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The Weak Commutation Matrices Of Matrix With Duplicate Entries In Its Main Diagonal

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Abstract – This article discusses the permutation matrix which is a weak commutation matrix. This weak commutation matrix is determined by compiling the duplicate entry in the main diagonal, i.e., two, three, and four which are the same in different positions. The method for determining this matrix is to use the property of the transformation vec matrix to vec transpose the matrix. Based on this, we have 24 Weak Commutation Matrices For The Matrices.

Keywords – Weak Commutation Matrix; Duplicate Entries; Diagonal Entry; Vec Matrix

I. Introduction

The vec matrix is a unique operation that can change a matrix into a column vector [1]. It can also be said to change the matrix into a vector by stacking the column vertically [2]. For example, the matrix A, $m \times n$, and its transpose is matrix A^T, then the vector $vec(A)$ and $vec(A^T)$ are matrix $mn \times 1$. Note that $vec(A)$ and $vec(A^T)$ have same entries, but the composition of the elements is different. The permutation matrix that changes the $vec(A)$ to $vec(A^T)$ is called the *commutation matrix* [3, 4].

A unique commutation matrix that transforms $vec(A)$ to $vec(A^T)$ for any matrix $m \times n$. However, then by [5], it was written that for the matrixs in the Kronecker quaternion group found in [6], several matrices are like a commutation matrix. Based on this, the authors define the *weak commutation matrix*. The weak commutation matrix is a matrix that transforms $vec(A)$ to $vec(A^T)$ by paying attention to the same entries in different positions or duplicate entries.

The research methods are based on the study of literature, which is related to the transformation of the permutation matrix on the vec matrix and vec transpose matrix. The first step of this research is to determine the variety of matrix, 4×4, with duplicate entries (see Section III, matrices in (3.1), (3.2), and (3.3)). Next, it presents the weak commutation matrix of the matrices.

II. Basic Theory

This section presents definitions, properties, and theorems related to the commutation matrix.

Definition 2.1 [9] *Let $A = [a_{ij}]$ be an $m \times n$ matrix, and A_j is the column of A. The $vec(A)$ is the n column vector, i.e*

$$vec(A) = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}$$

Let S_n denote the set of all permutations of the n element set $[n] := \{1,2,\dots,n\}$. A permutation is one-to-one function from $[n]$ onto $[n]$. The permutation of finite sets is usually given by listing of each element of the domain and its corresponding functional value. For example, we define a permutation σ of the set $[n] := \{1,2,3,4,5,6,7\}$ by specifying $\sigma(1) = 7$, $\sigma(2) = 1$, $\sigma(3) = 3$, $\sigma(4) = 6$, $\sigma(5) = 2$, $\sigma(6) = 4$, $\sigma(7) = 5$ A more convenient way to express this correspondence is towrite σ in array form as

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 1 & 3 & 6 & 2 & 4 & 5 \end{bmatrix} \qquad (2.1)$$

There is another notation commonly used to specify permutation. It is called cycle notation. Cycle notation has theoretical advantages in that specific essential properties of the permutation can be readily determined when cycle notation is used. For example, permutation in (2.1) can be written as $\sigma = (1\ 7\ 5\ 2)(4\ 6)$. For detail, see [5].

Theorem 2.2 [5] *Let π and σ be a permutation in S_n, then $P(\pi)P(\sigma) = P(\pi\sigma)$.*

If σ is a permutation, we have the identity matrix as follows:

Definition 2.3 [6] *Let σ be a permutation in S_n. Define the permutation matrix $P(\sigma) = [\delta_{i,\sigma(j)}]$, $\delta_{i,\sigma(j)} = entry_{i,j}(P(\sigma))$ where*

$$\delta_{i,\sigma(j)} = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{if } i \neq \sigma(j) \end{cases}$$

Example 2.1 Let $n := \{1,2,3,4\}$ and $\sigma = (1\ 4\ 3)$.

$P(143) = [\delta_{i,\sigma(j)}]$ and $\delta_{i,\sigma(j)} = \begin{cases} 1 \text{ if } i = \sigma(j) \\ 0 \text{ if } i \neq \sigma(j) \end{cases}$

(1 to 4; 2 to 2; 3 to 1; 4 to 3 ; $\sigma(1) = 4, \sigma(2) = 2, \sigma(3) = 1, \sigma(4) = 3$)

