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Elementary Logic for Philosophy of Science and Economic Methodology

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Elementary Logic for Philosophy of Science and Economic Methodology

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Abstract

This paper covers those parts of elementary logic that are needed in an introductory course on philosophy of science and the methodology of economics. With this aim in mind, much of the technical material from standard logic courses can be dropped. Specifically, the exposition emphasizes the semantic side, with only a few remarks on syntactical aspects. On the other hand, some philosophical and methodological remarks are in order to prepare the ground for methodological discussions. Where appropriate, exercises (including review questions and discussion questions) are provided. Solutions to selected exercises are given at the end of the paper.

1 Logic's Subject Matter

Traditionally, philosophers distinguish between deductive logic and inductive logic. Deductive logic is concerned with a specific relation between statements: the consequence relation. Inductive logic is concerned with learning from experience. In this paper, "logic" without qualifier always means "deductive logic". With the exception of a few remarks at the end of this section, this paper considers only deductive logic.

Let $A, B, C, \ldots$ denote statements. By definition, statement B is a consequence of statement A—or, equivalently, B follows from A, or A implies or entails B—if and only if, necessarily, B is be true if A is true. If B is a consequence of A, we write $A \Rightarrow B$.

Let us assume that, indeed, $A \Rightarrow B$. The condition that, necessarily, B is true if A is true is taken to mean that, no matter how the world looks like—now, in the past, and in the future—, if it is such that A is true then B is also true. The consequence relation holds independently of the facts. It is purely a matter of language; it holds because of the meaning of the statements. Since the analysis of meanings is called semantics, we can say that the consequence relation is a semantic relation.

Here is an example. Let A be the statement "There is a dog in my fridge." Let B be the statement "There is an animal in my fridge". The meaning of these statements ensures that $A \Rightarrow B$. It does not matter at all what the facts are.

It is, of course, possible that $A \Rightarrow B$ and $B \Rightarrow A$. In this case, A and B have the same meaning and are said to be equivalent, symbolically $A \Leftrightarrow B$. For example, let A be the statement "Adam is the father of Bob", and let B be the statement "Bob is the son of Adam." Then we have $A \Leftrightarrow B$.

The consequence relation is important for argumentation. An argument is said to be valid if and only if the conclusion of the argument is a consequence of its premises. Here is a valid argument:

There is a dog in my fridge.
There is an animal in my fridge.

The statement above the line is the premise, the statement below the line is the conclusion. The line could be read as "therefore".

In more complicated arguments, there are several premises. Here is a valid argument with two premises:

There is a dog in my fridge. My fridge is in my house.
There is an animal in my house.

Several premises can be joined by "and" to make one premise, which allows us to apply the definition of consequence. In this case, the conclusion obviously follows from "There is a dog in my fridge and my fridge is in my house."

A good argument should at least be valid. However, since an argument is, typically, used to argue that the conclusion is true, validity is not enough. The premises should also be true. A valid argument with true premises is called "sound". An argument supports a conclusion if and only if it is sound. This, at least, is the position of deductivism, the view that the only logic we need in science, and anywhere else, is deductive logic.

Inductivists accept deductive logic but want to extend it. There are two versions of inductivism, both relevant to economics. The first version uses probability theory as an extension of deductive logic. In philosophy, this probabilistic inductivism is known as Bayesianism; in economics, it is known as subjective expected utility theory. Even a short discussion of Bayesianism would lead too far away from the topic of this paper; however, one possible way of forging a connection between logic and probability will at least be mentioned when we discuss possible worlds.

The second version, a non-probabilistic inductivism, has prominent exponents in, among others, the methodology of experimental economics. These inductivists reject the view that good arguments should be valid. They believe that there are invalid arguments that lend some support to their conclusion if their premises are true. Such arguments are sometimes called inductively valid arguments (in contrast to deductively valid arguments). For instance, reasoning by analogy or generalization from observations to theories are said to proceed by such deductively invalid but inductively valid arguments. However, we do not use this terminology here; subsequently, "validity" always means "deductive validity".

In contrast to these inductivists, deductivists hold that an invalid argument is just an incomplete argument: it is lacking a premise whose addition would make it valid. Confronted with an invalid argument, a deductivist would either reject it or, more charitably, try to rescue it by supplying the missing premise and then consider the question of whether the resulting valid argument is sound.

Inductive logic is much more controversial than deductive logic. Since it aims at providing a foundation for learning from experience, it must be discussed from the perspective of epistemology (that is, the theory of knowledge). The present paper aims at providing the basics of deductive logic that are necessary for an introduction to epistemology.

Exercises to Section 1

1. (a) Let A be "Henry VIII. of England was a bachelor all his life." Let B be "Henry VIII. of England did never marry." Does $A \Rightarrow B$ hold?
   (b) Let A be "Socrates is human." Let B be "Socrates is mortal." Does $A \Rightarrow B$ hold?
   (c) Let A be "Melons are bigger than apples and apples are bigger than cherries." Let B be "Melons are bigger than cherries." Does $A \Rightarrow B$ hold?
2. Consider the following inductive argument.

   On day 1, the sun rises in Giessen.
   On day 2, the sun rises in Giessen.
   ———————————————
   On day 3, the sun rises in Giessen.

   Note that, for simplicity, we use the present tense whether the specified date is in the past, the present or the future.

(a) Let day 1 be January 1, 2015. Premises and conclusion of the argument are true. Explain why the argument is nevertheless invalid.
(b) Let day 2 be the day you are working on this exercise. The argument is, of course, still invalid. Supply a further premise so that the new argument is valid. Is your new argument sound?

2 Truth and Falsity

The following sections are based on some assumptions that have often been challenged in philosophy. Here, not much can be said about the literature on these topics. I restrict myself to some short remarks.

2.1 Truth as Agreement with the Facts

In our discussion of logic, we will restrict considerations to statements that are either true or false. Statements that are either true or false obey the "law of the excluded middle"—the idea that there is no third possibility beyond truth and falsity.

The notion of truth presupposed here is the commonsense notion: a statement is true if it corresponds to, or agrees with, the facts. For instance, the statement "There is a dog in my fridge" is true if and only if, indeed, there is a dog in my fridge, and false otherwise. In philosophy, the idea that truth is agreement with the facts is called the correspondence theory of truth.

There are other notions of truth. An example is the consensus theory of truth: a statement is true if and only if people agree with it. Truth in the sense of a consensus has not much to do with truth in the usual sense of the word. Instead of redefining "truth", it makes more sense to stick to the correspondence theory and call a consensus a consensus.

Although it seems reasonable enough, the correspondence theory has its problems. One source of worry are statements that refer to themselves. Consider the statement "The statement you are just reading is true" (X). Assume that X is true; since it says that it is true, it seems, indeed, to be true. Now assume that X is false; since it says that it is true, it seems to be false. It seems, then, that X can be either true or false—but there are no facts that could determine which.

Even worse is the statement "The statement you are just reading is false" (Y), which can have no truth value at all. Assume that Y is true; since it says that it is false, it seems to be false. Now assume that Y is false; since it says that it is false, it seems to be true.

Such difficulties can be avoided. In our case, we just have to assume that the statements $A, B, C, \ldots$ whose truth or falsity we consider are never statements about statements. That is, we may discuss the truth of statement A, but statement A does not talk about any statement, be it itself or a different statement.

2.2 Positive and Normative Statements

The restriction to statements that are either true or false rules out further statements. Economists usually distinguish between positive and normative statements. Statements that are true or false are positive statements. Normative statements are value judgments in the narrow sense ("Democracy is good"), general norms ("You should not kill"), recommendations, and so on. Normative statements need not express moral evaluations; for instance, "Venice is beautiful" is also a normative statement.

In contrast to positive statements, normative statements are neither true nor false. This, at least, is the position of non-cognitivism in ethics, which is also the position taken by most economists and the position taken in this paper.

Many normative statements cannot agree or disagree with the facts. Some persons may agree with a given normative statement and others might reject it, but there is no truth about the matter. However, in some case, normative statements contain implicit or explicit references to facts. Consider the normative statement "Adam ought to help Eve." One may argue that Adam ought to help Eve only if he is actually able to do so. This is often expressed in form of the maxim "Ought implies can". The normative statement "Adam ought to help Eve" might then be considered as false if Adam is in fact unable to help Eve.

However, in such cases, it seems more reasonable to say not that "Adam ought to help Eve" is false, but that "Adam ought to help Eve" has a false positive consequence while the statement itself is neither true nor false. "Adam ought to help Eve" may entail two statements: "Adam can help Eve" and "If Adam can help Eve, then Adam must help Eve". The positive statement "Adam can help Eve" is either true or false; the normative statement "If Adam can help Eve, then Adam must help Eve" is neither—it can be accepted or rejected but there is no truth about the matter.

This analysis assumes, very plausibly, that the consequence relation can also hold between normative statements and even between normative and positive statements. In the latter case, it is usually assumed that positive statements can follow from normative statements, as in the example above, but not the other way around. The claim that a normative statement follows from positive statements is usually rejected as an instance of the so-called “naturalistic fallacy”.

While these considerations are very reasonable, one should be aware that they presuppose a wider definition of consequence than the one used in this paper because the latter definition refers to truth. The logic of normative statements, deontic logic, is an important topic in ethics but it goes beyond the kind of logic considered here, which is based on the assumption that the statements we consider are either true or false.

2.3 Vagueness and Context

There are statements that are too vague for being either true or false, for instance, the statement “It rains.” In order to make sure that the law of the excluded middle holds, we must, at least, say when and where it is supposed to rain.

Quite often, much of the vagueness of a statement vanishes once one takes the context into account. If I look out of the window and say “It is raining”, time and place are implicitly given.

However, even then, some vagueness usually remains because the meaning of many words is also somewhat vague. For instance, “It is raining” suggests that more than one drop of water is falling per minute and square meter. However, it is unclear how many drops must be falling for the statement to be true.

Subsequently, I assume that the statements we consider are precise enough to obey the law of the excluded middle, even if this might be an idealization. Since we can usually make statements of scientific interest more precise when the need arises, this idealization may be harmless. Moreover, even a statement that is slightly vague, like “On January 1, 2015 at noon, it rains in Giessen” can be unambiguously true or false: if, at the relevant time, it pours everywhere in Giessen, the statement is true; and if, at the relevant time, no water at all falls anywhere in Giessen, it is false. Once a certain level of precision is reached, it is a matter of the facts whether the remaining vagueness of a statement becomes relevant.

When considering examples of statements, I will not try very hard to be precise. For the sake of shortness, I may even consider a statement like “It rains” as being either true or false.

2.4 Unknown Facts and Unknowable Facts

According to the correspondence theory of truth, it is irrelevant for the truth or falsity of statements whether we know or ever can find out whether a statement is true or false. The theory does not rule out the possibility of unknown and even unknowable facts. The statement "There exist little green people in one of our neighboring galaxies" is either true or false but we may never find out which.

