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A remark on Schottky representations and Reidemeister torsion

The present paper establishes a formula of Reidemeister torsion for Schottky representations. The theoretical results are applied to 3−manifolds with boundary consisting orientable surfaces with genus at least 2.

Keywords: Schottky representations, Reidemeister torsion, representation varieties, Atiyah-Bott-Goldman symplectic form, Thurston symplectic form.

Introduction

It is well-known that the representation varieties are important in many branches of mathematics and physics. For instance, let Σ be a compact Riemann surface of genus at least 2, Teichmüller space $\text{Teich}(\Sigma)$ of Σ is the space of deformation classes of complex structures on it. By the uniformization Theorem, it is the space of hyperbolic metrics, namely Riemannian metrics on Σ with Gaussian curvature constant (-1). Furthermore, Teichmüller space of Σ can be interpreted as discrete faithful representations of the fundamental group $\pi_1(\Sigma)$ of the surface to $\text{PSL}(2,\mathbb{R})$. It is well-known that some certain geometric structures on Σ can also be identified as certain surface group variety [1–6] and the references therein.

Representation varieties have a large number of applications in many branches of mathematics and physics such as in 3−manifold topology (in Bass-Culler-Shalen theory [7, 8], in A-polynomial [9], in hyperbolic geometry [10], in Casson invariant theory [11]), in Yang-Mills and Chern-Simons quantum field theories [12, 13], in skein theory of quantum invariants of 3-manifolds [14, 15], in the moduli spaces of flat connections, holomorphic bundles, and Higgs bundles [16].

Reidemeister torsion(R-torsion) is a topological invariant and was introduced by K. Reidemeister [17]. Using this invariant, he classified 3−dimensional lens spaces. W. Franz extended the R-torsion and classified the higher dimensional lens spaces [18]. R-torsion has many applications in several branches of mathematics and theoretical physics such as topology [19], differential geometry [20], representation spaces [21] dynamical systems [22], 3-dimensional Seiberg-Witten theory [23], algebraic K-theory [24], Chern-Simon theory [13], knot theory [24], theoretical physics and quantum field theory [13]. See Refs. [25] and [26] and the references therein for further information.

Real symplectic chain complex is a algebraic topological instrument and was introduced by E. Witten [21]. Combining this and R-torsion, he evaluated the volume of several moduli space of $\text{Rep}(\Sigma, G)$, which is the set of all conjugacy classes of homomorphisms from the fundamental group $\pi_1(\Sigma)$ of a Riemann surface Σ to the compact gauge group $G \in \{\text{SU}(2), \text{SO}(3)\}$.

In paper [27], we considered the set $\text{Rep}(\Sigma, G)$ of $G-$valued representations from the fundamental group $\pi_1(\Sigma)$ of the surface Σ to the exceptional groups G_2, F_4, and E_6. We proved the well-definiteness of R-torsion of such representations. We also established a formula for computing R-torsion of such representations in terms of the well known symplectic structure on $\text{Rep}(\Sigma, G)$, namely, Atiyah-Bott-Goldman symplectic form for the Lie group G. Then, we applied to $G-$valued Hitchin representations.

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In paper [28], we investigated $G-$valued representations of free or surface group with genus > 1 for $G \in \{\mathrm{GL}(n,\mathbb{C}), \mathrm{SL}(n,\mathbb{C})\}$. We also established a formula for computing R-torsion of such representations in terms of Atiyah-Bott-Goldman symplectic form for G. Moreover, we applied the obtained results to hyperbolic $3-$manifolds.

In the present paper, we prove a formula of R-torsion for Schottky representations. The theoretical results are applied to $3-$manifolds with boundary consisting orientable surfaces with genus at least 2.

1 Preliminaries

In this section, we provide the necessary definition and basic facts about the topological invariant R-torsion and the symplectic chain complex. For further information the reader is referred to [21,25,26,29] and the references therein.

Let $C_* = \left(0 \to C_n \overset{\partial_n}{\to} C_{n-1} \to \cdots \to C_1 \overset{\partial_1}{\to} C_0 \to 0\right)$ be a chain complex of finite dimensional vector spaces over the field $\mathbb{C}$ of complex numbers. For $p = 0, \dots, n$, we denote the kernel of ∂_p, the image of ∂_{p+1}, and the pth homology group of the chain complex C_* by $Z_p(C_*)$, $B_p(C_*)$, and $H_p(C_*)$, respectively. From the definition of $Z_p(C_*)$, $B_p(C_*)$, and $H_p(C_*)$ it follows

$$0 \longrightarrow Z_p(C_*) \hookrightarrow C_p \twoheadrightarrow B_{p-1}(C_*) \longrightarrow 0$$

and

$$0 \longrightarrow B_p(C_*) \hookrightarrow Z_p(C_*) \twoheadrightarrow H_p(C_*) \longrightarrow 0.$$

For $p = 0, \dots, n$, if $\mathbf{c}_p$, $\mathbf{b}_p$, and $\mathbf{h}_p$ are bases of C_p, $B_p(C_*)$, and $H_p(C_*)$, respectively and if $\ell_p : H_p(C_*) \to Z_p(C_*)$, $s_p : B_{p-1}(C_*) \to C_p$ are sections of $Z_p(C_*) \to H_p(C_*)$, $C_p \to B_{p-1}(C_*)$, respectively, then with the help of above short-exact sequences we have the basis $\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1})$ of C_p. Here, $\sqcup$ denotes the disjoint union.

Let $\mathbf{c}_p$, $\mathbf{b}_p$, $\mathbf{h}_p$, ℓ_p, and s_p be as above. Then, *R-torsion* of the chain complex C_* with respect to bases $\{\mathbf{c}_p\}_{p=0}^n$, $\{\mathbf{h}_p\}_{p=0}^n$ is defined by

$$\mathbb{T}\left(C_*, \{\mathbf{c}_p\}_0^n, \{\mathbf{h}_p\}_0^n\right) = \prod_{p=0}^{n} \left[\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1}), \mathbf{c}_p\right]^{(-1)^{(p+1)}},$$

where $[\mathbf{e}_p, \mathbf{f}_p]$ denotes determinant of the change-base-matrix from basis $\mathbf{f}_p$ to $\mathbf{e}_p$ of C_p.

R-torsion does not depend on the bases $\mathbf{b}_p$ and sections s_p, ℓ_p [24].

Let $\mathbf{c}'_p$, $\mathbf{h}'_p$ be also bases of C_p, $H_p(C_*)$, respectively. Then, the following change-base-formula is valid [24]:

$$\mathbb{T}\left(C_*, \{\mathbf{c}'_p\}_0^n, \{\mathbf{h}'_p\}_0^n\right) = \prod_{p=0}^{n} \left(\frac{[\mathbf{c}'_p, \mathbf{c}_p]}{[\mathbf{h}'_p, \mathbf{h}_p]}\right)^{(-1)^p} \mathbb{T}\left(C_*, \{\mathbf{c}_p\}_0^n, \{\mathbf{h}_p\}_0^n\right).$$

Let

$$0 \longrightarrow A_* \overset{\imath}{\longrightarrow} B_* \overset{\jmath}{\longrightarrow} D_* \longrightarrow 0 \tag{1}$$

be a short-exact sequence of chain complexes, and let $\mathbf{c}_p^A$, $\mathbf{c}_p^B$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, $\mathbf{h}_p^B$, and $\mathbf{h}_p^D$ are bases of A_p, B_p, D_p, $H_p(A_*)$, $H_p(B_*)$, and $H_p(D_*)$, respectively. Let us consider the corresponding Mayer-Vietoris long-exact sequence of vector spaces

$$\mathcal{C}_* : \ \cdots \longrightarrow H_p(A_*) \overset{\imath_p}{\longrightarrow} H_p(B_*) \overset{\jmath_p}{\longrightarrow} H_p(D_*) \overset{\delta_p}{\longrightarrow} H_{p-1}(A_*) \longrightarrow \cdots$$

associated to short-exact sequence (1). Note that $\mathcal{C}_{3p} = H_p(D_*)$, $\mathcal{C}_{3p+1} = H_p(A_*)$, and $\mathcal{C}_{3p+2} = H_p(B_*)$ then we can consider the bases $\mathbf{h}_p^D$, $\mathbf{h}_p^A$, and $\mathbf{h}_p^B$ for $\mathcal{C}_{3p}$, $\mathcal{C}_{3p+1}$, and $\mathcal{C}_{3p+2}$, respectively.

