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Kinematics of Fluids

Andrei Ludu
Embry-Riddle Aeronautical University, ludua@erau.edu

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Andrei Ludu

Nonlinear Waves and Solitons on Contours and Closed Surfaces

Third Edition

Springer

Springer Series in Synergetics

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Andrei Ludu

Nonlinear Waves and Solitons on Contours and Closed Surfaces

Third Edition

Springer

Andrei Ludu
Department of Mathematics
Embry-Riddle Aeronautical University
Daytona Beach, FL, USA

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To my family, the most important presence.

Foreword

The story of solitary waves traces back to John Scott Russel. Approaching 200 years ago he wrote:

> *I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.*

Russel went on to conduct experiments and published his findings in 1845 (check this). Initially, major figures such as Stokes and Airy denied the existence of what we would now call a traveling wave on the surface of water in a channel. In the second half of the nineteenth century, one sees in the correspondence between Stokes and Raleigh that Stokes had changed his mind and this fact even appears in published work. In the period of this correspondence, Rayleigh found an approximate relation between the amplitude and speed of a solitary wave in a channel. However, it was left to Boussinesq in the 1870s to write down evolution equations that approximated the motion of disturbances on the surface of water and which featured exact solitary-wave solutions. One of these was the celebrated Korteweg-de Vries equation of water wave theory that was rederived by Joseph Korteweg and his student Gustav de Vries in 1895. The issue of existence of these so-called solitary waves having been settled, at least as far as the nineteenth century hydrodynamicists were concerned, the subject went moribund.

It came back to life, though in disguise, in work of Fermi, Pasta, Ulam and Tsingou on a lattice and spring model for heat conduction in the 1950s. Later, by taking an appropriate continuum limit of this mass and spring model, Kruskal and Zabusky came again to the Korteweg-de Vries equation. This time, however, the subject did not die. In 1967, the inverse scattering theory for this equation was discovered by Gardner, Greene, Miura and Kruskal. Peter Lax took the first step in putting this formalism into a very imaginative mathematical structure. Since then, the subject rapidly achieved industrial proportions, with tens of thousands of journal pages and with many, many applications of the theory.

As Andrei Ludu, the author of the present monograph writes in his introduction, considering the large literature on solitary waves, why yet another book? There are several things that set this text apart from others in the field. First is the overall focus upon solitary waves defined on compact spaces. Of course, one thinks initially of the classical cnoidal-wave solutions of the Korteweg-de Vries equation, but as Ludu ably shows, this is the tip of a very large iceberg. Another aspect of the text that strikes a new chord is the differential geometric perspective; the view that solitary waves can be realized as the motion of a planar or three-dimensional curve under particular flow conditions and with suitable initial conditions. This is not original to the text in question, but an overall assessment of these ideas and a comprehensive review of its applications is not to be found elsewhere in the literature. And, speaking of applications, the text ends with a large number of very diverse and interesting applications.

The text breaks into four parts. Parts I and II, which comprise the first eight chapters, contain a sketch of the relevant topology and especially the differential geometry of curves and surfaces in two and three spatial dimensions. It should be acknowledged that this material is not for beginners. Someone without prior knowledge of at least portions of this material will not find it easy going. However, as a reminder to those with some knowledge, and a focus on exactly what is needed from differential geometry in what follows, it is very helpful. Especially the material in Chapter 6 will be useful even for the cognoscenti.

Chapter 7 works out the connection between the motion of curves in two and three dimensions and integrable systems. Chapter 8 does the same thing for the motion of surfaces. Technically, this is the heart of the script. This will be new material to many readers; indeed, it is a developing subject in the mathematical firmament.

Ludu's exposition in Parts I and II is technically sound, but it makes much of its headway by way of appealing to our intuition. Not every theorem is proved in detail, which is quite okay given the overall goal of the text.

In Parts III and IV, the text becomes more concrete. It begins with a more or less standard discussion of the kinematics of fluid motion in Chapter 9. Knowledgeable readers may well skip this, but for folks a little rusty, it is helpful. Some of the notation is laid out in this chapter as well.

Chapters 10 and 11 find us deriving the Euler and Navier-Stokes equations. This includes a very detailed discussion of surface tension from a geometrical perspective. He goes on to derive many of our favorite approximate models, such as the Korteweg-de Vries equation, the modified Korteweg-de Vries equation, the Boussinesq equation and the cubic Schrodinger equation. He examines the well-known solitary-wave solutions of these equations by way of the mathematical structure developed in Part I. He also derives what he terms the GKdV equation (Generalized Korteweg-de Vries equation) that results from carrying out the formal asymptotics in the shallow water parameter and the nonlinear parameter to higher order. This equation specializes to the various more familiar equations. Again, what is distinctly non-standard is his concentration upon solitary waves defined on compact spaces that can be obtained via the motion of curves whose theory was developed in Part II. This part is also not for a beginner. Without prior background in these sorts of derivations, it will be hard going. Hard going, but worth the effort.

Chapters 12–15 might well have been lumped into Part II of the text. While they enlarge upon the theory, they emerge from physical considerations. Chapters 12–14 are concerned with the fascinating shape oscillations of liquid drops in two and three space dimensions. Chapter 15 presents another quite different point of view that yields some of the same fascinating shapes that appeared earlier in droplets.

In the fourth portion of the text, Ludu shows his scientific upbringing. He started life as a physicist and throughout his career he has been closely tied to real-world phenomena. He admirably shows off his breadth in Part IV of the text. Here we find him dealing with a whole stable of solitons that arise in some unlikely places. There are solitons on filaments of various sorts, solitons on stiff chains, solitons on the boundaries of microscopic structures, solitons at stellar scales.

The text finishes with a mathematical annex that includes some interesting remarks that didn't fit anywhere else in the text.

This book is not to be read in an armchair. As Ludu states in his opening remarks, it is meant to be studied with pencil and paper at hand and with an algebraic manipulation program up on the screen of a computer. It is a text dense with ideas and methods, both mathematical and scientific, and a serious addition to the literature. The fact that it is going into a third edition attests to its impact.

Chicago, USA

Hongqiu Chen
Jerry Bona

Preface to the Third Edition

In order to offer as much content as possible from all chapters of the book to readers with various prerequisites in mathematics, we present below a reader's map that can help readers to navigate through the book without being stuck in sections with denser mathematical content. Pretty much like on a skiing course, we introduce three possible paths to meet the interest of all our readers:

- No * asterisk is the path that doesn't request special prerequisites in mathematics, except calculus and first level course in mathematical physics. For these readers, we recommend the following path:

$$Introduction \rightarrow 2.1 \rightarrow 3.1 \rightarrow 3.2 \rightarrow 3.3 \rightarrow 3.12 \rightarrow 3.13 \rightarrow 4.1 \rightarrow 4.2 \rightarrow$$

$$5 \rightarrow 6.1 \rightarrow 6.5 \rightarrow 7.1 \rightarrow 7.3 \rightarrow 9.1 \rightarrow 9.3 \rightarrow 9.5 \rightarrow 9.6.1 \rightarrow 10.1 \rightarrow 10.2 \rightarrow$$

$$10.3 \rightarrow 10.4.1 \rightarrow 10.5 \rightarrow 10.6.1 \rightarrow 11.1 \rightarrow 12.1 \rightarrow 12.6 \rightarrow$$

$$13.1 \rightarrow 14.1 \rightarrow 14.2 \rightarrow 14.3 \rightarrow 17.3 \rightarrow 18.2 \rightarrow 18.3 \rightarrow 19$$

- Sections labeled with one asterisk * request some previous knowledge in real analysis, differential systems and elements of geometry. For these readers we recommend in addition to the "No * asterisk path" to add the following sections:

$$3.4 \rightarrow 3.5 \rightarrow 3.6 \rightarrow 3.7 \rightarrow 3.10 \rightarrow 6.3 \rightarrow 6.4 \rightarrow 7.5 \rightarrow 9.4 \rightarrow 9.6.2 \rightarrow$$

$$10.4.3 \rightarrow 10.6.2 \rightarrow 11.2 \rightarrow 11.5 \rightarrow 12.2 \rightarrow 12.3 \rightarrow 12.4 \rightarrow 12.5 \rightarrow 12.6 \rightarrow$$

$$13 \rightarrow 14.4 \rightarrow 14.5 \rightarrow 15.2 \rightarrow 18.1 \rightarrow 18.4.$$

- Sections labeled with two asterisks ** address to mathematicians or theoretical physicists, or anyone who finds useful de dedicate some time practicing a higher level of mathematics, like algebraic topology, differential geometry, or nonlinear differential systems. For these readers, we recommend in addition to the "* asterisk path" to add the following sections:

$$2.2 \to 2.3 \to 3.8 \to 3.9 \to 3.11 \to 6.2 \to 7.2 \to 7.4 \to 7.6 \to 8 \to 9.2$$

$$\to 10.4.2 \to 10.4.4 \to 10.4.5 \to 10.4.6 \to 10.4.7 \to 10.6.3 \to 11.3 \to$$

$$11.4 \to 15.1 \to 15.3 \to 16 \to 17.$$

Besides corrections made in the previous editions, the goal of this third edition is to implement latest results on solitons traveling on closed, compact surfaces or curves. We cover again mathematical and physical problems ranging from nuclear to astrophysical scales. The third edition provides additional examples of systems and models where the interaction between nonlinearities and the compact boundaries is essential for the existence and the dynamics of solitons.

The first historic mention of what we call today *soliton* was made in 1834 by John Scott Russell following his discovery of a new type of *waves of translation* [1]. The mathematical model for such waves, the Korteweg-de Vries (KdV) equation, was first introduced by Boussinesq in 1877, and it was rediscovered in 1895 by Diederik Korteweg and Gustav de Vries [2]. Relations between nonlinear differential equations and differential geometry, without any reference yet to solitons, were first discovered by Edmond Bour in 1862 in the course of the study of surfaces of constant negative curvature, like the Gauss—Codazzi equation for surfaces of curvature -1 in $\mathbf{R}^3$. This is the first mention of the sine-Gordon equation $u - \sin u = 0$. The equation was rediscovered by Frenkel and Kontorova in 1939 in their study of crystal dislocations [3]. Only starting with 1970, this equation attracted a lot of attention due to the presence of soliton solutions and its mathematical connection with differential geometry. Therefore, it is the main goal of this book to focus on such interesting and/or recent aspects of relations between nonlinear integrable systems with their soliton solutions and differential geometry, mainly defined on compact manifolds.

The book consists of 19 chapters organized in four parts, a mathematical annex, and a bibliography. The first part contains the fundamentals of topology, differential geometry, and analysis approaches. To render this book accessible to students in all STEM disciplines, Chap. 2 recalls some basic elements of topology with emphasis on the concept of being compact. Chapter 3 introduces the reader to calculus on differentiable manifolds, vector fields, differentiable forms, and various types of derivatives. We take the reader from the definition of the differential manifold all the way to the Poincaré lemma. Next, in this chapter, we introduce different types of fiber bundles, the Cartan theory of frames, and the theory of connection and mixed covariant derivatives. Without always presenting the proofs, we tried though to keep a high level of rigorousness (relying on classical mathematical textbooks) all across the text while we still introduce intuitive comments for each definition or affirmation. In Chap. 4, we review various representation formulas for various dimensions. These formulas justify how the information about the evolution of smooth physical fields inside a bounded region can be recovered only from the information on the region's boundary.

The second part of the book is devoted to applications of differential geometry in the theory of curves and surfaces. Chapter 5 lays the basis for the differential geometry of curves in spaces with three dimensions. We introduce special sections for the theory of closed curves and curves lying on surfaces. Complementary to these, in Chap. 6, we introduce elements of the geometry of surfaces with applications to the action of differential operators on these surfaces. In Chap. 7, we derive the theory of motion of curves in two and three dimensions, and we emphasize the relationships between theory of motion of curves and solitons. We devoted a section on the axiomatic deduction of the theory of curve motions based on differentiable forms and Cartan connection theory. We describe the relationships between some special motions of curves and solitons. We describe nonlinear integrable systems that can be represented by such motions. In Chap. 8, we discuss the theory of motion of surfaces and again relate such motions to nonlinear integrable systems and solitons.

The third part of the book is dedicated to applications of soliton theory, especially solitons on closed curves and surfaces, in fluid dynamics. The working frame of hydrodynamics, which is also the main content of Part III, is presented in Chap. 9. In Chap. 10, we discuss problems related to liquid surface tension effects and the associated representation theories for fluid dynamics models. Chapter 11 describes one-dimensional integrable systems on compact intervals, together with their periodic solutions. In this chapter, we introduce the most common and most used subject in nonlinear waves, the Korteweg-de Vries equation and system. In Chaps. 12–14, we approach the same type of problems except in higher dimensions. We describe and analyze nonlinear shape excitations for two and three-dimensional compact fluid systems, like liquid drops, liquid shells, etc.

Chapter 15 is devoted to other applications of soliton theory on compact surfaces in one to three dimensions like nonlinear shapes of layered liquid drops, compact supported solitons, or the relationship between solitons and collective motions of nonlinear dynamical systems with boundary.

In the fourth part of the book, as a closure for the first three parts, we present novel and interesting physical (and even biological) applications of the theory of nonlinear systems and their soliton solutions. We describe several physical systems at different space-time-energy scales. In Chap. 16, we study the vortex filaments and other one-dimensional flows. In Chap. 17, we describe microscopic applications of solitons and instantons in the theory of elementary particles and quantum fields, in description of exotic shapes of heavy nuclei, the phenomenon of exotic radioactivity and relationships between solitons on closed curves and quantum Hall drops.

Chapter 18 contains macroscopic scale applications of compact supported solitons in magnetohydrodynamic, plasma systems, elastic solids with surface, nonlinear surface diffusion, and neutron stars.

The book is completed by a mathematical annex, including an original section on the theory and applications of nonlinear dispersion relations, and their use for the qualitative description of the soliton solutions of nonlinear partial differential equations.

A legitimate question of the potential reader would be: “Why one more book on solitons?” First of all, we have to acknowledge the importance of the interactions between compact boundary manifolds and the dynamics of particles and fields in mathematical in physical models. Historically, the solitons are observed in sort of “infinite” systems like infinite long lines or curves, planes or open surfaces, or unbounded space. However, there is more and more evidence of the existence solitons or of localized patterns (like vortices) in compact lower dimensional spaces, like closed curves and/or surfaces. As examples, we can mention the unprecedented information technology advances in optical communication (light bullets and ultra-short optical pulses), solid-state spectroscopy, ultra-cold atom studies, soliton molecules, spinning solitons, quantum computers, spintronics and mass memory systems, femtosecond laser pulses, mesoscopic superconductivity, etc. Consequently, the reasons for writing this book are generated by a constantly increasing number of new challenges, vivid topics and hundreds of published articles. As one last comment, we mention that this book is not devoted to the teaching of general theory of solitons, or the Inverse Scattering Transform, and other traditional methods to obtain nonlinear solutions. This book opens a new direction in the field of nonlinear system, namely about nonlinear waves and solitons evolving in compact spaces, like closed curves, contours, and closed surfaces, etc.

If a substantial percentage of users of this book feel that it helped them to enlarge their outlook in the intersection between the fascinating worlds of nonlinear waves and compact surfaces and closed curves, its purpose has been fulfilled.

During the production of this third edition of this book, I have received the best support and uninterrupted encouragement from my family. I have also greatly benefited from discussions with my colleagues, and I am particularly grateful to Adrian S. Carstea and Denys Dutykh who provided valuable help in the elaboration of this edition.

Daytona Beach, FL, USA Andrei Ludu

Preface to the Second Edition

Nonlinear phenomena represent intriguing and captivating manifestations of nature. The nonlinear behavior is responsible for the existence of complex systems, catastrophes, vortex structures, cyclic reactions, bifurcations, spontaneous phenomena, phase transitions, localized patterns and signals, and many others. The importance of studying nonlinearities has increased over the decades and has found more and more fields of application ranging from elementary particles, nuclear physics, biology, wave dynamics at any scale, fluids, plasma to astrophysics. The soliton is the central character of this 178-year-old story. A soliton is a localized pulse traveling without spreading and having particle-like properties plus an infinite number of conservation laws associated to its dynamics. In general, solitons arise as exact solutions of approximate models. There are different explanation, at different levels, for the existence of solitons. From the experimenter's point of view, solitons can be created if the propagation configuration is long enough, narrow enough (like long and shallow channels, fiber optics, electric lines, etc.), and the surrounding medium has an appropriate nonlinear response providing a certain type of balance between nonlinearity and dispersion. From the numerical calculations point of view, solitons are localized structures with very high stability, even against collisions between themselves. From the theory of differential equations point of view, solitons are cross-sections in the jet bundle associated to a bi-Hamiltonian evolution equation (here Hamiltonian pairs are requested in connection to the existence of an infinite collection of conservation laws in involution). From the geometry point of view, soliton equations are compatibility conditions for the existence of a Lie group. From the physicist point of view, solitons are solutions of an exactly solvable model having isospectral properties carrying out an infinite number of non-obvious and counter-intuitive constants of motion.

The progress in the theory of solitons and integrable systems has allowed the study of many nonlinear problems in mathematics and physics: non-local interactions, collective excitations in heavy nuclei, Bose—Einstein condensates in atomic physics, propagation of nervous pulses, swimming of motile cells, nonlinear oscillations of liquid drops, bubbles, and shells, vortices in plasma and in atmosphere, tides in neutron stars, only to enumerate few of possible applications. A number of other applications of soliton theory also lead to the study of the dynamics of boundaries.

In that, the last three decades have seen the completion of the foundation for what today we call nonlinear *contour dynamics*. The subsequent stage of development along this topic was connected with the consideration of an almost *incompressible* systems, where the boundary (contour or surface) plays the major role.

Many of the integrable nonlinear systems have equivalent representations in terms of differential geometry of curves and surfaces in space. Such geometric realizations provide new insight into the structure of integrable equations, as well as new physical interpretations. That is why the theory of motions of curves and surfaces, including here filaments and vortices, represents an important emerging field for mathematics, engineering and physics.

The first problem about such compact systems is that shape solitons, which usually exist in infinite long and shallow propagation media, cannot survive on a circle or sphere. That is because such compact manifolds cannot offer the requested type of environment (long and narrow), even by the introduction of shallow layers and rigid cores. However, there is another basic idea that supports, in a natural way, the existence of nonlinear solutions on compact spaces. Because of its high localization, a soliton is not a unique solution for the partial differential system. Its position in space is undetermined because, far away from its center, the excitation is practically zero. On the other hand, all linear equations provide uniqueness properties for their solutions. It results that strongly localized solutions and almost compact supported solutions can be generated only within nonlinear equations. There is an exception here: the finite difference equations with their compact supported wavelet solutions, but in some sense, a finite difference equation is similar to a nonlinear differential one.

Despite the many applications and publications on nonlinear equations on compact domains, there are still no books introducing this theory, except for several sets of lecture notes. One reason for this may be that the field is still undergoing a major development and has not yet reached the perfection of a systematic theory. Another reason is that a fairly deep knowledge of integrable systems on compact manifolds has been required for the understanding of solitons on closed curves and compact surfaces.

The goal of the second edition of this book is to analyze the existence and describe the behavior of solitons traveling on closed, compact surfaces or curves. The approach of the physical problems ranging from nuclear to astrophysical scales is made in the language of differential geometry. The text is rather intended to be an introduction to the physics of solitons on compact systems like filaments, loops, drops, etc., for students, mathematicians, physicists, and engineers. The author assumes that the reader has some previous knowledge about solitons and nonlinearity in general. The book provides the reader examples of systems and models where the interaction between nonlinearities and the compact boundaries is essential for the existence and the dynamics of solitons.