There is a further, related problem. Consider the statement "The next throw of this die will be a six". If the result of throwing a die is a real chance experiment, all the facts now do not determine the outcome of the throw. The event that a six will be thrown has a certain probability but whether it will occur or not is undetermined; it is impossible in principle, and not just impossible for us, to know before the event what is going to happen. One might argue, then, that, before the event, the statement is neither true nor false.

However, since we admit the possibility of unknowable facts, the problem posed by indeterminism can be solved by a simple convention. We assume that all statements have the truth value, true or false, that they are eventually going to have once the relevant facts are determined.

2.5 Analytic Truth

There are limiting cases of truth and falsity according to the correspondence theory. Consider the statement "All bachelors are unmarried". Given the meaning of the words occurring in it, this statement is true independently of the facts. This is a limiting case of correspondence to the facts: the statement agrees with all conceivable facts. Statements that are true just because of the meaning of words occurring in them, independently of the facts, are called analytic truths (or analytically true statements).

There are also analytic falsehoods or analytically false statements like "Some bachelors are married" (colloquially for "There exists a least one married bachelor"). Analytically false statements are also called contradictions; they are false just because of the meaning of the words occurring in them. Again, this can be viewed as a limiting case of the correspondence theory: no matter what the facts are, a contradiction is never in agreement with them.

Analytically true statements and analytically false statements together form the set of analytic statements. Statements that, in contrast, are either true or false depending on the facts are called synthetic statements. For instance, whether the statement "Currently, 275.328 bachelors are living in Germany" is true depends on a fact, namely, the number of bachelors currently living in Germany.

If one negates an analytic statement—by adding "It is not the case that" as a prefix—, the result is, again, an analytic statement. For instance, the statement "It is not the case that all bachelors are unmarried" is analytic and, of course, false, while "It is not the case that some bachelors are married" is analytic and true.

Likewise, if one negates a synthetic statement, they result is again a synthetic statement. For instance, "Currently, 275.328 bachelors are living in Germany" is true if and only if "It is not the case that, currently, 275.328 bachelors are living in Germany" is false. One of these synthetic statements is true but which one depends on the facts.

Negation, then, changes the truth value of a statement but not its classification as analytic or synthetic. These categories are semantic categories since it is a question of meanings whether a statement is analytic or synthetic.

There is an intimate connection between analytic statements and the consequence relation. Again, let A be the statement "There is a dog in my fridge" and B be the statement "There is an animal in my fridge", so that $A \Rightarrow B$. Consider, then, the compound statement "If there is a dog in my fridge, there is an animal in my fridge." This statement is analytically true. On the other hand, the compound statement "There is a dog in my fridge and it is not the case that there is an animal in my fridge" is a contradiction.

Indeed, to say that A entails B is the same as saying that "If A, then B" is analytically true or—which amounts to the same thing—that "A and it is not the case that B" is a contradiction. However, in order to understand exactly why this is so, we have to analyze the conditions under which compound statements made with the help of the so-called propositional connectives like "it is not the case that", "and", and "if-then" are true. Once we have done this, we can define the consequence relation in terms of the analyticity of statements.

The distinction between analytic and synthetic statements, then, is very basic, but it has nevertheless been disputed that the distinction exists. It is certainly not a trivial distinction. As already discussed, meanings are sometimes vague, and so it can be unclear whether a statement is analytically true or synthetic.

Assuming that statements clearly fall into one of two categories, analytic or synthetic, may be an idealization that is similar to the "law" of the excluded middle. Idealizations are, of course, false assumptions. Yet, in philosophy as in science, idealizations might sometimes be harmless—to wit, if replacing them by true assumptions does not make much of a difference to the conclusions of interest. So even if some of our assumptions turn out to be false, we can still hope that, for present purposes, these assumptions are harmless idealizations.

2.6 Meaning and Sense

The analyticity of statements as well as the consequence relation between statements rests on the meaning of the words occurring in the relevant statements. This seems to imply that, unless statements contain vague terms or are too complex, competent speakers of the language in question should be able to distinguish analytic and synthetic statements and to ascertain whether an argument is valid. This, however, is not the case. The problem is the definition of meaning in logic.

The meaning of a statement is, among others, determined by the meaning of the words (or phrases) occurring in the statement. According to the usual definition, the meaning of a word is the thing or set of things it refers to, its reference. This is also called the extension of the word. For instance, the extension of the name of a person is the person. The extension of the phrase "the planets of our solar system" is {Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune}. Since Pluto is a dwarf planet and not a planet, it is not in the extension of the phrase "the planets of our solar system"; accordingly, the statement "Pluto is one of the planets of our solar system" is false.

Traditionally, the planet Venus has two further names, "morning star" and "evening star", because sometimes the planet is visible in the morning, while at other times it is visible in the evening. To an astronomically uneducated observer, it is not obvious that "morning star" and "evening star" both refer to the same object. Indeed, there was a time when nobody new that "morning star" and "evening star" refer to the same object and it was firmly believed that the two names refer to different objects. Let us consider the relevance of such a state of knowledge for the identification of analytical statements and the consequence relation.

The statements "Socrates observes the morning star" (M) and "Socrates observes the evening star" (E) are equivalent because "morning star" and "evening star" both refer to the same thing, the planet Venus. The statement "If M, then E" is analytically true and $M \Rightarrow E$, that is, E is a consequence of M (and vice versa, of course). However, when it was unknown that "morning star" and "evening star" both refer to the same thing, nobody was able to recognize the analyticity of "If M, then E" or the consequence relations between the two statements. This was not a question of language competence or vagueness of language. People just did not, and could not, know that the statements M and E have the same meaning.

This example shows that questions of meaning cannot always be resolved by analyzing conventions of naming and language. It is a convention to refer to the star sometimes seen in the morning as the "morning star", and it is also a convention to refer to the star sometimes seen in the evening as the "evening star". However, these conventions lead to the equivalence of the statements M and E above, and this cannot be discovered by analyzing language; it can only be discovered by astronomical research. There is a further source of uncertainty about analyticity and the consequence relation beyond the vagueness of words or the complexity of statements.

If we want to talk about the meaning the two names had to the people who invented these names, we use the word "sense" instead of meaning. At the time, the statements M and E already had the same meaning but a different sense. The meaning of a word is also called its extension or its reference, that is, the thing or set of things the word refers to. The sense of a word is called its intension. However, it is not easy to say what sense or intension is.

In the rest of this paper, we consider only meanings, that is, extensions. The logic we consider is also called extensional logic because it rests on meanings as extensions. Extensional logic requires that we consider the statements M and E above as equivalent. However, we do not go into the details of analyzing meanings of different classes of words and of statements. Meanings are treated in a field called semantics, which is a part of the philosophy of language (with strong relations to linguistics, that is, the science of language).

3 Propositional Logic

3.1 Three Easy-to-understand Connectives

Again, we denote statements by $A, B, C, \ldots$ (with indices, if necessary: $A_1, A_2, \ldots$). In different examples, A may stand for different statements, as before.

In this subsection, we introduce three propositional connectives: the negation ("not", symbolically $\neg$), the conjunction ("and", symbolically $\wedge$), and the disjunction ("or", symbolically $\vee$). With the help of these connectives and, if necessary, parentheses, we can build complex statements like $A \wedge \neg(B \vee C)$. We discuss such complex statements below in subsection 3.3. Here we begin with defining the meaning of the three connectives mentioned above. This is done with the help of so-called truth tables (see table 1).

p	$\neg p$
t	f
f	t

p	q	$p \wedge q$
t	t	t
t	f	f
f	t	f
f	f	f

p	q	$p \vee q$
t	t	t
t	f	t
f	t	t
f	f	f

Table 1: Truth tables for the negation ("not", $\neg$), the conjunction ("and", $\wedge$) and the disjunction ("or", $\vee$).

In order to read the truth tables, note that we use statement variables $p, q, \ldots$, not statements. A formula like $p \wedge q$ has no truth value, just as the algebraic formula $x > y$, which also has no truth value. A truth value results only when variables are replaced by constants.

This can be done in two ways. In algebra, we could replace x and y in $x > y$ by numbers, say, 4 and 5, which leads to the statement $4 > 5$ (which is false). Or we could replace the variables by constants a, b, which yields $a > b$. This latter statement is true or false depending on which numbers a and b are standing for—something that may be left open.

In propositional logic, we could replace p and q in $p \wedge q$ either by specific statements or by constants A and B, which are standing for specific statements—which ones, may also be left open. However, in order to determine truth values, we only need the truth values, t (true) or f (false), of the statements replacing p and q. For this reason, we could replace statement variables by truth values instead of statements. This is how the truth table works. If we write out the second line of the truth table of the conjunction, it would be $t \wedge f = f$.

A truth table states that, whenever the variables are replaced by statements with the truth values given in a specific line, the compound statement has the truth value given in the same line. The connectives are, actually, functions (called "truth functions"). For instance, the formula $p \wedge q$ describes a function from $\{t, f\} \times \{t, f\}$ to $\{t, f\}$, with $\wedge$ as the name of the function and p, q as the variables.

The truth tables are tables of values for these functions that tell us, line by line, the truth value of the function given the truth value of its arguments. For this reason, the propositional connectives are called truth-functional connectives. The theory of truth-functional connectives is also called propositional logic.

The meaning of the truth-functional connectives is defined by their truth tables. In the case of the negation, the meaning is the same as "it is not the case that" in ordinary English. The conjunction has the same meaning as "and" if used as a connective between two statements.

The disjunction is a special case of the ordinary "or": the "inclusive or" and not the "exclusive or" ("either-or").

There are other connectives that are not truth-functional. For instance, the connective $\Rightarrow$ is not truth-functional because the truth value of $A \Rightarrow B$ is not determined by the truth values of A and B alone. Let A stand for "There is a living dog in my fridge", B for "There is a living being in my fridge", and C for "There is milk in my fridge." We assume that the statements refer to the time as I write this. Given that bacteria are living beings, B is, unfortunately, true, as is C, while A is false. However, $A \Rightarrow B$ is true while $A \Rightarrow C$ is false, although the truth values in $A \Rightarrow B$ and $A \Rightarrow C$ are the same.

3.2 The Conditional

This section introduces a further connective, the conditional ("if-then", symbolically $\rightarrow$). Depending on the context, "conditional" refers either to the connective or to an if-then statement like $A \rightarrow B$; that is, "conditional" is also used as an abbreviation for "conditional statement".

The conditional stands out among propositional connectives because in most ordinary uses if-then statements are fraught with additional meanings that are not captured by the conditional of propositional logic. This often leads to false interpretations of this connective.

p	q	$p \rightarrow q$
t	t	t
t	f	f
f	t	t
f	f	t

Table 2: Truth table for the conditional ("if-then", $\rightarrow$).

