The Enigmas of Fluctuations of the Universal Quantum Fields

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The primary ingredients of reality are the universal quantum fields, which fluctuate persistently, spontaneously, and randomly. The general perception of the scientific community is that these quansal quantum fields, which fluctuate persistently, confirms: spontaneously, and randomly. The general pertum fluctuations are due to the uncertainty principle. Here, we present cogent arguments to show that the uncertainty principle is a consequence of the quantum fluctuations, but not their cause. This poses a conspicuous enigma as to how the universal fields remain immutable with an expectation value so accurate that it leads to experimental results, which are precise to one part in a trillion. We discuss some reasonable possibilities in the absence of a satisfactory solution to this enigma.

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1 Introduction

The ultimate ingredients of reality, unveiled so far by science, are the abstract quantum fields that permeate all space of our unimaginably vast universe for all times. All of material manifestations arise from these universal quantum fields. Each elementary particle is a quantum of its underlying field, which comprises the fundamental building blocks of physical reality and are entrenched in our cherished Standard Model of particle physics.

he primary ingredients of reality are the univer- The esteemed Physics Nobel Laureate Steven Weinberg

The Standard Model provides a remarkably unified view of all types of matter and force (except for gravitation) that we encounter in our laboratories, in a set of equations that can fit on a single sheet of paper. We can be certain that the Standard Model will appear as at least an approximate feature of any better future theory. [\[1\]](#page-9-0)

Another distinguished Physics Nobel Laureate Frank Wilczek asserts: "The standard model is very successful in describing reality—the reality we find ourselves inhabiting" [\[2,](#page-9-1) p. 96]. Wilczek additionally enumerates: "The primary ingredient of physical reality, from which all else is formed, fills all space and time. Every fragment, each space-time element, has the same basic properties as every other fragment. The primary ingredient of reality is alive with quantum activity. Quantum activity has special characteristics. It is spontaneous and unpredictable" [\[2,](#page-9-1) p. 74].

These innately spontaneous and totally unpredictable activities of the quantum fields are known as *quantum fluctuations*. Thus, unlike the stable classical fields, the quantum fields are distinctly different in that they are incessantly teeming with intrinsic, spontaneous, random activity all taking place locally in all space time elements, from the infinitesimal to the infinite everywhere in this unimaginably vast universe making it looking like an extremely busy place with activities having infinite randomness.

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These deeper properties of the quantum fields arise from the need to introduce infinitely many degrees of freedom, and the possibility that all these degrees of freedom are excited as quantum mechanical fluctuations [\[3,](#page-9-2) pp. 338-339]. "Loosely speaking, energy can be borrowed to make evanescent virtual particles. Each pair passes away soon after it comes into being, but new pairs are constantly boiling up, to establish an equilibrium distribution" [\[3,](#page-9-2) p. 404].

Even though, we do not perceive its lively reality, indisputable evidence of its existence can be found everywhere in nature with the help of appropriate equipment because of the existence of the net equilibrium distribution. Some of the distinct manifestations of the ubiquitous quantum fluctuations will be presented following a brief discussion of the discovery of the universal quantum fields.

2 Discovery of the quantum fields

The existence of the quantum fields of nature came to light, totally unexpectedly, from Paul Dirac's brilliant ef-forts [\[4\]](#page-9-3) to combine Schrödinger's equation with special relativity, beginning in 1928. In addition to unveiling other important secrets of nature, Dirac's continuing, arduous work eventually pointed in 1931 to the possible existence of antiparticles like positrons with the same mass and spin, but opposite charge of the electrons. Indeed, such a particle was observed in August 1932 by Carl David Anderson [\[5\]](#page-9-4) in the cosmic ray tracks in a cloud chamber that led to his receiving the Nobel Prize in Physics for 1936 [\[6\]](#page-9-5).

This discovery ultimately led to the concept of the underlying space filling electron quantum field. When sufficient energy is provided, the underlying quantum field would simultaneously create an electron-positron pair. Such creation and annihilation of electron-positron pair was copious in the early universe. However, because of some yet to be completely understood leptogenesis process in the early universe [\[7\]](#page-9-6), a slight excess of leptons over anti-leptons were produced that left a net excess of electrons that we observe today.

Nevertheless, the existence of an underlying universal electron quantum field is established beyond any reasonable doubt because of the observed fact that all electrons have exactly the same properties no matter where and when they are produced, be in the early universe, in any laboratory on earth today or even in the transient jets ejected in astrophysical processes throughout the universe [\[8\]](#page-9-7). This phenomenon appears to be true for all the particles listed in the Standard Model [\[9\]](#page-9-8) providing convincing existence of the underlying quantum fields throughout the universe.