We focused on interesting and recent aspects of relations between integrable systems and their solutions and differential geometry, mainly on compact manifolds. The book consists of 17 chapters, a mathematical annex, and a bibliography. First part contains the fundamental differential geometry and analysis approach. To render this book accessible to students in science and engineering, Chapter 2 recalls some basic elements of topology with emphasis on the concept of being compact. In Chapter 3, we review the representation formulas for different dimensions. The formulas express how a lot of information about the evolution of differentiable forms and fields inside a compact domain can be recovered only from its boundary. Chapter 4 introduces the reader to the calculus on differentiable manifolds, vector fields, forms, and various types of derivatives. We take the reader from map all the way to the Poincaré lemma. Next we introduce different types of fiber bundles, including the Cartan theory of frames, and the theory of connection and mixed covariant derivative (for immersions). Without always presenting the proofs, we tried though to keep a high level of rigorousness (relying on classical mathematical textbooks) all across the text while we still introduce intuitive comments for each definition or affirmation. Chapter 5 lays the basis for the differential geometry of curves in $\mathbf{R}_3$ We devote here special sections to closed curves and curves lying on surfaces. Complementary, in Chapter 7, we introduce the elements of differential geometry of the surfaces with applications to the action of differential operators on surfaces. In Chapter 6, we derive the theory of motion of curves, both in two dimensions, and in the general case. We devoted a section on the axiomatic deduction of the theory of motions based on differentiable forms and Cartan connection theory. We relate these motions with soliton solutions and find the nonlinear integrable systems that can be represented by such motions of curves. In Chapter 8, we discuss the theory of motion of surfaces, and we also relate it to integrable systems.

The second part of the monograph contains an exposition of the basic branches of nonlinear hydrodynamics. The working frame of hydrodynamics is the main content of the first part of the monograph, namely Chapter 9. In Chapter 10, we discuss the problems on surface tension effects and representation theorems for fluid dynamics models. Chapter 11 concentrates with one-dimensional integrable systems on compact intervals, and their periodic solutions. Chapters 12 and 13 deal with nonlinear shape excitations of two-dimensional and three-dimensional liquid drops and bubbles. Chapter 14 is devoted to various applications of three-dimensional nonlinear drops and also to compact supported solitons.

In the third part of the book, as a final goal for the first two parts, we present additional physical applications of nonlinear systems and their soliton solutions on various systems of different scales. In Chapter 15, we study the vortex filaments and other one-dimensional flows. In Chapter 16, we describe microscopic applications like elementary particles as solitons, instantons, exotic shapes in heavy nuclei, exotic radioactivity and quantum Hall drops. Chapter 17 deals with macroscopic applications like magnetohydrodynamic plasma systems, elastic spheres, nonlinear surface diffusion, and neutron stars.

The book is closed by a mathematical annex including a section on nonlinear dispersion relations and their use for nonlinear systems of partial differential equations.

A legitimate question of the potential reader would be: "Why one more book on solitons?" First of all, we have to acknowledge the importance of the interactions between compact boundary manifolds and the dynamics of particles and fields in mathematical in physical models. Historically, the solitons are observed in sort of “infinite” systems like infinite long lines or curves, planes or open surfaces, or unbounded space. However, there is more and more evidence of the existence solitons or of localized patterns (like vortices) in compact lower dimensional spaces, like closed curves and/or surfaces. As examples, we can mention the unprecedented information technology advances in optical communication (light bullets and ultra-short optical pulses), solid-state spectroscopy, ultra-cold atom studies, soliton molecules, spinning solitons, quantum computers, spintronics and mass memory systems, femtosecond laser pulses, mesoscopic superconductivity, etc. Consequently, the reasons for writing this book are generated by a constantly increasing number of new challenges, vivid topics and hundreds of published articles.

If a substantial percentage of users of this book feel that it helped them to enlarge their outlook in the intersection between the fascinating worlds of nonlinear waves and compact surfaces and closed curves, its purpose has been fulfilled.

While writing the second edition of this book, I have greatly benefited from discussions with my colleagues. I am particularly grateful to Ivailo Mladenov, Thiab Taha, Annalisa Calini, Adrian Stefan Carstea who provided an inspirational and valuable help in the elaboration of this second edition. For the best advices and uninterrupted encouragement, I am indebted to my family.

May 2011

Andrei Ludu
Department of Mathematics
Embry-Riddle Aeronautical University
Daytona Beach, USA

Preface to the First Edition

Everything the Power of the World
does is done in a circle. The sky is
round and I have heard that the earth
is round like a ball and so are all the stars.
The wind, in its greatest power, whirls.
Birds make their nests in circles,
for theirs is the same religion as ours.
The sun comes forth and goes down
again in a circle. The moon does the
same and both are round. Even the
seasons form a great circle in their
changing and always come back again
to where they were. The life of a man
is a circle from childhood to childhood.
And so it is everything where power moves.
Black Elk (1863-1950)

Nonlinearity is a captivating manifestation of the observable Universe, whose importance has increased over the decades, and has found more and more fields of application ranging from elementary particles, nuclear physics, biology, wave dynamics at any scale, fluids, plasmas to astrophysics. The central character of this 172-year-old story is the soliton. Namely, a localized pulse traveling without spreading and having particle-like properties plus an infinite number of conservation laws associated to its dynamics. In general, solitons arise as exact solutions of approximative models. There are different explanations, at different levels, for the existence of solitons. From the experimentalist point of view, solitons can be created if the propagation configuration is long enough, narrow enough (like long and shallow channels, fiber optics, electric lines, etc), and the surrounding medium has an appropriate nonlinear response providing a certain type of balance between nonlinearity and dispersion. From the numerical calculations point of view, solitons are localized structures with very high stability, even against collisions between themselves. From the theory of differential equations point of view, solitons are cross-sections in the jet bundle associated to a bi-Hamiltonian evolution equation (here Hamiltonian pairs are requested in connection to the existence of an infinite collection of conservation laws in involution). From the geometry point of view, soliton equations are compatibility conditions for the existence of a Lie group. From the physicist point of view, solitons are solutions of an exactly solvable model having isospectral properties carrying out an infinite number of non-obvious and counter-intuitive constants of motion.

The progress in the theory of solitons and integrable systems has allowed the study of many nonlinear problems in mathematics and physics: elementary particle nonlocal interactions, collective excitations in heavy nuclei, Bose-Einstein condensates in atomic physics, propagation of nervous influxes, nonlinear oscillations of liquid drops, bubbles, and shells, vortexes in plasma and in atmosphere, tides in neutron stars, etc., only to enumerate few of possible applications. A number of other applications of soliton theory also lead to the study of the dynamics of boundaries. In that, the last three decades have seen the completion of the foundation for what today we call nonlinear *contour dynamics*. The subsequent stage of development along this topic was connected with the consideration of a almost *incompressible* systems, where the boundary (contour or surface) plays the major role.

The first problem about such compact systems is that shape solitons, that usually exist in infinite long and shallow propagation media, can not survive on a circle or sphere. That is because such compact manifolds can not offer the requested type of environment (long and narrow), even by the introduction of shallow layers and rigid cores. However, there is another basic idea that supports, in a natural way, the existence of nonlinear solutions on compact spaces. Because of its high localization, a soliton (or a compacton) is not a unique solution for the partial differential system. Its position in space is undetermined because, far away from its center, the excitation is practically zero. On the other hand, all linear equations provide uniqueness properties for their solutions. It results that strongly localized solutions and almost-compact supported solutions can be generated only within nonlinear equations. There is an exception here: the finite difference equations with their compact supported wavelet solutions, but in some sense, a finite-difference equation is similar to a nonlinear differential one.

Despite the many applications and publications on nonlinear equations on compact domains, there are still no books introducing this theory, except for several sets of lecture notes. One reason for this may be that the field is still undergoing a major development and has not yet reached the perfection of a systematic theory. Another reason is that a fairly deep knowledge of integrable systems on compact manifolds has been required for the understanding of solitons on closed curves and compact surfaces.

The main aim of this book is to present models of nonlinear phenomena that occur mainly on closed, compact surfaces or curves, especially where solitons and solitary waves are involved. The approach of the physical problems ranging from nuclear to astrophysical scales is made in the language of differential geometry. The text is rather intended to be an introduction to the physics of solitons on compact systems like filaments, loops, drops, etc., for students, mathematicians, physicists, and engineers. However, the book does not elaborate on the general theory of solitons, or the inverse scattering problem, for example. The author assumes that the reader has some previous knowledge about solitons, integrable systems and nonlinearity in general. The book furnishes the reader with models related to compact boundaries and their nonlinear dynamics, and, if available, with soliton-like solutions. This is a book to be read with pencil, paper, and a symbolic computer program at hand. Our intention is to furnish readers with enough knowledge to be able to identify, understand, and model such nonlinear systems.

This text is still far from being a comprehensive study on the topic of solitons on compact systems. It consists of 18 chapters, an appendix, and a bibliography. First part contains the fundamental differential geometry and analysis approach. To render this book accessible to students in science and engineering, Chapter 2 recalls some basic elements of topology. In Chapter 3, we review some representation formulas for different dimensions, as expressions of the comprehensive information contained in the boundaries. Chapter 4 introduces the reader in the calculus on differentiable manifolds, vector fields, forms, and various type of derivatives. Chapter 5 lays the basis for the differential geometry of curves in $\mathbf{R}_3$. In Chapter 6, we derive the theory of motion of curves, and we relate these motions with soliton solutions. In Chapter 7, we recall some elements of differential geometry of the surfaces, with applications on the action of differential operators on surfaces. In Chapter 8, we discuss the theory of motion of surfaces.

The second part of the monograph contains an exposition of the basic branches of nonlinear hydrodynamics. The working frame of hydrodynamics is the main content of the first part of the monograph, namely, Chapter 9. In Chapter 10, we discuss the problems on surface tension effects and representation theorems for fluid dynamics models. Chapter 11 concentrates with one-dimensional integrable systems on compact intervals, and their periodic solutions. Chapters 12 and 13 deal with nonlinear shape excitations of two-dimensional, and three-dimensional liquid drops and bubbles. Chapter 14 is devoted for various applications of three-dimensional nonlinear drops, and also to compact supported solitons.

In the third part of the book, as a final goal for the first two parts, we present additional physical applications of nonlinear systems and their soliton solutions on various systems of different scales. In Chapter 15, we study the vortex filaments and other one-dimensional flows. In Chapter 16, we describe microscopic applications like exotic shapes in heavy nuclei, exotic radioactivity, and quantum Hall drops. Chapter 17 deals with macroscopic applications like magnetohydrodynamic plasma systems, elastic spheres, neutron stars, etc.

The book is closed by a mathematical annex including a section on nonlinear dispersion relations and their use for nonlinear systems of partial differential equations.

The last comment of this preface would be: Why one more book on solitons, and why on compact spaces? A first answer is that there are already a large number of application on these vivid topics and hundreds of published articles. On the other hand, there is the importance of compact manifolds themselves in physics.

If a substantial percentage of users of this book feel that it helped them to enlarge their outlook in the intersection between the fascinating worlds of nonlinear waves and compact surfaces and closed curves, its purpose has been fulfilled.

I have greatly benefited from discussions with my colleagues and students, and I am particularly grateful to Thiab Taha for his sedulous and constant effort to provide the frame for such discussions through his nonlinear waves meetings. I should like to thank to whom gave me help and support to write this book: Randall J. Webb, Austin L. Temple, and the National Science Foundation (through the grant PHYS-0140274). For interesting and helpful conversations, I am indebted to many friends. For discussions and constant encouragement, I am indebted to my family. During the completion of the manuscript, Bob Odom has given me valuable suggestions. The last but not at all the least I am thankful to the Watson Library and the group working with the Illiad interlibrary borrowing who offered me the chance to cover all the necessary references.

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Andrei Ludu
Department of Physics and Astronomy
Louisiana State University
Baton Rouge, USA

Contents

Symbols

In general the spaces ($\mathbb{R}^3$ for example), the vectors ($\boldsymbol{v}$), and the matrices are denoted with bold letters, and the dimension is represented as a subscript.

$A \lhd B$	$A \subset B$ and A has the same structure as B (it is a sub-structure)
A, dA	Element of area, surface element
$\boldsymbol{a}, \boldsymbol{A}, \boldsymbol{v}$	Vector field
$\alpha, \beta, \delta, \ldots$ (not γ)	Unspecified labels in general (or labels from 1 to 2)
$\boldsymbol{b}$	Binormal in the Serret–Frenet frame
$C^k(M)$	Class of differentiable functions of order k defined on M
$C^\infty(M)$	Class of infinite-differentiable functions defined on M, also called *smooth* in this book
$D_{\boldsymbol{x}}$	Directional derivative
$Diff(A, B)$, $Hom(A, B)$, etc.	Diffeomorphisms, homeomorphisms, etc. from A to B.
$\blacktriangledown$	Covariant differential
∂M	Boundary of the domain M
E, F, G, e, f, g	Second fundamental form coefficients
f, df, f^*	Mapping, its differential, and the pull-back
g	Metrics
γ, Γ	Parametrized curve, or
Γ, ω	Connection, connection form
H	Mean curvature
$i, j, k, l, \ldots$	Labels in general (or labels in the $1, 2, \ldots, n \geq 3$)
$i = \sqrt{-1}$	If specified in the context
I	First fundamental form
K	Gaussian curvature
κ	Curvature of a curve
$\kappa_{1,2}$	Principal curvatures

κ_n	Normal curvature
κ_g	Geodesic curvature
M, X, Y	Manifold
ν	Viscosity
$\nabla_v, \nabla_{\alpha'}$	Covariant derivative
$\nabla_\Sigma, \nabla_\Sigma\cdot, \nabla_\Sigma\times, \Delta_\Sigma$	Surface gradient, surface divergence, surface curl, surface Laplacean
$\boldsymbol{n}, \boldsymbol{N}$	Normal to a curve in the Serret–Frenet frame, normal to a surface
ODE, PDE	Ordinary or partial (system of) differential equation(s)
Ω, ω	Differential form
Π	Second fundamental form of a surface
Σ	Surface
s, ds	Arc-length
$TM, TX, TY, \ldots$	Tangent space
t	Time
t	Unit tangent to a curve
$\boldsymbol{t}, \boldsymbol{N}, \boldsymbol{t}^\perp = \boldsymbol{N} \times \boldsymbol{t}$	Darboux frame associated to a given curve lying on a surface
τ	Torsion
$\hat{A}$	Tensor in general
τ_g	Geodesic torsion
u, v	Surface parameters
u, t	Curve parameters
v	Volume, only if results from context
$dv, d^n x, d^3 x$	Element of volume
$\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{V}, \boldsymbol{U}, \boldsymbol{W}$	Velocities or vorticities
$\boldsymbol{v}(\boldsymbol{w})$	Lie derivative of $\boldsymbol{w}$ with respect to (or along) $\boldsymbol{v}$
w.s.	Without summation
$x^i = \{x, y, z\}$	Specific three-dim coordinate notation
$x^\sigma = \{u, v\}$	Specific two-dim coordinate notation.

In general, the notations below represent the same expression

$$\int^x f(x)dx = \int^x f(x')dx' = \int_0^x f(x)dx + \mathcal{C}.$$

Chapter 9
Kinematics of Fluids

The goal of this chapter is to discuss the general frame of hydrodynamics, like particle trajectories (path lines), stream lines, streak lines, free surfaces, and fluid surfaces, and to compare their behavior in the Eulerian and Lagrangian frames. The following sections and chapters proceed on the assumption that the fluid is practically continuous and homogeneous in structure. Of course, the concept of continuum is an abstraction that does not take into account the molecular and nuclear structure of matter. In that, we assume that the properties of the fluid do not change if we consider smaller and smaller amounts of matter [1]. May be the wisest point of view while we remain at the level of general laws of fluid dynamics (or fluid mechanics) is to keep the physical scales rather vague [2]. This aspect is in direct relation with the fact that these laws can be made dimensionless in a large variety of situations.

9.1 Lagrangian Verses Eulerian Frames

In fluid dynamics there are two possible approaches for the dynamical equations: the Lagrangian (also called material or convected) frame and the Eulerian (also called the spatial) frame. In the Lagrangian frame we identify and label individual particles of fluid, and we setup the frame such that particles retain their coordinate labels in time. In this approach, it is more likely to use topology and group continuous transformation tools. The Eulerian frame describes the fluid from a stationary lab frame. The motion of fluid is recorded at a fixed point verses time. In this approach the mathematical tools are more related to geometry and field theory. In the following, we use the Eulerian approach, unless an explicit statement is made to the contrary. The fields that characterize the fluid are defined on some domains in the three-dimensional Euclidean space and they have a certain degree of mathematical smoothness. The degree of smoothness is chosen for a given fluid model such that the

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coarse grain structure of the infinitesimal fluid particles introduced above is not seen by the differential equations (i.e., the molecular structure of the matter). In other words, the fluid particle is small enough to allow the existence of smooth space–time differentials, but large enough to average the molecular and quantum properties over its volume. The fields under consideration are the velocity field $\boldsymbol{v}(\boldsymbol{r}, t)$, the non-negative defined mass density $\rho(\boldsymbol{r}, t)$, and the pressure field $P(\boldsymbol{r}, t)$. Of course, function of necessity, we can add the distribution of energy, free energy, enthalpy, entropy, force density, or other fields of interest [1, 3] to these fields. We assume, unless otherwise specified, that these fields are smooth enough so that the standard calculations may be performed on them.

9.1.1 Introduction

In practice we consider $\boldsymbol{r} = (x, y, z) \in D$ a point in domain D filled with fluid, and consider the particles moving in space and time. In the Lagrangian approach, at every moment of time t we defined the spatial velocity of a certain particle of fluid as $\boldsymbol{V} = \frac{d\boldsymbol{r}}{dt}$.

The Eulerian velocity field (spatial velocity field) $\boldsymbol{V}(\boldsymbol{r}, t)$, in principle not constant in time, is the velocity of a fluid particle that passes at moment t through the point $\boldsymbol{r}$. The Lagrangian frame is attached to that fluid particle, and it records the changes in velocity, density, etc., happening with this particle verses its own local time, measured with a clock attached to it. In such a Lagrangian system, physical quantities have a complex time dependence. While traveling, the fluid particle has its physical quantities measured in the local frame, so they experience a global time variation (also called total or Lagrangian or material time derivative) denoted by $\frac{d}{dt}$, or identified by placing a dot on the top of the quantity (sometimes it is also denoted $\frac{D}{Dt}$). A part of this time variation happens because the particle travels through different domains of space, hence experiencing different constraints. Such a partial variation is called Eulerian, or partial, and it is denoted $\frac{\partial}{\partial t}$ or simply by the subscript t. For example, we choose a fluid particle moving according to the law $\boldsymbol{r}_L(t)$, and we measure the scalar quantity $q(t) \equiv q(\boldsymbol{r}_L(t), t)$ associated to this particle, in this frame. The same quantity can be described in a fixed Eulerian frame, $Q(\boldsymbol{r}, t)$. The relation between these two formal approaches is given by

$$\dot{q} = \frac{dq}{dt}(\boldsymbol{r}_L(t), t) = \frac{\partial Q}{\partial t}(\boldsymbol{r}, t) + \boldsymbol{V}(\boldsymbol{r}, t) \cdot \nabla Q(\boldsymbol{r}, t), \tag{9.1}$$

where ∇ is the gradient operator $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, and $\cdot$ represents the usual Euclidean scalar product. Equation (9.1) is a well-known transformation law in hydrodynamic literature, yet is valid in a very restricted sense, namely only for scalar quantities and for the fluid velocity vector. If we try to apply the transformation (9.1) to a general vector field or to a covariant tensor field, the result fails, because the resulting quantity is not anymore a geometrical object of the same type. To keep the geometrical properties intact, we need a generalization of (9.1) for arbitrary covariant/contravariant geometrical objects ω. This is the *covariant time derivative* (also called convected or material time derivative) and it is defined by

$$\frac{d_c\omega}{dt} = \frac{\partial\omega}{\partial t} + \boldsymbol{v}(\omega), \tag{9.2}$$

where $\boldsymbol{v}(\omega)$ is the Lie derivative with respect to the flow $\boldsymbol{v}$. This generalization is introduced in Sect. 9.2.6.

