International Journal of Progressive Sciences and Technologies (IJPSAT)
ISSN: 2509-0119.

Vol. 32 No. 1 April 2022, pp. 105-111

Stability Analysis and Numerical Simulation of Prey-Predator System with Age Structure

Nila Destia Nasra*, Arrival Rince Putri, Mahdhivan Syafwan

Departement of Mathematics,
Faculty of Mathematics and Natural Sciences, Andalas University,
Kampus Unand LimauManis Padang, Indonesia, 25163

Abstract—This study developed a four-dimensional prey-predator model consisting four groups, i.e. immature prey, mature prey, immature predator, and mature predator. The reproduction of the prey depends on the mature group. Only mature predators take part in predation and prefer to consume mature prey populations. Analysis on the system by determining the equilibrium point of the prey-predator model. The results of the stability analysis obtained three equilibrium point i.e. the extinction of all species, the extinction of predator, and the existence of all species. Stability of model is analyzed based on the eigenvalues of the Jacobian matrix and by using Routh's criteria. Numerical simulations are carried out to confirm analytical results. Furthermore, numerical simulation by different parameter is provided. There are three cases at the equilibrium point of the existence of all species where the parameters α and β are different. The results show that the prey and predator populations will be stable or will not experience extinction for the all species equilibrium point.

Keywords—Stability, Jacobian Matrix, Eigenvalue, Routh's Criteria, Prey-Predator Model

I. Introduction

Every life phenomenon can be presented in the form of a mathematical model. The first mathematical model describing the dynamics of the population between prey and predators is known as the Lotka-Volterra model. The Lotka-Volterra model was introduced by the American biophysicist Alfred Lotka and the Italian mathematician Vito Volterra in 1926 [2]. The model is often used to describe the dynamics of a system consisting of two populations, i.e. population that is preyed on called as prey-population and the population that preys on called predator-population and both of them are known as the prey-predator model. The prey-predator model states that the prey population will increase if it is little or no prey population; on the contrary, the predator population will increase if the prey population is large and will decrease if no more prey is preyed on. Moreover, the increase in prey and predator populations depends on time [4].

Interactions between prey and predator do not only occur in two populations but it can be more than one prey and predator. There is predation involving one prey and two predators in some ecosystems. An example of such predation is the brown planthopper (*Nilaparvata lugens*) in rice plants which is preyed upon by natural enemies such as beetles (*Menochilus sexmaculatus*) and ladybugs (*Cyrtorhinus lividipennis*). The Lotka-Volterra model developed by involving the interaction between one prey and two predators with the assumption that the growth of prey and predator will follow the growth of logistics and the competition between the two predators has been discussed by Taufik and Denik (2018) [6]. Satria et al (2020) discussed prey-predator model about holling type II functional response [5].

The prey-predator model by considering the growth rate that depends on the population density in immature and mature prey groups has been discussed by Abrams and Quince (2005) [1]. The study analyzed the stability conditions and the impact of death on predators. Ghosh et al (2020) discussed prey-predator models for several groups, i.e. immature and mature prey and immature and mature predators [3]. They analyzed the stability of the model and the mortality impact of each population group. This study discusses the prey-predator model which refers to the model [3].

II. PREY-PREDATOR SYSTEM

2.1 Mathematical Model

The prey-predator model with age structure is presented in the form of a system of differential equations. This model is developed based on the following assumptions [3]:

a. Prey and predator population have four groups, i.e. the immature prey and predator groups and the mature prey and predator groups.

b. The rate of reproduction depends on the mature groups in each groups.

c. Only mature predators take part in predation and prefer to consume mature prey populations.

Based on the above assumptions, the prey-predator model consisting of groups of immature and mature prey as well as immature and mature predators is presented in the following system of differential equations :

$$\begin{aligned} \frac{dx_1}{dt} &= r_1 x_2 - b_1 x_1 - \mu_1 x_1, \\ \frac{dx_2}{dt} &= b_1 x_1 - \gamma x_2^2 - \frac{\alpha x_2 y_2}{h + x_2} - \mu_2 x_2, \\ \frac{dy_1}{dt} &= r_2 y_2 - b_1 y_1 - m_1 y_1, \\ \frac{dy_2}{dt} &= b_1 y_1 + \frac{\beta \alpha x_2 y_2}{h + x_2} - m_2 y_2, \end{aligned} \tag{1}$$

where x_1, x_2, y_1 and y_2 are the population of immature prey, mature prey, immature predator,and mature predator at timet; r_1 is the maximum growth rate per capita of mature prey; b_1 is the transition rate from immature prey to mature groups; r_2 is the reproductive rate for mature predator. Then, b_2 is the maximum growth rate per capita of mature predators; μ_i and m_i are the natural mortality rate of prey and predator where $i = 1,2$; then γ is the strength of competition between mature prey; γx_2^2 the crowding effect among mature prey classes; the functional responseof Holling type II $\frac{\alpha x_2}{h+x_2}$ is predation where α is attack rate, β is the conversion coefficient, and h is the half-saturation constant.