3 Discovery of quantum fluctuations

The idea that empty space can have an intrinsic activity is seemingly unintuitive. However, unlike the static classical fields, surprisingly the quantum fields always fluctuate. The first indication of this came from the father of the discovery of quanta, Max Planck himself. After his exposition in 1900 that the energy of a single radiating oscillator or a vibrating atomic unit comes in quanta, Planck proposed [\[10\]](#page-10-0) a new hypothesis for radiation. In a series of papers from 1911 to 1913, he reasonably established [\[11\]](#page-10-1) that the energy oscillators contained an additional term of $\frac{1}{2}\hbar\omega$, which marked the birth of the
concent of a zero point energy, as labeled by Einstein concept of a *zero-point energy*, as labeled by Einstein. In 1916, Walter Nernst proposed [\[12\]](#page-10-2) that even empty space was filled with zero-point electromagnetic radiation. Needless to say, the concept of this vacuum energy remained controversial. Even Einstein once proclaimed that the zero-point energy is dead as a doornail. However, this perception changed in 1924, when Robert Mulliken [\[13\]](#page-10-3) provided direct evidence. Using the band spectrum of the boron monoxide isotopes 10 BO and 11 BO, he showed that in contrast to the observed spectra, the transition frequencies between the ground vibrational states would disappear if there was no vacuum energy.

With the advent of the quantum mechanics, by the summer of 1926 the zero-point energy was no longer controversial, at least not in so far as it concerned material systems. Calculation [\[14,](#page-10-4) pp. 155-156] using the Schrödinger equation shows that the ground state of a quantum harmonic oscillator has the minimum energy of $\frac{1}{2}\hbar\omega$, which is attributable solely to the zero-point en-
ergy of vacuum fluctuation. Because of these vacuum ergy of vacuum fluctuation. Because of these vacuum fluctuations alone, the *standard deviations*, σ_x and σ_p , of the Fourier-conjugated Gaussian position and momentum wave functions of the ground state of a quantum harmonic oscillator cannot become zero but satisfy the relation $\sigma_x \cdot \sigma_p = \frac{1}{2}$
sufficient for quant $\frac{1}{2}\hbar$. Thus, the uncertainty relation is not sufficient for quantum fluctuations and hence they should be spontaneous.

4 Fluctuations of the radiation fields

Despite the origin of the concept of a quantum in the theory of thermal radiation, quantum mechanics in its early stages remarkably dealt only with material particles and not with radiation itself. The initial application of the new quantum theory to fields rather than particles came in 1926 in a paper by Born, Heisenberg and Jordan [\[15\]](#page-10-5). By applying the same mathematical techniques used

for material oscillator, they were able to show that the energy of each mode of oscillation of an electromagnetic field was quantized with the basic unit of $\hbar\omega$. But they dealt only with the EM radiation in empty space without interaction with any matter and as such their efforts did not lead to any significant prediction.

Paul Dirac's fundamental paper in 1927 [\[16\]](#page-10-6) radically changed the situation. In the presence of atoms or other system of charged particles, Dirac calculated the interaction energy between the field and an atom and used it as a perturbation upon the energy of the atom leading to some spectacular results. However, nature was still conceived to be composed of two very different ingredients of *particles* and *fields* that needed description in terms of quantum mechanics but in quite different ways. Again, as described earlier in Section [2,](#page-1-0) the existence of the all-pervading quantum fields of nature came to light from Dirac's brilliant efforts [\[4\]](#page-9-3) to combine Schrödinger's equation with special relativity, beginning in 1928.

In spite of all the outstanding successes of these efforts, some serious difficulty turned up and it took quite a while to resolve it. The problem essentially arises from the existence of the spontaneous vacuum fluctuations without application of any energy. These vacuum fluctuations produce copious amount of virtual particle and antiparticle pairs that significantly creates screening of the intrinsic mass and charge of a particle to provide their measured values. The self-energy of the electron, especially considering the higher-order perturbative calculations in quantum electrodynamics (QED) always turned out to be infinite as was first pointed out by J. R. Oppenheimer in 1930 [\[17\]](#page-10-7).

An infinite self-energy appears not only when the electron is moving in an orbit, but also when it is at rest in empty space. Therefore, the electron mass and charge listed in tables of physical data could not be just the *bare mass* and *bare charge*, the quantities that appear in our equations for the electron field, but should be identified with the bare mass and charge together with the infinite self-mass and self-charge, produced by the interaction of the electron with its own virtual cloud. Eliminating the infinities by a redefinition of physical parameters has come to be called *renormalization*. Although Dirac and others were totally against such a procedure, no inconsistency arises since we can never turn off the electron's virtual photon cloud because of the ubiquitous spontaneous fluctuations of the quantum fields.