9.1.2 Geometrical Picture for Lagrangian Verses Eulerian

We introduce the working space $(t, \boldsymbol{r}) \in \mathbb{R} \times \mathbb{R}^3$. From the Lagrangian point of view, the fluid particle motions are non-intersecting regular curves Γ_L in this base space, parameterized by time and described by equations $\boldsymbol{r}_L(t, \boldsymbol{r}_0)$. They are called *paths* or *material lines* [4] or *lines of motion* [1]. Since they do not intersect, each such curve is labeled by one of its points, $\boldsymbol{r}_0$, for example the position of the particle when $t = 0$. The tangent to this curve is

$$\boldsymbol{t}_L = \frac{(1, \boldsymbol{v}_L)}{\sqrt{1 + v_L^2}},$$

where $\boldsymbol{v}_L = \partial\boldsymbol{r}_L(t, \boldsymbol{r}_0)/\partial t$ is the Lagrangian velocity of the particle along the path. All these paths do not intersect and completely fill the base space when $\boldsymbol{r}_0 \in \mathbb{R}^3$.

If we choose a fixed point in space $\boldsymbol{r}$, some of the paths $\boldsymbol{r}_0$ will intersect this fixed point, $\boldsymbol{r}_L(t, \boldsymbol{r}_0) = \boldsymbol{r}$, so that we can write the "list" of these particles vs. time: $\boldsymbol{r}_0 = \boldsymbol{r}_0(t, \boldsymbol{r})$. Now, we can define the Eulerian velocity at $(t, \boldsymbol{r})$ by substituting this $\boldsymbol{r}_0(t, \boldsymbol{r})$ list in the velocity expression

$$\boldsymbol{v}_E(t, \boldsymbol{r}) = \boldsymbol{v}_L(t, \boldsymbol{r}_0(t, \boldsymbol{r})). \tag{9.3}$$

Example 1 We can illustrate the relation between Lagrangian and Eulerian velocities (9.3) with a simple one-dimensional example. Water is dripping downward from a hole in gravitational field, and different water molecules depart from the hole at different initial moments of time t_0. So the Γ_L curves are vertical parallel lines. Their laws of motion are

$$z(t) = \frac{g(t - t_0)^2}{2}.$$

In terms of some initial position z_0 their Lagrangian equations of motion read

$$z_L(t, z_0) = \frac{g}{2}\left(t - \sqrt{\frac{2z_0}{g}}\right)^2,$$

with

$$v_L(t, z_0) = g\left(t - \sqrt{\frac{2z_0}{g}}\right).$$

If we choose a reference level at z and equate $z = z_L$, we obtain

$$z_0 = \frac{g}{2}\left(t - \sqrt{\frac{2z}{g}}\right)^2$$

with the following signification: What is the initial position z_0 (at $t = 0$) of a particle to pass through the level z at the moment t? The resulting Eulerian velocity is, according to (9.3),

$$v_E(t, z) = v_L(t, z_0(z, t)) = \sqrt{2zg} = \text{const.},$$

as it should be from mechanics.

Now, we introduce a physical quantity Q defined for any fluid particle. For the particle labeled by $\boldsymbol{r}_0$ the Lagrangian value $Q_L(t, \boldsymbol{r}_0)$ is defined along Γ_L. Suppose this Γ_L intersects a fixed line $\boldsymbol{r} =$ constant at $\boldsymbol{r}_L(t, \boldsymbol{r}_0) = \boldsymbol{r}$. By solving this equation with respect to $\boldsymbol{r}_0$, we have $\boldsymbol{r}_0 = \boldsymbol{r}_0(t, \boldsymbol{r})$. We can define now the Eulerian value of Q by

$$Q_E(t, \boldsymbol{r}) = Q_L(t, \boldsymbol{r}_0(t, \boldsymbol{r})). \tag{9.4}$$

While following the particle in its motion, the quantity Q_L has a variation $dQ_L(t, \boldsymbol{r}_0) = (dQ_L/dt)dt$. At $\boldsymbol{r} =$ const., the quantity Q_E has another variation $dQ_E = (\partial Q_E/\partial t)dt$. By differentiation of (9.4) we have $dQ_L = dQ_E + (d\boldsymbol{r}_L \cdot \nabla Q_E)dt$. Since we follow the particle in its motion we have $d\boldsymbol{r}_L = \boldsymbol{v}_L dt$. Since all these relations are infinitesimal, and all are taken at $(t, \boldsymbol{r})$, we can use either $\boldsymbol{v}_E$ or $\boldsymbol{v}_L$ in them. In the end we obtain the classical relation between the Lagrangian and Eulerian variations of a physical quantity

$$\frac{dQ_L}{dt} = \left(\frac{\partial Q_E}{\partial t} + (\boldsymbol{v}_E \cdot \nabla)Q_E\right). \tag{9.5}$$

In local (Eulerian) coordinates $(t, \boldsymbol{r})$, this equation reads

$$(t, \boldsymbol{r}) \to \frac{dQ_L}{dt}(t, \boldsymbol{r}_0(t, \boldsymbol{r})) = \left(\frac{\partial Q_E}{\partial t} + (\boldsymbol{v}_E \cdot \nabla)Q_E\right)_{(t,\boldsymbol{r})}. \tag{9.6}$$

In the Lagrangian coordinates $(t, \boldsymbol{r}_0)$, same equation reads

$$(t, \boldsymbol{r}_0) \rightarrow \frac{dQ_L}{dt}(t, \boldsymbol{r}_0) = \left(\frac{\partial Q_E}{\partial t} + (\boldsymbol{v}_E \cdot \nabla) Q_E \right)_{(t, \boldsymbol{r} = \boldsymbol{r}_L(t, \boldsymbol{r}_0))}. \qquad (9.7)$$

The Lagrangian motion of particles is represented by a family of curves Γ_L filling the base space, and the Lagrangian velocity is a vector field defined on this base space, parametrized by the flow lines. The Eulerian velocity is the same differential vector field, except is parametrized by local coordinates, like any regular field. Consequently, a Lagrangian physical quantity Q_L is represented by a family of curves Γ_Q lying in a base space $\mathbb{R} \times \mathbb{R}^3 \times \hat{Q}$, where $Q \in \hat{Q}$. The Eulerian value of the same quantity is a regular surface $Q_E(t, \boldsymbol{r})$ parametrized by the base space and immersed in $\mathbb{R} \times \mathbb{R}^3 \times \hat{Q}$. The Eulerian derivative is the partial derivative of Q_E. The particle paths Γ_L have tangents

$$\boldsymbol{t}_L = \frac{1}{\sqrt{1 + v_L^2}}(1, \boldsymbol{v}_L).$$

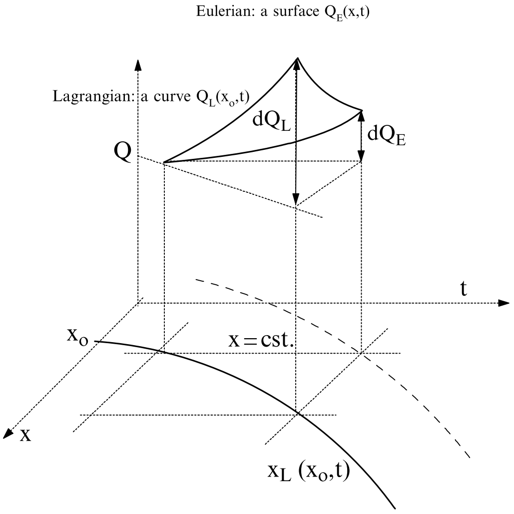


Fig. 9.1 The Lagrangian–Eulerian point of view for a one-dimensional flow. The path of a fluid particle is represented in the base horizontal plane by the curve $x_L(x_0, t)$; all such fluid paths are labeled by their x_0 initial points. The mapping of the fluid path into the base space of a physical observable Q is a curve $x_Q(x_0, t)$, i.e., the Lagrangian value of the physical quantity $Q_L(x_0, t)$. The Lagrangian variation along the fluid path is dQ_L in a certain dt. But, if we measure Q at a constant position x, we have its Eulerian value, and consequently its Eulerian variation dQ_E for the same time interval dt. The Eulerian value $Q_E(x, t)$ actually represents the Lagrangian value associated to another particle (*dashed line*) that actually moves through the same spot x at $t + dt$. When fluid particles fill up the space x and move, the Lagrangian values of the physical quantities associated to the particles of fluid generate curves, but the Eulerian values generate a surface

The curves for Q_L lying in the base space have tangents

$$\hat{t}_Q = \frac{1}{\sqrt{1 + v_L^2 + \dot{Q}_L^2}}(1, \boldsymbol{v}_L, \dot{Q}_L),$$

where the dot means time differentiation. In this geometrical context, the relation between Lagrangian and Eulerian variations (9.5) reads

$$\dot{Q}_L = D_{t_{\Gamma_L}} Q_E, \quad \text{or} \quad \dot{Q}_L(t) = (Q_E \circ \Gamma_L)'(t).$$

The Lagrangian derivative is just the directional derivative of the function Q_E along the particle path, see Fig. 9.1.

9.2 Fluid Fiber Bundle

9.2.1 Introduction

Hydrodynamics studies the motion of fluid particles. The combination between the discrete labeling of the system of particles on one hand, and the smooth dependence of physical quantities on time on the other hand enhances the importance of families of curves for hydrodynamical systems. Somehow, this fact has a geometrical background arriving from the importance of compact submanifolds (closed curves, closed surfaces) for vector fields and flows (see Sect. 3.5) and [5]. Curves of special interest, parametrized by time, are the path lines, stream lines, streak lines, and vorticity lines, studied from both Lagrangian and Eulerian points of view (Sect. 9.1.2). Moreover, there are the fluid particle lines (also called material lines, particle contours, or circuit lines) and filaments especially important in conservation laws. We can raise the question if such particle contours are stable or they break at a certain point, or if they are invariant, etc. For example, to use the Kelvin or Ertel's theorems for closed contours (Theorem 10.3) related to invariants of the fluid dynamics, we need to have rigorous definition for the material lines of fluid particles than just intuition.

Example 2 To exemplify such a possible situation, when a particle contour can deform up to a breaking point (because of a stagnation point of the flow, for example) we choose an incompressible inviscid irrotational two-dimensional flow past a cylinder. To solve the flow we use a conformal mapping procedure. The velocity field is represented by $v(z) = \phi_x + i\phi_y$, $z = x + iy$, and it is tangent to the curves $\phi =$ const. because of the Riemann–Cauchy conditions. We build the holomorphic function $H(z) = \Phi(x, y) + i\Psi(x, y)$ where Φ is the potential function and Ψ is the stream function, i.e., the harmonic conjugate function to Φ. We have

$$v = \frac{dH^*}{dz},$$

and the cylinder contour Γ equation is $x^2 + y^2 = 1$. We perform the transformation $u + iv = \omega = f(z) = z + z^{-1}$. The cylinder contour transforms into $f(\Gamma) = \{z|v = 0\}$. A solution of the Laplace equation in the ω coordinates and for the boundary condition $\omega = 0$ on $f(\Gamma)$ is $G(\Phi) = \Phi_0\omega$. We have

$$H(z) = G \circ f(z) = A\left(z + \frac{1}{z}\right).$$

For example, in polar coordinates the stream lines ($\Psi = \text{const.}$) become

$$\Psi_0\left(r - \frac{1}{r}\right)\sin\phi = C = \text{const.}$$

The equation of the stream lines becomes

$$r(\phi) = \frac{r_0 + \sqrt{r_0^2 + 4\sin^2\phi}}{2\sin\phi}$$

and the Eulerian velocity is

$$\boldsymbol{v} = \Psi_0\left(\frac{-y\cos\phi(x^2 + y^2 - 1) + x\sin\phi(x^2 + y^2 + 1)}{(x^2 + y^2)^{3/2}},\right.$$
$$\left.\frac{x\cos\phi(x^2 + y^2 - 1) + y\sin\phi(x^2 + y^2 + 1)}{(x^2 + y^2)^{3/2}}\right).$$

From the Euler equation the pressure becomes

$$P = \Psi_0^2\rho\frac{2(x^2 - y^2) - 1}{2(x^2 + y^2)^2},$$

where ρ is the density. In Fig. 9.2 we present the pressure distribution around the cylinder contour. The Lagrangian paths of fluid particles are obtained by numerical integration of the equations

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial x_0} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial x_0} = -\frac{1}{\rho}\frac{\partial P}{\partial x_0}, \ldots \text{etc.}$$

In Fig. 9.3 we present the isobaric and stream lines, and the evolution of a particle contour line (thick line). Initially we choose all particles of this contour line to lie along a vertical segment. Then, we calculate their Lagrangian positions at a later moment of time. We notice the tendency of the contour line to spread and tear. In an extreme example this line may even be broken by possible abrupt changes in

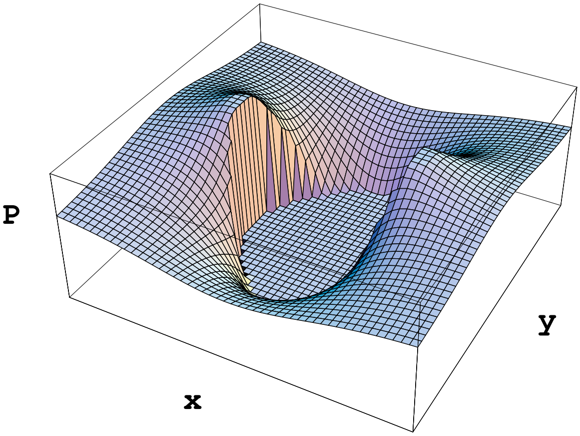


Fig. 9.2 Pressure distribution for a two-dimensional incompressible inviscid irrotational flow past a cylinder

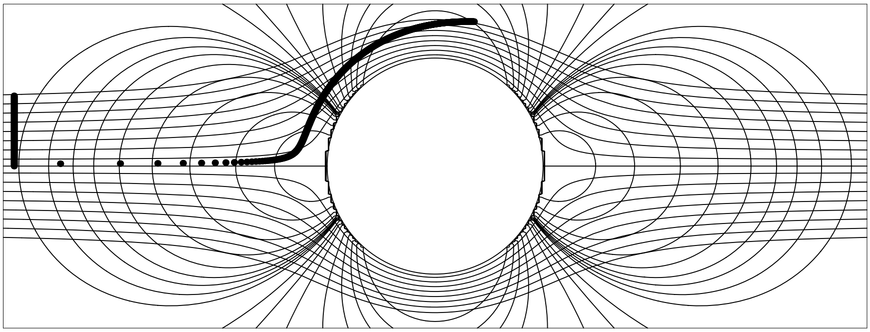


Fig. 9.3 Stream lines and isobaric lines (*thin lines*) for a two-dimensional incompressible inviscid irrotational flow past a cylinder. *Thick lines*: a finite particle contour at $t = 0$ (the vertical segment), and its Lagrangian flow at a later moment of time

the Lagrangian velocities. This example shows that it makes sense to analyze the geometry and stability of particle contours for a general flow.

9.2.2 Motivation for a Geometrical Approach

We can always present a fluid using the following traditional picture of the flow, also introduced in Sect. 9.1.2. We introduce the available space for the fluid (the reference fluid container [6, 7]) as a domain D of $\mathbb{R}^3$, and add an extra dimension for time to form a base space $D \times \mathbb{R}$. The particle paths $\boldsymbol{r}_L(\boldsymbol{r}_0, t)$ are smooth time-parametrized curves in this base space. The projection on the horizontal planes (projections perpendicular

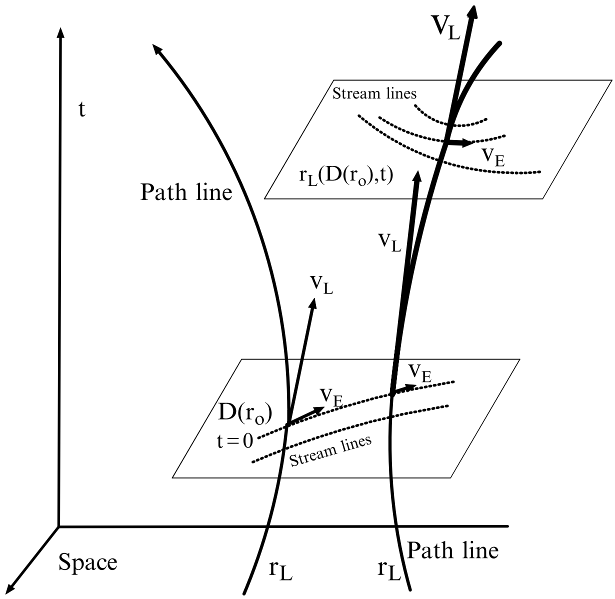


Fig. 9.4 A two-dimensional fluid domain $D(\boldsymbol{r}_0)$ shown at two moments of time 0, t, and two path lines $\boldsymbol{r}_L(t)$ whose tangents are the Lagrangian velocities $\boldsymbol{v}_L$. The projection of the Lagrangian velocity field on the tangent space of the fluid domain is the Eulerian velocity field $\boldsymbol{v}_{\mathcal{E}}$. The integral curves of the Eulerian vector field in the fluid domain, at a given moment of time t, are the stream lines at that moment (*dotted lines*). The projections of the path lines on the fluid domain do not coincide with path lines in general

on the time axis) of the tangent vectors to these curves represents the velocity fields of the particles. The two velocities, i.e., the Lagrangian (material) and Eulerian (spatial) velocities, have the same value at the same point of the base space. The only difference between these two types of velocities consists in the parametrization of the vector fields. The Lagrangian velocity field is defined along the particle paths in the base space, while the Eulerian velocity field is defined on the horizontal plane, in points where these paths intersect it, at a moment of time t. The integral curves of the Eulerian velocity field contained in any "horizontal" plane are the stream lines at that moment of time. However, the path lines do not identify with the lift of the stream lines in the base space. Namely, if we choose a point $\boldsymbol{r}$ in some horizontal plane t and we compare the path line crossing through this point, and the vertical lift of the stream line crossing the same point, these two curves are different in general. An example is presented in Fig. 9.4. In Fig. 9.5 we show another example of path lines and stream lines, when the particle moves along an open path, but locally the stream lines may appear to be closed.

For any given fixed point $\boldsymbol{r}_0$ in the initial plane, we can draw all paths crossing this at different moments of time (Fig. 9.6). The intersections of all these paths with a certain horizontal plane t generate a streak line initiated by a "nozzle" placed at $\boldsymbol{r}_0$.