As for all other propositional connectives, the meaning of the conditional is defined by its truth table (see table 2). The table states that a statement like $A \rightarrow B$ is false if and only if A—the antecedent or if-clause—is true and B—the consequent or then-clause—is false. This seems to coincide with many uses of "if-then" in ordinary English.

Here is an example. Let A stand for "It rains." Let B stand for "The street is wet." The statement $A \rightarrow B$ then stands for "If it rains, then the street is wet." It seems reasonable to say that this statement is only false if it rains and the street is not wet.

However, this example can be quite misleading because the statement "If it rains, then the street is wet" usually conveys more than the conditional of propositional logic can express. Note first that, actually, we should be more precise. "It rains" is too vague for being true or false, and, in this case, this matters. Let us replace it by "It rains today in our town". The then-clause should also be made more precise: "Our street is wet today." We then have "If it rains today in our town, then our street is wet today." This makes clear that the conditional we consider does not describe a connection of generic events, rain in general and the wetness of streets in general, but connects two statements that, taken on their own, are either true or false.

Still, it seems to coincide with ordinary English usage to say that the statement is false if and only if it rains today in our town and our street is not wet today. After all, the if-then statement does not say anything about the case where it does not rain today in our town. Hence, it cannot be false in this case, which implies—if we are prepared to apply the law of the excluded middle—that it must be true.

However, let us consider a different example, $C \to D$, with C standing for "I jump out of the window here and now" and D standing for "I fly away like a bird here and now," where "here and now" refers to the time and place where I write these lines. As I write this, I sit at my desk on the second floor of my house, and I can ensure you that C is false (as is D, but this does not matter). The truth table, then, implies that the conditional $C \to D$, standing for "If I jump out of the window here and now, then I fly away like a bird here and now", is true. This, most speakers of ordinary English would say, is ridiculous.

What is going on? In ordinary English (as well as in other languages like German), if-then statements often express a causal or law-like connection between the facts stated in the if-clause and the then-clause, a connection that is supposed to hold also in hypothetical cases. That is, the claim "If I jump out of the window here and now, then I fly away like a bird here and now" might be meant to imply also the counterfactual conditional "If I jumped out of the window here and now (which, actually, I do not do), then I would fly away like a bird here and now." The counterfactual conditional, however, is false since it contradicts the laws of physics. If I jumped out of the window here and now, I would drop more or less like a stone (since, here and now, there are no intervening factors that could prevent me from dropping).

However, the conditional of propositional logic cannot be interpreted as a counterfactual conditional. We can use it to connect two arbitrary statements, and the resulting compound statement is always true if the if-clause is false. The conditional does not cover counterfactual cases. Its meaning is defined by the truth table, and this meaning is much weaker than the meaning of if-then in most statements of ordinary English.

Why, then, do we use the misleading words "if-then" for the connective? Because the conditional of propositional logic captures an important part (but only a part) of the meaning of the ordinary-language if-then, and because it can be used in the definition of stronger conditionals that cover more (or maybe even all) of the meanings of the ordinary-language if-then. However, defining these stronger conditionals requires so-called modal logics, that is, logics that consider connectives that are not truth functional.

In this introduction, we consider just one connective that is not truth functional, namely, $\Rightarrow$. This connective only captures semantic relations between statements. In contrast, the conditional in "If I jumped out of the window here and now, I would drop more or less like a stone" describes a nomological relation between events, that is, a relation based on laws of nature.

3.3 Well-formed Formulas

Not all formulas make sense. Those that do are called well-formed formulas (often abbreviated wff). For instance, $\neg p \wedge q$ is a wff, while $\wedge pq\neg$ is not.

Well-formed formulas are recursively defined as follows: (1) A statement variable is a wff. (2) If α and β are wffs, then so are (a) $(\neg\alpha)$, (b) $(\alpha \wedge \beta)$, (c) $(\alpha \vee \beta)$, and (d) $(\alpha \rightarrow \beta)$.

Here is an example for the application of these rules: p, q and r are wffs according to rule 1. Therefore, $(\neg p)$ is a wff according to rule 2a. Therefore, $((\neg p) \wedge q)$ is a wff according to rule 2b. Therefore, $(((\neg p) \wedge q) \rightarrow r)$ is a wff according to rule 2d.

The outermost parentheses of a wff can be dropped. Many more parentheses become superfluous if one introduces rules of operator precedence.

Rules of operator precedence are well-known from elementary algebra. For instance, a term like $-3 + 4/2$ equals -1, because the minus sign is evaluated first and applies to the shortest well-formed term following it (which is 3), and the division must be evaluated before the addition. In order to change the sequence of evaluations, parentheses must be used, as in $-(3{+}4)/2$, which equals -3.5. Writing down something like $4/2{\cdot}2$ is incorrect because multiplication and division pull the same rank in terms of operator precedence and the value of $4/2{\cdot}2$ would depend on the sequence of evaluation: $1 = 4/(2 \cdot 2) \neq (4/2) \cdot 2 = 4$. Parentheses can, of course, be used even where they are not necessary, as in $(-3) + (4/2)$.

The rules of operator precedence in propositional logic are very similar. The relevant operators are the propositional connectives. The negation behaves like the minus sign and is evaluated first. Conjunction and disjunction behave like multiplication and division; they are evaluated next. The conditional behaves like addition and is evaluated last.

Let us consider some examples that are analogous to the algebraic examples above. We consider formulas were statement variables have been replaced by truth values so that we can evaluate the formula's truth value. A formula is well-formed if and only if its truth value is uniquely determined once we assign truth values to its variables.

A formula like $\neg f \to t \wedge t$ evaluates to t, because the negation is evaluated first and applies to the shortest well-formed formula following it (which is f), and the conjunction must be evaluated before the conditional. In order to change the sequence of evaluations, parentheses must be used, as in $\neg(f \to t) \wedge t$, which evaluates to f. A formula like $t \vee f \wedge t$ is incorrect because its truth value would depend on the sequence of evaluation: $t = t \vee (f \wedge f) \neq (t \vee f) \wedge f = f$. Parentheses can, of course, be used even where they are not necessary, as in $(\neg f) \to (t \wedge t)$.

Using the definitions of the connectives by their truth tables and the rules of operator precedence, we can set up the truth table of any well-formed formula. For beginners, it is useful to note down the truth values of parts of a complex formula under the symbol of the relevant connective (see table 3).

p	q	r	p	$\wedge$	$\neg q$	$\to$	$(r$	$\to$	$q)$
t	t	t		f	f	$\underline{t}$		t	
t	t	f		f	f	$\underline{t}$		t	
t	f	t		t	t	$\underline{f}$		f	
t	f	f		t	t	$\underline{t}$		t	
f	t	t		f	f	$\underline{t}$		t	
f	t	f		f	f	$\underline{t}$		t	
f	f	t		f	t	$\underline{t}$		f	
f	f	f		f	t	$\underline{t}$		t	

Table 3: Truth table for a complex formula with truth values of its parts listed below the relevant connective. The truth values of the complete formula are underlined.

Replacing all variables by statements turns a well-formed formula (wff) into a statement. Replacing all statements by variables turns a statement in a wff. For instance, $\neg A \wedge B$ is a statement, while $\wedge AB \neg$ is not. If we replace only some variables in a wff by statements (or some statements in a statement by variables), the result is a wff like, for instance, $\neg A \wedge q$; however, we will not need to consider such mixed cases where both variables and statements occur in a wff.

An important form of replacement is uniform replacement (or uniform substitution). It means that a specific symbol, say p, is always replaced by the same symbol, say, A. If we uniformly replace p by A and q by B in the wff $p \wedge q \to p$, we get $A \wedge B \to A$ but not $A \wedge B \to B$. We sometimes denote uniform replacement by $\mapsto$: for instance, with $p \mapsto A$ and $q \mapsto B$, we get $p \wedge q \to p \mapsto A \wedge B \to A$.

Two well-formed formulas with the same truth tables denote the same truth function. For instance, the truth function defined by $\neg(\neg p \wedge \neg q)$ is the same as the one defined by $p \vee q$: both formulas evaluate to f if and only if p and q take on the value f. Since $\neg(\neg p \wedge \neg q)$ and $p \vee q$ are the same truth function, all statements we derive from these formulas by uniform substitution of variables by statements have the same meaning. Consider, for instance, $\neg(\neg A \wedge \neg C)$ and $A \vee C$. The same facts make these statements true, namely, those facts that make A or C or both true.

It does not matter whether two truth functions are defined in terms of the same variables. Consider the truth function defined by the formula $\neg(\neg p \wedge \neg q)$. With $p \mapsto r, q \mapsto s$ we get a formula whose truth table is the same as $r \vee s$. Therefore, $\neg(\neg p \wedge \neg q)$ and $r \vee s$ define the same truth function. In general, if the truth tables of two formulas coincide after some uniform replacement of variables by variables, the two formulas define the same truth function.

A truth table completely reveals the meaning of a truth function. Arbitrarily complex truth functions are just like simple connectives: they define complicated connectives whose meaning is stated by their truth tables.

Negation and conjunction or, alternatively, negation and disjunction are sufficient for defining any complex truth function. This also means that we can define the conditional with the help of the other simple connectives (see table 4). These definitions are useful as reminders of the restricted meaning of the conditional of propositional logic: the temptation to read something into the conditional which is not there (like, for instance, a causal meaning) vanishes if one considers such an alternative definition.

p	q	$p \to q$	$\neg p \vee q$	$\neg(p \wedge \neg q)$
t	t	t	t	t
t	f	f	f	f
f	t	t	t	t
t	t	t	t	t

Table 4: Definitions of the conditional in terms of the other connectives.

3.4 Valid Formulas and Tautologies

Some truth functions are constant functions: they evaluate always to the same truth value. Table 5 lists three examples of truth functions that always evaluate to t. The formulas defining these truth functions are called valid formulas.

p	q	$p \vee \neg p$	$p \wedge q \to p$	$p \to p \vee q$
t	t	t	t	t
t	f	t	t	t
f	t	t	t	t
f	f	t	t	t

Table 5: Three valid formulas.

If we uniformly replace variables by constants in valid formulas, we get analytically true statements, for instance, $A \vee \neg A$, $A \wedge B \to A$ or $A \to A \vee B$. These statements are true just because of the meaning of the propositional connectives occurring in them. Analytic truths of this kind are called tautologies.

By negating a tautology, we get a contradiction whose falsity derives only from the meaning of the propositional connectives. We call such a contradiction "contradiction of propositional logic". Contradictions of propositional logic can of course also be found by uniform replacement in truth functions with the constant value f. The formulas defining these truth functions are called unsatisfiable. Table 6 lists three examples of unsatisfiable formulas.

p	q	$p \wedge \neg p$	$\neg(p \wedge q \to p)$	$p \vee \neg p \to q \wedge \neg q$
t	t	f	f	f
t	f	f	f	f
f	t	f	f	f
f	f	f	f	f

Table 6: Three unsatisfiable formulas.