III. RESULT AND DISCUSSION

3.1 Stability Analysis

There are three equilibrium points of the system of equations (1), i.e.:

1. The Extinction of All Species $S^0 = (0,0,0,0)$.

   The Jacobian matrix of equilibrium point S^0 is

$$J = \begin{bmatrix} -(b_1+\mu_1) & r_1 & 0 & 0 \\ b_1 & -\mu_2 & 0 & 0 \\ 0 & 0 & -(b_1+m_1) & r_2 \\ 0 & 0 & b_2 & -m_2 \end{bmatrix} \tag{2}$$

   Eigenvalues from matrix Jacobian (2) is $\lambda_{1,2} = -\frac{1}{2}\left(\mu_2 + \mu_1 + b_1 \pm \sqrt{D_1}\right)$ and $\lambda_{3,4} = -\frac{1}{2}\left(m_2 + m_1 + b_2 \pm \sqrt{D_2}\right)$ where

   $D_1 = (\mu_2 + b_1 + \mu_1)^2 - 4(b_1\mu_2 + \mu_1\mu_2 - r_1 b_1)$,

   $D_2 = (m_2 + b_2 + m_1)^2 - 4(b_2 m_2 + m_1 m_2 - r_2 b_2)$.

   The equilibrium point S^0 is asymptotically stable if

- $D_1 < 0$ so $Re(\lambda_{1,2}) \in C$ with $\lambda_{1,2} < 0$
- $D_1 \geq 0$ so $\lambda_{1,2} \in \mathcal{R}$. If $(\mu_2 + \mu_1 + b_1) \geq \sqrt{D_1}$ so $\lambda_{1,2} < 0$.

2. The Extinction of Predator $\overline{S} = (\overline{x_1}, \overline{x_2}, 0,0)$

where

$$\overline{x_1} = \frac{r_1 \overline{x_2}}{b_1 + \mu_1},$$

$$\overline{x_2} = \frac{r_1 b_1 - \mu_2(b_1 + \mu_1)}{\gamma(b_1 + \mu_1)}.$$

The Jacobian matrix of equilibrium point $\overline{S}$ is

$$J = \begin{bmatrix} -(b_1+\mu_1) & r_1 & 0 & 0 \\ b_1 & 2\gamma x_2 - \mu_2 & 0 & -\frac{\alpha x_2}{h+x_2} \\ 0 & 0 & -(b_1+m_1) & r_2 \\ 0 & 0 & b_2 & \frac{\beta\alpha x_2}{h+x_2} - m_2 \end{bmatrix} \tag{3}$$

Eigenvalues from matrix Jacobian (3) is $\lambda_{1,2} = -\frac{1}{2}\left(2\gamma\overline{x_2} + \mu_2 + b_1 + \mu_1 \pm \sqrt{D_1}\right)$ and $\lambda_{3,4} = -\frac{1}{2}\left(-\frac{\beta\alpha\overline{x_2}}{h+\overline{x_2}} + m_2 + b_2 + m_1 \pm \sqrt{D_2}\right)$ where,

$$D_1 = (2\gamma\overline{x_2} + \mu_2 + b_1 + \mu_1)^2 - 4(2\gamma\overline{x_2}b_1 + \mu_2 b_1 + 2\gamma\overline{x_2}\mu_1 + \mu_1\mu_2 - r_1 b_1),$$

$$D_2 = \left(-\frac{\beta\alpha\overline{x_2}}{h+\overline{x_2}} + m_2 + b_2 + m_1\right)^2 - 4\left(-\frac{\beta\alpha\overline{x_2}b_2}{h+\overline{x_2}} + b_2 m_2 - \frac{\beta\alpha\overline{x_2}m_1}{h+\overline{x_2}} + m_1 m_2 - r_2 b_2\right).$$

The equilibrium point $\overline{S}$ is asymptotically stable if

- $D_1 < 0$ so $Re(\lambda_{1,2}) \in C$ with $\lambda_{1,2} < 0$
- $D_1 \geq 0$ so $\lambda_{1,2} \in \mathcal{R}$. If $(2\gamma\overline{x_2} + \mu_2 + b_1 + \mu_1) \geq \sqrt{D_1}$ so $\lambda_{1,2} < 0$.