Theorem 1. [24] Suppose $\mathbf{c}_p^A$, $\mathbf{c}_p^B$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, $\mathbf{h}_p^B$, and $\mathbf{h}_p^D$ are as above. Suppose also $\left[\mathbf{c}_p^B, \mathbf{c}_p^A \oplus \widetilde{\mathbf{c}_p^D}\right] = \pm 1$, where $j\left(\widetilde{\mathbf{c}_p^D}\right) = \mathbf{c}_p^D$. Then, it follows

$$\begin{aligned} \mathbb{T}\left(B_*, \{\mathbf{c}_p^B\}_0^n, \{\mathbf{h}_p^B\}_0^n\right) &= \mathbb{T}\left(A_*, \{\mathbf{c}_p^A\}_0^n, \{\mathbf{h}_p^A\}_0^n\right) \\ \times \quad \mathbb{T}\left(D_*, \{\mathbf{c}_p^D\}_{p=0}^n, \{\mathbf{h}_p^D\}_0^n\right) &\mathbb{T}\left(C_*, \{\mathbf{c}_{3p}\}_0^{3n+2}, \{0\}_0^{3n+2}\right). \end{aligned}$$

Theorem 1 yields the sum-lemma.

Lemma 1. Assume A_*, D_* are chain complexes of vector spaces and $\mathbf{c}_p^A$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, and $\mathbf{h}_p^D$ are bases of A_p, D_p, $H_p(A_*)$, and $H_p(D_*)$, respectively. Then, the following equality

$$\mathbb{T}(A_* \oplus D_*, \{\mathbf{c}_p^A \sqcup \mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^A \sqcup \mathbf{h}_p^D\}_0^n) = \mathbb{T}(A_*, \{\mathbf{c}_p^A\}_0^n, \{\mathbf{h}_p^A\}_0^n)\mathbb{T}(D_*, \{\mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^D\}_0^n)$$

is valid.

The proof of Lemma 1 can also be found in [30].

$(C_*, \partial_*, \{\omega_{*,q-*}\})$ is said to be $\mathbb{C}-$*symplectic chain complex* of length q, if

1 $C_* : 0 \to C_q \stackrel{\partial_q}{\to} C_{q-1} \to \cdots \to C_{q/2} \to \cdots \to C_1 \stackrel{\partial_1}{\to} C_0 \to 0$ is a chain complex of length q, where $q \equiv 2 \pmod 4$,

2 for $p = 0, \ldots, q$, $\omega_{p,q-p} : C_p \times C_{q-p} \to \mathbb{C}$ is a $\partial-$compatible non-degenerate anti-symmetric bilinear form. Namely,

$$\omega_{p,q-p}\left(\partial_{p+1}a, b\right) = (-1)^{p+1}\omega_{p+1,q-(p+1)}\left(a, \partial_{q-p}b\right)$$

and

$$\omega_{p,q-p}(a,b) = (-1)^{p(q-p)}\omega_{q-p,p}(b,a).$$

From the fact that $q \equiv 2 \pmod 4$ we have $\omega_{p,q-p}(a,b)$ is $(-1)^p\omega_{q-p,p}(b,a)$. From $\partial-$compatibility of $\omega_{p,q-p}$ we obtain the non-degenerate pairing $[\omega_{p,q-p}] : H_p\left(C_*\right) \times H_{q-p}\left(C_*\right) \to \mathbb{C}$.

For the rest of the paper, if the $\mathbb{C}-$*symplectic chain complex* $(C_*, \partial_*, \{\omega_{*,q-*}\})$ is clear, then $\Delta\left(\mathbf{h}_p, \mathbf{h}_{q-p}\right)$ is the determinant of the matrix of the non-degenerate pairing

$$[\omega_{p,q-p}] : H_p\left(C_*\right) \times H_{q-p}\left(C_*\right) \to \mathbb{C}$$

in the bases $\mathbf{h}_p$, $\mathbf{h}_{q-p}$.

Assume C_* is a $\mathbb{C}-$symplectic chain complex of length q and $\mathbf{c}_p$, $\mathbf{c}_{q-p}$ are bases of C_p, C_{q-p}, respectively. We say $\omega-$*compatible*, if the matrix of $\omega_{p,q-p}$ in $\mathbf{c}_p$, $\mathbf{c}_{q-p}$ is equal to the $k \times k$ identity matrix $\mathrm{Id}_{k\times k}$ when $p \neq q/2$ and $\begin{pmatrix} 0_{l\times l} & \mathrm{Id}_{l\times l} \\ -\mathrm{Id}_{l\times l} & 0_{l\times l} \end{pmatrix}$ when $p = q/2$, where $k = \dim C_p = \dim C_{q-p}$ and $2l = \dim C_{q/2}$.

For computing R-torsion in terms of intersections pairings, we have the following result suggests a formula. Namely,

Theorem 2. [31] If $(C_*, \partial_*, \{\omega_{*,q-*}\})$ is a $\mathbb{C}-$symplectic chain complex with the $\omega-$compatible bases $\mathbf{c}_p$, $p = 0, \ldots, q$ and if $\mathbf{h}_p$ is a basis of $H_p\left(C_*\right)$, $p = 0, \ldots, q$, then the following formula holds:

$$\left|\mathbb{T}\left(C_*, \{\mathbf{c}_p\}_0^q, \{\mathbf{h}_p\}_0^q\right)\right| = \prod_{p=0}^{(q/2)-1} |\Delta(\mathbf{h}_p, \mathbf{h}_{q-p})|^{(-1)^p} \sqrt{\left|\Delta\left(\mathbf{h}_{q/2}, \mathbf{h}_{q/2}\right)\right|}^{(-1)^{q/2}}. \qquad (2)$$

In case $\mathbf{h}_p = \mathbf{h}_{q-p} = 0$, the convention $0 = 1.0$ is used and hence $\Delta(\mathbf{h}_p, \mathbf{h}_{q-p}) = 1$. Let us also note that equation (2) can be improved as:

$$\mathbb{T}\left(C_*, \{\mathbf{c}_p\}_0^q, \{\mathbf{h}_p\}_0^q\right) = \prod_{p=0}^{(q/2)-1} \Delta(\mathbf{h}_p, \mathbf{h}_{q-p})^{(-1)^p} \sqrt{\Delta\left(\mathbf{h}_{q/2}, \mathbf{h}_{q/2}\right)}^{(-1)^{q/2}}. \tag{3}$$

For details of (3), we refer the reader to [28; Remark 2.4]. See [27, 28, 30], for further applications of Theorem 2.