3.5 Defining the Consequence Relation

We can now state the relation between the consequence relation and analytic statements more precisely.

The meaning of words as well as laws of nature both lead to statements that are sometimes called "necessary". Necessarily, all bachelors are unmarried, and, necessarily, people jumping out of second-floor windows drop more or less like stones unless special factors intervene. However, the first case is a case of semantic necessity, and the second case is a case of nomological necessity, that is, the necessity due to laws of nature. These two kinds of necessity are very different and should not be confused.

When we talk about analytic statements and the consequence relation, we are talking about semantic necessity. The connective $\Rightarrow$ expresses semantic necessity: $A \Rightarrow B$ means that, by semantic necessity, B is true if A is true. The connective $\to$ expresses no necessity at all. If, however, the meaning of A and B ensures that B must be true if A is true, $A \to B$ is analytically true since the only case where $A \to B$ would be false, A true and B false, cannot occur due to the meanings of A and B.

There is a connection, then, between the consequence relation and the conditional. We can use the conditional to define consequence: by definition, for any two statements p and q, $p \Rightarrow q$ if and only if $p \to q$ is analytically true.

Since $\neg(p \to q)$ is the same truth function as $p \wedge \neg q$, and since the negation of an analytically true statement is a contradiction, there is an equivalent definition of the consequence relation: by definition, for any two statements p and q, $p \Rightarrow q$ if and only if $p \wedge \neg q$ is a contradiction.

3.6 Proofs Using Truth Tables

Truth tables can be used to prove that a statement is a tautology or a contradiction of propositional logic.

Let A be the statement under consideration. The first step in such a proof is to find a formula from which A can be derived through uniform replacement of variables by statements. Once the formula is found, setting up the truth table reveals whether the statement is a tautology or a contradiction of propositional logic. The problem is that, in general, there are several formulas to be found, which may yield different results.

Here is an example. Let A stand for "If the sun shines and the tourists come, prices go up or the sun shines." A possible formula is $p \to q$, from which we can derive A by substituting "The sun shines and the tourists come" for p and "Prices go up or the sun shines" for q. This formula is neither valid nor unsatisfiable.

Another possible formula is $p \wedge q \to r \vee p$, from which we can derive A by substituting "The sun shines" for p, "The tourists come" for q and "Prices go up" for r. This formula is valid, which proves that A is a tautology.

Finding the right formula can be made more difficult by the fact that a language like English has different connectives that, for the purposes of propositional logic, translate into the same connective. For instance, the statement "The sun shines but prices do not go up" is, when it comes to determine its truth value, equivalent to "The sun shines and prices do not go up".

If, however, a statement is already given in the language of propositional logic, like $A \vee B \to B$, proofs are easy under two assumptions. The first assumption is that different constants stand for different statements. The second assumption is that the constants stand for "atomic" statements: statements that themselves do not consist of sub-statements connected by propositional connectives. For instance, "The sun shines" is atomic in this sense while "The sun shines but prices do not go up" is not. Under these assumptions, the right formula for proving the status of the statement can be found through uniform replacement of constants by variables where, additionally, different constants are replaced by different variables. Let us call this form of replacement "one-to-one replacement".

For instance, consider $A \vee B \to B$, where A and B are different atomic statements. Through one-to-one replacement of the constants by variables, we can get $p \vee q \to q$, which is neither valid nor unsatisfiable. Therefore, $A \vee B \to B$ is neither a tautology nor a contradiction of propositional logic.

Nevertheless, $A \vee B \to B$ can be analytically true, for instance, if $A \Leftrightarrow B$, as when A is "Adam is the father of Bob" and B is "Bob is the son of Adam". Or it could be analytically false, if A itself is analytically true ("All bachelors are unmarried") and B is a contradiction ("Some bachelors are married"). But since A and B are different atomic statements, A or B are not analytic because of the meaning of propositional connectives; therefore, even in these cases, $A \lor B \to B$ cannot be a tautology or a contradiction of propositional logic.

The same method can sometimes be used to prove that an argument is valid. An argument with premises $A_1, \ldots, A_n$ and conclusion B is valid if $A_1 \land \ldots \land A_n \to B$ is a tautology or, equivalently, $A_1 \land \ldots \land A_n \land \neg B$ is a contradiction of propositional logic.

Arguments can, of course, be valid even if this cannot be shown by means of propositional logic. Here is an example.

(A)	All bachelors are unhappy.
(B)	John is a bachelor.
(C)	John is unhappy.

This argument is valid: $A \land B \to C$ is analytically true and, equivalently, $A \land B \land \neg C$ is a contradiction. However, there is no way to prove this by means of propositional logic. The validity of this argument depends, among others, on the meaning of the logical word "all", which is not a propositional connective but a so-called "quantifier". In order to construct proofs involving quantifiers like "all" or "some", we would have to consider predicate logic, an extension of propositional logic that is beyond the scope of the present paper.

3.7 Rules of Inference

Instead of proving the validity of arguments with the help of truth tables, one can use rules of inference, that is, blueprints for valid arguments.

Here is an example of a valid rule of inference called *and-elimination:*

$$\frac{p \land q}{p}$$

We have just taken the if-clause of the valid conditional formula $p \land q \to p$ as the premise and the then-clause as the conclusion. Uniform replacement of variables by statements in such a valid rule of inference yields a valid argument:

$$\frac{A \land B}{A}$$

This argument is valid because $A \land B \to A$ is a tautological conditional, which means that A must be true if $A \land B$ is true.

By introducing rules of inference for propositional logic, it is possible to devise a propositional calculus, that is, a set of rules of inference that allows us to draw conclusions without explicitly considering the definitions of the connectives, that is, truth tables.

Here is an example. In addition to and-elimination, we introduce *modus ponens,* another valid rule of inference:

$$\frac{\begin{array}{l} p \to r \\ p \end{array}}{r}$$

Remember that two premises can be read as a conjunction. The valid formula behind the above rule of inference is $p \wedge (p \to r) \to r$.

Consider the two statements $A \wedge B$ and $B \to C$. And-elimination allows us to infer B from $A \wedge B$. *Modus ponens* allows us to infer C from B and $B \to C$. The two rules, then, allow us to infer the conclusion C from the two premises $A \wedge B$ and $B \to C$. In other words, we have proved that the following argument is valid:

$$\frac{\begin{array}{l} A \wedge B \\ B \to C \end{array}}{C}$$

Every valid conditional formula yields a valid rule of inference. We could have proved the validity of the last argument by applying another valid rule of inference:

$$\frac{\begin{array}{l} p \wedge q \\ q \to r \end{array}}{r}$$

In principle, there exist infinitely many valid rules of inference. However, only a few simple valid rules of inference are needed for constructing proofs in propositional logic: anything that can be achieved with complicated rules of inference can also be achieved by combining a few simple valid rules of inference.

Here is a further important valid rule of inference, traditionally called *modus tollens:*

$$\frac{\begin{array}{l} p \to q \\ \neg q \end{array}}{\neg p}$$

There are also invalid rules of inference. For instance, the following rule of inference, usually called "affirming the consequent", describes a frequently occurring error in reasoning:

$$\frac{\begin{array}{l} p \to q \\ q \end{array}}{p}$$

This rule is invalid because the formula $(p \to q) \wedge q \to p$ is not valid. Uniform replacement of variables by statements in an invalid rule yields invalid arguments except in special cases.

Exercises to Section 3

1. Check the following formulas. If they are well-formed, supply the relevant truth table and decide whether the formula is valid, unsatisfiable, or neither. In some cases, thinking about the truth table may help you to decide whether a formula is well-formed.
   (a) $\neg(\neg p \vee \neg p)$
   (b) $\neg(\neg p \vee)\neg p$
   (c) $\neg\neg p \vee \neg p$
   (d) $p \vee \neg q \to \neg p \vee q$
   (e) $p \to \neg q \to \neg p \vee q$
   (f) $(p \to \neg q) \to \neg p \vee q$
   (g) $(p \to \neg r) \to (\neg p \vee q) \wedge r$
   (h) $(p \to q) \vee (r \to q) \to r \wedge \neg q$
   (i) $(p \to q) \vee (r \to q) \vee \neg(p \wedge r)$
2. Define the "exclusive or" ("either-or", symbolically $\underline{\vee}$) with the help of a truth table.
3. Define the connective "if and only if" (called "biconditional", symbolically $\leftrightarrow$) with the help of a truth table.
4. How many different truth functions $\{t, f\} \times \{t, f\} \to \{t, f\}$ exist?
5. Can you state a general procedure for finding a wff that generates a given truth table and uses only the connectives $\wedge$, $\vee$ and $\neg$?
6. For each of the following wffs, find a wff that describes the same truth function but uses only the connectives $\wedge$ and $\neg$.
   (a) $p \vee q$
   (b) $p \underline{\vee} q$
   (c) $(p \to q) \to r$
   (d) $(p \to q) \vee \neg(\neg q \to r)$
7. Can you state a general procedure for finding a wff that generates a given truth table and uses only the connectives $\wedge$ and $\neg$?
8. For each of the following wffs, find a wff that describes the same truth function but uses only the connectives $\vee$ and $\neg$.
   (a) $p \leftrightarrow q$
   (b) $(p \to q) \to r$
   (c) $p \vee q \to p \wedge r$
9. Can you state a general procedure for finding a wff that generates a given truth table and uses only the connectives $\vee$ and $\neg$?
10. Assume that A, B, C are different atomic statements. Which of the following statements, then, is a tautology or a contradiction of propositional logic (or neither)?

(a) $\neg(\neg A \lor A)$
(b) $\neg(\neg A \land A)$
(c) $A \lor \neg B \to \neg B \lor A$
(d) $(A \to B) \land (B \to C) \to (A \to C)$
(e) $(A \to B) \land \neg B \to \neg A$
(f) $(A \to B) \land A \to B$
(g) $(A \to B) \land B \to A$
(h) $(A \to B) \land A \to \neg B$
(i) $\neg((A \to B) \land A \to B)$
(j) $(A \land B \to C) \leftrightarrow (A \to \neg B \lor C)$

11. Find two special cases where "affirming the consequent" turns, through substitution of variables by constants, into a valid argument.

4 Possible Worlds

4.1 Possible Worlds for a Set of Statements

For many purposes, it is useful to illustrate relations between statements with set-theoretic diagrams. This can be done with the help of the idea of possible worlds.

Let us consider a set of three statements: "It rains" (A), "It is cold" (B), and "It is hot and humid" (C). We assume that these statements all refer to the time and place where I am writing these lines and are either true or false. I know their truth values but you do not. However, you can certainly speculate about the different possibilities of these statements being true or false. These possibilities can be listed in a truth table (see table 7).

A	B	C	possible world
t	t	t	–
t	t	f	ω_1
t	f	t	ω_2
t	f	f	ω_3
f	t	t	–
f	t	f	ω_4
f	f	t	ω_5
f	f	f	ω_6

Table 7: Possible worlds based on three statements A, B, C, with B and C contradicting each other.