3. The Existence of All Species $S^* = (x_1^*, x_2^*, y_1^*, y_2^*)$

where

$$x_1^* = \frac{r_1 x_2^*}{b_1 + \mu_1},$$

$$x_2^* = \frac{Ah}{\beta\alpha - A},$$

$$y_1^* = \frac{r_2 y_2^*}{b_2 + m_1},$$

$$y_2^* = \frac{h + x_2^*}{\alpha}\left(\frac{r_1 b_1}{b_1 + \mu_1} - \mu_2 - \gamma x_2^*\right),$$

and

$$A = m_2 - \frac{r_2 b_2}{b_2 + m_1}.$$

The Jacobian matrix of equilibrium point S^* is

$$J = \begin{bmatrix} -(b_1+\mu_1) & r_1 \\ & 2\gamma x_2 - \frac{\alpha y_2}{h+x_2} + \frac{\alpha x_2 y_2}{(h+x_2)^2} - \mu_2 & 0 & -\frac{\alpha x_2}{h+x_2} \\ b_1 & 0 & 0 & 0 \\ 0 & 0 & -(b_1+m_1) & r_2 \\ 0 & \frac{\beta\alpha y_2}{h+x_2} - \frac{\beta\alpha x_2 y_2}{(h+x_2)^2} & b_2 & \frac{\beta\alpha x_2}{h+x_2} - m_2 \end{bmatrix} \quad (4)$$

Therefore, the characteristic equation of the Jacobian matrix (1) is

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 \quad (5)$$

where

$a_3 = -(a_{11} + a_{22} + a_{33} + a_{44})$,

$a_2 = a_{11}a_{22} + a_{11}a_{33} + a_{11}a_{44} - a_{12}a_{21} - a_{13}a_{31} - a_{14}a_{41} + a_{22}a_{33} + a_{22}a_{44} - a_{23}a_{32} - a_{24}a_{42} + a_{33}a_{44} - a_{34}a_{42}$,

$$\begin{aligned} a_1 = {} & -a_{11}a_{22}a_{33} - a_{11}a_{22}a_{44} + a_{11}a_{23}a_{32} + a_{11}a_{24}a_{42} - a_{11}a_{33}a_{44} + a_{11}a_{34}a_{43} + a_{12}a_{21}a_{33} + a_{12}a_{21}a_{44} \\ & - a_{12}a_{23}a_{31} - a_{12}a_{24}a_{41} - a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31} + a_{13}a_{31}a_{44} - a_{13}a_{34}a_{41} - a_{14}a_{21}a_{42} \\ & + a_{14}a_{22}a_{41} - a_{14}a_{31}a_{43} + a_{14}a_{33}a_{41} - a_{22}a_{33}a_{44} + a_{22}a_{34}a_{43} + a_{23}a_{32}a_{44} - a_{23}a_{34}a_{42} \\ & - a_{24}a_{32}a_{43} + a_{24}a_{33}a_{42}, \end{aligned}$$

$$\begin{aligned} a_0 = {} & a_{11}a_{22}a_{33}a_{44} - a_{11}a_{22}a_{34}a_{43} - a_{11}a_{23}a_{32}a_{44} + a_{11}a_{23}a_{34}a_{42} + a_{11}a_{24}a_{32}a_{43} - a_{11}a_{24}a_{33}a_{42} - a_{12}a_{21}a_{33}a_{44} \\ & - a_{12}a_{21}a_{34}a_{43} + a_{12}a_{23}a_{31}a_{44} - a_{12}a_{33}a_{34}a_{41} - a_{12}a_{23}a_{31}a_{43} + a_{12}a_{24}a_{33}a_{41} + a_{13}a_{21}a_{32}a_{44} \\ & - a_{13}a_{21}a_{34}a_{42} - a_{13}a_{22}a_{31}a_{44} + a_{13}a_{22}a_{34}a_{41} + a_{13}a_{24}a_{31}a_{42} - a_{13}a_{24}a_{32}a_{41} - a_{14}a_{21}a_{32}a_{43} \\ & + a_{14}a_{21}a_{33}a_{42} + a_{14}a_{22}a_{31}a_{43} - a_{14}a_{22}a_{33}a_{41} - a_{14}a_{23}a_{31}a_{42} + a_{14}a_{23}a_{32}a_{41}. \end{aligned}$$

Determination of the eigenvalues sign $\lambda_1, \lambda_2, \lambda_3$ and λ_4 in the equation (5) using Routh's criteria table is presented in Table 1.