5 Effects due to quantum fluctuations

5.1 Lamb shift

In the historic Shelter Island conference in 1947, perhaps the most exciting report was presented by Willis Lamb. Using the great advances in microwave technology developed during the war, Lamb and Retherford [\[18\]](#page-10-8) presented convincing evidence showing that the two supposedly degenerate energy levels ²S_¹ and ²P₁</sub> of the hydrogen atom are separated by $1057.\overline{862}$ MHz compared to the calculated value of 1057.864 MHz.

This was not predicted by the Dirac equation, according to which these states should have the same energy. The observed shift now known as the Lamb Shift was named after Willis Lamb that led to the essential validation of renormalization. Shortly after its discovery, Hans Bethe [\[19\]](#page-10-9) was the first one to derive the Lamb shift by implementing the idea of mass renormalization, which allowed him to present a somewhat preliminary calculation, without relativity, in support of the observed energy shift.

Eventually, a very erudite group of physicists including Julian Schwinger, Shin-Ichiro Tomonaga, Richard Feynman and Freeman Dyson [\[20\]](#page-10-10) developed a reliable way by incorporating infinite "counter terms" in the Hamiltonian to compensate for the infinite mass and charge. The Lamb shift currently provides a measurement of the fine-structure constant α to better than one part in a billion [\[21\]](#page-10-11), allowing a precision test of quantum electrodynamics and a robust testimonial to the existence of ubiquitous quantum fluctuations.

5.2 Anomalous electron g-factor

Another subject that drew attention at the Shelter island conference is the anomalous g-factor of the electron spin. The spin magnetic moment of the electron was derived by Dirac [\[4,](#page-9-3) [22\]](#page-10-12) in his pivotal work in 1928 to have a value of 2. In advanced QED, the anomalous magnetic moment of the electron spin arises due to the effect of quantum fluctuations as in Lamb Shift. The value was first measured by Polykarp Kusch and Henry M. Foley [\[23\]](#page-10-13) shortly before the conference. A more accurate value of $\frac{9}{2}$ = 1.00119 was presented by Foley and Kusch [\[24,](#page-10-14) [25\]](#page-10-15). Detailed discussion of the anomalous magnetic moment has been provided by Kusch in his Nobel Lecture [\[26,](#page-10-16)[27\]](#page-10-17). Julian Schwinger [\[28\]](#page-10-18) was the first to present a theoretical derivation of the anomalous g-factor caused by vacuum fluctuations to be $\frac{g}{2} = 1.001162$.
The latest measurement of the

The latest measurement of the anomalous magnetic dipole moment of the electron spin is provided by Fan et al. [\[29\]](#page-10-19) to be $\frac{g}{2} = 1.00115965218059(13)$, a precision

*Figure 1: A schematic outline for direct demonstration of the source of a physical force in the Casimir e*ff*ect arising from quantum fluctuations of the electromagnetic quantum field.*

better than one part in a trillion. This represents one of the most accurate measurements in all of physics, again providing a glorious testimonial to the existence of quantum fluctuations.

Willis Lamb and Polycarp Kusch shared the 1955 Nobel Prize in Physics for their work that crucially depended upon the existence of quantum fluctuations [\[26,](#page-10-16)[27,](#page-10-17)[30,](#page-10-20)[31\]](#page-10-21).

5.3 The Casimir effect

As predicted by Hendrik Casimir [\[32\]](#page-10-22) in 1948, there is an attractive force created between two uncharged perfectly parallel metal plates inserted in a quantum vacuum. This effect occurs because the plates would be slightly pushed towards each other as some of the quantum fluctuations of the electromagnetic quantum field, with wavelength longer than the distance between the plates, would not be sustainable in the space between the metal plates. Although simple in principle, the actual experiment turned out to be quite difficult.