The truth table 7 is different from the truth tables we had before: we consider statements A, B, C, not variables p, q, r. For variables, all combinations of truth values are possible; for statements, this need not be the case. In our case, it is impossible that B and C are both true; it cannot be hot and cold at the same time and place (at least if, as it is assumed here, "hot" and "cold" refer to non-overlapping intervals on a temperature scale). Or in other words: the assumption that B and C are both true is a contradiction.

A possible world is a non-contradictory assignment of truth values to a set of statements. In our example, we find just six possible worlds, which form the set $\Omega = \{\omega_1, \ldots, \omega_6\}$ of all possible worlds based on our three statements. Since we assume that the statements refer to the time and place where I am writing these lines, I can tell you their truth values: B is true while A and C are false, meaning that ω_4 is the correct assignment of truth values. It is customary to say, then, that ω_4 is the actual world. All the other worlds are fictions; they are unrealized possibilities.

Given that each of the three statements is either true or false, and given that we consider all non-contradictory assignments of truth values, the actual world must always be included in the set of possible worlds. It does not matter that the set of statements we consider is clearly insufficient to describe all features of reality.

We must distinguish clearly between A being true in a possible world, say, ω_1, and A being true, full stop. In this example, A is true in possible worlds ω_1, ω_2 and ω_3, but since the actual world is ω_4, A is false.

We can relate the three statements A, B, C to subsets of the set Ω of possible worlds. Let Ω_p for $p = A, B, C$ be the subset of possible worlds in which p is true. We have $\Omega_A = \{\omega_1, \omega_2, \omega_3\}$, $\Omega_B = \{\omega_1, \omega_4\}$, and $\Omega_C = \{\omega_2, \omega_5\}$. Statement p is true if and only if the actual world is an element of Ω_p.

We can, of course, consider richer sets of statements. First of all, let us consider complex statements build from A, B, C with the help of the truth-functional connectives. These complex statements add no further possible worlds since their truth values are already determined by the truth values of A, B, C. However, complex statements may correspond to new subsets of Ω. For instance, table 7 shows that $\Omega_{A \vee B} = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $\Omega_{B \wedge C} = \emptyset$ and $\Omega_{A \vee B \vee C} = \Omega$.

Adding statements whose truth values are not already determined by the truth values of previous statements yields more possible worlds. Sets of statements are, of course, always countable because statements are of finite length and composed from a finite repertoire of signs (in written language) or phonemes (in spoken language). However, sets of statements that are rich enough for scientific purposes generate uncountably many possible worlds.

Here is an example. We consider the statements "It rains here on day s" for $s = 1, 2, \ldots$, where day 1 is January 1, 2015, day 2 is the next day, and so on. Each of these infinitely many statements can be true or false; any combination is possible. This set of statements is, of course, countable, but the set of possible worlds we get from them is uncountable. This can be shown by a diagonalization argument similar to the argument demonstrating the uncountability of the real numbers (see also exercise 3 below).

4.2 The Possible-worlds Diagram

We consider a countably infinite set of statements that are each either true or false as a basis for an uncountable set Ω of possible worlds. We depict Ω as the set of all points of a square (see table 1): each point of the square is a possible world.

Each statement p from our set of statements corresponds to the set Ω_p, $p = A, B, C, \ldots$ of possible worlds where p is true.

We introduce some further notation. The complement of Ω_A in Ω is denoted $\overline{\Omega_A}$. It follows that $\Omega_A \cup \overline{\Omega_A} = \Omega$, where $\cup$ denotes union of sets. The set-theoretic difference $\Omega_A \backslash \Omega_B$ denotes the set of all possible worlds in Ω_A that are not in Ω_B. It follows that $\Omega_A \backslash \Omega_B = \Omega_A \cap \overline{\Omega_B}$, where $\cap$ denotes intersection of sets. If Ω_A is a proper subset of Ω_B, we write $\Omega_A \subset \Omega_B$; if the possibility $\Omega_A = \Omega_B$ is not excluded, we write $\Omega_A \subseteq \Omega_B$.

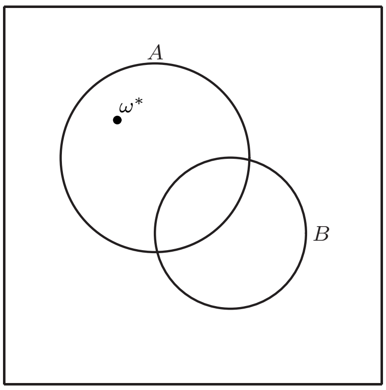


Figure 1: The set Ω of possible worlds, the sets Ω_A and Ω_B corresponding to statements A and B, and the actual world (point ω^*). Here, A is true and B is false.

Propositional connectives correspond to set operations (see figure

2). Let ω^* denote the actual world.

Negation: The statement $\neg A$ is true if and only if A is false, that is, if and only if $\omega^* \notin \Omega_A$ and, therefore, $\omega^* \in \overline{\Omega_A}$. Therefore, we have $\Omega_{\neg A} = \overline{\Omega_A}$.

Conjunction: The statement $A \wedge B$ is true if and only if A and B are true, that is, if and only if $\omega^* \in \Omega_A \cap \Omega_B$. Therefore, we have $\Omega_{A \wedge B} = \Omega_A \cap \Omega_B$.

Disjunction: The statement $A \vee B$ is true if and only if either A or B or both are true, that is, if and only if $\omega^* \in \Omega_A \cup \Omega_B$. Therefore, we have $\Omega_{A \vee B} = \Omega_A \cup \Omega_B$.

Conditional: The statement $A \rightarrow B$ is equivalent to $\neg(A \wedge \neg B)$ (see table 4). Statement $A \wedge \neg B$ is true if and only if $\omega^* \in \Omega_A \backslash \Omega_B$ (that is, the set of possible worlds that are in Ω_A but not in Ω_B). Therefore, we have $\Omega_{A \rightarrow B} = \overline{\Omega_A \backslash \Omega_B}$.

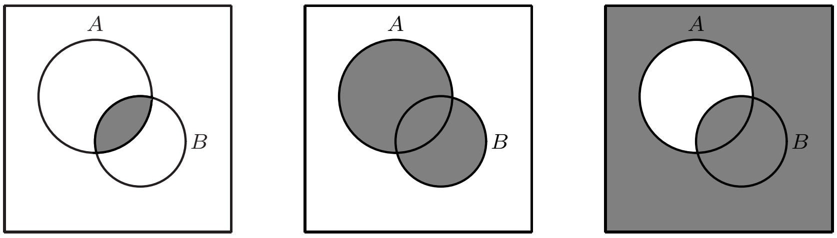


Figure 2: Set-theoretic illustration of $A \wedge B$, $A \vee B$, and $A \rightarrow B$ (from left to right). Shaded areas show the sets of possible worlds where the complex statements are true.

Analytically true statements are true in all possible worlds and correspond, therefore, to the complete square. Consider, for instance, the tautology $A \vee \neg A$: we have $\Omega_{A \vee \neg A} = \Omega_A \cup \overline{\Omega_A} = \Omega$.

Analytically false statements (contradictions) are false in all possible worlds. Consider, for instance, the contradiction $A \wedge \neg A$: we have $\Omega_{A \wedge \neg A} = \Omega_A \cap \overline{\Omega_A} = \emptyset$. Since every point of the square is a possible world, the empty set cannot be drawn.

In contrast to analytic statements, synthetic statements are true in some and false in other possible worlds. Their set-theoretic counterpart is a proper subset of Ω. However, not every proper subset of Ω corresponds to a statement since the set of statements is countable while Ω is uncountable and therefore has uncountably many proper subsets. This means that there are more proper subsets of Ω than statements.

Subsequently, we assume that our set of statements contains with any statement also its negation, and with any two statements also their conjunction. Since negation and conjunction are sufficient for defining any other connective, our set of statements contains all statements that can be constructed with the help of the propositional connectives from the basic set of statements.

The corresponding set Σ of subsets of Ω contains the empty set (for contradictions), the set Ω itself (for tautologies and other analytically true statements), with any subset of Ω its complement (because negations are included), and with any two subsets of Ω their union (because conjunctions are included). In mathematics, the pair (Ω, Σ) is called an algebra.

At this point, we could go on and define a (finitely additive) probability measure $\mathcal{P}$ on (Ω, Σ) and assign each statement a probability P which is equal to the probability $\mathcal{P}$ of the corresponding subset: $P(p) = \mathcal{P}(\Omega_p)$ for all statements p in our set of statements. Such a definition yields a connection between logic and probability theory, which is a starting point for developing Bayesianism. However, in this paper, I cannot pursue this line of thought any further.

4.3 Possible Worlds and the Consequence Relation

We consider the possible-worlds diagram for the case that A entails B. $A \Rightarrow B$ means that B must be true if A is true. This requires that the actual world must be in Ω_B if it is in Ω_A, that is, $\Omega_A \subseteq \Omega_B$ (see figure 3). If it is not the case that $A \Rightarrow B$ (symbolically, $A \nRightarrow B$), there must be possible worlds that are in Ω_A but not in Ω_B, that is, $\Omega_A \nsubseteq \Omega_B$.

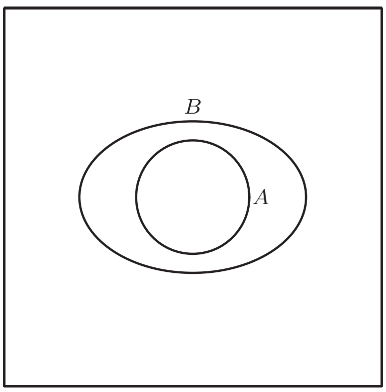


Figure 3: Possible-worlds diagram for A and B where A entails B ($A \Rightarrow B$).

The limiting case is $\Omega_A = \Omega_B$, where each set is a subset of the other so that we have $A \Leftrightarrow B$.

The familiar language of set theory makes it easy to prove several properties of the consequence relation.

1. All analytical truths are equivalent.
   Proof: Let A and B both be analytically true. Then $\Omega_A = \Omega = \Omega_B$. Therefore, $A \Leftrightarrow B$.

2. An analytical truth entails only analytical truths.
   Proof: Let A be analytically true and $A \Rightarrow B$. Then $\Omega_A = \Omega \subseteq \Omega_B$. Therefore, $\Omega_B = \Omega$, which means that B is analytically true.

3. Any analytical truth follows from any statement.
   Proof: Let B be analytically true. Then $\Omega_B = \Omega$ and, therefore, $\Omega_A \subseteq \Omega_B$ for any statement A. Therefore, $A \Rightarrow B$.

4. All contradictions are equivalent.
   Proof: Let A and B both be contradictions. Then $\Omega_A = \Omega_B = \emptyset$. Therefore, $A \Leftrightarrow B$.