2 Main results

Let Σ be a closed orientable surface of genus at least 2 with the universal covering $\widetilde{\Sigma}$. Let G be the Lie group $\mathrm{PSL}(2,\mathbb{C})$ and $\mathcal{G}$ be the Lie algebra of G with the non-degenerate symmetric bilinear form B. Here, B is the Killing form.

Assume $\varrho : \pi_1(\Sigma) \to G$ is a homomorphism from the fundamental group $\pi_1(\Sigma)$ of Σ to G. Let $E_\varrho = \widetilde{\Sigma} \times \mathcal{G}/\sim$ be the corresponding adjoint bundle over Σ. Here, $(x_1, t_1) \sim (x_2, t_2)$, if $(x_2, t_2) = (\gamma \cdot x_1, \gamma \cdot t_1)$ for some $\gamma \in \pi_1(\Sigma)$, the action of γ in the first component by deck transformation $(\gamma \cdot x_1 = \gamma(x_1))$ and in the second component by the adjoint action $\left(\gamma \cdot t_1 = \mathrm{Ad}_{\varrho(\gamma)}(t_1) = \varrho(\gamma)\, t_1 \varrho(\gamma)^{-1}\right)$.

Let K be a cell-decomposition of Σ for which the adjoint bundle E_ϱ is trivial over each cell and $\widetilde{K}$ be the lift of K to the $\widetilde{\Sigma}$. Denote by $\mathbb{Z}[\pi_1(\Sigma)]$ the integral group ring. Let $C_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) = C_*\left(\widetilde{K}; \mathbb{Z}\right) \otimes \mathcal{G}/\sim$, where for all $\gamma \in \pi_1(\Sigma)$, $\sigma \otimes t \sim \gamma \cdot \sigma \otimes \gamma \cdot t$, the action of γ by the first component is by deck transformation and in the second is by adjoint action. We have the following chain complex:

$$0 \longrightarrow C_2\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \stackrel{\partial_2 \otimes \mathrm{id}}{\longrightarrow} C_1\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \stackrel{\partial_1 \otimes \mathrm{id}}{\longrightarrow} C_0\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \longrightarrow 0. \tag{4}$$

Here, ∂_p denotes the usual boundary operator. Denote by $H_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and $H^*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ the homologies and cohomologies of the chain complex (4), respectively, where $C^*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ denotes the set of $\mathbb{Z}[\pi_1(\Sigma)]$-module homomorphisms from $C_*\left(\widetilde{K}; \mathbb{Z}\right)$ to $\mathcal{G}$. See [25] for details and unexplained subjects.

Clearly, for conjugate $\varrho, \varrho' : \pi_1(\Sigma) \to G$ i.e. $\varrho'(.) = A\varrho(.)A^{-1}$ for some $A \in G$, we have isomorphic $C_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and $C_*\left(K;\ \mathcal{G}_{\mathrm{Ad}_{\varrho'}}\right)$. Similarly, the corresponding cochains $C^*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and $C^*\left(K;\ \mathcal{G}_{\mathrm{Ad}_{\varrho'}}\right)$ are isomorphic.

Consider chain complex (4). Assume $\left\{e_j^p\right\}_{j=1}^{m_p}$ is a basis of $C_p(K; \mathbb{Z})$. For $j = 1, \ldots, m_p$, fix a lift $\widetilde{e}_j^p$ of e_j^p. Then, $c_p = \left\{\widetilde{e}_j^p\right\}_{j=1}^{m_p}$ of $C_p\left(\widetilde{K}; \mathbb{Z}\right)$ is a $\mathbb{Z}[\pi_1(\Sigma)]-$basis. Assume $\mathcal{A} = \{a_k\}_{k=1}^{\dim \mathcal{G}}$ is a $B-$*orthonormal basis* of the Lie algebra $\mathcal{G}$. Namely, the matrix of the form B equals to the identity matrix of size $\dim \mathcal{G}$. Hence, we obtain a $\mathbb{C}-$basis $\mathbf{c}_p = c_p \otimes_\varrho \mathcal{A}$ of $C_p\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$. We call such a basis a *geometric basis* for $C_p\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$.

If $\mathbf{c}_p = c_p \otimes_\varrho \mathcal{A}$ and $\mathbf{h_p}$ are respectively the geometric basis of $C_p\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and a basis of $H_p\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$, then $\mathbb{T}\left(C_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right), \{c_p \otimes_\varrho \mathcal{A}\}_{p=0}^2, \{\mathbf{h}_p\}_{p=0}^2\right)$ is said to be the *R-torsion* of the triple K, Ad_ϱ, and $\{\mathbf{h}_p\}_{p=0}^2$.

Theorem 3. [28; Theorem 3.1] If Σ, K, ϱ, $\mathbf{c}_p = c_p \otimes_\varrho \mathcal{A}$, and $\mathbf{h}_p$, $p = 0, 1, 2$, are as above, then $\mathbb{T}\left(C_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right), \{c_p \otimes_\varrho \mathcal{A}\}_{p=0}^2, \{\mathbf{h}_p\}_{p=0}^2\right)$ does not depend on the basis $\mathcal{A}$, lifts $\widetilde{e}_j^p$, conjugacy class of ϱ, and the cell-decomposition K.

From Theorem 3, we have the well-definiteness of R-torsion of such representations, and hence we write $\mathbb{T}(\Sigma, \{\mathbf{h}_p\}_{p=0}^2)$ rather than $\mathbb{T}\left(C_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right), \{c_p \otimes_\varrho \mathcal{A}\}_{p=0}^2, \{\mathbf{h}_p\}_{p=0}^2\right)$.

Assume Σ, K, G, $\mathcal{G}$, ϱ, $\mathbf{c}_p = c_p \otimes_\varrho \mathcal{A}$ are as above. Let us consider the dual cell-decomposition K' of Σ corresponding to the cell-decomposition K. Consider the lifts $\widetilde{K}$ and $\widetilde{K'}$ of K and K', respectively. For $i = 0, 1, 2$, we have the intersection form

$$(\cdot,\cdot)_{i,2-i} : C_i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times C_{2-i}\left(K'; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \longrightarrow \mathbb{C} \tag{5}$$

defined by $(\sigma_1 \otimes t_1, \sigma_2 \otimes t_2)_{i,2-i} = \sum_{\gamma\in\pi_1(\Sigma)} \sigma_1.\left(\gamma \bullet \sigma_2\right)\ B\left(t_1, \gamma \bullet t_2\right)$. Here, "." denotes the intersection number pairing, the action of γ on σ_2 by deck transformation and on t_2 is by the adjoint action.

Using the anti-symmetric, $\partial-$compatible $(\cdot,\cdot)_{i,2-i}$, we have the non-degenerate anti-symmetric form

$$[\cdot,\cdot]_{i,2-i} : H_i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times H_{2-i}\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \longrightarrow \mathbb{C}. \tag{6}$$

Note that if $D_i = C_i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \oplus C_i\left(K'; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$, and if we consider the bilinear form $\omega_{i,2-i} : D_i \times D_{2-i} \to \mathbb{C}$ defined by extending the intersection form (5) zero on $C_i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times C_{2-i}\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and $C_i\left(K'; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times C_{2-i}\left(K'; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$, then D_* becomes a $\mathbb{C}-$symplectic chain complex. Note also that the bases c_i of $C_i\left(\widetilde{K}; \mathbb{Z}\right)$ and c_i' of $C_i\left(\widetilde{K'}; \mathbb{Z}\right)$ corresponding to c_i result an $\omega-$compatible basis for D_*.