With a very carefully designed experiment, the effect of the small force was conclusively demonstrated by Steve Lamoreaux [\[33\]](#page-10-23) in 1997. This is considered as a simple but direct proof of the existence of quantum fluctuations in the vacuum as illustrated in Fig. [1.](#page-3-0)

5.4 Spontaneous emission of radiation

According to the Schrödinger equation, any stationary excited state by itself should have an infinite lifetime if nothing disturbs it. With no light in the universe whatsoever, it would be hard to imagine the existence of intelligent life in it! However, Dirac's 1927 masterpiece [\[16\]](#page-10-6), revealed that the *spontaneous emission* of radiation from an excited state of an atom or a molecule appears as a

Figure 2: Chip ion trap for quantum computing at NIST. Credit: [Y. Colombe, National Institute of Standards and Tech](https://www.nist.gov/image/quantumcomputingiontrappingjpg)[nology, U.S. Department of Commerce.](https://www.nist.gov/image/quantumcomputingiontrappingjpg)

forced emission caused by the vacuum fluctuations of the electromagnetic field.

Thus, to explain spontaneous transitions, quantum mechanics must be extended, whereby the electromagnetic field is quantized at every point in space. The quantum field theory of electrons and electromagnetic fields was labeled by Dirac as *quantum electrodynamics* (QED). In QED, the spontaneously emitted photon has infinite different modes to propagate into, thus the probability of the atom re-absorbing the photon and returning to the original state is negligible, making the atomic decay practically irreversible. If one were to keep track of all the vacuum modes, the combined atom-vacuum system would undergo unitary time evolution, making the decay process reversible.

However, in cavity QED the decay rate, transition energy, and radiation pattern of spontaneous emission can all be altered by modifying the vacuum field fluctuations by a cavity wall. The coupling of atom and vacuum fields was first formulated by Jaynes and Cummings [\[34\]](#page-11-0) in 1963, in which it was predicted that spontaneous emission becomes even reversible if an atom is put in a high-Q single-mode cavity.

In recent times, the subject of cavity QED has advanced immensely. The 2012 Nobel Prize for Physics was awarded to Serge Haroche and David Wineland for their work on controlling such quantum systems [\[35](#page-11-1)[–38\]](#page-11-2) and thus attesting yet again the existence and importance of vacuum quantum fluctuations. The techniques developed to create and measure cavity QED states are now being applied to the development of quantum computers. An example is shown in Fig. [2.](#page-3-1)

*Figure 3: Schematic snapshot of screening of the bare electron charge e*⁰ *is performed by virtual electron-positron pairs, which are e*ff*ective dipoles as discussed in Ref. [\[8\]](#page-9-7).*

5.5 Vacuum polarization

In quantum field theory (QFT), the bare mass or charge of an elementary particle like electron is the ultimate upper limit of its mass or charge, which is presumed to be infinite. It differs from the measured mass or charge because the latter includes the *screening* of the particle by pairs of virtual particles that are temporarily created by the quantum fluctuation around the particle. This is depicted in Fig. [3](#page-4-0) and is known as *vacuum polarization*. At smaller distances as we begin to penetrate the polarization cloud, we come closer to the bare charge or mass. The range of the correction term is roughly the Compton wave length. Usually, the bare mass and bare charge are included in the Lagrangian while only the physical mass and charge are taken as observables. This is known as *renormalization*, which had played an essential role in the Standard Model.

5.6 Running of coupling constants

As mentioned earlier, the bare charge is *infinite* as is the bare mass. The effective charge and effective mass change with the energy scale as we penetrate the screening cloud and are determined by the local surrounding screening of charge and mass by virtual particles. While virtual particles obey conservation of energy and momentum, they can have any energy and momentum, even one that is not allowed by the relativistic energy–momentum relation for the observed mass of that particle. Such a particle is called off-shell.

In QFT, a coupling constant or gauge coupling parameter is a number that determines the strength of the force exerted in an interaction. The coupling constants are not really constant as they depend on the energy scale. One can change the energy scale and thus observe different

Figure 4: The inverse of the running coupling constants α_1^{-1} **,** α^{-1} **and** α^{-1} **, respectively for the electromagnetic, weak and** α *strong force, as a function of energy. The explicit formulas for* σ_1^{-1} and α_3^{-1} , respectively for the electromagnetic, weak and
trong force, as a function of energy. The explicit formulas for *the running coupling constants can be found in Ref. [\[39\]](#page-11-3).*

values for the coupling constant as one penetrates the surrounding shielding created by vacuum fluctuations of appropriate fields with higher energy. A special role is played in relativistic quantum theories by couplings that are dimensionless, i.e., are pure numbers. An example of a dimensionless constant is the fine-structure constant α .

The value of the coupling constant is said to *run* with energy, and the constants themselves are usually referred to as running coupling constants. The inverse of the running coupling constants of the electromagnetic, weak and strong force are plotted in Fig. [4.](#page-4-1) This is another testament of the existence of the vacuum fluctuations of three different force fields.