5. Any statement follows from a contradiction.
   Proof: Let A be a contradiction and B be any statement. Then $\Omega_A = \emptyset \subseteq \Omega_B$. Therefore, $A \Rightarrow B$.

6. A contradiction follows only from a contradiction.
   Proof: Let B be a contradiction and assume $A \Rightarrow B$. Then $\Omega_B = \emptyset$ and $\Omega_A \subseteq \Omega_B$. Therefore, $\Omega_A = \emptyset$ and A is a contradiction, too.

7. Let $A \Rightarrow B$. Let ω^* denote the actual world.
   (a) If A is true, B must be true.
   Proof: If $\omega^* \in \Omega_A$ and $\Omega_A \subseteq \Omega_B$, then $\omega^* \in \Omega_B$.
   (b) If B is false, A must be false.
   Proof: If $\omega^* \notin \Omega_B$ and $\Omega_A \subseteq \Omega_B$, then $\omega^* \notin \Omega_A$.
   (c) B can be true even when A is false.
   Proof: If $\Omega_A \subset \Omega_B$, then $\omega^* \in \Omega_B$ and $\omega^* \notin \Omega_A$ is possible.
   (d) A and B can both be false.
   Proof: If $\Omega_A \neq \Omega$ and $\Omega_B \neq \Omega$, it is possible that $\omega^* \notin \Omega_A$ and $\omega^* \notin \Omega_B$.

   These properties are illustrated in figure 4.

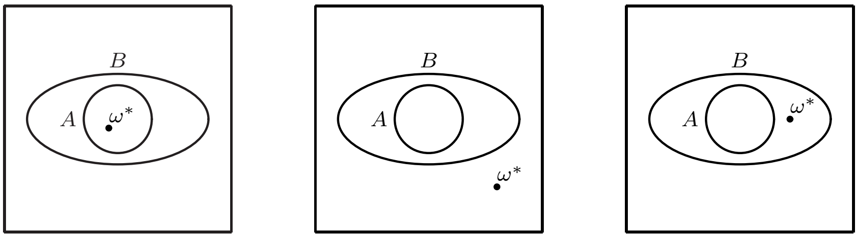


Figure 4: Let A entail B ($A \Rightarrow B$). Let point ω^* depict the actual world. The diagrams illustrate, from left to right, properties 7a-7c of the consequence relation. The middle diagram also illustrates property 7d.

4.4 Strength of Statements

In economics, we often distinguish between stronger and weaker statements or theories. Statement A is stronger than statement B, and B is weaker than A, if and only if $A \Rightarrow B$ but not the other way round, so that $A \not\Leftrightarrow B$.

This definition is, at first sight, slightly counterintuitive. After all, if A is stronger than B, Ω_A must be a proper subset of Ω_B and, therefore, a smaller set than Ω_B. Intuitively, one tends to identify "larger" with "stronger". However, look at the relation between A and B in the following way. The statement B claims that the actual world is in the subset Ω_B of all possible worlds. The statement A makes a stronger claim: not only is the actual world in Ω_B, it is in the smaller set Ω_A.

A statement can be strengthened by conjunction with another statement and weakened by disjunction with another statement. For instance, $A \wedge C$ is stronger than A unless $A \not\Rightarrow C$ (which implies $A \wedge C \Leftrightarrow A$). And $A \vee C$ is weaker than A unless $C \not\Rightarrow A$ (which implies $A \vee C \Leftrightarrow A$)

In the discussion of scientific hypotheses, it is often important to recognize which of two similar hypotheses is stronger. Scientific hypotheses are usually synthetic conditionals (or rather universal synthetic conditionals, but the difference does not matter here). Let us, therefore, consider a synthetic conditional $A \to B$ (that is, $A \not\Rightarrow B$). If $A \wedge C$ is stronger than A, $A \to B$ is stronger than $A \wedge C \to B$: strengthening the if-clause weakens a synthetic conditional. If $B \vee C$ is weaker than B, $A \to B$ is stronger than $A \to B \vee C$: weakening the then-clause strengthens a synthetic conditional.

Here is an example for these relations. Consider three statements: "The tourists arrive" (A), "The sun shines" (B) and "Prices go up" (C). "If the tourists arrive, prices go up" ($A \to C$) is a stronger claim than "If the tourists arrive and the sun shines, prices go up" ($A \wedge B \to C$) since $A \wedge B$ is stronger than A. And since $\neg B \vee C$ is weaker than C, "If the tourists arrive, prices go up or the sun doesn't shine" ($A \to \neg B \vee C$) is weaker than "If the tourists arrive, prices go up" ($A \to C$). Indeed, we have $A \wedge B \to C \Leftrightarrow A \to \neg B \vee C$; the same weakening of a conditional can be achieved by strengthening the if-clause or weakening the then-clause.

Let A be stronger than B. Then every statement that follows from B also follows from A but not necessarily the other way round. This can be seen from the possible-worlds diagram: any superset of Ω_B is also a superset of Ω_A but there are supersets of Ω_A that are not supersets of Ω_B. Of course, not every set of possible worlds corresponds to a statement. It is easy, however, to find statements that follow from A but not from B. For instance, trivially, $A \Rightarrow B$ but $B \nRightarrow A$.

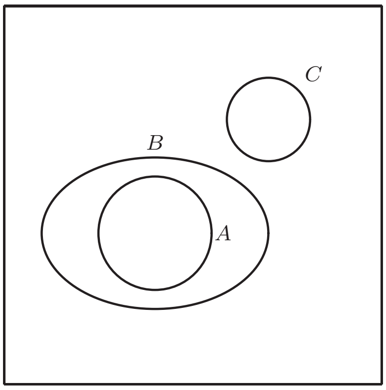


Figure 5: A is stronger than B but A and C cannot be compared with respect to strength.

The set of consequences of a statement is often called its content. Stronger statements, then, have higher content. It is not possible, however, to rank all statements in terms of strength or content. This is obvious from the possible-worlds diagram (see figure 5).

Statement A is stronger than statement B if and only if A rules out more possible worlds than B and, therefore, has a higher content. In other words: a statement is the stronger, the more is ruled out or "forbidden" by it. Analytically true statements are the weakest statements: all other statements are stronger. The content of analytically true statements is just the set of analytically true statements, which are all equivalent. Since analytically true statements rule out no possible world at all, they are sometimes called "empty" (or "empty truths").

Contradictions are the strongest statements: all other statements are weaker. The content of contradictions is the set of all statements; they have maximal content and rule out all possible worlds.

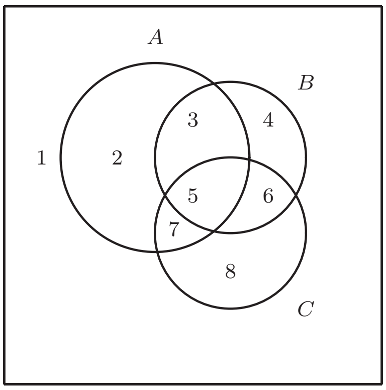
The conclusion of a valid argument is as strong as the conjunction of its premises at best; in most cases, the conclusion is weaker. The conclusion of a valid argument is already contained in the content of (the conjunction of) its premises. The conclusion of an invalid argument, in contrast, leads to a statement that is not already in the content of its premises. Such arguments are called "content increasing" or "ampliative" because the conclusion adds content to the premises.

When speaking about strength and content, it seems natural to use the term "information". Stronger statements are more informative, and the content of a statement might be considered as the informational content of the statement. If one uses "information" in this sense, one can say that valid arguments, in contrast to invalid arguments, do not lead to new information.

However, "information" is also used in the sense of "knowledge". If the word is used in this sense, valid arguments can certainly yield new information: before seeing the argument and recognizing its validity, one may not know that the conclusion follows from the premises.

Exercises to Section 4

1. Consider the following box illustrating the set of possible worlds. Subsets correspond to three statements A, B, C. Numbers refer to the largest surrounding area not intersected by lines. Thus, statement A corresponds to the union of the areas 1, 2, 3 or, shortly, $A \widehat{=} 2, 3, 5, 7$. With the same notation, $B \widehat{=} 3, 4, 5, 6$ and $C \widehat{=} 5, 6, 7, 8$.



For each of the following statements, indicate the area in the diagram where the statement is true.

(a) $A \wedge B \wedge \neg C$

(b) $\neg A \wedge \neg B \wedge \neg C$

(c) $\neg A \vee \neg B \vee \neg C$

(d) $A \wedge B \to C$

(e) $(A \to B) \to C$

2. Draw possible-world diagrams in accordance with the following assumptions.

   (a) A, B and $A \vee B$ are synthetic statements. $A \wedge B$ is a contradiction.

   (b) The set of the three synthetic statements A, B, C is inconsistent (that is, $A \wedge B \wedge C$ is a contradiction) but any pair of them is consistent.

   (c) The three statements A, B, C are logically independent in the following sense: each statement can be true or false independently of the truth value of the other statements.

   (d) A, B, C are synthetic. $A \to B$ and $\neg A \to C$ are both analytically true statements. $B \to C$ is a synthetic statement.

   (e) A, B, C are synthetic. $\neg A \vee B$ and $\neg(A \wedge \neg C)$ are both analytically true statements. $B \to C$ is a synthetic proposition.

3. Consider the countably infinite set of statements "It rains here on day s" (A_s) for $s = 1, 2, \ldots$, where day 1 is January 1, 2015, day 2 is the next day, and so on. Show that this set gives rise to an uncountable set of possible worlds. **Hint:** Use Cantor's diagonal(ization) argument, which you may have seen in a proof that the set of real numbers is uncountable.

4. Let A and B be synthetic statements with A stronger than B. Find a statement C that is weaker than A with $B \not\Rightarrow C$.

5. Consider the following invalid argument.

(A)	On January 1, 2015, the sun shines.
(B)	On January 2, 2015, the sun shines.

   Supply the weakest premise C such that the resulting new argument is valid. Show with the help of the possible-worlds diagram that, indeed, C cannot be weakened further.

6. Consider a probability measure $\mathcal{P}$ on (Ω, Σ) and assign each statement a probability P which is equal to the probability $\mathcal{P}$ of the corresponding subset: $P(p) = \mathcal{P}(\Omega_p)$ for all statements p.

   (a) Use the familiar properties of probabilities to show that analytical truths have probability 1, contradictions have probability 0, and $P(A) = 1 - P(\neg A)$.

   (b) Define the conditional probability for statements and show that $A \Rightarrow B$ implies $P(B|A) = 1$ if $P(A) > 0$.

   (c) Let $A \wedge B$ be a contradiction. Let $P(A) = 0.2$ and $P(B) = 0.1$. Compute $P(A \vee B)$.

   (d) Let $P(A) = 0.2$, $P(B) = 0.1$ and $P(A \wedge B) = 0.05$. Compute $P(A \vee B)$.