Kronecker pairing $\langle\cdot,\cdot\rangle : C^i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times C_i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \to \mathbb{C}$ is defined by $\langle \theta, \sigma \otimes_\varrho t\rangle = B\left(t, \theta\left(\sigma\right)\right)$. It has natural extended to $\langle\cdot,\cdot\rangle : H^i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times H_i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \to \mathbb{C}$.

Recall the cup product $\cup : C^i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times C^j\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \to C^{i+j}\left(\widetilde{\Sigma}; \mathbb{C}\right)$ is defined by $\left(\theta_i \cup \theta_j\right)\left(\sigma_{i+j}\right) = B\left(\theta_i\left(\left(\sigma_{i+j}\right)_{\text{front}}\right), \theta_j\left(\left(\sigma_{i+j}\right)_{\text{back}}\right)\right)$. Here, σ_{i+j} is in $C_{i+j}(\widetilde{K}; \mathbb{Z})$ and $\widetilde{K}$ denotes the lift of K to $\widetilde{\Sigma}$ $\theta_i : C_i\left(\widetilde{K}; \mathbb{Z}\right) \to \mathcal{G}$, $\theta_j : C_j\left(\widetilde{K}; \mathbb{Z}\right) \to \mathcal{G}$ are $\mathbb{Z}[\pi_1(\Sigma)]$-module homomorphisms. This yields the cup product

$$\smile_B : C^i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times C^j\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \longrightarrow C^{i+j}\left(K; \mathbb{C}\right)$$

with natural extension

$$\smile_B : H^i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times H^j\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \longrightarrow H^{i+j}\left(\Sigma; \mathbb{C}\right),$$

where $[\theta_i] \smile_B [\theta_j] = [\theta_i \smile_B \theta_j]$.

Using the isomorphisms by (6) and the Kronecker pairing, we get the Poincare duality isomorphisms

$$\mathrm{PD} : H_i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \cong H_{2-i}\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)^* \cong H^{2-i}\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right).$$

For $i = 0, 1, 2$ we have the

$$\begin{array}{ccccccc} H^{2-i}\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) & \times & H^i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) & \overset{\smile_B}{\longrightarrow} & H^2\left(\Sigma; \mathbb{C}\right) \\ \uparrow \mathrm{PD} & & \uparrow \mathrm{PD} & \circlearrowleft & \uparrow \\ H_i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) & \times & H_{2-i}\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) & \overset{[\cdot,\cdot]_{i,2-i}}{\longrightarrow} & \mathbb{C}. \end{array}$$

Here, $\mathbb{C} \to H^2\left(\Sigma; \mathbb{C}\right)$ sends $1 \in \mathbb{C}$ to the fundamental class of $H^2(\Sigma; \mathbb{C})$ and the inverse of this is integration over Σ.

Clearly, we have the following pairing

$$\Omega_{i,2-i} : H^i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times H^{2-i}\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \overset{\smile_B}{\longrightarrow} H^2\left(\Sigma; \mathbb{C}\right) \overset{\int_\Sigma}{\longrightarrow} \mathbb{C}. \tag{7}$$

$\Omega_{1,1}$ is called Atiyah-Bott-Goldman symplectic form for G on the representation variety $\mathrm{Rep}(\Sigma, G)$.

In [28], we established a formula for computing Reidemeister torsion of representations in terms of $\Omega_{1,1}$ Atiyah-Bott-Goldman symplectic form for the Lie group G. More precisely, *Theorem 4.* [28; Theorem 3.2] Let Σ, K, K', ϱ be as above. Let $\mathbf{c}_p$ and $\mathbf{c}'_p$ be the corresponding geometric bases of $C_p\left(K;\mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and $C_p\left(K';\mathcal{G}_{\mathrm{Ad}_\varrho}\right)$, respectively, $p = 0,1,2$. If $\mathbf{h}_p$ is a basis of $H_p\left(\Sigma;\mathcal{G}_{\mathrm{Ad}_\varrho}\right)$, $p = 0,1,2$, then the following formulas are valid

i. $\mathbb{T}\left(\Sigma,\{\mathbf{h}_p\}_{p=0}^2\right) = ie^{\frac{i\theta}{2}}\dfrac{\Delta\left(\mathbf{h}_0,\mathbf{h}_2\right)}{\sqrt{\Delta\left(\mathbf{h}_1,\mathbf{h}_1\right)}}$,

ii. $\mathbb{T}\left(\Sigma,\{\mathbf{h}_p\}_{p=0}^2\right) = ie^{\frac{i\theta}{2}}\dfrac{\sqrt{\delta\left(\mathbf{h}^1,\mathbf{h}^1\right)}}{\delta\left(\mathbf{h}^2,\mathbf{h}^0\right)}$.

Here, $\Delta\left(\mathbf{h}_p,\mathbf{h}_{2-p}\right)$ is the determinant of the matrix of (6) in $\mathbf{h}_p$ and $\mathbf{h}_{2-p}$, $\Delta\left(\mathbf{h}_0,\mathbf{h}_2\right) = \left|\Delta\left(\mathbf{h}_0,\mathbf{h}_2\right)\right| e^{i\theta}$, where $i = \sqrt{-1}$ and $-\pi < \theta \leq \pi$. $\delta\left(\mathbf{h}^{2-p},\mathbf{h}^p\right)$ is the determinant of the matrix of (7) in $\mathbf{h}^p$ and $\mathbf{h}^{2-p}$, and $\mathbf{h}^p$ denotes the Poincare dual basis of $H^p(\Sigma;\mathcal{G}_{\mathrm{Ad}_\varrho})$ corresponding to $\mathbf{h}_p$ of $H_p(\Sigma;\mathcal{G}_{\mathrm{Ad}_\varrho})$, $p = 0,1,2$.

Note that in case $H_0\left(\Sigma;\mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and thus $H_2\left(\Sigma;\mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ are zero, by Theorem 4 we get

$$\mathbb{T}\left(\Sigma,\{0,\mathbf{h}_1,0\}\right) = i\,\sqrt{\Delta\left(\mathbf{h}_1,\mathbf{h}_1\right)}^{\,(-1)} = i\,\sqrt{\delta\left(\mathbf{h}^1,\mathbf{h}^1\right)}.$$

3 *Applications*

Schottky representation and Thurston symplectic form

Before stating our application, let us recall Thurston symplectic form. For more information and unexplained subjects, we refer [32] and the references therein.

Let Σ_g, $g \geq 2$, be a closed orientable surface. We say that $\lambda \subset \Sigma_g$ is a *geodesic lamination*, if it is closed and also consists of disjoint complete geodesics without any self-intersection points, called *leaves* of λ (see Figure 1 (a)). We say that the geodesic lamination λ is *maximal*, if the complement $\Sigma_g - \lambda$ consists of finitely many ideal triangles, that is, triangles with vertices at infinity (see Figure 1 (b)).