Table 1:Criteria's Routh

λ^n	1	a_{n-2}	a_{n-4}	$\cdots$
λ^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
λ^{n-2}	b_1	b_2	b_3	$\cdots$
λ^{n-3}	c_1	c_2	c_3	$\cdots$
$\vdots$	$\vdots$	$\vdots$		

3.2 Numerical Simulation

In this section, a numerical simulation with several parameter values is given to confirm the analytical solution obtained in the previous section. In model (1), it is assumed that the parameter value is $r_1 = 1, r_2 = 0.5$, $b_1 = 0.5$, $b_2 = 0.2$, $\mu_1 = 0.1$, $\mu_2 = 0.1$, $m_1 = 0.4$, $m_2 = 0.2$, $\gamma = 0.55$, and $h = 1$ [3].

Using the given parameters, the eigenvalues for equilibrium point on the existence of all species of the system (1) are as follows:

Case 1. $\alpha = 0.2$ and $\beta = 0.6$

There is equilibrium point which is all species are exist, $S^* = (x_1^*, x_2^*, y_1^*, y_2^*) = (0.64093, 0.38456, 3.0105, 3.6126)$.

Substitute this equilibrium point in to equation (4), so that,

$$J = \begin{bmatrix} -0.6 & 1 & 0 & 0 \\ 0.5 & -0.89993 & 0 & -0.055548 \\ 0 & 0 & -0.6 & 0.5 \\ 0 & -0.22614 & 0.2 & -0.16667 \end{bmatrix} \quad (6)$$

By using Routh's criteria to determine the roots of the polynomial equation, then,

$$b_1 = 1.2005$$

$$c_1 = 0.037162$$

$$d_1 = 0.000074142$$

Table 2: Routh's Criteria for case 1

λ^4	1	1.2025	0.0045221
λ^3	2.2666	0.04570	0
λ^2	1.2005	0.0045222	0
λ	0.037162	0	0
1	0.000074142	0	0

All elements in column one are positive so system (1) has a negative real part where the eigenvalues for the equilibrium point $S^*$are $\lambda_1 = -1.4664, \lambda_2 = -0.016319 + 0.061250i, \lambda_3 = -0.016319 - 0.061250i$, and $\lambda_4 = -0.76755$. Therefore, the equilibrium point $S^* = (0.64093, 0.38456, 3.0105, 3.6126)$ is stable.

The solution of the model (1) for the initial value $x_1(0) = 2, x_2(0) = 1, y_1(0) = 2$, and $y_2(0) = 4$ can be seen in Figure 1 below.

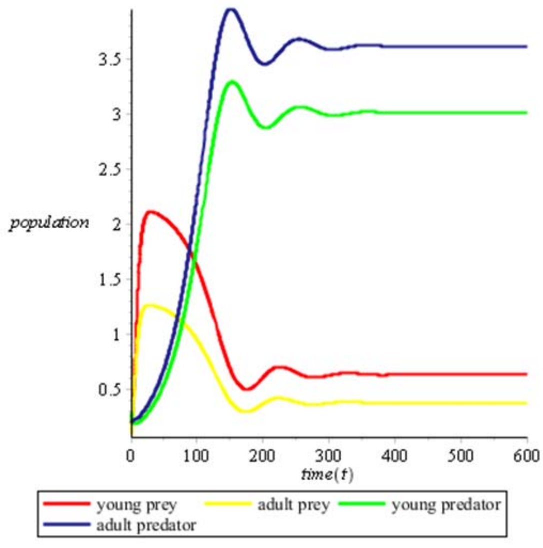

Figure 1: Solution of System (1) for $\alpha = 0.2$ and $\beta = 0.6$.

Figure 1 shows that the population of immature predator and mature predator is increasing until it reaches a stable population. Meanwhile, the population of immature and mature prey decreases. When it reaches a certain time, the population of immature and mature prey, as well as immature and mature predator, will stabilize towards the equilibrium point $S^* = (0.64093, 0.38456, 3.0105, 3.6126)$.

Case 2. $\alpha = 0.6$ and $\beta = 0.5$

The solutions of system (1) are presented in Figure 2.