5.7 Intermingling of quantum fields

Each of the quantum fields whose respective particles are ensconced in the Standard Model of particle physics contains a small amount of all the other fields. This is primarily due to the fact that a quantum fluctuation, which is an irregular disturbance of a field causes similar disturbances in all the other fields in succession (Fig. [5\)](#page-5-0).

Quoting Frank Wilczek: "The electromagnetic field gets modified by its interaction with a spontaneous fluctuation in the electron field—or, in other words, by its interaction with a virtual electron-positron pair. [...] They lead to complicated, small but very specific modifications of the force you would calculate from Maxwell's equations. Those modifications have been observed, precisely, in accurate experiments" [\[2,](#page-9-1) p. 89]. Emphasizing Wilczek's critical observation again that despite the precipitous transitory characteristics of the virtual particles, there is a residual equilibrium distribution [\[3,](#page-9-2) p. 404].

Figure 5: The electromagnetic force between electrically charged particles. (a) Exchange of a virtual photon γ *between two electrons e* − *. (b) The electromagnetic field can be a*ff*ected by spontaneous activity in the electron field through transient creation of electron-positron pairs, e[−] + e⁺, as discussed in Ref. [\[2,](#page-9-1) p. 90]. In the Feynman diagrams, time flows vertically such that the bottom of the diagram is the past and the top is the future, whereas a single spatial dimension is depicted horizontally.*

So, we ascertain that particles arising out of the quantum fields are not just simple objects, and although sometimes people naively describe them as simple ripples in a single field, that is far from true. Only in a universe with no spontaneous activities—with no interactions among quantum fields at all—are particles merely ripple in a single field! In fact, we know quite explicitly what the fields are, out of which a physical particle is built, at least order by order in perturbation theory.

The irregular disturbances of the fields relate to virtual particles since their respective energy-momentum does not correspond to the physical mass of a particle. One says that these particles are off-shell. However, in the process, the total energy-momentum is exactly conserved at all times. Because of the self-interaction of the quantum fields, such an energy-momentum eigenstate of the field can be expressed as a specific Lorentz covariant superposition of field shapes of the electron field along with all the other quantum fields of the Standard Model of particle physics. It is particularly important to emphasize again that the quantum fluctuations are transitory but new ones are constantly boiling up to establish an equilibrium distribution so stable that their contribution to the screening of the bare charge provide the measured charge of the electron to be accurate up to nine decimal places [\[40\]](#page-11-4) and the electron q -factor results in a measurement accuracy of better than a part in a trillion [\[29\]](#page-10-19).

5.8 Seeding of galaxies from quantum fluctuations

The interpretation of the observed temperature anisotropies in the Cosmic Microwave Background (CMB) shown in Fig. [6](#page-6-0) is the result of density perturbations which seeded the formation of the large-scale structures of galaxies, clusters, and superclusters that we observe today. The discovery of temperature anisotropies by COBE, WMAP, and Planck satellites provides evidence that such density inhomogeneities existed in the early universe, most likely caused by quantum fluctuations in the scalar inflaton field [\[41\]](#page-11-5). Since it requires a lot more energy for the primordial photons to overcome the gravitational pull and exit the denser potential wells, these areas actually end up having less energy and are colder as shown in blue speckles than the less dense regions of higher energy shown in red.

When we look at the CMB radiation, we are looking at roughly 40000 causally disconnected regions of the universe. How is it, then, that each of these has the same temperature to one part in $10⁵$? Furthermore, if large-scale structure grew via gravitational infall from tiny inhomogeneities in the early universe, where did these primordial perturbations come from? Such perturbations are produced naturally during inflation, a period of exponential expansion in the early universe that makes it simpler and smoother.

Figure 6: Cosmic microwave background radiation observed by the Planck satellite. This is a very graphic demonstration of colossally enlarged universal quantum fluctuations. Credit: [ESA and the Planck Collaboration.](https://www.esa.int/ESA_Multimedia/Images/2013/03/Planck_CMB)

In 1981, a young post-doctoral fellow, Alan Guth at the Stanford University presented a visionary article [\[42\]](#page-11-6) that revolutionized our concept about the very early universe. He persuasively pointed out that to satisfactorily explain some of the mysterious aspects such as the homogeneity, flatness, etc., of the early universe, there had to be an extraordinarily fast expansion of the space itself by a factor of nearly 10³⁰ by a process he dubbed *inflation*. It was presumably brought about by the scalar inflaton quantum field, details of which are yet to be clearly understood.