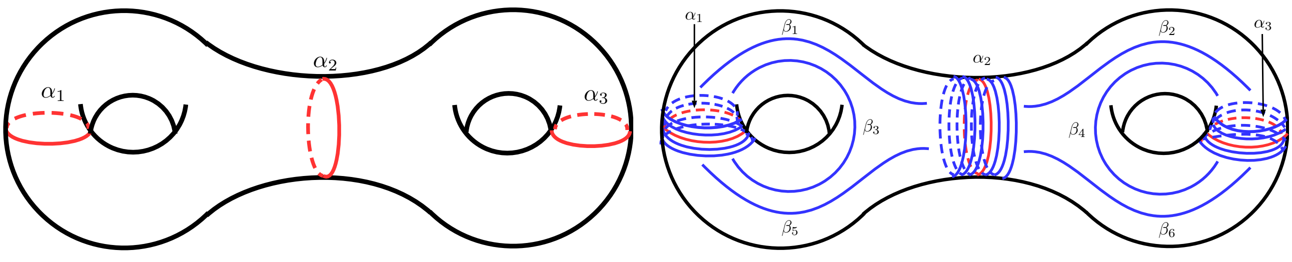

Figure 1. (a) Geodesic lamination with 3 closed leaves (b) Maximal geodesic lamination with 3 closed leaves and 6 infinite leaves spiraling towards closed leaves.

Let $\lambda \subset \Sigma_g$ be a geodesic lamination and G be an abelian group. A $G-$valued *transverse cocycle* σ for λ is a function from the set of all transverse arcs to the leaves of λ to G so that σ is finitely additive and invariant under the homotopy of arcs transverse to λ. To be more precise, $\sigma(k) = \sigma(k_1) + \sigma(k_2)$, when the arc k transverse to leaves of λ is decomposed into two subarcs k_1, k_2 with disjoint interiors, and $\sigma(k) = \sigma(k')$ when the transverse arc k is deformed to arc k' through arcs transverse to the leaves of the geodesic lamination λ (Fig. 2). Let us denote the group of $G-$valued transverse cocycles for λ by $\mathcal{H}\left(\lambda;G\right)$. In the case λ is a maximal geodesic lamination and $G = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{R}/2\pi\mathbb{Z}$, $\mathcal{H}(\lambda;G)$ is isomorphic to G^{6g-6} [33]. For example, by using a (fattened) train-track $\Phi \subset \Sigma_g$ carrying the lamination λ, one gets the isomorphism $\mathcal{H}(\lambda;\mathbb{R}) \cong \mathbb{R}^{6g-6}$.

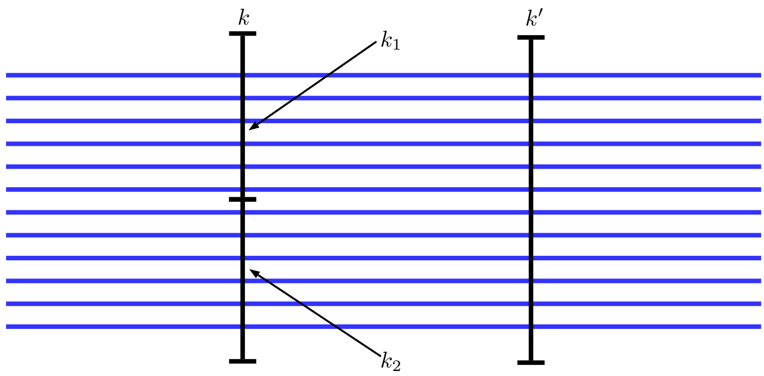

Figure 2. The arcs k and k' are transverse to the leaves of lamination λ. The arc k is deformed to k' through arcs transverse to the leaves of the geodesic lamination. Moreover, k is splitted into two transverse subarcs k_1, k_2 with disjoint interiors.

Recall that a *train-track* $\Phi \subset \Sigma_g$ is composed of finitely many "long" rectangles $e_1, \ldots, e_n$, called *edges* of Φ, foliated by arcs parallel to the "short" sides and meeting only along arcs (possibly reduced to a point) lying in their short sides. Furthermore, each point of the "short" side of a rectangle is also contained in another rectangle, each component of the union of the short sides of all rectangles is an arc, as opposed to a closed curve, and finally since the closure $\overline{\Sigma_g - \Phi}$ of the complement $\Sigma_g - \Phi$ has a certain number of "spikes", corresponding to the points where at least 3 rectangles meet, it is also required that no component of $\overline{\Sigma_g - \Phi}$ be a disc with $0, 1$ or 2 spikes or an annulus with no spike.

Note that foliating the edges of the train-track Φ by using the short sides, we get a foliation of Φ, and the leaves are called the *ties* of Φ. The finitely many ties where several edges meet are said to be the *switches* of Φ. If a tie is not a switch, then it is called a *generic* tie. If λ lies entirely in the interior of Φ and if, moreover, the leaves of λ are transverse to the ties of Φ, then λ is said to be *carried* by Φ (Fig. 3). We refer [34] for constructions of a train-track.

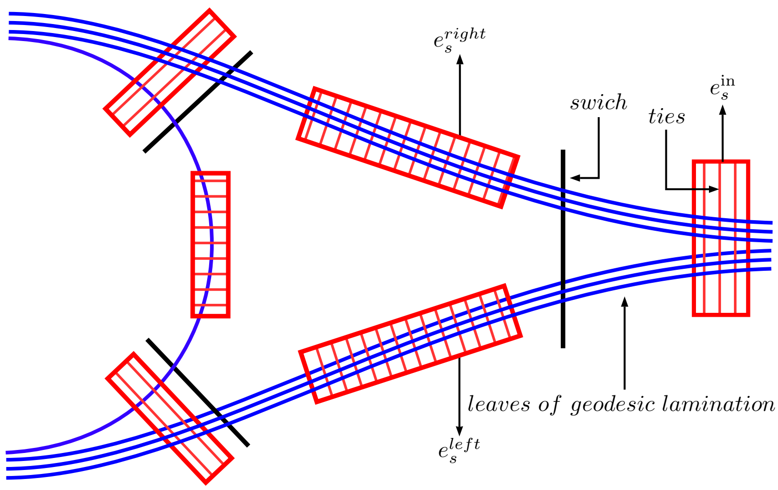

Figure 3. Locally a train-track carries a geodesic lamination.

Suppose $\Phi \subset \Sigma_g$ is a train-track. A real-valued function from the set of edges of Φ is called an *edge* *weight system* for Φ, if it satisfies the *switch relation*. Namely, for each switch s of Φ, let $e_1, \ldots, e_p$ be the edges adjacent to one side of s and let $e_{p+1}, \ldots, e_{p+q}$ be the edges adjacent to the other side, we have $\sum_{i=1}^{p} a(e_i) = \sum_{j=p+1}^{p+q} a(e_j)$. Let us denote the real vector space of all edge weight systems for Φ by $\mathcal{W}(\Phi;\mathbb{R})$.

Let $\lambda \subset \Sigma_g$ be a geodesic lamination carried by the train-track Φ. Consider the injective map associating each transverse cocycle $\sigma \in \mathcal{H}(\lambda;\mathbb{R})$ to the edge weight system $a_\sigma \in \mathcal{W}(\Phi;\mathbb{R})$ defined by $a_\sigma(e) = \sigma(k_e)$. Here, k_e is a tie of e. In the case of maximal lamination λ, the map is an isomorphism $\mathcal{H}(\lambda;\mathbb{R}) \cong \mathcal{W}(\Phi;\mathbb{R})$ [33].