   (e) One may wonder whether the probability of a conditional is the conditional probability of the then-clause given the if-clause. In order to show that this is usually not true, consider the following probabilities: $P(A \wedge \neg B) = 0.1$, $P(A \wedge B) = 0.2$, $P(\neg A \wedge B) = 0.3$, $P(\neg A \wedge \neg B) = 0.4$. Show that $P(B|A) \neq P(A \rightarrow B)$.

5 Semantic and Logical Consequence

5.1 Logical Words and Logical Truths

The propositional connectives belong to the so-called “logical words”. These are words (or phrases) that refer neither to things nor to properties or relations. There are certainly more logical words than just the propositional connectives. Consider statements like “All humans are rational” or “There exists a rational human”. The logical words in these statements are “all”, “are” and “there exists”.

There is no complete agreement, however, about the list of logical words, and there is no exact definition. Nowadays, although some logicians and philosophers object, words like “possible” or “necessary” are often included.

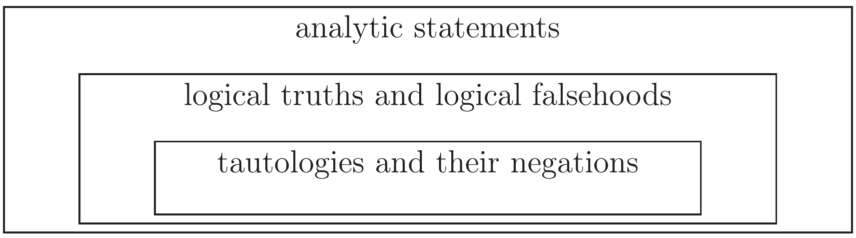


Figure 6: Subsets of analytic statements.

Analytic statements whose truth value is determined by the meaning of logical words alone are called “logical truths” or “logical falsehoods”, respectively. Logical falsehoods are also called “logical contradictions”.

The set of logical truths and logical falsehoods contains the tautologies and their negations, the contradictions of propositional logic, whose truth value depends only on the logical connectives. Further logical words give rise to further logical truths and falsehoods, for instance, “All bachelors are bachelors”, which, in contrast to “All bachelors are unmarried”, is true because of the meaning of the logical words “all” and “are” while the meaning of “bachelor” is irrelevant. The set of logical truths and logical falsehoods, then, is a superset of the tautologies and their negations (see figure 6). However, the boundary between analytic statements and logical truths and falsehoods is disputed. According to one’s list of logical words, a statement like “What is necessary is also possible" is either an analytical or a logical truth.

An analytic statement can be transformed into a logical truth or logical falsehood by substituting synonyms of non-logical words. For instance, "All bachelors are unmarried" is an analytical truth but not a logical truth since it is true in virtue of the meaning of "bachelor". Substituting "unmarried men" for "bachelors" yields "All unmarried men are unmarried", which is a logical truth since the meaning of the non-logical words, "men" and "unmarried", is irrelevant for the statement's truth.

5.2 Logical Consequence and Logically Valid Arguments

An analogous distinction follows for the concept of consequence. We had the following definition: by definition, for any two statements p and q, $p \Rightarrow q$ if and only if $p \rightarrow q$ is analytically true (or if and only if $p \wedge \neg q$ is analytically false). This concept of consequence is usually called "semantic consequence". In contrast, logical consequence is defined more narrowly: by definition, for any two statements p and q, $p \Rightarrow q$ if and only if $p \rightarrow q$ is logically true (or if and only if $p \wedge \neg q$ is logically false).

All the other semantic notions defined so far—validity of arguments, content and strength of statements—have their "logical" counterparts: logical validity, logical content, logical strength. We can always write "semantic validity", "semantic content" and "semantic strength" if we want to stress the distinction between the logical and the semantic concepts.

It is not always necessary to distinguish between semantic and logical concepts. After all, analytic truths or falsehoods can be converted into logical truths or falsehoods, respectively. Nevertheless, as the following example shows, the distinction is important in science.

Consider the following argument.

(A) Adam prefers peaches to apples.
(B) Adam prefers apples to pears.

(C) Adam prefers peaches to pears.

If one's meaning of "to prefer" implies that preference is transitive, this argument is semantically valid; otherwise, it is invalid. For economists' purposes, this would be a highly unsatisfactory situation. Assume that Bob, a behavioral economist, claims that A and B are true while Nora, a neoclassical economist, claims that C is false. Nora argues that, therefore, either A or B or both must be false. Bob admits that C is false but, nevertheless, defends his position that A and B are true. Of course, both can be right if they attach different meanings to "preference".

In order to avoid fruitless disputes that arise just because different meanings are given to the same words ("semantic disputes"), our two economists could try to pin down meanings more exactly. For instance, Nora might add a further premise to the argument:

(T)	For all x, y, z, if Adam prefers x to y and y to z, then he prefers x to z.
(A)	Adam prefers peaches to apples.
(B)	Adam prefers apples to pears.
(C)	Adam prefers peaches to pears.

This extended argument is logically valid: its truth no longer depends on the meaning of "to prefer" (or on the meaning of "apples", "peaches", "pears" or "Adam") but only on the meaning of logical words. Of course, the meaning of the logical words is also disputed; just remember that the logical connectives are defined by the truth tables, which refer to truth and falsity and, therefore, must be interpreted using some theory of truth. But if our two economists agree with respect to the theory of truth—or, more realistically, have never heard that one can disagree about the meaning of "true" and "false"—, they can go on to argue about economics. Bob might argue that, while C is false, A and B are true and T is false. The addition of premise T might prompt him to state his own theory of preference, for instance, a stochastic theory.

It must be noted that adding the premise T does not dissolve all potential disputes about the meaning of "preference". For instance, Bob might ask how Nora came to the conclusion that C is false. Let us assume that Nora observed that Adam had to choose between a peach and a pear and chose the pear. Nora argues that this shows that C is false:

(D)	Adam chooses a pear when he could have chosen a peach.
$(\neg C)$	Adam does not prefer peaches to pears.

Nora considers this argument as valid. Bob disagrees. In order to clear up the matter, Nora could again supply two premises:

(R)	For all x, y, if Adam chooses x when he could have chosen y, then he prefers x to y.
(S)	For all x, y, if Adam prefers x to y, then he does not prefer y to x.
(D)	Adam chooses a pear when he could have chosen a peach.
$(\neg C)$	Adam does not prefer peaches to pears.

Both will now agree that this argument is logically valid. Bob might now argue that either R (preference revelation) or S (asymmetry of the preference relation) or both are false. This would allow him to reject Nora's conclusion that C is false and, if he so wishes, to retain the transitivity assumption T.

These simple examples demonstrate that progress in scientific disputes can be achieved, among others, by restating arguments such that they are logically valid: this clarifies the meaning of non-logical words and, thereby, reduces the scope for fruitless disputes caused by undetected differences in meanings attached to the same words. Instead, disputes can focus on scientific hypotheses.

However, it is impossible to eliminate in advance all potential causes of disputes that originate in differences of meanings. Scientific theories must, after all, refer to things and their properties and relations with other things. Therefore, we have to use non-logical words, and the meaning of non-logical words may not only be unclear—it can be the case that we have false theories about meanings, as the example of the morning star and the evening star in 2.6 shows.

In practice, nobody insists that scientific arguments are logically valid. In addition to the logical words, we also use mathematical words like "set" and "is an element of", which serve the same purpose as the logical words. We state our theories using the language of logic and mathematics, assuming (or hoping) that there are no relevant disagreements about the meaning of the logical and mathematical words. Again, problems caused by unclear meanings do not necessarily disappear completely. Nevertheless, we often achieve greater compactness and clarity by using the mathematical language. And we can try to avoid semantic disputes by using mathematically valid arguments. As far as we achieve this aim, disputes will be disputes about the truth or falsity of scientific hypotheses.

Exercises to Section 5

1. Criticize the following argument.

   > Contradictions are maximally strong statements. In a certain sense, they are maximally informative. For instance, the theory of general relativity as well as all economic theories follow from "Some bachelors are married." Therefore, contradictions are highly fruitful in science. It is a mistake to reject theories that contain contradictions. Indeed, one should use contradictions as a starting point in science.

2. Criticize the following argument.

   > An economic model is a set of assumptions. Economists derive theorems from these assumptions. Economic theory is just the set of theorems that have been found. The theorems are analytically true (or tautological, as economists

often say) and, therefore, do not need to be tested empirically. For instance, let $A_1, \ldots, A_n$ be the assumptions of the standard general equilibrium model with perfect competition, no externalities and so on. Let B be the statement "Any general equilibrium is efficient." The first fundamental theorem of welfare economics is $A_1 \wedge \ldots \wedge A_n \to B$. This statement is an analytical truth; there is no need to test it. Whatever empirical economists do, they are not concerned with testing economic theory.

6 Solutions to Selected Exercises

Exercises to Section 1

1. (a) Yes. It does not matter that both statements are false.
   (b) No. Humans are mortal but this is usually not considered to be part of the meaning of the word "human". Otherwise, science fiction stories featuring immortal humans would make no sense, like stories featuring married bachelors.
   (c) Yes. The meaning of comparative words like "bigger" or "higher" is usually taken to involve transitivity.
2. (a) It is not purely a matter of language that, after two sunrises in Giessen, there must necessarily be a third.
   (b) The new premise might be "In Giessen, the sun rises every day." Or it might be less general: "If the sun rises in Giessen on January 1, 2015 and on January 2, 2015, then it also rises in Giessen on January 3, 2015." Adding one of these premises makes the argument valid. If one adds the first premise, the two old premises are not needed to make the argument valid. Since the old premises are true, the extended argument is sound if the new premise is true. For all we know, the more general premise is not true since our sun will not shine forever. The less general premise is true since the conclusion is true (see 3.2 on if-then statements below).

Exercises to Section 3

1. (a) Neither valid nor unsatisfiable:

p	$\neg(\neg p \vee \neg p)$
t	t
f	f

   (b) Not a wff.
   (c) Valid formula:

p	$\neg\neg p \vee \neg p$
t	t
f	t

   (d) Neither valid nor unsatisfiable:

p	q	$p \vee \neg q \to \neg p \vee q$
t	t	t
t	f	f
f	t	t
f	f	t

(e) Neither valid nor unsatisfiable:

p	q	r	$(p \to \neg r) \quad \to \quad (\neg p \vee q) \wedge r$
t	t	t	t
t	t	f	f
t	f	t	t
t	f	f	f
f	t	t	t
f	t	f	f
f	f	t	t
f	f	f	f

2. Definition of "either-or":

p	q	$p \underline{\vee} q$
t	t	f
t	f	t
f	t	t
f	f	f

3. Definition of biconditional:

p	q	$p \leftrightarrow q$
t	t	t
t	f	f
f	t	f
f	f	t

The biconditional is the negation of the "exclusive or".