The 2014 Kavli Prize in Astrophysics was awarded to Guth for his pioneering contributions to the theory of cosmic inflation. The predictions of the simplest versions of the theory have been so successful that most cosmologists accept that some form of inflation truly did occur in the very early universe. Assuming the veracity of cosmic inflation at the dawn of the universe, the imprint of the disturbances of the inflaton quantum field could manifestly be discernible as immensely enlarged vacuum fluctuations of the field in the cosmic microwave background radiation anisotropy, as described earlier, both by the WMAP and Planck satellite [\[43\]](#page-11-7). Thus, a very graphic demonstration of the existence of the fluctuations of a cosmic quantum field is cogently demonstrated.

In fact, quantum fluctuations are more omnipresent than we perceive. Each fundamental particle is a quantum of its respective underlying quantum field. As pointed out earlier, each of the quantum fields contains a small amount of all the other fields due to mutual interaction of the fields caused by spontaneous vacuum fluctuations. Thus, every elementary particle is a witness to the universal spontaneous quantum fluctuations. How much more ubiquitous anything can be?

6 Fluctuation of quantum fields and uncertainty principle

Despite the overabundance of evidences listed above, demonstrating that the fluctuations of the universal quantum fields are inherently spontaneous, there seems to be a pervasive view in much of the scientific community that quantum fluctuations are caused by the uncertainty principle. This could perhaps be attributed to the fact that the importance of quantum fluctuations was not properly appreciated until 1970s [\[44,](#page-11-8) p. 124] while the idea of the uncertainty principle originated during the very early days of quantum mechanics in 1927. Consequently, the uncertainty principle also gloriously known as the "famous Heisenberg uncertainty principle," with its somewhat of a volatile history, has become almost a slogan in quantum physics.

A conspicuous example can be found in a review [\[45\]](#page-11-9) by the famed physicist Victor Weisskopf where he states:

In quantum mechanics an oscillator cannot be exactly at its rest position except at the expense of an infinite momentum, according to Heisenberg's uncertainty relation. [\[45,](#page-11-9) p. 71]

The contention that an "oscillator cannot be exactly at its rest position" to avoid an "infinite momentum" is rather misleading. In Section [3,](#page-1-1) we have confirmed that the ground state of a quantum harmonic oscillator has the minimum energy of $\frac{1}{2}\hbar\omega$, which is attributable solely to
the zero-point energy of vacuum fluctuations the zero-point energy of vacuum fluctuations.

The uncertainty relation is merely a relation between the standard deviations, σ_x and σ_k , of position and wavenumber wave functions, which are Fourier transforms of each other. For a Gaussian wave packet with *standard deviations,* σ_x and σ_k , respectively computed in position and wavenumber basis, the minimum uncertainty relation is obeyed

$$
\sigma_x \cdot \sigma_k = \frac{1}{2} \tag{1}
$$

since for a Gaussian, $\sigma_x = \frac{1}{2\sigma}$
In modern quantum mach $\frac{1}{2\sigma_k}$.
.han

In modern quantum mechanical literature, it is common to define *standard deviations* using ∆ symbol and expectation values [\[46\]](#page-11-10)

$$
\Delta x \equiv \sigma_x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2}
$$
 (2)

$$
\Delta k \equiv \sigma_k = \sqrt{\langle \hat{k}^2 \rangle - \langle \hat{k} \rangle^2}
$$
 (3)

Following the original formulaion of the uncertainty principle by Heisenberg [\[47,](#page-11-11) [48\]](#page-11-12), the *standard deviations* ∆*x* and ∆*k* were interpreted by Kennard [\[49\]](#page-11-13) as "measures of indetermination" of the corresponding physical observables. The mathematical form of the minimum uncertainty relation is not affected by the interpretation of the standard deviations as physical uncertainties

$$
\Delta x \cdot \Delta k = \frac{1}{2} \tag{4}
$$

Such an uncertainty relation is valid for both classical and quantum mechanical functions. However, it becomes very significant in quantum mechanics as we introduce, for example, the quantum mechanical observation of Louis de Broglie of a matter wave with length $\lambda = h/p$, or equivalently $p = \hbar k$, leading to the quantum uncertainty principle between position \hat{x} and momentum \hat{p} observables to be

$$
\Delta x \cdot \Delta p = \frac{1}{2}\hbar \tag{5}
$$

Thus, the uncertainty relation between position and momentum follows from the simple quantum mechanical relationship $p = \hbar k$ or more meticulously from the Hamiltonian of the quantum harmonic oscillator, which governs the temporal dynamics (quantum fluctuations) of the ground state (see also the [Appendix\)](#page-9-9). It is a consequence of the spontaneous quantum fluctuations of the ground state that the position and momentum observables obey the uncertainty relation, but it is not the other way around. Contrary to the apparent general notion, the uncertainty principle by itself cannot cause anything to fluctuate. It merely gives the relationship of an effect and has nothing to do with the cause.