One can arrange the train-track Φ so that at each switch s of Φ, there are one incoming edge e_s^{in} touching the switch s on one side and two outgoing edges e_s^{left}, e_s^{right} touching s on the other side, where as seen from the incoming edge e_s^{in} and for the orientation of the surface Σ_g, e_s^{left} branches out to the left and e_s^{right} branches out to the right. *Thurston symplectic form* on $\mathcal{W}(\Phi)$ is the anti-symmetric bilinear form $\omega_{\text{Thurston}} : \mathcal{W}(\Phi;\mathbb{R}) \times \mathcal{W}(\Phi,\mathbb{R}) \to \mathbb{R}$ defined by

$$\omega_{\text{Thurston}}(a,b) = \frac{1}{2}\sum_s \det\begin{bmatrix} a\left(e_s^{\text{left}}\right) & a\left(e_s^{\text{right}}\right) \\ b\left(e_s^{\text{left}}\right) & b\left(e_s^{\text{right}}\right) \end{bmatrix},$$

where the summation is over all switches of Φ.

By using the isomorphism $\mathcal{H}(\lambda;\mathbb{R}) \cong \mathcal{W}(\Phi;\mathbb{R})$, we have the *Thurston symplectic form* $\omega_{\text{Thurston}} : \mathcal{H}(\lambda;\mathbb{R}) \times \mathcal{H}(\lambda;\mathbb{R}) \to \mathbb{R}$. As is well known that ω_{Thurston} is an algebraic intersection number and is independent of Φ [32, 34].

Recall that Teichmüller space $\text{Teich}(\Sigma_g)$ of the surface Σ_g is the space of isotopy classes of complex structures on Σ_g. By The Uniformization Theorem, it is the space of isotopy classes of Riemannian metrics with constant Gaussian curvature (-1), that is, hyperbolic metrics on Σ_g. One can also identify it with the space of conjugacy classes of all discrete faithful homomorphisms from the fundamental group $\pi_1(\Sigma_g)$ to $\text{PSL}(2,\mathbb{R})$. With the help of a maximal geodesic lamination $\lambda \subset \Sigma_g$ and sending to each hyperbolic metric $m \in \text{Teich}(\Sigma_g)$ the corresponding *shearing cocycle* $\sigma_m \in \mathcal{H}(\lambda;\mathbb{R})$, F. Bonahon embedded $\text{Teich}(\Sigma_g)$ as an open cone $\mathcal{C}(\lambda) \subset \mathcal{H}(\lambda;\mathbb{R})$ [32]. If k is an arc transverse to λ, the shearing cocycle $\sigma_m(k)$ measures the "shift to the left" between the two ideal triangles in $\mathbb{H}^2/\varrho_m(\pi_1(\Sigma_g))$ corresponding to the components of $\Sigma_g - \lambda$ containing the endpoints of k. Here, $\varrho_m : \pi_1(\Sigma_g) \to \text{PSL}(2,\mathbb{R})$ is the discrete faithful representation associated to m.

Recall that for a homorphism $\varrho : \pi_1(\Sigma_g) \to \text{PSL}(2,\mathbb{C})$, there is the following commutative diagram

$$\begin{array}{ccccc} H^1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{\text{Ad}_\varrho}\right) & \times & H^1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{\text{Ad}_\varrho}\right) & \overset{\smile_B}{\longrightarrow} & H^2(\Sigma_g;\mathbb{C}) \\ \uparrow\text{PD} & & \uparrow\text{PD} & \circlearrowleft & \uparrow \\ H_1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{\text{Ad}_\varrho}\right) & \times & H_1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{\text{Ad}_\varrho}\right) & \overset{[\cdot,\cdot]_{1,1}}{\longrightarrow} & \mathbb{C}. \end{array} \tag{8}$$

Here, $\mathbb{C} \to H^2(\Sigma_g;\mathbb{C})$ is the isomorphism sending $1 \in \mathbb{C}$ to the fundamental class of $H^2(\Sigma_g;\mathbb{C})$.

Recall also that

$$\omega_{\text{PSL}(2,\mathbb{C})} : H^1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{\text{Ad}_\varrho}\right) \times H^1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{\text{Ad}_\varrho}\right) \overset{\smile_B}{\to} H^2(\Sigma_g;\mathbb{C}) \overset{\int_{\Sigma_g}}{\to} \mathbb{C}$$

is called Atiyah-Bott-Goldman symplectic form for $\text{PSL}(2,\mathbb{C})$ [35]. It is known that $\omega_{\text{PSL}(2,\mathbb{C})}$ is related with the Goldman symplectic form on $\text{Teich}(\Sigma_g)$

$$\omega_{\text{Goldman}} : H^1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{R})_{\text{Ad}_\varrho}\right) \times H^1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{R})_{\text{Ad}_\varrho}\right) \overset{\smile_{B_\mathbb{R}}}{\to} H^2(\Sigma_g;\mathbb{R}) \overset{\int_{\Sigma_g}}{\to} \mathbb{R}.$$

Here, $B_{\mathbb{R}}$ is the Killing form of the set $\mathfrak{sl}(2,\mathbb{R})$, which is 2×2 trace zero matrices over $\mathbb{R}$.

In [31], considering the isomorphism $T_{\varrho}\mathrm{Teich}(\Sigma_g)\cong\mathcal{H}(\lambda;\mathbb{R})$, which is obtained by the real-analytical parameterization of F. Bonahon [32] and complexfying $\omega_{\mathrm{Thurston}}$, it was proved that

$$\omega_{\mathrm{PSL}(2,\mathbb{C})}=2\omega_{\mathrm{T}}. \tag{9}$$

Here,

$$\omega_{\mathrm{T}}:\mathcal{H}(\lambda;\mathbb{C})\times\mathcal{H}(\lambda;\mathbb{C})\to\mathbb{C} \tag{10}$$

is the complexfied Thurston symplectic form.

For more information and unexplained subjects, we refer the reader to [31] and the references therein.

For a fixed $g\geq 2$, let us consider the free group $\mathrm{F_g}$ with generators $X=\{x_1,\dots,x_g\}$. The set $\mathrm{Hom}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ of all homomorphisms from $\mathrm{F_g}$ to $\mathrm{PSL}(2,\mathbb{C})$ can be identified with $\mathrm{PSL}(2,\mathbb{C})^g$ by considering the map $\varrho\mapsto(\varrho(x_1),\dots,\varrho(x_g))$.

Let $\chi(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ be the quotient $\mathrm{Hom}(\mathrm{F_g},G)//G$. As is well known that $\chi(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ naturally has the structure of an algebraic variety and it differs from the set theoretical quotient $\mathrm{Hom}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C})$ only at reducible points, namely, representations whose images fix a point on $\widehat{\mathbb{C}}$ [36]. Let $\mathcal{D}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ and $\mathcal{E}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ denote respectively the set of all discrete, faithful representations and those of representations with dense image in $\mathrm{PSL}(2,\mathbb{C})$. It is well known $\mathcal{E}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ is not empty and open, $\mathcal{D}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ is closed and outside of these representations in $\chi(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ has measure zero [37] and the references therein.

Let $A_i,B_i,\ i=1,\dots,g$, be $2g$ disjoint closed (topological) disks in $\partial\mathbb{H}^3$ and let $\gamma_1,\dots,\gamma_g\in\mathrm{PSL}(2,\mathbb{C})$ be the Möbiüs transformations of the Riemann sphere $\widehat{\mathbb{C}}$ so that $\gamma_i(A_i)$ is the closure of the complement of B_i. The set $\{\gamma_1,\dots,\gamma_g\}$ generate a free discrete group of rank g, called a *Schottky group*. The representation ϱ obtained by $x_i\mapsto\gamma_i$ is in $\mathcal{D}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$. Let $\mathcal{S}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ be the set of Schottky representations. As is well known that $\mathcal{S}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ lies in the interior of $\mathcal{D}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ [38].