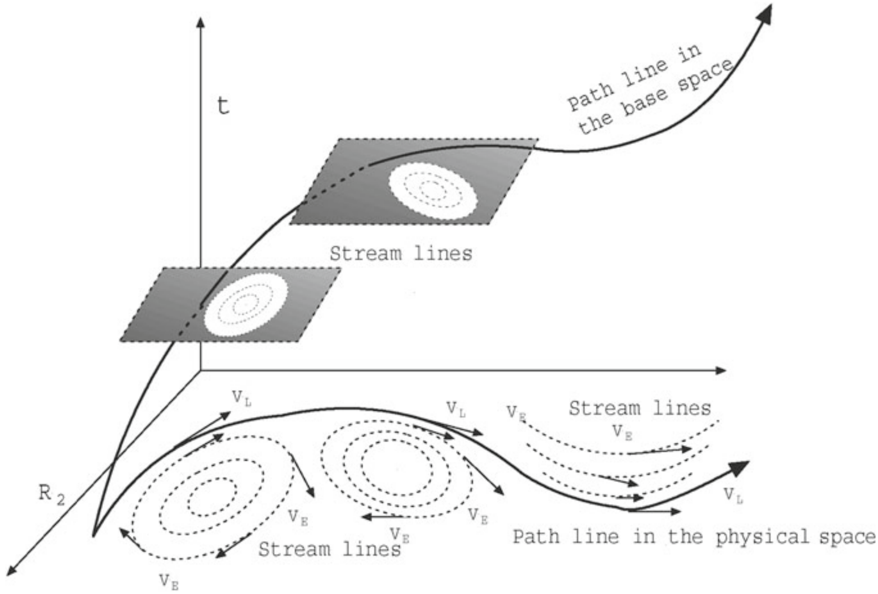


Fig. 9.5 A two-dimensional example. A path line in the physical space $\mathbb{R}^2$ (*horizontal solid curve*) and in the base space X (*lifted solid curve*), and associated stream lines at different moments of time (*dashed lines*)

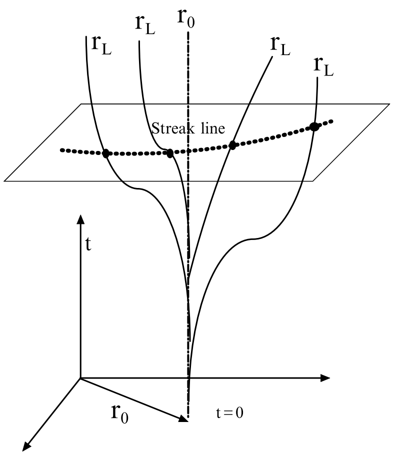


Fig. 9.6 Same space as in Fig. 9.4, except we present several paths emerging from the "nozzle" point $\boldsymbol{r}_0$ (*dashed-dotted axis*) at different moments of time. The intersections of all such paths with a horizontal plane t provide a streak line (*dotted*) generated by the "nozzle" at t

In traditional approaches, see for example [2, 8–10], the motion of the particles is described by a one-parameter (time) group of diffeomorphisms acting on the domain $D(\boldsymbol{r}_0)$. The Lagrange coordinate of a particle is the result of the action of this group on the corresponding element $\boldsymbol{r}_0$. If the motion is incompressible, the group of diffeomorphisms is volume preserving. In this formalism, the infinitesimal generator of the group is the Lagrangian field of velocities.

However, even practical, such a model is not quite perfect. That is because we tend to associate the same geometrical space to physical spaces with different signification, namely the material points (initial positions space), and the spatial points per se. Even if initially ($t = 0$) the positions $\boldsymbol{r}_0$ of all fluid particles, $\boldsymbol{r}_0 \in D$, belong to the position space during the motion, these vectors actually form a space of parameters, labeling the particles. On the other hand, the positions of the particles at any arbitrary moment of time (given by the Lagrangian equations of motion $\boldsymbol{r}_L(\boldsymbol{r}_0, t)$) belong to a space of positions. The above picture does not make this difference a geometrical difference, and in that is incomplete and difficult to generalize for more complicated flows. For example, in Fig. 9.4, we can see that the stream lines at different moments of time belong to different planes. We need to make the distinction between the material space and the space of positions from a geometrical perspective. This is possible by using a fiber bundle structure instead of a common space.

9.2.3 The Fiber Bundle

We present a formalism in which a fluid is described using cross-sections σ in a fiber bundle $\mathcal{F}$ over some base manifold X. For the definitions and properties of a fiber bundle, the reader can check Sect. 3.9 and its [6, 7, 11, 12]. An intuitive picture of a fiber bundle consists in taking a certain manifold called fiber F, and assigns a homeomorphic transformation of F to any point of a base manifold X, constructing a sort of a local cartesian product. In the case of a fixed container for the fluid (even the case of the whole space), the traditional model is to consider the base as the space of particles (usually labeled by their initial positions) and the fiber is the space available for particle positions (see Fig. 9.7, left). On the contrary, a free surface introduces one more freedom in the problem. We cannot construct it using the same pattern (see Fig. 9.7, center) because we allow different particles to belong to different shapes simultaneously, which is impossible. A possible choice to build a fiber bundle is borrowed from the mechanics of deformable bodies (see Fig. 9.7, right). The base space is the manifold of all possible shapes, and the standard fiber is particle position space. The role of the particle labeling space is taken over by the nontrivial structure group.

The base manifold (for the nonrelativistic case) is usually a space–time manifold built as a product between a smooth three-dimensional oriented Riemannian manifold (M, g), where g is the metric, and $\mathbb{R}$ for time, i.e., $X = M \times \mathbb{R}$. The coordinates in X are $x = (x^\mu) = (x^i, t) \in X$, with $i = 1, \ldots, 3$, $\mu = 1, \ldots, 4$. For fluid dynamics we can choose the fiber $F = M$ with coordinates $y \in F$ [6]. Consequently,

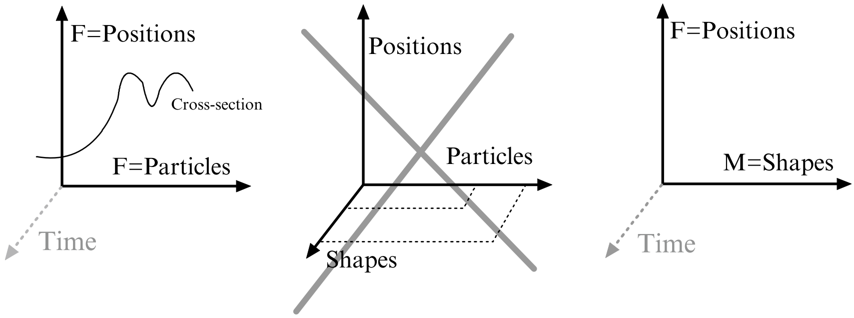


Fig. 9.7 Possible fiber bundle structures (M, F) for fluid dynamics problems. Left: In the case of no free surface the base space is the space of particles, and the fiber is the space available for the particles positions; Center: A free fluid surface introduces more freedom in the problem making the previous (Left) structure inoperable. It would allow different particles to belong to different shapes simultaneously, which is impossible; Right: Mechanics of deformable bodies model for the fiber bundle. The base space is the manifold of all possible shapes, and the standard fiber is particle position space. *Dotted line* means that time does not need necessarily to be included explicitly in the geometry picture

the local coordinates in this $\mathcal{F}$ bundle over X are (x, t, y), and the projection is $\Pi : \mathcal{F} \to X,\ (x, t, y) \to (x, t)$. Transformations and operations that affect only the base (spatial changes like rotations, etc.) are called fiber-preserving transformations. A lift of any geometrical object γ (a curve, surface, function, form, etc.) defined in the base space is a map of this object into the fiber bundle, $\gamma \to \gamma' \in \mathcal{F}$, such that it projects back down to the original object in M, $\Pi \circ \gamma' = \gamma$.

Cross-sections in this bundle $\sigma : X \to \mathcal{F}$ represent time-dependent *configurations*, i.e., particle position fields. The cross-section has the coordinates $\sigma(x) = (x^\mu, \sigma^i(x)) = (x^\mu, y^i)$. On the top of the configuration bundle $\mathcal{E}$, we can construct another fiber bundle $J^1\mathcal{F}$ over $\mathcal{F}$ called the first jet bundle [6, 13], with the fiber above (x, y) consisting of linear maps from the tangent space of the base space to the tangent space of the bundle, $\gamma : T_x X \to T_{(x,y)}\mathcal{F}$, satisfying $d\pi \circ \gamma = Id_{T_x X}$.

For any cross-section σ in $\mathcal{F}$ over X, the differential $d\sigma_x$ at x (also called tangent map, see Sect. 3.1) is an element of the jet bundle $J^1\mathcal{F}_{\sigma(x)}$. Consequently, the map $x \to d\sigma_x$ is a cross-section of the jet bundle over X. This section, denoted $j^1\sigma$, is called the first jet extension of σ. In coordinates, it is given by $j^1\sigma(x) = (x^\mu, \sigma^i(x), \partial_\mu \sigma^i)$, where $\partial_\mu = (\partial_i, \partial_t)$. It is this triple which represents the fluid motion. The first three base coordinates space components x^i, originally coming from the initial positions of the fluid particles, now represent the particle labeling. The $\sigma^i(x)$ components identify the position of the x particle in space, and the $\partial_t \sigma^i$ components represent the velocity of the particle x.

9.2.4 Fixed Fluid Container

For the case when the fluid moves in a fixed region, i.e., with fixed boundaries, the group structure of the fiber bundle $\mathcal{F}$ is the identity, and the bundle is trivial, $\mathcal{F} = X \times M$. The spatial part of the base manifold M represents the reference configuration (initial positions of all fluid particles). Actually, the coordinate x ceases to represent the initial position, but remains attached to the particle and labels it for the rest of the evolution. So, the space part of the base manifold x (the material points) labels the fluid particles through the one-to-one correspondence between particles and their initial positions in the reference fluid container. The time base X corresponds to the time evolution. The fiber over any base point is the same manifold, meaning that the space available for any particle is the same at any moment of time. Its coordinates y are called spatial points. The fiber at any point $F_{(x,t)}$ represents the available space for particle x at the moment t, and it is diffeomorphic with M, i.e., the *reference fluid container* [6, 7]. In the case of $\mathcal{F}$, the requirement for the existence of a projection $\Pi : \mathcal{F} \to X$ from the definition of a fiber bundle (Sect. 3.9, Definition 26) guaranties that all points of the fiber, at any point of the base, are filled with fluid.

The fluid motion is described by a cross-section $\sigma(x, t)$ of the bundle $\mathcal{F}$ representing the particle placement field. Not any cross-section can represent a real motion of the fluid, and some *minimal constraintts* are needed. First, σ is not allowed to create or annihilate fluid particles, and second, two different particles cannot hold the same spatial point at the same moment of time. In the traditional approach presented above (the one not using geometry of a fiber bundle) these two constraintts are fulfilled by requesting that the Lagrangian paths of the fluid particles represent a diffeomorphism of the reference fluid container. In the fiber bundle formalism, these two physical constraintts require a similar thing. The restriction of the cross-section $\sigma(x, t)|_{t=t_0}$ at a constant $t = t_0$ (for every moment of time t_0) needs to be a diffeomorphism of the manifold $F = M$. Of course, this is also possible because the bundle is trivial, and there is a canonical diffeomorphism between any two fibers at any two points.

Let us ignore for a second the deep geometrical implications of the existence of the group of diffeomorphisms, and let us just look at these conditions locally, in terms of coordinates. For some more insight into this topic, we recommend for example [6, 7, 9]. This condition is equivalent to the vector field to be divergence free. This means that the infinitesimal generator of this diffeomorphisms is a divergence-free vector field, or in other words that the flow is incompressible.

In addition, the specific cross-section form should result from a solution of the dynamic equations of motion, for example Euler (10.15) or Navier–Stokes (10.13) equations, under some additional boundary, initial or regularity conditions which may be required, too. This constraint will be addressed in the next chapters. For an explicit discussion of this topics, see for example [6, Theorem 2.1] and reference herein.

In the local coordinates of a given fiber, $y(x, t) \in F_{(x,t)}$ represents the spatial position of the particle x at moment t, $(x, t, y)\sigma(x, t)$. The path lines are the restrictions of the cross-section $\boldsymbol{r}_L(x_0, t) = \sigma|_{x=(x_0,t)}$ for fixed point in the space part of the base space. The tangent vectors to these curves can be expressed in two ways. If we write $\boldsymbol{v}_L(x,t) = \partial\sigma^v(x,t)/\partial t$ we have the Lagrangian (material) velocity field. The superscript v (as in vertical) represents the components of the cross-section along the fiber. The Lagrangian velocity field is actually represented by the last three components of the cross-section in the first jet bundle $d\sigma$. Namely $j^1\sigma = (\sigma, \partial_i\sigma, \boldsymbol{v}_L)$.

Conversely, if we invert the equation $y(x,t)$ with respect to y, we can express the velocity field in coordinates $\boldsymbol{v}_L(x(y),t) = \boldsymbol{v}_E(y,t)$, which is nothing but the Eulerian velocity field. So, even if locally the Eulerian and Lagrangian velocities coincide at the same point of the fiber bundle $\mathcal{F}$, they are vector fields in different spaces. The Eulerian velocity is a vector space defined on the standard fiber manifold F. Indeed, because the fiber at any point $F_{(x,t)}$ is diffeomorphic with the standard fiber F, according to the *minimal constraintts*, we can map vectors tangent to any fiber into vectors tangent to the standard fiber $F = M$. So, a cross-section σ in $\mathcal{F}$ generates a vector field on F at any moment of time, the Eulerian flow. The integral curves of this field are, at every moment of time, the collections of time-dependent stream lines, they lie in the standard fiber, and they have no special assigned parameter (the stream lines collection is also called flow net [14]). Contrary to the stream lines, the path lines are time parametrized, hence constant, and they lie in the fiber bundle. Again, the collection of path lines do not coincide with the flow net in general (they coincide if the flow is stationary). It is also true that the path lines never cross the flow net lines.

If we come back to Fig. 9.4, we understand now the trihedron presented there as the base space, and the horizontal planes as fibers at different points, with their associated Eulerian fields of velocities. The reunion of all path lines forms the cross-section σ.

Since $\sigma(M, t_0) \simeq M$ is a diffeomorphisms because of the *minimal constraintts*, the image of any compact set in M is a compact set in $F_{(x,t)}$. Such sets are the particle structures that remain "stable" to this extent. If such a set is a submanifold of dimension 1, we call it particle line or material line or circuit line, or filament. Once identified in the reference fluid container, this line conserves its topological proprieties in time. If the submanifold is two dimensional, it is a particle surface, or free fluid surface, etc., and so on. We noticed above that the particle paths are restrictions of the cross-sections describing the dynamics for constant x. Similarly, particle lines are restrictions of the cross-section for constant time, and on subsets of the M manifold: $\sigma(x,t)|_{(x\in D, t=t_0)} = \hat{\sigma}(x)|_{x\in D}$.

There is another interesting approach about the path lines as orbits of a group of diffeomorphisms of the spatial part of the base space. Actually, any such diffeomorphism (any flow) can be understood as a relabeling operation of the fluid particles. Such a relabeling operation is connected with a continuous symmetry of the system. If we consider the fluid a Lagrangian system and the flow is incompressible, the Noether current associated to this symmetry is the fluid momentum conservation, see Fig. 9.8.

In the following, we give an interpretation of the transformation between variation of Eulerian and Lagrangian quantities (9.1), (9.6), or (9.7) in terms of a connection.

Let us consider again the fiber bundle $\mathcal{F}$ representing a fluid confined in a fixed space domain identified by the manifold $M \ni (x^i)$, where $i, j = 1, \ldots, 3$ and $\mu =$

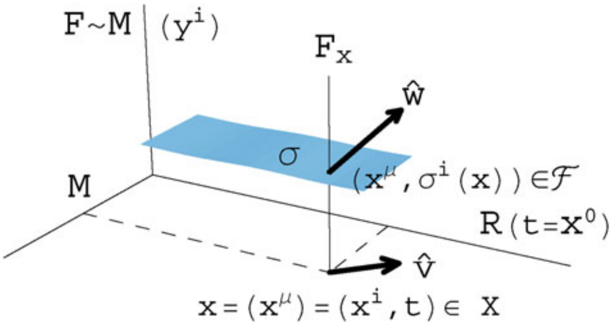


Fig. 9.8 Structure of the fiber bundle associated with a fluid. The axes here are the base space (M), the fiber (F), and the time ($t = x^0$)

$0, \dots, 3$. The base space is the direct product $X = M \times \mathbb{R} \ni (x^\mu) = (x^i, x^0 = t)$. We choose the fiber $F = M$, a trivial identity structure group $G = \{e\}$, the projection Π, $F_x = \Pi^{-1}(x)$ and a cross-section $\sigma : X \to \mathcal{F}$. The cross-section maps $x = (x^\mu) \to \sigma^a = (x, \sigma^j(x))$, and its differential $d\sigma : TX \to T\mathcal{F}$ maps $T_xX \ni \hat{v}(x) = (\boldsymbol{v}, v^0) = (v^i, v^0) = (v^\mu) \to \hat{w} = (\boldsymbol{w}, w^0, \bar{w}) = (w^i, w^0, \bar{w}^j) \in T_{\sigma(x)}\mathcal{F}$, with $a = (\mu, j)$. In components, the action of the differential, which is a section in the first jet fiber bundle over $\mathcal{F}$, reads

$$d\sigma(\hat{v}) = \left(\frac{\partial\sigma^a}{\partial x^\mu} v^\mu\right) = \left(\frac{\partial\sigma^\nu}{\partial x^\mu} v^\mu, \frac{\partial\sigma^j}{\partial x^\mu} v^\mu\right) = \left(\frac{\partial x^\nu}{\partial x^\mu} v^\mu, \frac{\partial\sigma^j}{\partial x^i} v^i + \frac{\partial\sigma^j}{\partial t} v^0\right) =$$

$$\left(v^\nu, \left(v^i \frac{\partial}{\partial_i}\right)\sigma^j + v^0 \frac{\partial\sigma^j}{\partial t}\right) = \left(\boldsymbol{v}, 1, (\boldsymbol{v}\cdot\nabla)\boldsymbol{\sigma} + v^0 \frac{\partial}{\partial t}\boldsymbol{\sigma}\right), \tag{9.8}$$

according to (3.4). If we restrict ourselves on curves being path lines in the time parametrization, the tangent vectors are $\hat{v} = (\boldsymbol{v}, 1)$, i.e., $v^0 = 1$. The interpretation of (9.8) is as follows. Spatial part $\boldsymbol{\sigma}$ of vectors in the tangent space to the base is in one-to-one correspondence with vectors in the tangent space to the fiber, by the triviality of $\mathcal{F}$. So $\boldsymbol{\sigma}$ is actually a fiber vector, i.e., an “Eulerian” vector in a local space frame. This Eulerian vector is mapped to a vector in the tangent space to the bundle, which is a “Lagrangian” vector

$$TM \ni \boldsymbol{v} \to \left[(\boldsymbol{v}\cdot\nabla) + \frac{\partial}{\partial t}\right]\boldsymbol{\sigma}, \text{ with } \hat{\sigma} = (x, \boldsymbol{\sigma}) \in T\mathcal{F}. \tag{9.9}$$

If we put $\boldsymbol{v}_E = \boldsymbol{\sigma}$, (9.9) reads $d\sigma(\boldsymbol{v}_E) = \boldsymbol{v}_L$, i.e., the well-known transformation between the partial time derivative and the material (total) derivative. In this sense, (9.9) describes a connection in $\mathcal{F}$ in the first jet bundle J^1 (for example, see Olver's book [13]). Coming down to the $\mathcal{F}$ bundle, we note that the only possible connection is a trivial one, with zero coefficients. This is because the bundle is trivial, so the only admissible infinitesimal transformations are translations. The situation is different if the shape of the fluid container is allowed to change in time.

Even if we used such a complicated fiber bundle construction for the transformation of the time derivatives, the Eulerian–Lagrangian transformation formula (9.9) is useful so far only for the tangent vectors (i.e., tangent to the path lines), and it cannot be applied to more general vector fields, not mentioning higher rank mixed tensorial fields.

9.2.5 Free Surface Fiber Bundle

If the shape of the reference fluid container changes with time (boundaries not fixed anymore), the fiber F_x depends on the point $(x^i, t) \in X$ through the time dependence and the bundle is not anymore a global cartesian product. Consequently, it has a nontrivial structure group G. If the fluid has only one compact free surface, the fiber bundle $\mathcal{F}$ has a different structure than the one described in Sect. 9.2.4.

We consider the fluid "drop" as a connected, simple-connected domain $\mathcal{D}_\Sigma \simeq D_3 \subset \mathbb{R}^3$ with smooth boundary (shape) $\partial\mathcal{D} = \Sigma$, and under no external forces or torques. By $\simeq D_3$ we mean a diffeomorphisms with the three-dimensional disc $x^2 + y^2 + z^2 <= 1$. The drop has a set of possible shapes. If we can parameterize the set of all possible shapes with coordinates, we could set the structure of a manifold M. The shape coordinates can be determined by the expansion in spherical harmonics, for example, and we can associate to M the $l_2(\mathbb{C})$ space structure with the topology induced by the norm. We call M the *shape space* of the drop. The base space will be, like in the previous case, $X = M \times \mathbb{R} \ni (\Sigma, t)$.