$ent_{11}(P(\sigma)) = \delta_{1,\sigma(1)} = 0$, $(\sigma(1) = 4)$; $ent_{12}(P(\sigma)) = \delta_{1,\sigma(2)} = 0$ $(\sigma(2) = 2)$; $ent_{13}(P(\sigma)) = \delta_{1,\sigma(3)} = 1$, $(\sigma(3) = 1)$; $ent_{14}(P(\sigma)) = \delta_{1,\sigma(4)} = 0$, $(\sigma(4) = 3)$; $ent_{21}(P(\sigma)) = \delta_{2,\sigma(1)} = 1$ $(\sigma(1) = 4)$; $ent_{22}(P(\sigma)) = \delta_{2,\sigma(2)} = 0$, $(\sigma(2) = 2)$; $ent_{23}(P(\sigma)) = \delta_{2,\sigma(3)} = 0$, $(\sigma(3) = 1)$; $ent_{24}(P(\sigma)) = \delta_{2,\sigma(3)} = 0$ $(\sigma(4) = 3)$; $ent_{31}(P(\sigma)) = \delta_{3,\sigma(1)} = 0$, $(\sigma(1) = 4)$; $ent_{32}(P(\sigma)) = \delta_{3,\sigma(2)} = 0$, $(\sigma(2) = 2)$; $ent_{33}(P(\sigma)) = \delta_{3,\sigma(3)} = 0$ $(\sigma(3) = 1)$; $ent_{34}(P(\sigma)) = \delta_{3,\sigma(3)} = 0$, $(\sigma(4) = 3)$; $ent_{41}(P(\sigma)) = \delta_{4,\sigma(1)} = 0$, $(\sigma(1) = 4)$; $ent_{42}(P(\sigma)) = \delta_{4,\sigma(1)} = 0$, $(\sigma(2) = 2)$; $ent_{43}(P(\sigma)) = \delta_{4,\sigma(1)} = 0$, $(\sigma(3) = 1)$; $ent_{44}(P(\sigma)) = \delta_{4,\sigma(4)} = 0$, $(\sigma(4) = 3)$.

So we have $P(143) = \begin{bmatrix} \delta_{14} & \delta_{12} & \delta_{11} & \delta_{13} \\ \delta_{24} & \delta_{22} & \delta_{21} & \delta_{23} \\ \delta_{34} & \delta_{32} & \delta_{31} & \delta_{33} \\ \delta_{44} & \delta_{42} & \delta_{41} & \delta_{43} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

The commutation matrix is a kind of permutation matrix of order mn expressed as a block matrix where each block is of the same size and has a unique 1 in it, as defined below:

Definition 2.4 [2] *A permutation matrix P is called a commutation matrix of matrix , $m \times n$, if it satisfies the following condition:*

a. *$P = [A_{ij}]$ is an $m \times n$ block matrix with each block A_{ij} be a $n \times m$ matrix.*

b. *For each $i \in [m]$, $j \in [n]$, $A_{ij} = \left(a_{s,t}{}^{(i,j)}\right)$ is a $(0,1)$ matrix with a unique 1 which lies at the position (j, i).*

We denote this commutation matrix by $K_{m,n}$ and thus a communication matrix is of size $mn \times mn$.

Example 2.2. Matrix $K_{3,2}$ is a 6×6 permutation matrix partitioned by a 3×2 block matrix, i.e:

$$K_{2,3} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

Where $A_{ij} = \left(a_{s,t}{}^{(i,j)}\right)$ is a 3×2 matrix whose unique non-zero entry is $a_{j,i}{}^{(i,j)} = 1$. Specifically

$$K_{2,3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The definition of the commutation matrix is given differently by [7], that is

$$K_{m,n} = \begin{bmatrix} I_m \otimes \boldsymbol{e}_{1n}{}^T \\ I_m \otimes \boldsymbol{e}_{2n}{}^T \\ \vdots \\ I_m \otimes \boldsymbol{e}_{mn}{}^T \end{bmatrix}$$

where I_m is an identity matrix and $\boldsymbol{e}_{in}$ is an $n-$dimensional column vector which has 1 in the i^{th} position and 0's elsewhere; that is

$\boldsymbol{e}_{in} = [0, 0, \dots, 0, 1, 0, \dots, 0]^T$

and

$I_m \otimes \boldsymbol{e}_{in}{}^T = [a_{ij}\boldsymbol{e}_{in}{}^T]$, $a_{ij} \in I_m$.

III. Result And Discussion

This paper aims to determine the weak commutation matrix of the following matrices, i.e.,

The arbitrary matrix 4×4 with two duplicate entries in its main diagonal:

$$\begin{bmatrix} * & & & \\ & * & & \\ & & & \\ & & & \end{bmatrix}, \begin{bmatrix} * & & & \\ & & & \\ & & * & \\ & & & \end{bmatrix}, \begin{bmatrix} * & & & \\ & & & \\ & & & \\ & & & * \end{bmatrix}, \begin{bmatrix} & & & \\ & * & & \\ & & * & \\ & & & \end{bmatrix}, \begin{bmatrix} & & & \\ & * & & \\ & & & \\ & & & * \end{bmatrix}, \begin{bmatrix} & & & \\ & & & \\ & & * & \\ & & & * \end{bmatrix} \quad (3.1)$$

The arbitrary matrix 4×4 with three duplicate entries in its main diagonal:

$$\begin{bmatrix} * & & & \\ & * & & \\ & & * & \\ & & & \end{bmatrix}, \begin{bmatrix} * & & & \\ & * & & \\ & & & \\ & & & * \end{bmatrix}, \begin{bmatrix} * & & & \\ & & & \\ & & * & \\ & & & * \end{bmatrix}, \begin{bmatrix} & & & \\ & * & & \\ & & * & \\ & & & * \end{bmatrix} \quad (3.2)$$