In [39], Y. Minsky proved the existence of an open set $\mathcal{M}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ of $\chi(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ which is strictly larger than $\mathcal{S}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ and on which $Out(\mathrm{F}_g)$ acts properly discontinuously.
We have

Theorem 5. Let $\mathrm{F_g}$ denote the fundamental group $\pi_1(H_g)$ of handle body $\mathrm{H_g}$ of genus $g\geq 2$ with boundary Σ_g, and let M denote the double of $\mathrm{H_g}$. Suppose $\lambda\subset\Sigma_g$ is a fixed maximal geodesic lamination and $\varrho\in\mathcal{M}(\mathrm{F_g},\mathrm{PSL}(2,\mathbb{C}))$ is such that $\varrho\circ r\in\mathrm{Teich}(\Sigma_g)$. Let $\mathbf{h}_i^{\mathrm{F}_g}$ be bases for $H_i\left(\mathrm{F}_g;\mathfrak{sl}(2,\mathbb{C})_{Ad_\varrho}\right)$, $i=0,1,2,3$. Then, there exist basis $\mathbf{h}_j^{\mathrm{M}}$ and $\mathbf{h}_k^{\Sigma_g}$ of $H_j\left(\mathrm{M};\mathfrak{sl}(2,\mathbb{C})_{Ad_\varrho}\right)$ and $H_k\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{Ad_{\varrho\circ r}}\right)$, $j=0,1,2,3$, $k=0,1,2$, respectively so that Reidemeister torsion of the corresponding Mayer-Vietoris long exact sequence $\mathcal{H}_\star$ in these bases is 1. In addition, the following formula holds:

$$\mathbb{T}\left(\mathrm{F}_g,\left\{\mathbf{h}_i^{\mathrm{F}_g}\right\}_0^3\right)\;=\;e^{\frac{\sqrt{-1}}{4}(-\beta_0+\pi-\theta_1)}\,2^{\frac{\chi(\Sigma_g;\mathfrak{sl}(2,\mathbb{C}))}{4}}\sqrt[4]{\Omega_{\mathrm{T}}}.$$

Here, $\beta_0=\dim H_0\left(\mathrm{M};\mathfrak{sl}(2,\mathbb{C})_{Ad_\varrho}\right)$, $\mathbf{h}_{1,1}^0$ is a basis of $H_1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{Ad_{\varrho\circ r}}\right)\oplus H_1\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{Ad_{\varrho\circ r}}\right)$ such that $\mathbb{T}\left(C_*\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{Ad_{\varrho\circ r}}\right)\oplus C_*\left(\Sigma_g;\mathfrak{sl}(2,\mathbb{C})_{Ad_{\varrho\circ r}}\right),\left\{\mathbf{h}_{1,1}^0\right\}\right)$ is equal to 1, $\left[\mathbf{h}_{1,1}^0,\mathbf{h}_1^{\Sigma_g}\oplus\mathbf{h}_1^{\Sigma_g}\right]$ $=\left|\left[\mathbf{h}_{1,1}^0,\mathbf{h}_1^{\Sigma_g}\oplus\mathbf{h}_1^{\Sigma_g}\right]\right|e^{\sqrt{-1}\theta_1}$. Here, $\chi(\Sigma_g;\mathfrak{sl}(2,\mathbb{C}))$ is $\chi(\Sigma_g)\dim_{\mathbb{C}}\mathfrak{sl}(2,\mathbb{C})$, Ω_{T} is determinant of the matrix of the symplectic form (10) in the basis $\mathfrak{h}\oplus\sqrt{-1}\,\mathfrak{h}$, $\mathfrak{h}$ is the basis of $\mathcal{H}(\lambda;\mathbb{R})$ associated with the isomorphism obtained by the embedding $\mathrm{Teich}(\Sigma_g)\hookrightarrow\mathcal{H}(\lambda;\mathbb{R})$[32], and $\mathbf{h}^1$ is the Poincare dual basis of $H^1\left(\Sigma_g;\mathfrak{sl}\left(2,\mathbb{C}\right)_{Ad_{\varrho\circ r}}\right)$ corresponding to $\mathbf{h}_1^{\Sigma_g}$. Here, $r:\pi_1\left(\Sigma_g\right)\to\pi_1\left(\mathrm{F}_g\right)$ is the homomorphism obtained by the embedding $\Sigma_g\hookrightarrow\mathrm{F}_g$.

The proof of Theorem 5 is based on combining Theorem 4 and [28; Theorem 4.2], and the above results, using the commutative diagram (8), Eq. (9), and the definition of $\omega_{\mathrm{PSL}(2,\mathbb{C})}$.

Let us now apply [28; Theorem 4.3]. As is well known that for a compact orientable 3−manifold H, the holonomy representation of the complete hyperbolic structure $\mathrm{Hol}:\pi_1(\mathrm{H})\to\mathrm{Isom}^+\mathbb{H}^3\cong\mathrm{PSL}(2,\mathbb{C})$ can be lifted to a representation $\widetilde{\mathrm{Hol}}:\pi_1(\mathrm{H})\to\mathrm{SL}(2,\mathbb{C})$ [40]. It is also well known that there is a one-to-one correspondence between the lifts and spin structures on H. Considering one of the lifts and composing one of a finite dimensional representation V of $\mathrm{SL}(2,\mathbb{C})$, we get a representation $\varrho:\pi_1(\mathrm{H})\to\mathrm{SL}(V)$. Recall that for every positive integer n there is a unique irreducible representation V_n of $\mathrm{SL}(2,\mathbb{C})$ of dimension n, namely, $(n-1)$-th symmetric power of the standard representation $V_2=\mathbb{C}^2$. Considering V_n and all above, we get $\varrho_n:\pi_1(\mathrm{H})\to\mathrm{SL}(n,\mathbb{C})$.

Let H be a compact orientable non-elementary hyperbolic 3−manifold with a boundary consisting of ℓ surfaces $\Sigma_{g_1},\ldots,\Sigma_{g_\ell}$ of genus at least 2, and $n\geq 2$. Recall that H is non-elementary if its holonomy is an irreducible representation in $\mathrm{PSL}(2,\mathbb{C})$.

In [40; Theorem 0.1], P. Menal-Ferrer and J. Porti prove that the inclusion $\partial\mathrm{H}\subset\mathrm{H}$ induces an injection, $H^1\left(\mathrm{H};\mathfrak{sl}\left(n,\mathbb{C}\right)_{Ad_{\varrho n}}\right)\hookrightarrow H^1\left(\partial\mathrm{H};\mathfrak{sl}\left(n,\mathbb{C}\right)_{Ad_{\varrho n}}\right)$ with $\dim H^1\left(\mathrm{H};\mathfrak{sl}\left(n,\mathbb{C}\right)_{Ad_{\varrho n}}\right)=(1/2)$ $\dim H^1\left(\partial\mathrm{H};\mathfrak{sl}\left(n,\mathbb{C}\right)_{Ad_{\varrho n}}\right)$, and an isomorphism $H^2\left(\mathrm{H};\mathfrak{sl}\left(n,\mathbb{C}\right)_{Ad_{\varrho n}}\right)\cong H^2\left(\partial\mathrm{H};\mathfrak{sl}\left(n,\mathbb{C}\right)_{Ad_{\varrho n}}\right)$.

