи формуласы.

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Семейство определенных интегралов, содержащих многочлены Лежандра

Основная цель настоящей статьи — дать аналитические решения (ранее не найденные и не опубликованные в литературе) некоторых задач, связанных с определенными интегралами, подынтегральными выражениями которых являются произведения производных полиномов Лежандра первого рода разного порядка, с помощью некоторых производных полиномов Лежандра первого рода $\mathrm{P}_n(x)$, формулы Родригеса, обобщенного правила Лейбница для последовательного интегрирования по частям и некоторых значений последовательных дифференциальных коэффициентов $(x^2-1)^r$ при $x=\pm 1$.

Ключевые слова: полиномы Лежандра, формула Родригеса, обобщенное правило Лейбница для последовательного интегрирования по частям, формула Мерфи для многочлена Лежандра.