For any shape we choose a trihedron fixed in this shape, for example the origin in the center of mass, and the axes directed toward the positions of some chosen zeros of the spherical harmonics. The configuration of the fluid within the given shape Σ will be referred to this trihedron. For a given shape Σ, all possible configurations of the fluid particles $\{\boldsymbol{r} | \boldsymbol{r} \in \mathcal{D}_\Sigma\}$ can be described by the set of diffeomorphic (shape invariant) transformation of $\mathcal{D}_\Sigma$ onto itself. These transformations form a Lie group of diffeomorphisms $Diff_\Sigma$. Any element g_Σ of this group maps some distribution of particles inside this shape into another distribution of particles within the same shape. So, by the *minimal constraintts*, the fiber over $x = (\Sigma, t) \in M$ is represented by the group of diffeomorphisms of the shape $\Pi^{-1}(\Sigma, t) = Diff_\Sigma$. The structure group is the group of diffeomorphisms of the three-dimensional disc, $Diff_{D_3}$, which is the group model for all the other diffeomorphisms groups. Consequently, $\mathcal{F}$ is a principal bundle, and the coordinate on the fiber over (Σ, t) is a certain group transformation $Diff_\Sigma \ni g_\Sigma : \mathcal{D}_\Sigma \to \mathcal{D}_\Sigma$.

This construction must be carried out for all possible shapes. Thus, the total configuration space of the fluid $\mathcal{F}$ is a fiber bundle over the base X, of fiber $Diff_\Sigma$. A shape evolution will be identified by a (time-like) curve $\gamma \in X$, i.e., a regular curve of shapes $\Sigma(t)$ parametrized by time. For any particular shape, we have to integrate a set of dynamical equations $\triangle(\Sigma, \boldsymbol{r}, t)$ to find the positions of the particles associated to that shape. The shape at any moment of time determines the position of particles within the fiber. Hence, a cross-section $\sigma : X \to \mathcal{F}$ represents the evolution of the drop, namely in components $t \to \Sigma(t) \to \boldsymbol{r}_L(\boldsymbol{r}_0, t) = g_{\Sigma(t)}(\boldsymbol{r}_0)$. From the geometrical point of view, the dynamical equations of the free surface fluid are equations for this section. These are basically the equation of continuity, equations for momentum conservation (Euler or Navier–Stokes equations), and energy transfer equation.

For any shape in M, we need to specify its fixed reference trihedron and its reference (we may call it initial) distribution of particles $\boldsymbol{r}_0$. This choice is not unique, and the freedom involved is a typical gauge freedom. A similar gauge freedom is encountered in electromagnetism when we study magnetic monopoles, in the dynamics of elastic bodies or in the study of the geometric phase change of the wave function for time variable Hamiltonian (Berry's phase). Making a choice for the trihedron orientation and the reference particle distribution with respect to any shape is nothing but a cross-section in $\mathcal{F}$. However, the physical results should be independent of this choice, i.e., gauge invariant.

Translation of the drop center of mass could be eliminated from the beginning, but the shapes should also conserve total angular momentum. Angular momentum can be changed by deformations (motion in the base space) and also by particle rotations (motions in the fiber). We need to "synchronize" the succession of deformations with a unique succession of rotations, such that total angular momentum to be constant. In that, we can introduce a new type of connection, different from that one introduced above between Eulerian and Lagrangian approach on tangent vectors (9.9).

For any given smooth curve γ in the base space M, we need to lift it to a curve γ' in the total space $\mathcal{F}$ in a unique way. Remember that a lift is a map $\gamma \in M \to \gamma' \in \mathcal{F}$ such that $\Pi(\gamma') = \gamma$. However, the lift of a path is not unique by definition. The mathematical tool needed to make it unique is the connection [12, 15]. A connection, or better said its differential expression, would assign to any tangent vector $\boldsymbol{v}(x) \in T_x M$, an element in TF_x, which is the Lie algebra of the group $Diff_{\Sigma}$. Globally, when we move along a closed path in M the corresponding lifted path in $\mathcal{F}$ may not be closed. That is for $\gamma(x_0) = \gamma x_1$ we may have $\gamma'(x_0) = \gamma' x_1$ in F. Two different points on the same fiber mean a relabeling of the particles, or a motion inside the drop. Such a relabeling could be associated with a finite nonzero rotation of the drop. The drop begins to move by changing its shape and ends up to the same initial shape after a finite amount of time. But during this motion, it actually undergoes a net rotation.

A similar situation happens when we build the configuration space of a deformable body. Again, we choose for any shape a trihedron fixed in this shape. The orientation of the body, ignoring free translations of the center of mass, could be described by a proper rotation matrix $\hat{R} \in SO(3)$ which maps the body-fixed trihedron to a space frame contained in the ambient space in which the drop is constrained to move, i.e., $\mathbb{R}^3$. Thus, the total configuration space $\mathcal{F}$ is a fiber bundle over the base $M \times \mathbb{R}$, of fiber $SO(3)$.

Like in the case of the drop, the angular momentum of the body can be changed by deformations (motion in the base space) and also by rotations (motions in the fiber). In this example, the connection assigns to any tangent vector $\boldsymbol{v}(x) \in TM$, an element in $TSO(3)$, which is nothing but the Lie algebra $so(3)$. When we move along a closed path in M the corresponding lifted path is not closed in general. Two different points on the same fiber mean a change in the orientation, a rotation. The body moves and changes its shape, but during this motion, it undergoes a rotation. However, because the $SO(3)$ Lie group is not commutative, there are problems in integrating this lifted path in the fiber. The problem is solved, for example in gauge field theory, by the so-called *Wilson integral*. In [12] there is an eloquent example, namely the falling cat problem. The cat is dropped from an upside down position, but it lands on its feet, even if it is isolated. The cat manages to deform its body during the flight, such that all in all involves a net rotation of the body, to conserve its angular momentum, see also [16]. Similar examples of free deformable compact shapes occur in the theory of swimming of microorganisms in zero Reynolds number [17]. In that case the systems are investigated by using the theory of a gauge field over the space of shapes. The topics of fiber bundles in hydrodynamics have plenty of online and printed resources out of which we mention for example [6, 7, 9, 18–20].

9.2.6 How Does the Time Derivative of Tensors Transform from Euler to Lagrange Frame?

In Sects. 9.2.4 and 9.2.5, we have seen that changing the frame from the Eulerian to Lagrangian is actually mapping vectors from the tangent space of the base space to the tangent space of the fiber. To transform higher-order tensors we need to introduce a new time derivative through a covariant formalism. Equations (9.1) and (9.5)–(9.7) are not covariant because the time is not explicitly included in the metric, yet the Lagrangian → Eulerian transformation $\omega(x, t) \to \Omega(\sigma, t)$ is a time-dependent coordinate change. Consequently, the partial time derivative does not transform like a tensor because of the time-dependent basis vectors, the same reason that ordinary derivatives are not covariant (see for example in Sect. 3.10 the comment right after (3.45)).

The traditional material derivative is covariant just for the coordinates, the velocity vector, and (obviously) for scalars, as we know from (9.1) and (9.5)–(9.7), and it was proved geometrically in (9.9), because the velocity belongs to the tangent space. Let us have an (r, s) Lagrangian tensor $\omega(x, t)$ depending on the Lagrangian coordinates (x, t). Its time derivative, i.e., the rate of change $d\omega/dt$ of the tensor while keeping the Lagrangian coordinates constant, does not transform into the time derivative of the corresponding Eulerian tensor, $\omega(x, t) \to \Omega(\sigma, t)$).

$$\frac{\partial \omega}{\partial t}(x, t) \nrightarrow \frac{\partial \Omega}{\partial t}(\sigma, t).$$

To provide a covariant time derivative for arbitrary vector fields and higher-order tensors, we need to calculate the pull-back transformation of (9.9), and make sure that the result is a tensor of the same type. That is, to introduce a *covariant time derivative* operator (e.g., [21] where it is called convected or convective) which describes the change in time for a certain geometrical quantity ω along (or with respect to) the flow lines of the fluid, in the Eulerian frame (σ^1, t). The covariant variation of this quantity is the sum of its internal time variation described by the partial derivative, and the Lie derivative of ω with respect to the flow described by the vector field $\boldsymbol{v}_E = (v^i)$

$$\frac{d_c \Omega(\sigma, t)}{dt} = \frac{\partial \Omega}{\partial t} + \boldsymbol{v}_E(\Omega). \tag{9.10}$$

For scalars, (9.10) reduces to the well-known formula (9.1) or (9.8). We will refer in the following to (3.19) and (3.20), describing the action of the Lie derivative on various geometrical objects.

For example, the time covariant derivative acts on a contravariant vector field $\boldsymbol{A}(\sigma, t) = (A^i)$ defined in the Eulerian frame, according to the form (3.19)

$$\frac{d_c \boldsymbol{A}}{dt} = \frac{\partial \boldsymbol{A}}{\partial t} + [\boldsymbol{v}_E, \boldsymbol{A}]. \tag{9.11}$$

The covariant time derivative action on a covariant vector $\omega = (A_i)$ is given by the sum between the partial derivative with respect to time and the Lie derivative with respect to $\boldsymbol{v}_E$ acting on the 1-form (3.20)

$$\frac{d_c \Omega_i}{dt} = \frac{\partial \Omega_i}{\partial t} + v^k \frac{\partial \Omega_i}{\partial \sigma^k} + \Omega_k \frac{\partial v^k}{\partial \sigma^i}, \tag{9.12}$$

The action on an Eulerian tensor of rank (0, 2) is

$$\frac{d_c \Omega_{ij}}{dt} = \frac{\partial \Omega_{ij}}{\partial t} + v^k \frac{\partial \Omega_{ij}}{\partial \sigma^k} + \omega_{kj} \frac{\partial v^k}{\partial \sigma^i} + \Omega_{ik} \frac{\partial v^k}{\partial \sigma^j}, \tag{9.13}$$

and so on. The physical signification of the covariant derivative on the LHS of all (9.11)–(9.13) is the following. First, we calculate the partial time derivative of a Lagrangian tensor, then we transform this quantity into the Eulerian frame. This transformed Eulerian object is not anymore the simple partial derivative of the Eulerian tensor, but the covariant time derivative of the Eulerian tensor.

To exemplify (9.10) in a direct and even more intuitive way, we obtain the transformation of the time derivative for a tensors of rank (1, 1) for example by a simple matrix transformation formalism based on formula (3.46). Similar calculations in components are done in [4, Chap. 8]. We write the tensor transformation of components of ω when changing frame from Lagrangian to Eulerian

$$\Omega = J \omega J^{-1}, \quad \text{that is} \quad \Omega^p_q = \frac{\partial \sigma^p}{\partial x^i} \frac{\partial x^j}{\partial \sigma^q} \omega^i_j. \tag{9.14}$$

By time differentiation of (9.14) with respect to time, we have

$$\frac{d\Omega}{dt}J + \Omega\frac{dJ}{dt} = \frac{dJ}{dt}\omega + J\frac{d\omega}{dt}.$$

Since Ω is Eulerian we have $\Omega(\boldsymbol{\sigma}, t)$ and further $\Omega(\boldsymbol{\sigma}(\boldsymbol{x}, t), t)$, so

$$\frac{d\Omega^p_q}{dt} = \frac{\partial\Omega^p_q}{\partial t} + v^j\frac{\partial\Omega^p_q}{\partial\sigma^j}.$$

Moreover, we can write

$$\frac{dJ^j_i}{dt} = \frac{\partial v^j}{\partial x^i} = \frac{\partial v^j}{\partial\sigma^k}\frac{\partial\sigma^k}{\partial x^i},$$

and define the matrix of gradients of velocity

$$\gamma^j_i = \frac{\partial v^j}{\partial\sigma^i}.$$

With these notations we have

$$\Omega\gamma J + \frac{d\Omega}{dt}J - \gamma J\omega = J\frac{d\omega}{dt},$$

and by using $dJ/dt = \gamma J$ and by multiplication with J^{-1} to the right, we obtain

$$J\frac{d\omega}{dt}J^{-1} = \frac{d\Omega}{dt} + [\Omega, \gamma] \equiv \frac{d_c\Omega}{dt}, \tag{9.15}$$

where the commutator on the RHS arises from $\Omega\gamma - \gamma(J\omega J^{-1})$. Equation (9.15) represents the transformation of the time derivative $d\omega/dt$, and since the RHS is an operator applied to the Eulerian tensor Ω, we define the LHS as the *covariant* (or convected) time derivative. In components it reads

$$\left(J\frac{d\omega}{dt}J^{-1}\right)^j_i \equiv \frac{d_c\Omega^j_i}{dt} = \frac{\partial\Omega^j_i}{\partial t} + v^k\frac{\partial\Omega^j_i}{\partial\sigma^k} + \Omega^j_k\frac{\partial v^k}{\partial\sigma^i} - \Omega^k_i\frac{\partial v^j}{\partial\sigma^k}, \tag{9.16}$$

where we used the notation d_c/dt for this covariant derivative. It is easy to check that (9.16) is in agreement with the general formulation from (9.12) and (9.13). For the action of the covariant time derivative on other types of tensors, see Exercises 4 and 5 at the end of the chapter. Also the action of d_c/dt can be expressed entirely in terms of covariant derivatives [4]. For example for a (0, 2)-tensor, we have

$$\frac{d_c\Omega_{ij}}{dt} = \frac{\partial\Omega_{ij}}{\partial t} + (v^k\nabla_k)\Omega_{ij} + (\nabla_j v^k)\Omega_{ik} + (\nabla_i v^k)\Omega_{kj}. \tag{9.17}$$

Let us choose a simple example to understand how (9.16) works. We consider a stationary viscous flow next to a rigid wall at $\sigma^3 = 0$ (or simply $z = 0$) with velocity $\boldsymbol{v}_E = (0, v, 0)$. The velocity is subjected to a boundary layer effect and it depends on the distance to the wall, $v = v(\sigma^3)$. In the Lagrangian (convected) frame the pressure is constant in time and so is its gradient, having nonzero component in the σ^3-direction, $\nabla P = (0, 0, \partial P/\partial \sigma^3) = (\alpha_1, \alpha_2, \alpha_3)$. The time derivative of this gradient, which is a (0, 1) covariant vector, is zero. However, in the Eulerian frame by using (9.16), we have a nonzero material time derivative

$$\frac{d_c \nabla P}{dt} = \left(0, \frac{\partial v_E}{\partial \sigma^3} (\nabla P)_3, 0\right).$$

There is a change in time for the gradient in the Eulerian frame even if the same gradient is constant in the Lagrangian frame, and this contribution comes from the last term in the RHS of (9.16), and not from the first two traditional terms on the same RHS. Physically, it means that the gradient is initially vertical, but because of the horizontal shearing of the layers of fluid, this gradient is "tilted" more and more horizontally.

This treatment presented above is not the only way to introduce a covariant time derivative. For example in [22] the authors introduce a *corotational* derivative where the local vorticity of the flow is incorporated into the derivative. However, the covariant time derivative defined by (9.15) and (9.16) is the most familiar one, and it was initially introduced in [23] in formulating rheological equations of state. This derivative was used in [24] to develop a theory of fluid motion on an interface, and later was geometrically extended in [4, 21]. In this last citation there are enumerated some disadvantages of the covariant time derivative. For example, it is not compatible with the metric tensor, and it involves gradients of the velocity so it is not directional. On the other hand, the importance of the covariant time derivative (9.15) and (9.16) is not only mathematical. Many nonlinear transport and mixing processes are described by advection–diffusion equations [21], consisting in a material time derivative for the concentration of the quantity advected, and a divergence of the diffusivity tensor. In the Lagrangian frame (along the direction of compression of fluid elements) the advected terms drop out, and the governing equation reduces to a simple diffusion equation, much more tractable. Moreover, because of the formalism presented in this section, this simplified diffusion equation is still covariant. This allows the introduction of a Riemannian metric on the tangent space to the coordinate space, and allows in principle the use of spectral approximation procedures.

9.3 Path Lines, Stream Lines, and Particle Contours

In this section, we present a parallel between the Eulerian and Lagrangian approaches from the point of view of the flow box theorem (see Sect. 3.4). We discus here only finite time flows with $t \in [t_1, t_2]$, $-\infty < t_1 < t_2 < \infty$. We begin our construction with the fluid initial reference container, i.e., a domain $D_0 \subset \mathbb{R}^3$. We construct the base space $X = \mathbb{R}^3 \times [t_1, t_2]$, and we assign a local coordinate system in $\boldsymbol{r}_0 \in D_0$. We assume that we are given the fluid flow as smooth homeomorphisms $\boldsymbol{r}_L : D_0 \times [t_1, t_2] \to \mathbb{R}^3$ such that the restriction $\boldsymbol{r}_L|_{D_0 \times \{t\}}$ is injective for any fixed $t \in [t_1, t_2]$. In coordinates this reads $(\boldsymbol{r}_0, t) \to \boldsymbol{r}_L(\boldsymbol{r}_0, t)$. The family of curves $L = \{\gamma_L \Rightarrow \boldsymbol{r}_L(\boldsymbol{r}_0, t) | \boldsymbol{r}_0 \in D_0\}$ is the particle paths, with tangents $\dot{\boldsymbol{r}}_L = \boldsymbol{v}_L$ and metric $g_L = v_L^2$. These curves can be lifted in the base space and mapped into a family $\tilde{L} = \{\gamma'_L \Rightarrow (\boldsymbol{r}_L(\boldsymbol{r}_0, t), t) \in \mathbb{R}^3 \times [t_1, t_2] | \boldsymbol{r}_0 \in D_0\}$. The metric of γ'_L is $\tilde{g}_L = v_L^2 + 1$. Both γ_L and γ'_L are Lagrangian path lines viewed in different spaces.

For any $t \in [t_1, t_2]$ we can construct $D_t = \boldsymbol{r}_L(D_0, t) \subset \mathbb{R}^3$. A particle contour is a parametrized curve $\Gamma_0 = \{\gamma_0(s) \subset D_0, s \in I\} \subset D_0$. The question is what happens to such a particle contour in time. Is $\Gamma_t = \{\gamma(s, t) = \boldsymbol{r}_L(\gamma_0(s), t)\}$ a regular curve with the same topology as Γ_0? We have the following result.

Lemma 7 *The set Γ_t defined by $\gamma(s, t)$ as above is a* regular parametrized curve *if*

$$\hat{J}(\boldsymbol{r}_L(\boldsymbol{r}_0, t))|_{t=const.} \cdot \boldsymbol{t}_{\Gamma_0} \neq 0,$$

for $\forall s \in I, t \in [t_1, t_2]$. *Here* **t** *is the tangent vector to a curve.*

Proof We have

$$\frac{\partial \boldsymbol{r}}{\partial s}(s, t) = \frac{\partial x_L^i}{\partial \gamma^j}\frac{d\gamma^j}{ds} = \frac{\partial x_L^i}{\partial x_0^j} \cdot \boldsymbol{t}_{\Gamma_0}^j(s), \tag{9.18}$$

which represents the requested inequality. □

In other words, a particle contour at the initial moment of the flow remains a regular curve while transported by the flow in time if the unit tangent of this initial curve is not in the kernel of the Jacobian matrix of the Lagrangian path function of the initial coordinates (the flow). If conditions in Lemma 7 are fulfilled, the particle contour Γ_0 remains a regular curve during the flow, so one can apply circulation or other types of theorems on it. The Jacobian matrix plays a basic role in hydrodynamics [25]. It allows the determination of the main flow parameters and the geometrical characteristics, in particular the metric properties.

As an application, we can use Lemma 7 criterium in Example 2. The initial vertical particle contour (for example $x_0 = 0$, $y_0 \in [-a, a]$) will breakup at a certain moment of time t if, according to (9.18),

$$\frac{\partial \boldsymbol{r}_L}{\partial y_0}(t) = 0,$$

where we consider y to be the vertical axis in Fig. 9.3. Obviously, from the continuity of the cylinder contour, the coordinates of all path lines depend on y, so (even it looks hard to believe) the above derivative is nonzero everywhere and consequently the path lines will not disrupt.