Theorem 6. Assume Σ_{g_i}, H, M, G, $\mathcal{G}$, ϱ, $\mathbf{h}_k^{\mathrm{H}}$, $\mathbf{h}_k^{\mathrm{M}}$, and $\mathbf{h}_j^{\Sigma_{g_i}}$ are as above. Then, the following formula is valid:

$$\begin{aligned}\mathbb{T}\left(\mathrm{H},\left\{\mathbf{h}_k^{\mathrm{H}}\right\}_0^3\right) &= e^{\frac{\sqrt{-1}}{4}\left(-\beta_0+\ell\pi-\sum_{i=1}^{\ell}\theta_1^{\Sigma_{g_i}}\right)}\prod_{i=1}^{\ell}\Delta\left(\mathbf{h}_1^{\Sigma_{g_i}},\mathbf{h}_1^{\Sigma_{g_i}}\right)^{-1/4}\\ &= e^{\frac{\sqrt{-1}}{4}\left(-\beta_0+\ell\pi-\sum_{i=1}^{\ell}\theta_1^{\Sigma_{g_i}}\right)}\prod_{i=1}^{\ell}\sqrt[4]{\delta\left(\mathbf{h}^{1,i},\mathbf{h}^{1,i}\right)}.\end{aligned}$$

Here, $\left[\mathbf{h}_{1,1}^{0,\Sigma_{g_i}},\mathbf{h}_1^{\Sigma_{g_i}}\oplus\mathbf{h}_1^{\Sigma_{g_i}}\right]=\left|\left[\mathbf{h}_{1,1}^{0,\Sigma_{g_i}},\mathbf{h}_1^{\Sigma_{g_i}}\oplus\mathbf{h}_1^{\Sigma_{g_i}}\right]\right|e^{\sqrt{-1}\theta_1^{\Sigma_{g_i}}}$, $r_i:\pi_1\left(\Sigma_{g_i}\right)\to\pi_1(\mathrm{H})$ denotes the homomorphism obtained by the embedding $\Sigma_{g_i}\hookrightarrow\mathrm{H}$, $\beta_0=\dim H_0\left(\mathrm{M};\mathcal{G}_{Ad_\varrho}\right)$ and $\mathbf{h}_{1,1}^{0,\Sigma_{g_i}}$ is a basis of $H_1\left(\Sigma_{g_i};\mathcal{G}_{Ad_{\varrho\circ r_i}}\right)\oplus H_1\left(\Sigma_{g_i};\mathcal{G}_{Ad_{\varrho\circ r_i}}\right)$ so that $\mathbb{T}\left(C_*\left(\Sigma_{g_i};\mathcal{G}_{Ad_{\varrho\circ r_i}}\right)\oplus C_*\left(\Sigma_{g_i};\mathcal{G}_{Ad_{\varrho\circ r_i}}\right),\left\{\mathbf{h}_{1,1}^{0,\Sigma_{g_i}}\right\}\right)=1$, $\mathbf{h}^{j,i}$ is the Poincare dual basis of $H^j\left(\Sigma_{g_i};\mathcal{G}_{Ad_{\varrho\circ r_i}}\right)$ corresponding to the basis $\mathbf{h}_j^{\Sigma_{g_i}}$ of $H_j\left(\Sigma_{g_i};\mathcal{G}_{Ad_{\varrho\circ r_i}}\right)$.

The proof of Theorem 6 is based on considering the short-exact sequence

$$0\to\bigoplus_{i=1}^{\ell}C_*\left(\Sigma_{g_i};\mathcal{G}_{Ad_{\varrho\circ r_i}}\right)\to C_*\left(\mathrm{H};\mathcal{G}_{Ad_\varrho}\right)\oplus C_*\left(\mathrm{H};\mathcal{G}_{Ad_\varrho}\right)\to C_*\left(\mathrm{M};\mathcal{G}_{Ad_\varrho}\right)\to 0$$

and combining [28; Theorem 4.1] and [28; Theorem 4.3].

Combining these and Theorem 6, we have

Theorem 7. Considering $n=2$ and for $i=1,\ldots,\ell$, fixing a maximal geodesic lamination $\lambda_i\subset\Sigma_{g_i}$, if $\varrho_2:\pi_1\left(\mathrm{H}\right)\to\mathrm{SL}(2,\mathbb{C})$ is such that $\varrho_2\circ r_i\in\mathrm{Teich}\left(\Sigma_{g_i}\right)$, $i=1,\ldots,\ell$, applying (ii) of Theorem 5, and using the notation there, we get

$$\mathbb{T}\left(\mathrm{H},\left\{\mathbf{h}_k^{\mathrm{H}}\right\}_0^3\right)=e^{\frac{\sqrt{-1}}{4}\left(-\beta_0+\ell\pi-\sum_{i=1}^{\ell}\theta_1^{\Sigma_{g_i}}\right)}\,2^{\frac{1}{4}\sum_{i=1}^{\ell}\chi\left(\Sigma_{g_i};\mathfrak{sl}(2,\mathbb{C})\right)}\prod_{i=1}^{\ell}\sqrt[4]{\Omega_{\mathrm{T},i}}.$$

Here, $\Omega_{\mathrm{T},i}$ is the matrix of the complex Thurston symplectic form $\omega_{\mathrm{T}} : \mathcal{H}(\lambda_i;\mathbb{C}) \times \mathcal{H}(\lambda_i;\mathbb{C}) \to \mathbb{C}$ in the basis $\mathfrak{h}^i \oplus \sqrt{-1}\,\mathfrak{h}^i$, and $\mathbf{h}^{j,i}$ is the Poincare dual basis of $H^j\left(\Sigma_{g_i};\mathfrak{sl}(2,\mathbb{C})_{Ad_{\varrho_2\circ r_i}}\right)$ corresponding to $\mathbf{h}_j^{\Sigma_{g_i}}$, and $\mathfrak{h}^i$ is the basis of $\mathcal{H}(\lambda_i;\mathbb{R})$ associated with the isomorphism obtained by the real analytical embedding $\mathrm{Teich}(\Sigma_{g_i}) \hookrightarrow \mathcal{H}(\lambda_i;\mathbb{R})$ [32]. Here, $r_i : \pi_1(\Sigma_{g_i}) \to \pi_1(\mathrm{H})$ is the homomorphism obtained by the embedding $\Sigma_{g_i} \hookrightarrow \mathrm{H}$.

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Шоттки көрсетілімі мен Рейдмейстер бұралуы жайында ескерту

Мақалада Шоттки көрсетілімі үшін Рейдмейстердің бұралу формуласы анықталған. Теориялық нәтижелер 2-ден кем емес текті бағдарланған беттерден тұратын жиекті 3−көпбейнелерге қолданылады.

Кілт сөздер: Шоттки көрсетілімі, Рейдмейстер бұралуы, көрсетілімнің көпбейнелері, Атьи-Ботта-Голдман симплектикалық формасы, Терстонның симплектикалық формасы.

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Замечание о представлениях Шоттки и кручении Рейдемейстера

В статье установлена формула кручения Рейдемейстера для представлений Шоттки. Теоретические результаты применены к 3−многообразиям с краем, состоящим из ориентируемых поверхностей рода не менее 2.

Ключевые слова: представления Шоттки, кручение Рейдемейстера, многообразие представлений, симплектическая форма Атьи-Ботта-Голдмана, симплектическая форма Терстона.