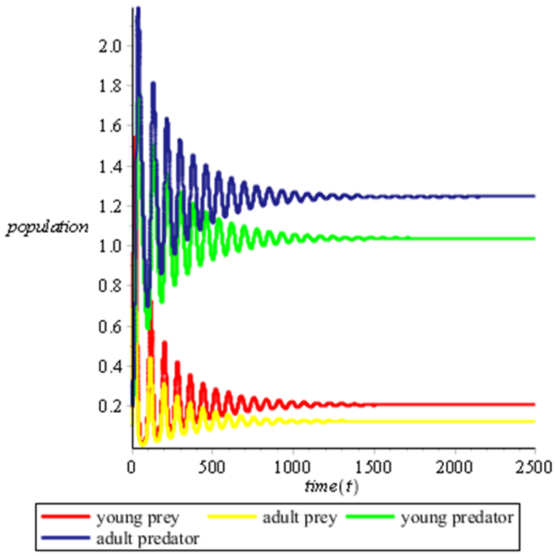

Figure 2: Solution of system (1) for $\alpha = 0.6$ and $\beta = 0.5$.

Figure 2 shows that if α and β are enlarged then the immature and mature prey populations increase and decrease. Meanwhile, the population of immature and mature predators has increased. After some time, the four populations reached the point 0.20832, 0.12499, 1.0384, and 1.2461,and will be stable at that point.

Case 3. $\alpha = 0.2$ and $\beta = 0.1$

The solutions of system (1) are presented in Figure 3.

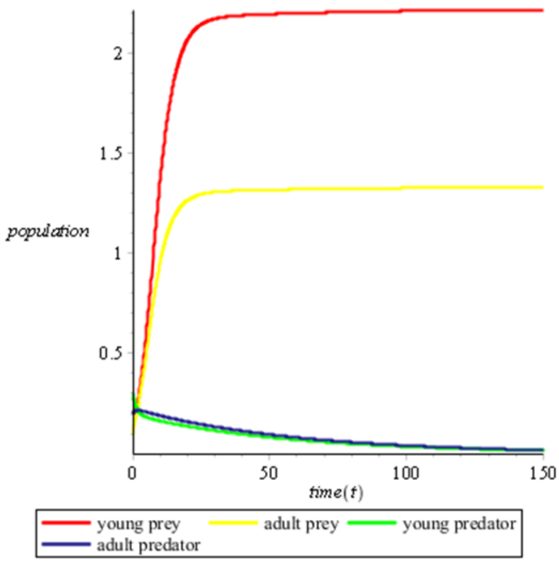

Figure 3: Solution of system (1) for $\alpha = 0.2$ and $\beta = 0.1$.

Figure 3 shows that α and β are reduced then the population is unstable and will experience extinction. It is affected by the small attack rate of mature prey and mature predator so the conversion coefficient of mature predator is also small. Therefore, mature predators do not get food to survive so they will experience extinction.

The parameters α and β affect the population of immature and mature prey as well as immature and mature predator. When the parameters α and $\beta$are enlarged, the population will remain stable at a certain time. Meanwhile, when the parameters α and $\beta$are reduced, the population of immature and mature predators will experience extinction at a certain time.

IV. Conclusion

The prey-predator model in the system (1) was analyzed by using the Routh's criteria. Numerical simulation of the system (1) were obtained by using the different parameter values. The analytical and numerical results concluded that the equilibrium point in the system is only stable at the equilibrium point for all species. The equilibrium point for the existence for all species will be stable when the parameters α and β are increased. However, when the parameters α dan β are reduced, the prey and predator populations will go to extinction.

References

[1] Ambrams. P. A dan Quince. C. 2005. *The Impact of Mortality on Predator Population Size and Stability in Systems with Stage-Structured Prey*. 68(4):253-266.

[2] Boyce. E. W dan Richard. C. D. 2009. *Elementary Differential Equations and Boundary value Problems*. United States :Jhon Wiley and Sons, Inc.

[3] Ghosh. B., Zhdanova. O. L., Barman. B., danFrisman. E. Y. 2020. *Dynamics of Stage-Structure Predator-Prey Systems UnderDensityDependent Effect and Mortality*. Ecology 41:100812.

[4] Latipah. K., Putri. A. R., and Syafwan. M. 2020. *Stability of Analysis Prey Predator Model Holling Type II with Infected Prey*. Journal of Physics: Conference Series 1554(1), 012038.

[5] Satria. A., Putri. A. R., and Syafwan. M. 2020. *Analytical Study and Numerical Simulation of Diffusive Prey-Predator Model with Holling Type II Functional Response.* Journal of Physics: Conference Series 1554(1), 012039.

[6] Taufiq. I danDenik. A. 2018. *Model Predator-Prey dengan Dua Predator danSatu Prey Terinfeksi*. 8-15.