The question is whether the set $\cup_{t\in[t_1,t_2]} D_t$ is a submanifold of $\mathbb{R}^3$. If it is, we can assign local coordinates for its points in the form $p = (\boldsymbol{r}_0, t)$. In other words, if the reunion of all path lines over a certain finite interval of time is dense enough to form a topological space. The answer can be given at least locally, by using the flow box theorem (Theorem 4). Obviously, the Lagrangian velocity field of any particle $\boldsymbol{v}_L$ fulfills the conditions for the existence of flow boxes on X. Indeed, for any $t \in [t_1, t_2]]$ and any point $p = (\boldsymbol{r}, t) \subset D_t$, we can find a neighborhood $V(\boldsymbol{r})$ and $t \pm \delta t$ such that it exists $a > 0$ and the triple

$$((V(\boldsymbol{r}), (t - \delta t, t + \delta t))a, \gamma_L(\boldsymbol{r}_L(\boldsymbol{r}_0, t), t + \lambda)),$$

is a flow box.

Moreover, we assume that the fluid flows in such a way that X is a topological space with the product topology of $\mathbb{R}^3 \times \mathbb{R}$. We also assume that the fluid flows in a bounded region (bounded fixed region or free compact surface), so the Lagrangian velocity field has compact support in X. Consequently $\gamma_L(\boldsymbol{r}_0, t)$ are maximal integral curves and form a foliation of X (see Sect. 3.4). Since the field of velocities of particles has compact support, according to Lemma 1, it is complete, and any of its integral curves can be extended so that its domain of parameter becomes $\mathbb{R}$. So the Lagrangian paths $\gamma_L(\boldsymbol{r}_0)$ form a foliation of the manifold D_t which is homeomorphic with D_0. We mention again that inside each D_t, we have $\boldsymbol{v}_E(\boldsymbol{r}_L(\boldsymbol{r}_0, t), t) \equiv \dot{\boldsymbol{r}}_L(\boldsymbol{r}_0, t)$, but inside the same D_t the integral curves of $\dot{\boldsymbol{r}}_L$ are not the γ_L curves.

There are of course differences and similarities between the stream and path lines.

Example 3 In Fig. 9.9 we present a cross-section into a spherical drop of incompressible inviscid fluid in oscillation with an $l = 2$ mode. The thin lines are the stream lines and the thick line is a path line.

Example 4 To illustrate better these differences, we present a simple example of a two-dimensional flow. We assume that we know the flow of this two-dimensional fluid in the Eulerian frame, and hence we know the Eulerian velocities $\boldsymbol{v}_E(\boldsymbol{r}, t)$ at every point and every moment of time. For example let us choose

$$\boldsymbol{v}_E(x, y, t) = (x, y + \epsilon t), \tag{9.19}$$

where ϵ is an arbitrary parameter. The stream lines, lying in the instantaneous plane $\mathbb{R}^2$, are obtained by integrating

$$\frac{dx}{x} = \frac{dy}{y + \epsilon t}, \tag{9.20}$$

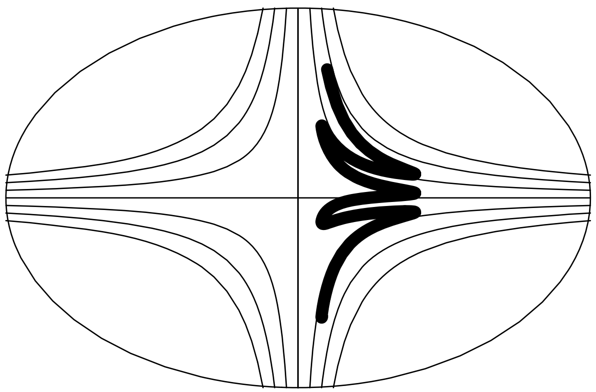


Fig. 9.9 Cross-section into a spherical drop of incompressible inviscid fluid in oscillation in an $l = 2$ mode. The *thin curves* are the stream lines, while the *thick curve* is an example of a path line

resulting in the implicit equation

$$y_E = \frac{y_0 + \epsilon t}{x_0} x_E - \epsilon t, \tag{9.21}$$

or in the parametric form $\boldsymbol{r}_E(s; x_0, y_0; t)$

$$x = \frac{s}{\sqrt{1 + \left(\frac{y_0+\epsilon t}{x_0}\right)^2}}$$

$$y = \frac{y_0 + \epsilon t}{x_0} \frac{s}{\sqrt{1 + \left(\frac{y_0+\epsilon t}{x_0}\right)^2}} - \epsilon t. \tag{9.22}$$

Equations (9.21) and (9.22) represent the stream line passing through a point (x_0, y_0). From the Eulerian velocity we obtain the Lagrangian velocity by integrating the equations

$$\frac{dx_L}{dt} = x_L(x_0, y_0, t)$$

$$\frac{dy_L}{dt} = y_L(x_0, y_0, t) + \epsilon t.$$

The lifted path lines in parametric form have the expression $\gamma_L(x_L(x_0, y_0, t), y_L(x_0, y_0, t), t)$ with

$$x_L(x_0, t) = x_0 e^t$$

$$y_L(x_0, y_0, t) = (y_0 + \epsilon t)e^t - \epsilon(t + 1), \tag{9.23}$$

and in implicit form read

$$y_L(x_0, y_0, t) = (y_0 + \epsilon)\frac{x_L}{x_0} - \epsilon\left(\ln\frac{x_L}{x_0} + 1\right). \tag{9.24}$$

Of course the path lines and the stream lines have different expressions, not forgetting the fact that they belong to different spaces. For a check, we notice that if we eliminate the time dependence by setting $\epsilon = 0$, these lines (9.21)–(9.24) have the same expression. In stationary flow the stream lines and the path lines coincide in the horizontal space. We can also check the definition condition $\boldsymbol{v}_L(t) = \boldsymbol{v}_E(\boldsymbol{r}_L(t), t)$. Indeed, we can write

$$v_{Ex} = x_E|_{\boldsymbol{r}_L(t)} = x_L(t) = x_0 e^t = v_{xL}(t),$$

and from (9.23)

$$v_{Ly}(t) = (y_0 + \epsilon)e^t - \epsilon = y_E|_{\boldsymbol{r}=\boldsymbol{r}_L(t)} + \epsilon t = v_{Ey}.$$

Another check is to verify the relation between the Eulerian and Lagrangian

$$\frac{dv_{Ly}}{dt} = (y_0 + \epsilon)e^t = y + \epsilon t + \epsilon$$

$$\frac{\partial v_{Ey}}{\partial t} + (\boldsymbol{v}_E \cdot \nabla)v_{Ey} = \epsilon + x\frac{\partial(y + \epsilon t)}{\partial x} + (y + \epsilon t)\frac{\partial(y + \epsilon t)}{\partial t} = y + \epsilon t + \epsilon, \tag{9.25}$$

and a similar equation for v_x.

For any t, the stream lines (9.22) form a family of curves $\gamma_E(s; \boldsymbol{r}_0; t)$ labeled by the points $\boldsymbol{r}_0 \in \gamma_E$, parameterized by the arc-length s. These curves provide foliations of each horizontal space $\mathbb{R}^2$, for each moment of time. The vector field $\boldsymbol{v}_E(\boldsymbol{r}, t)$ generates also a family of integral curves in the base space $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}^{time}$ determined by the equations

$$\frac{dx}{x} = \frac{dy}{y + \epsilon t} = \frac{dt}{1}. \tag{9.26}$$

At $t = 0$ we have

$$\gamma_E(s; \boldsymbol{r}_0; 0) = \frac{s}{\sqrt{x_0^2 + y_0^2}}(x_0, y_0) \tag{9.27}$$

and the solutions of (9.26) and (9.27) coincide modulo a reparameterization. This means that the Eulerian stream lines are the projections of the lifted Lagrangian path lines in the horizontal planes *only* at $t = 0$. The above example is also shown in Fig. 9.10.

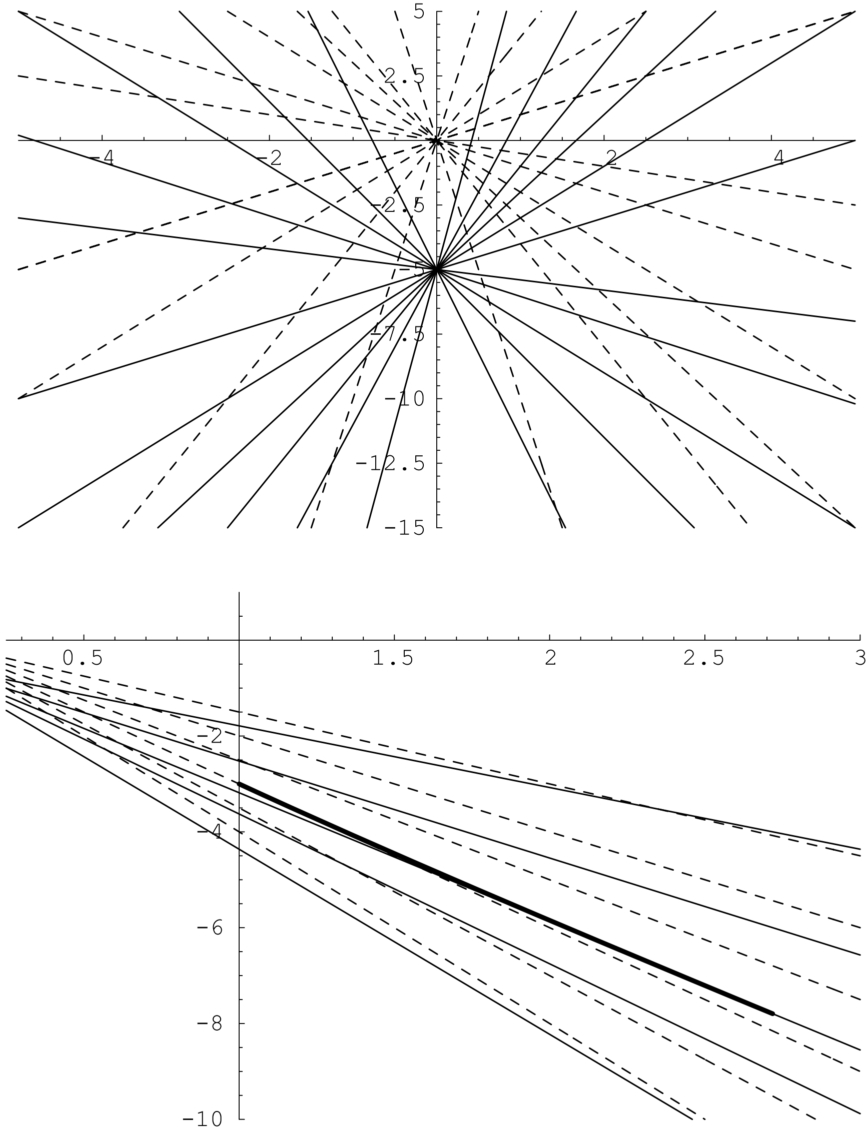


Fig. 9.10 Two dimensional plot $(x - y)$ of flow lines. *Upper graphic*: stream lines $\gamma_E(t)$ in the horizontal plane generated by (9.21) at $t = 0$ (*dashed lines*) and $t = 1$ (*continuous lines*). *Lower graphic*: a region of the same flow, with stream lines at $t = 0$ (*dashed*) and $t = 1$ (*smooth*), and a path line (*thick line*) of a particle moving from $t = 0$ to $t = 1$. The path line is tangent to $\boldsymbol{v}_E(t = 0)$ (*dashed line*) at its upper left end, and tangent to $\boldsymbol{v}_E(t = 1)$ (*smooth line*) at its lower right end, respectively

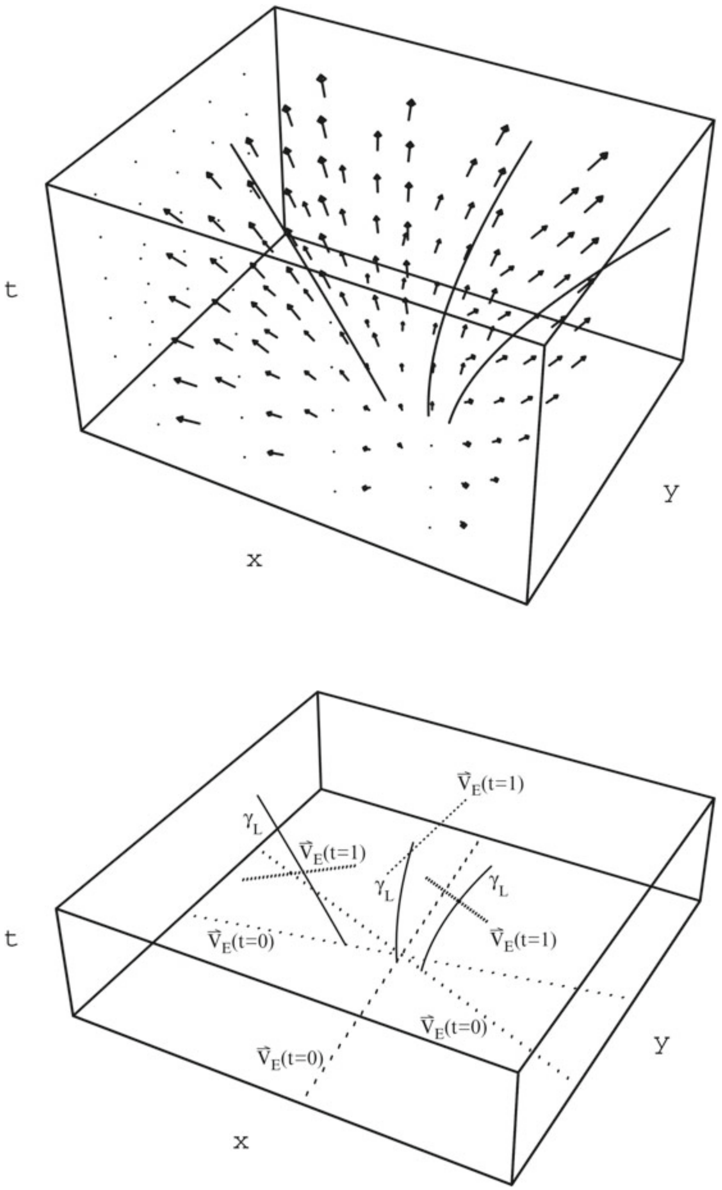


Fig. 9.11 *Upper box*: Lagrangian velocity field represented in the base space with *arrows*. Three Lagrangian paths as particular integral curves of this field are shown. *Lower box*: same Lagrangian paths γ_L (*continuous line*). If we project the unit tangent of each such Lagrangian path onto the horizontal plane, we obtain the Eulerian velocity field $\boldsymbol{v}_E$. The *dotted lines* are integral curves of this Eulerian field. The three longer dotted lines on the base of the box are three such stream lines, intersecting the three Lagrangian path lines at $t = 0$, respectively. The other three dotted (shorter) lines in the upper plane are other three stream lines, occurring at $t = 1$, and intersecting the same three Lagrangian path lines at $t = 1$, respectively

In Fig. 9.11, we present the same flow described by (9.21) and (9.23) in the base space (a three-dimensional representation, where time is the vertical axis).

9.4 Eulerian–Lagrangian Description for Moving Curves

This section is very short, and its purpose is to recall that the idea of establishing a Lagrangian–Eulerian change of frames in lower-dimensional flows is not quite trivial. We elaborated a little about Eulerian–Lagrangian coordinates and velocities in Sects. 8.2 and 8.3 together with the introduction of the convective velocity. Here we just mention one possibility to introduce Eulerian coordinates on a moving curve, like for example a thin vortex filament in motion. We can consider that the Lagrangian coordinates along a curve of length L are given by the arc-length parameterized form of the curve $\boldsymbol{r}(s,t)$. The curve is in motion, and the velocity can be expressed in its Serret–Frenet local frame $\{\boldsymbol{t}, \boldsymbol{n}\}$ in the form $\boldsymbol{V}(s,t) = U(s,t)\boldsymbol{n} + W(s,t)\boldsymbol{t}$. We introduce the mapping $e : [0, L] \to \mathbb{C}$

$$e(s,t) = \int^{s} e^{i\theta(s',t)} ds',$$

from the Lagrangian coordinate to the Eulerian one, where $\theta = \int^{s} \kappa(s',t) ds'$ is the tangent angle of the curve, and κ is its curvature (Sect. 5.1). In the Eulerian coordinate, we can express all the intrinsic properties of the curve, namely $\theta = -i \ln(e_s)$, $\kappa = -i e_{ss}/e_s$, and the dynamics of the transformation of coordinates is given by $e_{st} = [(W - iU)e_s]_s$ [26]. In terms of the new coordinate e and time, the dynamical equation for the velocity components is $\theta_t e^{i\theta} = e^{2i\theta}(W - iU)_e$. Let us choose now a curve motion with zero normal velocity and constant tangential velocity. Since such a motion is only a reparameterization of the curve, i.e., it is not a real motion, we expect the Eulerian coordinate to remain constant. Indeed, from the above relations we have $e_{st} = 0$ so $e = \text{const}$.

9.5 The Free Surface

Physically, free surface is the bounding surface of a certain amount of fluid under consideration. From the mathematical point of view, we consider the free surface Σ to be a piecewise smooth, orientable, regular surface. The free surface is described by the relation $S(\boldsymbol{r}, t) = 0$. This free surface has to fulfill the so-called *free surface kinematic condition*. In the Lagrangian description this equation reads

$$\frac{dS}{dt} = 0, \tag{9.28}$$

which means [1] that a particle lying in the surface can not have normal velocity with respect to this surface, otherwise will produce a normal flow of fluid across the surface, which contradicts the free surface definition. To use the Eulerian picture, and to express the kinematic condition in terms of the velocity field $\boldsymbol{v}$, we choose a particle P that moves *together with the moving surface* Σ. The particle has a velocity $\boldsymbol{v}_{P\Sigma}(t) = d\boldsymbol{r}_P(t)/dt$. If the particle P moves *together* with Σ, there is a relation between $\boldsymbol{v}$ and S given by

$$\boldsymbol{v}_{P\Sigma} \cdot \nabla S + \frac{\partial S}{\partial t} = 0. \tag{9.29}$$

It is easy to prove this equation if we assume that the particle is contained in the surface at an arbitrary moment t and also at $t + \delta t$. That is: if $S(\boldsymbol{r}_{P\Sigma}(t), t) = 0$, then $S(\boldsymbol{r}_{P\Sigma}(t + \delta t), t + \delta t) = 0$. Equation (9.28) can also be written as

$$\left(\boldsymbol{v} \cdot \nabla S + \frac{\partial S}{\partial t} \right)_{\Sigma} = 0,$$

and this is a possible form for the free surface kinematics condition. The Σ subscript means that this equation is taken only on Σ, or in other words that, in this equation $(\boldsymbol{r}, t)$ have to fulfill $S(\boldsymbol{r}, t) = 0$. This form is more useful if the surface equation S is provided explicitly. For example if $S = 0 \rightarrow z = \eta(x, y, t)$, we have

$$\frac{d\eta}{dt} = v_z = \frac{\partial \eta}{\partial t} + v_x \frac{\partial \eta}{\partial x} + v_y \frac{\partial \eta}{\partial y}. \tag{9.30}$$

We would like to comment that, in some literature, this free surface kinematics condition is explained as "a fluid particle originally on the boundary surface will remain on it." This is not, in general, true. The P particle may sink inside the fluid (like in the case of dragging of the capillary surface by adherence forces) or evaporate. A more general physical statement would be that, for any particle lying at moment t in the surface, its velocity is tangent to the surface at that moment. From the mathematical point of view, this problem is equivalent to the fact that $d\boldsymbol{r}/dt$ is not well defined at the surface, because the set of points forming a geometrical surface Σ admits many mappings into itself. To eliminate this ambiguity, one can use just the normal velocity, as it is suggested by Meyer [2]. We can define the unit normal to the regular surface $S(\boldsymbol{r}, t) = 0$ by $\boldsymbol{n} = \nabla S / |\nabla S|$. The normal component of the velocity of Σ is

$$v_n = \left(\boldsymbol{n} \frac{d\boldsymbol{r}}{dt} \right) \bigg|_{\Sigma} \cdot \boldsymbol{n} = -\boldsymbol{n} \frac{\partial S}{\partial t} \frac{1}{|\nabla S|}.$$

By using (9.28) for S, we have

$$\boldsymbol{v}_n = -\frac{\frac{\partial S}{\partial t}}{|\nabla S|} = -\frac{\frac{dS}{dt} - (\boldsymbol{V} \cdot \nabla) S}{|\nabla S|} = \frac{(\boldsymbol{V} \cdot \nabla) S}{|\nabla S|},$$

where the last RHS is nothing but the velocity field along the normal to the surface $\boldsymbol{V} \cdot \boldsymbol{n}$. So we have obtained

$$\boldsymbol{v}_n = \boldsymbol{V}_n, \tag{9.31}$$

which is the most compact (and precise) form of the free surface kinematic condition: the normal component of the Lagrangian fluid particle velocity is equal, in any point of the surface, with the normal component of the Eulerian velocity.