7 History of the uncertainty principle

The somewhat of a checkered history of the uncertainty principle started with the publication [\[47,](#page-11-11) [48\]](#page-11-12) of Werner Heisenberg in 1927.

(1) hagen with the legendary Niels Bohr. In spite of a request During this time Heisenberg was working in Copenfrom Bohr, who was on a vacation at the time, to hold off submission of the paper, Heisenberg went ahead and sent it for publication, immensely disturbing Bohr. This resulted in Heisenberg's submission of an "Addition in Proof" to the paper to correct for some valid objections of Bohr.

Using a thought experiment with a γ ray microscope, Heisenberg argued that the product in the *noise* in a position measurement and the momentum *disturbance* caused by that measurement should be no less than $\frac{1}{2}\hbar$. Masanao Ozawa [\[50\]](#page-11-14) pointed out the shortcoming of Heisenberg's derivation.

Shortly after Heisenberg's original publication, Earle Kennard [\[51\]](#page-11-15) provided the correct relationship relating the *standard deviation* of position σ_x and the standard deviation of momentum σ_p for any quantum wavefunction

$$
\sigma_x \cdot \sigma_p \ge \frac{1}{2}\hbar \tag{6}
$$

Heisenberg attempted to show that this relation is a straightforward mathematical consequence of the commutation relationship derived by Born and Jordan [\[52\]](#page-11-16)

$$
\hat{p}\hat{x} - \hat{x}\hat{p} = -i\hbar \tag{7}
$$

Based on the above equation, Heisenberg proposed an energy-time commutation relationship

$$
\hat{E}\hat{t} - \hat{t}\hat{E} = -i\hbar \tag{8}
$$

which would lead to

$$
\Delta E \cdot \Delta t \ge \frac{1}{2}\hbar \tag{9}
$$

According to Paul Busch, the time–energy relationship has been a controversial issue since its beginning [\[53\]](#page-11-17). He ascertains:

Different types of time–energy uncertainty relation can indeed be deduced in specific contexts, but there is no unique universal relation that could stand on equal footing with the position– momentum uncertainty relation. [\[53,](#page-11-17) p. 69]

In conclusion of his presentation, Busch summarizes the main types of time–energy uncertainty relations and their range of validity depending on the physical interpretation of the quantities ∆*E* and ∆*t*. In the case of vacuum fluctuations, it would be reasonable to derive an energy– time uncertainty relationship from the relativistic relation $E = hv$ since it treats space and time on an equal footing, as we would expect for a relativistic theory like QFT. To facilitate conservation of energy, we can then use the relationship $E \cdot t = h$ for quantum fluctuations, where $t = 1/\nu$.

In the absence of even a valid universal energy–time uncertainty relationship, how could we then even think that the quantum fluctuations are caused by the uncertainty principle? It can merely be a relationship between the effects and not the cause. One can only anticipate that the paradigm is changing. In fact, in a popular modern textbook [\[44\]](#page-11-8), we are starting to notice statements like the following:

"Incidentally, the vacuum in quantum field theory is a stormy sea of quantum fluctuations." [\[44,](#page-11-8) p. 20]

"The fluctuating quantum field is real." [\[44,](#page-11-8) p. 72]

"By definition, vacuum fluctuations occur even when there is no source to produce particles." [\[44,](#page-11-8) p. 124]

8 Concluding remarks

With the advent of the effective quantum field theory, we are now fortunate to be aware of the existence of universal quantum fields that fill all space all the time. These abstract fields and their respective quanta, as listed in the Standard Model of particle physics, constitute the ultimate ingredient of the universe disclosed to us so far.

The idea of the underlying vista of the quantum world germinated from Max Planck's proposal of the existence of indivisible packets of energy called quanta. Although Planck had difficulty in believing in their reality, Albert Einstein substantiated the veracity of quanta from experimental observation, which led to the development of quantum mechanics. Paul Dirac's masterful effort to combine quantum mechanics with Einstein's special relativity eventually led to the discovery of the quantum fields that are the ultimate ingredients of reality.

Within a decade of his proposal of the quanta, Planck also revealed the possible existence of a zero-point energy, causing immense controversy about their origin. With further development of quantum mechanics, its source was identified to be thoroughly random fluctuations of the quantum vacuum. With more advanced development, the fluctuations are now recognized to be most likely due to spontaneous activity of the ultimate reality of the quantum fields.