The arbitrary matrix 4×4 with four duplicate entries in its main diagonal:

$$\begin{bmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{bmatrix} \quad (3.3)$$

Note: Symbol $*$ is duplicate entries

Definition 3.1 *Let A be an $m \times n$ matrix. Then the weak commutation matrix of A, denoted by P^*, is a permutation matrix in which $P^*vec(A) = vec(A^T)$, where P^* is an $mn \times mn$ matrix.*

Proposition 3.2

Let A be a matrix in (3.1), (3.2), and (3.3).

a. For any matrix, the weak commutation matrix for A is

$$P(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

b. For $a_{11} = a_{22},\ a_{11} = a_{22} = a_{33},\ a_{11} = a_{22} = a_{44},\ a_{11} = a_{22} = a_{33} = a_{44}$; the weak commutation matrix for A is

$$P(1\ 6)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

c. For $a_{11} = a_{33}$, $a_{11} = a_{22} = a_{33}$, $a_{11} = a_{33} = a_{44}$, and $a_{11} = a_{22} = a_{33} = a_{44}$, the weak commutation matrix for A is

$$P(1\ 11)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

d. For $a_{11} = a_{44}$, $a_{11} = a_{22} = a_{44}$, $a_{11} = a_{33} = a_{44}$, and $a_{11} = a_{22} = a_{33} = a_{44}$, the weak commutation matrix for A is

$$P(1\ 16)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

e. For $a_{22} = a_{33}$, $a_{11} = a_{22} = a_{33}$, $a_{22} = a_{33} = a_{44}$, and $a_{11} = a_{22} = a_{33} = a_{44}$, the weak commutation matrix for A is

$$P(6\ 11)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

f. For $a_{22} = a_{44}$, $a_{11} = a_{22} = a_{44}$, $a_{22} = a_{33} = a_{44}$, and $a_{11} = a_{22} = a_{33} = a_{44}$, the weak commutation matrix for A is

$$P(6\ 16)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

g. For $a_{33} = a_{44}$, $a_{11} = a_{33} = a_{44}$, $a_{22} = a_{33} = a_{44}$, and $a_{11} = a_{22} = a_{33} = a_{44}$, the weak commutation matrix for A is

$$P(11\ 16)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

h. For $a_{11} = a_{22} = a_{33}$ and $a_{11} = a_{22} = a_{33} = a_{44}$, the weak commutation matrices for A are

$$P(1\ 6\ 11)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

$$P(1\ 11\ 6)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

i. For $a_{11} = a_{22} = a_{44}$, and $a_{11} = a_{22} = a_{33} = a_{44}$,, the weak commutation matrices for A are

$$P(1\ 6\ 16)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

$$P(1\ 16\ 6)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

j. For $a_{11} = a_{33} = a_{44}$, and $a_{11} = a_{22} = a_{33} = a_{44}$,, the weak commutation matrices for A are

$$P(1\ 11\ 16)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

$$P(1\ 16\ 11)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

k. For $a_{22} = a_{33} = a_{44}$, and $a_{11} = a_{22} = a_{33} = a_{44}$, the weak commutation matrices for A are

$$P(6\ 11\ 16)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

$$P(6\ 16\ 11)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

l. For $a_{11} = a_{22} = a_{33} = a_{44}$, the weak commutation matrices for A are

$$P(1\ 6)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(11\ 16)(12\ 15)$$

$$P(1\ 11)(2\ 5)(3\ 9)(4\ 13)(6\ 16)(7\ 10)(8\ 14)(12\ 15)$$

$$P(1\ 16)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(11\ 16)(12\ 15)$$

$$P(1\ 16\ 11\ 6)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

$$P(1\ 11\ 6\ 16)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

$$P(1\ 6\ 16\ 11)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

$$P(1\ 16\ 6\ 11)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

$$P(1\ 11\ 16\ 6)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

$$P(1\ 6\ 11\ 16)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

Proof.

Based on Definition 2.4, for arbitrary matrix in (3.1), (3.2) and (3.3), we have

$$K_{4,4} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}$$

Where A_{ij}, $i, j = 1,2,3,4$ and $A_{ij} = \left(a_{s,t}{}^{(i,j)}\right)$ *is a* $(0,1)$ matrix with a unique 1 which lies at the position (j, i). Then

$$K_{4,4} = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix} = P(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15) = P^*$$

Next, we have $P(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)vec(A) = vec(A^T)$, and proven a.

For b. – k., Since there is a duplicate entries, the position of the permutation in the matrix can move each other for the same entry and not change the results of $P^* vec(A) = vec(A^T)$.

We directly proven that $P(1\ 6)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)\ vec(A) = vec(A^T)$ for A with condition b., and so on.

IV. Conclusion

This article provides a way to determine the weak commutation matrix with certain conditions. In this case, we choose an arbitrary matrix, 4×4, by selecting the duplicate entries in different position in the main diagonal. It is found that 24 weak commutation matrices for the matrices.

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