9.6 Equation of Continuity

In Sects. 9.6.1 and 9.6.2, we analyze the equation of continuity. There are two reasons for choosing this topic. The first reason is that this equation provides a simple working application of the basic theorems of existence and uniqueness of the solutions of (linear or nonlinear) PDE. The second reason is that the equation of continuity has variable coefficients and it represents also a good toy model for such type of equations. However, it is still linear PDE, yet interesting in some of its particular solutions so it makes a "smooth" pedagogical transition from linear to nonlinear.

9.6.1 Introduction

In the nonrelativistic approximation mass is neither created nor destroyed, so we have the law of conservation of mass, i.e., a positive invariant

$$m = \int_D \rho d\mathcal{V} > 0,$$

integrated on the closure of the domain D filled with fluid. From its invariance we find the so-called equation of continuity integral or differential form

$$\int_D \left(\frac{\partial \rho}{\partial t} + div(\rho \boldsymbol{V}) \right) d\mathcal{V} = 0, \quad \frac{\partial \rho}{\partial t} + div(\rho \boldsymbol{V}) = 0, \tag{9.32}$$

in either integral or differential form. $\boldsymbol{V}(\boldsymbol{r}, t)$ is the velocity field and $\mathcal{V}$ is the volume. In fluid mechanics, the equation of continuity is coupled with other equations for conservation of momentum (Euler or Navier–Stokes) and for energy or entropy transfer, such that in total we have five scalar PDEs for the five scalar fields for the problem: ρ, $\boldsymbol{V}$, and p the pressure (by scalar we mean here also a component of a vector field). The continuity equation alone is not useful for physics, and some of its solutions do not have physical signification, unless coupled with the other dynamical equations. However, we present in the followings a theorem of existence and uniqueness, and some applications for (9.32). Such examples are not usually analyzed in books of fluid dynamics, but they can work as a good exercise of mathematical physics.

We study the equation of continuity when the velocity field is given, and we integrate it to find the density distribution. The continuity equation (9.32) is a homogenous linear PDE of order 1, with variable coefficients, defined in a certain domain $\mathcal{D} \subset \mathbb{R}^4$ of space–time. The main tool we need is the Cauchy–Kovalevskaya theorem for existence and uniqueness of the solutions of a general (not necessarily linear) PDE [27]. According to this theorem, the continuity equation has one unique real analytic solution $\rho(\boldsymbol{r}, t)$ for a given analytic velocity field $\boldsymbol{V}(\boldsymbol{r}, t)$ and given Cauchy condition provided by $\rho(\boldsymbol{r}, t)|_{\Sigma} = g(\xi_1, \xi_2, \xi_3)$, where g is an analytic function defined on a regular hypersurface $\Sigma \subset \mathbb{R}^4$. The Cauchy–Kovalevskaya theorem can be applied to any nonlinear PDE, for arbitrary Cauchy conditions expressed in terms of analytic functions, if one of the highest order derivative of the PDE can be explicitly written as an analytic function depending on the other terms and variables in the PDE. For example in (9.32), PDE of order 1, we can write the time derivative of the unknown function ρ on the LHS, and express it as an analytic function of the variable coefficients $\boldsymbol{V}_i$ and partial derivatives of ρ with respect to the other coordinates x_i, on the RHS (named generically $f(\boldsymbol{r}, t, \rho, \partial\rho/\partial x_i, \ldots)$)

$$\frac{\partial \rho}{\partial t} = f - \sum_{i=1}^{3} \frac{\partial(\rho V_i)}{\partial x_i}.$$

The function f is analytical because the finite sum and multiplication preserve analyticity, so we are in the frame of the Cauchy–Kovalevskaya theorem, which we discussed earlier, see Theorem 2. In general, if the PDE is of order m we need m Cauchy conditions, one for each derivative of order 0 to $m-1$ of the unknown function, with respect to a non-tangent direction on the Cauchy hypersurface.

In order to ease reader's search, we briefly write the Cauchy–Kovalevskaya theorem below. If a PDE of order m in the unknown function $u(x_1, \ldots, x_n)$ can be written in the form

$$\frac{\partial^m u}{\partial x_1^m} = f\left(x_1, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial^m u}{\partial x_1^{m_1} \ldots \partial x_n^{m_n}}\right), \tag{9.33}$$

where $m = m_1 + \cdots + m_n$ and where the term $\frac{\partial^m u}{\partial x_1^m}$ does not appear on the RHS, then the Cauchy problem attached to this PDE:

$$\left.\frac{\partial^j u}{\partial l^j}\right|_{\Sigma} = g_j, \quad j = 0, 1, \ldots, m-1 \tag{9.34}$$

with functions g_j defined on the $(n-1)$-dimensional regular hypersurface $\Sigma \subset \mathbb{R}^n$, where l is an arbitrary not tangent direction on Σ, admits a unique analytical solution u, if the functions f, g_j are analytical on their domains of definition.

This theorem states the existence and uniqueness of an analytic solution, but this does not exclude the existence of other, non-analytical solutions of the same Cauchy problem. However, if the PDE is linear (Holmgren uniqueness theorem) there are no solutions except the analytical ones. This last result shows that possible compact supported solutions or very localized solutions (like solitons, compactons, peakons, etc.), which of course are not analytical functions, could not arise from a linear PDE. High localization is strictly related, or generated, by the nonlinearity in the PDE.

We remind here that there is one special case in which linear equations provide compact supported solutions, i.e., the discrete wavelets 2-scale equation [28]. For example, the Haar scaling function (the step function), defined as 1 on [0, 1] and zero in the rest of real axis, is a solution of the finite difference equation $\Phi(x/2) = \Phi(x) + \Phi(x-1)$. This result reveals a possible deeper connection between linear finite difference equations (or infinite-order linear PDE equations) and nonlinear PDE.

Returning to the continuity equation we prove the existence and uniqueness theorem for its Cauchy problem. In the course of this proof we use special Cauchy condition defined on the hyperplane $t = t_0$. However, it is easy to generalize the following proof for general Cauchy conditions on an arbitrary hypersurface. This is because any arbitrary Cauchy hypersurface is regular, and hence we can find a local change of coordinates $(\boldsymbol{x}, t) \to (\boldsymbol{x}', t')$, such that the hypersurface in the new coordinates is determined by the equation $t' = t_0'$, without any loss of generality or analyticity. Choosing the Cauchy condition on the hyperplane $t = t_0$ means knowing the density at the initial moment in the whole space, or in the domain of definition of the position vector. In the general Cauchy hypersurface case, the condition can be both initial condition and boundary condition, for example if Σ is defined by $\Sigma = \{(\boldsymbol{x}, t) | t = t_0 \text{ and } \boldsymbol{x} \in D\} \bigcup \{(\boldsymbol{x}, t) | t \leq t_0 \text{ and } \boldsymbol{x} \in \partial D\}$, etc. Moreover, we can always reduce any Cauchy condition to a null Cauchy condition. If the function $\tilde{\rho}$ is a solution of the equation

$$\frac{\partial \tilde{\rho}}{\partial t} = -div(\tilde{\rho}\boldsymbol{V}) - div(g\boldsymbol{V}) \tag{9.35}$$

under the null Cauchy condition $\tilde{\rho}(\boldsymbol{r}, t_0) = 0$, then $\rho = \tilde{\rho} + g(\boldsymbol{r})$ is a solution of the continuity equation (9.32) for the same $\boldsymbol{V}$, and the general Cauchy condition $\rho(\boldsymbol{r}, t_0) = g(\boldsymbol{r})$. The analyticity of the functions involved is not changed by this functional substitution. In the following, we use a generic function f instead of the RHS of the PDE under consideration, no matter if it is (9.32), (9.33), or (9.35).

The sketch of the proof of existence and uniqueness of the solution of the continuity equation can be presented briefly as follows. We construct the Taylor series of a hypothetic analytic solution ρ of (9.32), by using the initial condition and the equation itself. If such a solution exists, then by construction it is unique. To prove its existence, we construct an upper bound function f^{ub} for the RHS of (9.32). Such a construction is always possible, and the good news is that its associate solution, i.e., the solution of $\partial\rho/\partial t = f^{ub}$, is an upper bound function for ρ. By using the comparison criterium, $\rho \ll \rho^{ub}$, it results that ρ is uniformly convergent, hence analytical. This concludes the proof. Now we proceed with the detailed discussion.

To construct the Taylor series we use the following.

Lemma 8 *If the velocity field $\boldsymbol{V}(\boldsymbol{r}, t)$ and the Cauchy condition $\rho(\boldsymbol{r}, t_0) = g(\boldsymbol{r})$ are analytic in a neighborhood $\mathcal{V}(\boldsymbol{r}_0, t_0)$, then the Cauchy problem for* (9.32) *admits one unique analytic solution in $\mathcal{V}$.*

Proof Since this hypothetic solution is analytic, we can construct it as a Taylor series in the form

$$\rho(\boldsymbol{r}, t) = \rho(\boldsymbol{r}_0, t_0) + (t - t_0)\frac{\partial \rho}{\partial t}\bigg|_0 + \sum_{i=1}^{3}(x_i - x_{i0})\frac{\partial \rho}{\partial x_i}\bigg|_0$$

$$+\frac{1}{2!}\left[(x_i - x_{i0})(x_j - x_{j0})\sum_{i,j=0}^{3}\frac{\partial^2 \rho}{\partial x_i \partial x_j}\bigg|_0 + \sum_{i}^{3}(x_i - x_{i0})(t - t_0)\frac{\partial^2 \rho}{\partial x_i \partial t}\bigg|_0\right.$$

$$\left. + (t - t_0)^2\frac{\partial^2 \rho}{\partial t^2}\bigg|_0\right] + \frac{1}{3!}\left[\sum_{i,j,k=0}^{3}(x_i - x_{i0})(x_j - x_{j0})(x_k - x_{k0})\frac{\partial^3 \rho}{\partial x_i \partial x_j \partial x_k}\bigg|_0 + \cdots\right] + \cdots, \tag{9.36}$$

where by subscript 0 we understand that the value is taken in the point $(\boldsymbol{r}_0, t_0)$. Substitute in this series the initial Cauchy and the equation itself

$$\rho(\boldsymbol{r}_0, t_0) = g(\boldsymbol{r}_0)$$

$$\frac{\partial \rho}{\partial t}\bigg|_0 = -div(\rho \boldsymbol{V})|_0 = -div(g\boldsymbol{V})$$

$$\frac{\partial \rho}{\partial x_i}\bigg|_0 = \left(\frac{\partial}{\partial x_i}\rho(\boldsymbol{r}, t_0)\right)_{\boldsymbol{r}_0} = \frac{\partial g}{\partial x_i}(\boldsymbol{r}_0)$$

$$\frac{\partial^I \rho}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_n}}\bigg|_0 = \frac{\partial^I g}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_n}}(\boldsymbol{r}_0)$$

$$\frac{\partial^2 \rho}{\partial x_i \partial t}\bigg|_0 = -div\frac{\partial}{\partial x_i}(g\boldsymbol{V}(\boldsymbol{r}, t_0))_{\boldsymbol{r}_0}, \text{ etc.}, \tag{9.37}$$

and so on, for all terms. The hypothetic analytic solution is now fully determined, which proves its uniqueness. To prove its existence, we need to introduce the concept of *upper bound function* in general in $\mathbb{R}^n$. □

Definition 61 Let $x_0 \in \mathbb{R}^n$ and f is an analytic function defined on a neighborhood $\mathcal{V}(x_0)$, such that

$$f(x) = \sum_{i_1, i_2, \dots i_n} F_{i_1, i_2, \dots i_n} (x_1 - x_{01})^{i_1} \cdots (x_n - x_{0n})^{i_n},$$

for $x \in \mathcal{V}(x_0)$. We define an analytic function on $\mathcal{V}(x_0)$

$$f^{ub}(x) = \sum_{i_1, i_2, \dots i_n} G_{i_1, i_2, \dots i_n} (x_1 - x_{01})^{i_1} \cdots (x_n - x_{0n})^{i_n},$$

called *upper bound* of f, if $\forall i_1, \dots i_n$ we have:

1. $|F_{i_1, \dots i_n}| < G_{i_1, \dots i_n}$.
2. $0 \leq G_{i_1, \dots i_n}$.

The notation is $f \ll f^{ub}$. The next step is to find an upper bound function for the RHS term of the continuity equation.

Theorem 23 *For any function*

$$f = \sum_{i_1, i_2, \dots i_n} F_{i_1, i_2, \dots i_n} (x_1 - x_{01})^{i_1} \cdots (x_n - x_{0n})^{i_n},$$

analytic on a neighborhood $\mathcal{V}(x_0)$, there is a neighborhood $\mathcal{W}(x_0) \subset \mathcal{V}(x_0)$ where f has an analytic upper bound function of the form

$$f^{ub}(x) = \frac{M}{1 - \frac{\sum_{i=1}^{n} (x_i - x_{0i})}{\alpha}} + C, \tag{9.38}$$

where $M > 0$, $\alpha \in \mathbb{R}$, and C is a constant.

Proof Obviously, $\exists \xi \in \mathcal{W}$ such that the numeric series

$$\sum_{i_1, i_2, \dots i_n} F_{i_1, i_2, \dots i_n} (\xi_1 - x_{01})^{i_1} \cdots (\xi_n - x_{0n})^{i_n},$$

is uniformly convergent, which implies that the sequence $F_{i_1, i_2, \dots i_n} (\xi_1 - x_{01})^{i_1} \cdots (\xi_n - x_{0n})^{i_n} \to 0$, so it is bounded, i.e., $\exists M > 0$ such that

$$|F_{i_1, i_2, \dots i_n} (\xi_1 - x_{01})^{i_1} \cdots (\xi_n - x_{0n})^{i_n}| < M.$$

Then

$$M \sum_{i_1, \dots i_n} \frac{(x_1 - x_{01})^{i_1} \cdots (x_n - x_{0n})^{i_n}}{|(\xi_1 - x_{01})^{i_1} \cdots (\xi_n - x_{0n})^{i_n}|},$$

is an upper bound for f on $\mathcal{W}$, according to Definition 3. Since the above series is also a geometric progression, we can calculate its sum. Then we can find an upper bound function f^{ub} for this progression in the form

$$\frac{M}{\left(1-\frac{x_1-x_{01}}{|\xi_1-x_{01}|}\right)\cdots\left(1-\frac{x_n-x_{0n}}{|\xi_n-x_{0n}|}\right)} < \frac{M}{1-\frac{\sum_{i=1}^{n}(x_i-x_{0i})}{\alpha}} + \text{cst.} = f^{ub}(x), \qquad (9.39)$$

with $\alpha = \min\{|\xi_1 - x_{01}|, \ldots |\xi_n - x_{0n}|\}$. The next step is to take this type of upper bound function in $n = 4$ and use it in the RHS of the continuity equation, instead of its original RHS, with an appropriate choice of the arbitrary constant cst.

$$\frac{\partial \rho^{ub}}{\partial t} = \frac{M}{1-\frac{t+x+y+z+\rho+\sum_{i=1}^{3}\frac{\partial \rho}{\partial x_i}}{\alpha}} - M. \quad \square \qquad (9.40)$$

Lemma 9 *The null Cauchy problem for (5.14) has a unique analytic solution* ρ^{ub} *in a neighborhood of* 0*, whose Taylor series has all coefficients nonnegative.*

Proof We introduce the variable $\chi = t + x + y + z$ and we look for solutions of (5.14) of the form $\rho(t, x, y, z) = u(\chi)$ under the initial condition $u(0) = 0$. The PDE (5.14) reduces to an ODE

$$u'(\alpha - \chi - 3M) - uu' - 3(u')^2 - Mu - M\chi = 0,$$

and according to the Peano theorem (remember, it is based on the fixed point theorem [29]) this equation has a unique analytical solution in the initial condition $u(0) = 0$. When $\chi = 0$ we have a possible solution $u'(0) = 0$. By differentiating the ODE one more time, and by calculating it again in $\chi = 0$, we have $u''(0) = M/(\alpha - 3M)$. If we choose $\alpha \geq 3M$ it results $u^{(k)}(0) \geq 0$ for $k = 0, 1, 2$. In general, after n successive differentiations, we have

$$u^{(n)}(0) = \frac{1}{\alpha - 3M}\left(\sum_{k,j=0}^{n} |C_{kj}|u^{(k)}(0)u^{(j)}(0) + (\alpha M + n)u^{(n)}(0)\right).$$

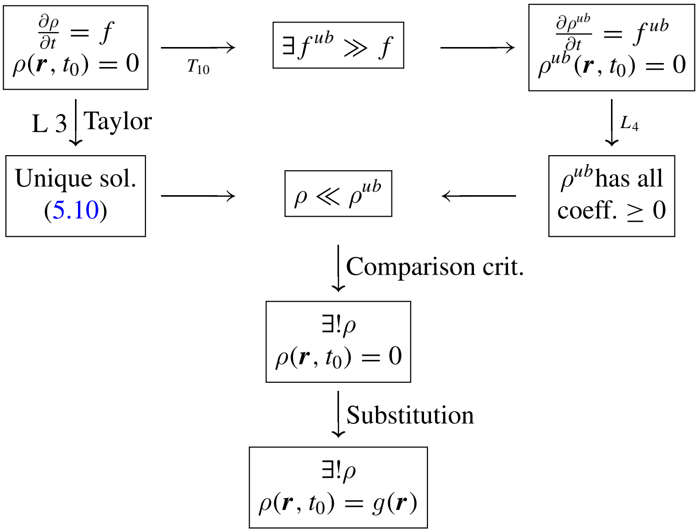
It results, by induction, that $\forall k$, $u^{(k)}(0) \geq 0$ if $\alpha > 3M$. This result proves that the null Cauchy problem for (9.40) has always an unique analytic solution, whose Taylor series coefficients are nonnegative:

$$\rho^{ub}(\boldsymbol{r}, t) = \sum |C_{i_0,i_1,i_2,i_3}| t^{i_0} x^{i_1} y^{i_2} z^{i_3}. \qquad (9.41)$$

There is no loss of generality by choosing null Cauchy conditions in Lemma 3. We proved in (9.35) that any null Cauchy conditions can be changed into arbitrary Cauchy conditions, so Lemma 3 is general. Now we attack the final step of our proof.

The uniqueness of the Cauchy problem for (9.32) was proved in Lemma 2, so we just need to prove the existence of analytic solution ρ. Since the actual RHS term of the continuity equation is analytic in all its variables, we can find an upper bound function for the PDE in the form of (9.38). We solved this auxiliary PDE (Lemma 9) and its solution ρ^{ub} has the property: $\rho \ll \rho^{ub}$. This is true because we build the solutions term by term, by using the functions f, f^{ub}, and the Cauchy data g (like we did in (9.36) and (9.37)). The upper bound property transfers from the fs to the ρs. Consequently, all the coefficients (partial derivatives in 0) of the Taylor series for ρ are upper bounded by the corresponding coefficients (corresponding partial derivatives in 0) of ρ^{ub}. Since the series in (9.41) is analytic, by the comparison criterium, it results the analyticity of the series ρ (see (9.36) and (9.37)). But this is the actual solution of (9.32), which proves the whole theorem.