4. Sixteen, because there are two truth values and four positions where truth values can be inserted in a truth table to define a truth function:

p	q	p<connective>q
t	t	<truth value>
t	f	<truth value>
f	t	<truth value>
f	f	<truth value>

There are $2^4 = 16$ different possible ways to fill out this truth table.

5. Assume that the truth table has n statement variables $p_1, p_2, \ldots, p_n$. Assume further that there are m lines for which the truth table yields t; for all other lines, it yields f. The sought-after wff is a disjunction of m conjunctions of the form $\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n$, one for each line that evaluates to t. Each of these m conjunctions is to be constructed as follows: if, in the relevant line, p_i takes on the value t, then $\alpha_i = p_i$; if, on the other hand, p_i takes on the value f, then $\alpha_i = \neg p_i$.

   It may, however, be possible to simplify the wff found by this procedure.

6. (a) $\neg(\neg p \wedge \neg q)$

   (b) $\neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q)$

(c) $\neg(\neg(p \wedge \neg q) \wedge \neg r)$

7. Use the procedure of exercise 5 above and turn the disjunction into a (negated) conjunction.

8. (a) $\neg(p \vee q) \vee \neg(\neg p \vee \neg q)$

   (b) $\neg(\neg p \vee q) \vee r$

9. Use the procedure of exercise 5 above and turn the conjunctions into (negated) disjunctions.

10. (a) Contradiction (negation of the tautology $\neg A \vee A$).

    (b) Tautology (negation of the contradiction $\neg A \wedge A$).

    (c) The formula $p \vee \neg q \to \neg q \vee p$, which results from $A \vee \neg B \to \neg B \vee A$ through one-to-one replacement of constants by variables, is valid. Therefore, the statement is a tautology.

    (d) The formula $(p \to q) \wedge q \to p$, which results from $(A \to B) \wedge B \to A$ through one-to-one replacement of constants by variables, is neither valid not unsatisfiable. This can be shown with the help of its truth table. Therefore, the statement $(A \to B) \wedge B \to A$ is neither a tautology nor a contradiction of propositional logic.

11. $p \mapsto A, q \mapsto A$ or $p \mapsto A \vee B, q \mapsto A$

Exercises to Section 4

1. (a) $A \wedge B \wedge \neg C \mathrel{\widehat{=}} 2, 3, 4$

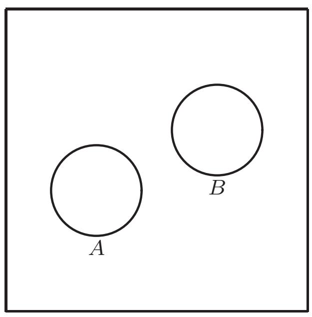
   (b) $\neg A \wedge B \wedge \neg C \mathrel{\widehat{=}} 1$

   (c) $\neg A \vee \neg B \vee \neg C \mathrel{\widehat{=}} 1, 2, 3, 4, 6, 7, 8$ (all but 5)

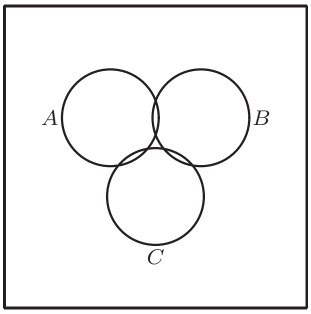
   (d) $A \wedge B \to C \mathrel{\widehat{=}} 1, 2, 4, 5, 6, 7, 8$ (all but 3)

   (e) $(A \to B) \to C \mathrel{\widehat{=}} 2, 5, 6, 7, 8$ (all but 1, 3, 4)

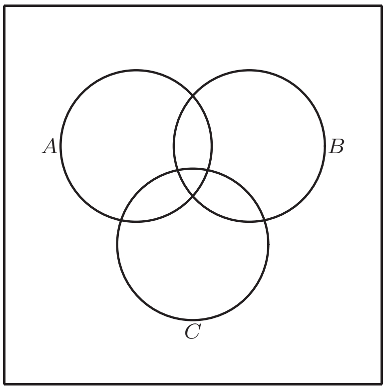
2. (a)



   (b)



(c)



(d) A, B, C are proper nonempty subsets of Ω. A is a subset of B. $\neg C$ is a subset of A.

(e) B and C intersect, but none is a subset of the other. A is a subset of the intersection.

3. A possible world ω assigns to all statements A_s, $s = 1, 2, \ldots$ a truth value. It can therefore be represented as an infinite sequence of truth values: $v_1(\omega), v_2(\omega), v_3(\omega), \ldots$ where $v_s(\omega) \in \{t, f\}$, $s = 1, 2, \ldots$ is the truth value of A_s in possible world ω. Any assignment of truth values is possible without contradiction. Assume, then, that the set of all these possible worlds is countable. This means that all possible worlds can be listed in a sequence, without leaving out any possible world: $\omega_1, \omega_2, \ldots$. Consider such a sequence:

$$\begin{array}{lcccccccc}
\omega_1: & \underline{t} & t & f & t & f & f & t & \ldots \\
\omega_2: & f & \underline{t} & t & t & f & t & t & \ldots \\
\omega_3: & t & f & \underline{f} & f & f & f & f & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

Construct a possible world θ by taking the first truth value from ω_1, the second truth value from ω_2 etc. and negating this truth value: $v_s(\theta) = \neg v_s(\omega_s)$ for $s = 1, 2, \ldots$. From the list above, we get $f, f, t, \ldots$: a possible world θ that differs from each of the listed possible worlds $\omega_1, \omega_2, \ldots$ at least with respect to the underlined truth values (which are on the diagonal of the list). Therefore, θ is not in the list. But this is a contradiction: we assumed that the list contains all possible worlds. Hence, our assumption that we can list all possible worlds is false; the set of possible worlds is not countable.

4. $C \Leftrightarrow A \vee \neg B$

5. $A \wedge C \wedge \neg B$ must be a contradiction. This holds if and only if C is false in all possible worlds where $A \wedge \neg B$ is true. The weakest statement satisfying this condition is false only in those possible worlds where $A \wedge \neg B$ is true, that is, $C \Leftrightarrow \neg(A \wedge \neg B) \Leftrightarrow A \to B$: C is the statement "If the sun shines on January 1, 2015, then the sun shines on January 2, 2015." In order to see that C is indeed the weakest premise, consider the diagram for $\Omega_C = \Omega_{A \to B} = \Omega_A \backslash \Omega_B$. All possible worlds not in Ω_C are worlds where $A \wedge \neg B$ is true. Hence, if we weaken C to C', $A \wedge C' \wedge \neg B$ is not a contradiction since it is true at those worlds that are in $\Omega_{C'}$ but not in Ω_C. Therefore, if C' is weaker than C, $A \wedge C' \nRightarrow B$.

6. (a) Let A be an analytical truth. Then $\Omega_A = \Omega$. $\mathcal{P}(\Omega) = 1$ since $\Omega \in \Sigma$ is the sure event. Therefore, $P(A) = 1$.
   Let A be a contradiction. Then $\Omega_A = \emptyset$. $\mathcal{P}(\emptyset) = 0$ since $\emptyset \in \Sigma$ is the impossible event. Therefore, $P(A) = 0$.
   We have $\Omega_{\neg A} = \overline{\Omega_A}$. We have $P(A \vee \neg A) = 1 = \mathcal{P}(\Omega_A \cup \overline{\Omega_A})$. Probability theory implies $\mathcal{P}(\Omega_A \cup \overline{\Omega_A}) = \mathcal{P}(\Omega_A) + \mathcal{P}(\Omega_A)(\overline{\Omega_A})$ since $\mathcal{P}(\Omega_A \cap \overline{\Omega_A}) = \emptyset$. Therefore, $P(A \vee \neg A) = P(A) + P(\neg A) = 1$, implying $P(A) = 1 - P(\neg A)$.

   (b) $P(p|q) := \mathcal{P}(\Omega_p \cap \Omega_q)/\mathcal{P}(\Omega_q)$ for all statements p, q with $\mathcal{P}(\Omega_q) > 0$. From this definition, we get $P(p|q) = P(p \wedge q)/P(q)$ for all statements p, q with $P(q) > 0$.
   If $A \Rightarrow B$, then $\Omega_A \subseteq \Omega_B$, implying $\Omega_A \cap \Omega_B = \Omega_A$. Therefore, $P(B|A) = \mathcal{P}(\Omega_B \cap \Omega_A)/\mathcal{P}(\Omega_A) = \mathcal{P}(\Omega_A)/\mathcal{P}(\Omega_A) = 1$ if $P(A) = \mathcal{P}(\Omega_A) > 0$.

Further Readings

Papineau (2012) is highly recommend for further reading. It is a useful and very clear introduction to several topics that are relevant for philosophy, including elementary logic and possible worlds. It also contains an introduction to set theory with some philosophical background. Those who would like to acquire more of the mathematics needed for philosophy might want to look into Steinhart (2009), an excellent book that covers some of the same ground (but not logic), in more depth and with additional topics that are of interest in connection with logic (for instance, formal semantics). Steinhart (2009: 168-170) also presents Cantor's diagonal(ization) argument.

There are many logic textbooks, each with its own strengths and weaknesses. They all contain much more material than one needs for an introductory course on philosophy of science and economic methodology. Two classics, also recommended by Papineau (2012), are Hodges (2001) and Lemmon (1978). However, each of these textbooks uses its own terminology and approach, which also differs considerably from the terminology and approach of this paper.

Below the textbook level, Priest (2000) provides a first introduction that is fun to read and emphasizes the links between logic and philosophy.

Many aspects of logic are also covered in two internet encyclopedias: *The Stanford Encyclopedia of Philosophy* (SEP) and the *The Internet Encyclopedia of Philosophy* (IEP). Although these encyclopedias contain specialized surveys rather than didactic expositions, the entry on propositional logic in the IEP (Klement n.d.) and the entry on classical logic in the SEP (Shapiro 2013) together could replace a textbook on logic. However, the expositions are rather advanced. The same goes for further entries on fundamental ideas mentioned in section 2 like the correspondence theory of truth or the analytic/synthetic distinction, just to mention two important ones. Nevertheless, it is quite instructive to see how the discussion of seemingly harmless and commonsense ideas quickly becomes difficult. This, of course, does not discredit these ideas. The alternatives might always be worse.

Musgrave (2011) provides a thorough criticism of inductivism, starting from the analysis of simple arguments and leading on to a discussion of the basic theory of knowledge. This paper should be read again after an introductory course on philosophy of science. Musgrave's (2009) volume of essays, based on talks for non-philosophers, could be read as an easy and pleasurable introduction to such a course (especially ch. 3, Is science a rational enterprise?, which nicely complements Musgrave 2011).

Conditionals are an advanced topic in logic as well as grammar (with grammatical differences between, for instance, English and German). A survey of counterfactual and subjunctive conditionals that also covers language aspects is von Fintel (2012).

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