These facts pose at least two enigmas in a universal scale:

- 1. What causes the totally unpredictable fluctuations of the quantum fields without involvement of a net energy? Quantum fields appear to be nature's universal credit facility. Energy can be borrowed if it is paid back on time. The more the amount of energy is borrowed, the quicker it must be reimbursed. This spontaneous activity increases tremendously at the fundamental distance scale, which corresponds to higher energies, making the universe look like an extremely busy place with activities having infinite randomness.
- 2. Yet, despite the intense activities of the quantum fields with obvious sheer randomness, the expectation value or equivalently the average value of the energy of the universal quantum fields have remained immutable from almost the beginning of time. The fact that an elementary particle of a field like an electron has exactly the same values irrespective of when or where in the universe they are created provides conspicuous evidence. Despite all the extreme chaos of the activity, measurement of the value of the electron's anomalous q -factor gives an incredible accuracy of one part in a trillion!

The obvious question is: what could these enigmas possibly reveal? Because of the lack of any other obvious option, would it be persuasive to conclude that the incessant, spontaneous, and totally random fluctuations of the universal quantum fields as well as the observation that the fields remain ever immutable in spite of them, appear to be an intrinsic property of the universe? What could it signify? Only time will tell. Could it perhaps have something to do with Max Plank's intriguing reply in a prominent newspaper interview, "I regard consciousness as fundamental. I regard matter as derivative from consciousness" [\[54\]](#page-11-18)?

9 Appendix

The quantum harmonic oscillator has the characteristic Hamiltonian

$$
\hat{H} = \frac{1}{2} \frac{\hat{p}^2}{m} + \frac{1}{2} m \omega^2 \hat{x}^2
$$
 (10)

For time-independent potential energy function in the Hamiltonian, such as $V(x) = \frac{1}{2}m\omega^2 \hat{x}^2$, the time-
dependent Schrödinger equation can be solved by sepadependent Schrödinger equation can be solved by separation of space *x* and time *t* variables [\[55,](#page-11-19) pp. 25-29], as a result of which the spatial and temporal dependence of the ground state wave function can be factorized as

$$
\Psi_0(x,t) = \psi_0(x) \exp\left(-\iota \frac{E_0}{\hbar}t\right) \tag{11}
$$

where $E_0 = \frac{1}{2}$ $\frac{1}{2}\hbar\omega$ is the energy of the ground state. Thus,
olve only the time-independent Schrödinger we need to solve only the time-independent Schrödinger equation for the ground state position wave function

$$
\hat{H}\psi_0(x) = E_0\psi_0(x) \tag{12}
$$

Explicitly solving the time-independent Schrödinger equation for the position wavefunction $\psi_0(x)$ of the ground state with zero quanta, $n = 0$, gives [\[55,](#page-11-19) eq. (2.60)]

$$
\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \tag{13}
$$

The position probability density $|\psi_0(x)|^2$ can be rewritten
in standard Gaussian form as in standard Gaussian form as

$$
|\psi_0(x)|^2 = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{x^2}{2\sigma_x^2}\right) \tag{14}
$$

where

$$
\sigma_x = \sqrt{\frac{\hbar}{2m\omega}}\tag{15}
$$

 γ^2 γ $2m\omega$
The momentum wavefunction of the ground state is the Fourier transform of $\psi_0(x)$, namely

$$
\psi_0(p) = (\pi m \hbar \omega)^{-\frac{1}{4}} \exp\left(-\frac{p^2}{2m \hbar \omega}\right) \tag{16}
$$

The momentum probability density $|\psi_0(p)|^2$ in standard
Gaussian form is Gaussian form is

$$
|\psi_0(p)|^2 = \frac{1}{\sqrt{2\pi\sigma_p^2}} \exp\left(-\frac{p^2}{2\sigma_p^2}\right) \tag{17}
$$

where

$$
\sigma_p = \sqrt{\frac{m\hbar\omega}{2}}\tag{18}
$$

The energy of the ground state can be computed as

$$
E_0 = \langle \psi_0(x) | \hat{H} | \psi_0(x) \rangle = \frac{1}{2} \hbar \omega \tag{19}
$$

Thus, the uncertainty relation between position and momentum follows from the Hamiltonian of the quantum harmonic oscillator [\(10\)](#page-9-10), which governs the temporal dynamics (quantum fluctuations) of the ground state. It is a consequence from the quantum fluctuations of the ground state that the position and momentum observables obey the uncertainty relation, but it is not the other way around.

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