We briefly present the above proof in the diagram (9.42)



□ (9.42)

9.6.2 *Solutions of the Continuity Equation on Compact Intervals*

In Sect. 9.6.1 we discussed the general conditions under which the continuity equation has a unique analytical solution. In this section we investigate some special one-dimensional situations having exact solutions. That is a Cauchy one-dimensional problem for $\rho(x, t)$ for given $V(x, t)$. We focus especially on the behavior of the solutions at the boundaries of a compact interval of length $2L$. The one-dimensional version of the continuity equation reads

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial V}{\partial x} + V \frac{\partial \rho}{\partial x} = 0, \tag{9.43}$$

for $x \in [-L, L]$, $t \geq 0$. At the boundaries of the interval, we should have no flow of matter so we impose the BC $v(\pm L, t) = 0$, in addition to the Cauchy condition. It is easy to build the general solution from the Fourier expansions

$$\rho(x,t) = \sum_{n \geq 0} \rho_n(t) e^{\frac{i\pi n x}{L}}, \quad V(x,t) = \sum_{n \geq 0} V_n(t) e^{\frac{i\pi n x}{L}}, \tag{9.44}$$

and from the BC we have

$$\sum_{n \geq 0} (-1)^n V_n(t) = 0. \tag{9.45}$$

If we plug the formulas from (9.44) in the continuity equation (9.43), we obtain a recursion relation

$$\rho'_k(t) = -\frac{i\pi k}{L} \sum_{n=0}^{k} \rho_n V_{k-n}. \tag{9.46}$$

With the notation

$$\mathbb{V}^k(t) \equiv e^{-\frac{i\pi k}{L} \int_0^t V_0(t')dt'},$$

we have (9.46), the new recursion relation

$$\rho_k(t) = \mathbb{V}^k(t) \left(\rho_k(0) - \frac{i\pi k}{L} \int_0^t \mathbb{V}^{-k}(t') \sum_{n=0}^{k-1} \rho_n(t') V_{k-n}(t') dt' \right), \tag{9.47}$$

where $\rho_k(0)$ are determined by the initial condition through the inverse Fourier transform

$$\rho_n(0) = \frac{1}{2\pi} \int_{-L}^{L} \rho_{initial}(x) e^{-\frac{in\pi x}{L}} dx. \tag{9.48}$$

We choose a simple physical example, where the initial density is the same everywhere within the compact $[-L, L]$, and zero outside. That is $\rho(x, 0) = m/(2L)$, where m is the total mass of the fluid inside the bounded segment. It results $\rho_0(0) = m/(2L)$ and $\rho_n(0) = 0$ for $n > 0$. We also choose a simple configuration for the velocity, namely $V(x,t) = a \sin(\omega t) \left(e^{\frac{i\pi x}{L}} + e^{\frac{2i\pi x}{L}} \right)$. That is $V_1(t) = V_2(t)$. This is a stationary (longitudinal) oscillation in velocity along the segment, with zero velocity in the ends. We have $V_n(t) = 0$ for $n = 0, 3, \ldots$. By substituting these expressions for the velocity components in (5.23), we obtain $\mathbb{V}^{\pm k} = 1$ and

$$\rho_k(t) = -\frac{i\pi ka}{L}\int_0^t \sin(\omega t')(\rho_{k-1} + \rho_{k-2})dt', \quad k = 1, 2, \ldots \tag{9.49}$$

This recursion provides the unique solution for $k \geq 1$.

Apparently, finding general solutions for the continuity equation in one-dimensional, $\rho_t + \rho V_x + \rho_x V = 0$, is a simple procedure (subscripts represent, again, differentiation). However, there is a hidden problem at the boundaries, produced by the zeros of the coefficients in the PDE. At the ends of the interval, we have to assume no flow of fluid, so $V(\pm L, t) = 0$. In a neighborhood $(L - \epsilon, L)$ of the right boundary for example, we can test the behavior of a Fourier component of the solution $\rho_\omega(x, t) = r(x)e^{i\omega t}$, and we obtain

$$\frac{d(\ln\ r_\omega)}{dx} V_\omega = -\left(\frac{dV_\omega}{dx} + i\omega\right), \tag{9.50}$$

which means that in this neighborhood, even if $V_x = 0$, we still have the RHS nonzero. But, when $V \to 0$, it seems that $d(ln\ r_\omega)/dx \to \infty$. So, the zeros of velocity at boundaries may introduce singularities in density (by reciprocity, in the inverse problem, isolated zeros of density can also introduce singularities in velocity). Let us suppose that the velocity approaches the zero as a power law $V(L - \epsilon, t) \simeq \epsilon^a$, $a > 0$. If $a < 1$ we have $\lim_{x\to L}(\rho) < +\infty$. But if $a > 1$ we expect $\lim_{x\to L}(\rho) = +\infty$. If V is a rapidly decreasing function in that neighborhood, we can neglect the third term in (9.43) and use the approximation

$$\frac{\partial\rho}{\partial t} \simeq -\rho\frac{\partial V}{\partial x},$$

to investigate the behavior of ρ. By direct integration we obtain

$$\rho(L - \epsilon, t) \simeq \rho_L e^{-\int_0^t} V_x(L - \epsilon, t')dt',$$

where ρ_L is a constant. This asymptotic solution is a very rapidly increasing function toward L, but it is not anymore a singularity.

Let us illustrate with examples. We take a simple form for velocity in a compact interval $x \in [-L, L]$

$$v(x, t) = V_0 \sin\omega t \cos kx,$$

as stationary oscillations, where $k = (2n + 1)\pi/(2L)$, n arbitrary integer and V_0, ω are constants. The solution can be easily obtained by the procedure indicated above or by simple separation of variables. The general solution is a real integral over the label λ of the following components

$$\rho(x,t,\lambda) = \rho_0 e^{-\frac{\lambda}{\omega}\cos\omega t} \frac{\left(\cos\frac{kx}{2} + \sin\frac{kx}{2}\right)^{a-1}}{\left(\cos\frac{kx}{2} - \sin\frac{kx}{2}\right)^{a+1}},$$

where $a = -\lambda/(kV_0)$, and ρ_0 are constants. Obviously this solution has singularities within $[-L, L]$, provided by the trigonometric zeros of the denominator. The reason is the cancellation of velocity in different points (function of how large is n) including the boundaries. Velocity approaches zero by following a quadratic law: $V(L-\epsilon, t) \simeq k^2\epsilon^2/2$.

What can be done to eliminate these singularities? Of course, by coupling the continuity equation with Euler and energy conservation equations, the nonphysical solutions will be eliminated. However, one simple possibility to eliminate the singularity in density is to introduce an artificial constant term in velocity

$$V = V_0(\sin\omega t \cos kx + V_1).$$

From the physical point of view, it means that we have a little ($V_1 \ll 1$) constant "leakage" of fluid at the boundaries. With this new expression for velocity we have

$$\rho(x,t,\lambda) = \rho_0 e^{-\frac{\lambda}{\omega}\cos\omega t} \left(\frac{1 + \frac{V_1 - 1}{\sqrt{1-V_1^2}} \tan\frac{(2n+1)\pi x}{4L}}{1 - \frac{V_1 - 1}{\sqrt{1-V_1^2}} \tan\frac{(2n+1)\pi x}{4L}} \right)^{\frac{2L\lambda}{(2n+1)\pi V_0 \sqrt{1-V_1^2}}} \cdot$$

$$\cdot \frac{1}{V_1 + \cos\frac{(2n+1)\pi x}{2L}}.$$

The solution is not anymore singular in $\pm L$ and it is illustrated in Fig. 9.12. Global longitudinal oscillations of the fluid induce oscillations in the amount of fluid accumulated to the right end of the domain.

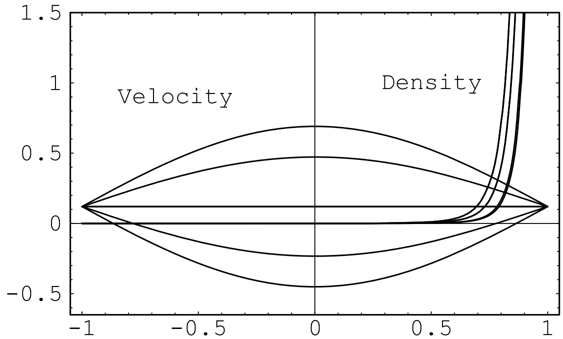


Fig. 9.12 Plot of velocity and density from one-dimensional continuity equation on an interval $[-1, 1]$. Velocity has stationary oscillations – up and down in this figure means motion of the fluid to right and left – and the fluid is accumulating in the right end. The density has itself push–pull oscillations

It is interesting to check the reverse phenomenon, namely if zeros in density provide singularities in velocity. For the stationary oscillating density inside $[-L, L]$

$$\rho(x,t) = \rho_1 \sin kx \sin \omega t,$$

with ω, ρ_1 constants and k defined as above, we compute the velocity in the form

$$V(x,t) = V_) \cot \omega t \frac{C_1 + \omega \rho_1 \cos kx}{k(\rho_0 + \rho_1 \sin kx)},$$

where V_0, C_1, and ρ_0 are constants. In Fig. 9.13 we plot both the velocity and the density for this example for $L = 1$. Indeed, the density-isolated zeros provided by $\sin kx$ result in singularity in velocity given by the cot function. Another example is presented for a semi-infinite domain $x \in (-\infty, 0]$. We choose the velocity of the form

$$V(x,t) = -\frac{ax}{at + \rho_0 \cosh \frac{tx}{b}},$$

where a, b, and ρ_0 are arbitrary constants. Around zero the velocity behaves like $V(0) \simeq x$ which provides a "milder" type of singularity for ρ. The corresponding solution for density is

$$\rho(x,t) = \rho_0 + at\text{sech}\frac{tx}{b}.$$

The results are presented in Fig. 9.14. In the last example, we present some localized traveling wave solutions along the axis. We assume the propagation of a KdV solitary wave on the free surface of a one-dimensional channel

$$\eta(x,t) = A\text{sech}_2 \frac{x - vt}{L},$$

where A is the wave amplitude, L the half-width, and v the group velocity. The tangent velocity of the fluid at the free surface is given by

$$V(x,t) = -\frac{2A}{L}\text{sech}^2 \frac{x - vt}{L} \tanh \frac{x - vt}{L}.$$

We neglect that the KdV equation for shallow water was deduced in the incompressibility approximation, at least for a very thin layer on the surface [30]. Let us presume that this layer is compressible (like a surfactant layer on the surface of the incompressible fluid) and the density in it is the solution of the continuity equation for the velocity given above. The density reads

$$\rho(x,t) = \rho_0 \frac{1}{v - V(x,t)},$$

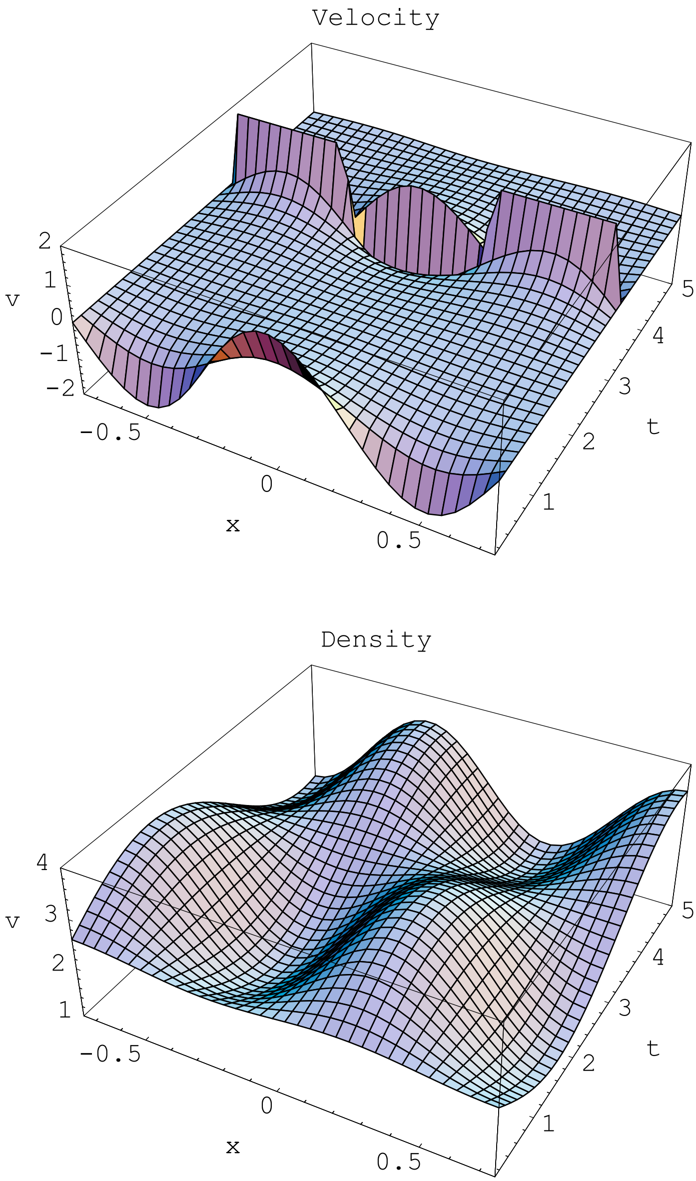


Fig. 9.13 At $t = 0$ density is uniformly distributed, and velocity has a positive maximum centered around $x = 0$, and two symmetric negative minima. Initially, the matter is pushed from left and right into two points, placed with approximation at $x = 0.25$ and $x = -1$. Around $t = 2$ one can see in the density plot the resulting accumulation of fluid in these two points. At this moment the velocity is almost zero and we have quasiequilibrium. Next, the velocity changes the sign, and the fluid is pushed toward two other centers, namely $x = 0$ and $x = 1$. As a result, at $t = 5$ we have more accumulation of fluid in these points. About $t = 4$ velocity has its singularity

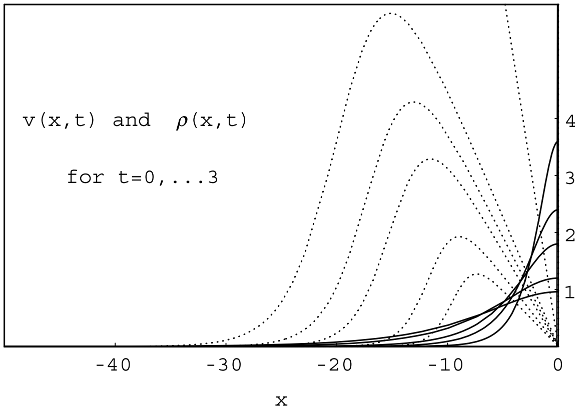


Fig. 9.14 Velocity (*dotted lines*) and density (*continuous lines*) for a one-dimensional semi-infinite axis. The velocity has a localized bump which pushes the fluid against the right wall, creating a fluid accumulation

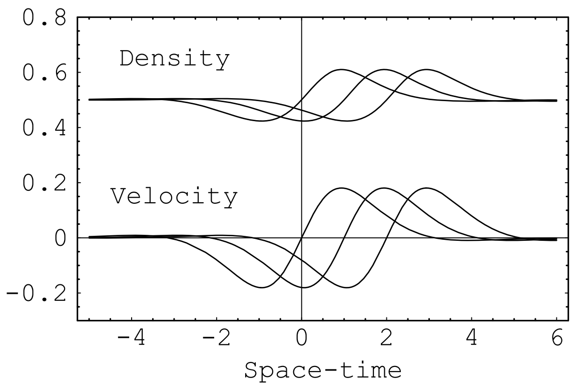


Fig. 9.15 Surface density and tangent velocity at the free surface for an MKdV soliton

where ρ_0 is the equilibrium density in the absence of the wave. Density has no singularities in this example. We present the results in Fig. 9.15. We can obtain a similar result for an MKdV soliton. We choose the velocity profile as a modulated breather [30]

$$V(x,t) = V_0 \text{sech}\frac{x-vt}{L}\sin\omega(x-vt).$$

The density profile is given by a similar equation as in the KdV case

$$\rho(x,t) = \frac{\rho_0}{V(x)-v},$$

see Fig. 9.15.

9.7 Problems

1. Show that the free surface condition, i.e., the path of a fluid particle $\boldsymbol{r}_L$ does not leave a surface Σ (see (9.5), (9.28), and (9.29)), is the equivalent of requesting the Lagrangian path of the particle to belong to the time variable surface, both described in extended space $\mathbb{R} \times \mathbb{R}^3$ for time and positions.
2. Consider a sphere of radius R at rest surrounded by inviscid, incompressible, and irrotational fluid of density ρ. The fluid moves past the sphere such that the velocity at infinite distance from the sphere is a constant and uniform field $\boldsymbol{v}_\infty = (0, 0, -u)$. Find the Eulerian velocity, the pressure field and the stream lines. Find the Lagrangian paths and compare them with the stream lines.
3. Let us have the following field of Eulerian velocity

$$\boldsymbol{v}_E(\boldsymbol{r}, t) = (a_1(t)x^{\alpha_1}, a_2(t)y^{\alpha_2}, a_3(t)z^{\alpha_3}),$$

where $a_i(t)$ are arbitrary smooth functions and $\alpha_i \in \mathbb{R}$. Find the equations of the stream lines and the path lines. Show that if $a_i(t)$ are constant, the stream and path lines coincide for an appropriate choice of integration constants.
4. Consider the Lagrangian paths of some fluid particles $\boldsymbol{r}_L(\boldsymbol{r}_0, t)$ as a one-parameter t group of diffeomorphisms mapping the initial positions of the particles into the current ones $\boldsymbol{r}_0 \rightarrow \boldsymbol{r}_L$, acting in $\mathbb{R}^3$. Consider a time-dependent physical quantity Ω described by a differentiable 1-form ω defined on $T^*_{\boldsymbol{r}_L}\mathbb{R}^3$. Prove that the Lie derivative of this 1-form with respect to the tangent directions to the diffeomorphism transformations

$$L_{\boldsymbol{r}_L(\boldsymbol{r}_0,t)}(\omega) = \lim_{dt\rightarrow 0} \frac{d\boldsymbol{r}^*_L(\omega) - \omega}{dt} = \frac{d}{dt}\left(\omega_j \frac{\partial x^j_L}{\partial x^i_0} - \omega_i\right) dx^i$$

provides the Eulerian–Lagrangian law of transformation for Ω.
5. Equations (9.12) and (9.13) were obtained by using the Lie derivative with respect to the fluid flow. Try to find the same equations from a different approach, namely a new law of covariant differentiation on a four-dimensional manifold (σ^0, σ^i) with a linear connection. The last two and three terms, respectively, in the RHS of (9.12) and (9.13) could be understood as connection coefficients with the Christoffel symbols of the second kind fulfilling

$$\Gamma^i_{k0} = -\frac{\partial v^i}{\partial \sigma^k}.$$

Hint: we need to introduce a metric on this manifold, $g_{\mu\nu}$, with $\mu, \nu = 0, 1, \ldots, 3$. The Christoffel symbols of first and second kind are related by $\Gamma^\alpha_{\beta\gamma} = g^{\delta\alpha}\Gamma_{\beta\delta\gamma}$, and the last one is defined by the metric

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\gamma\beta}}{\partial \sigma^\alpha} + \frac{\partial g_{\beta\alpha}}{\partial \sigma^\gamma} - \frac{\partial g_{\alpha\gamma}}{\partial \sigma^\beta}\right),$$

see for example [4, 31–34]. A possible hypothesis could be $g_{i0} = 0$, $g_{00} = \text{const}$. The remaining PDE equations for g_{ij} may result in an exponential matrix solution. It is interesting to relate the skew-symmetry property of this PDE in the metric coefficients with the fact that the integral curves of a rotational flow are singular.

6. Prove that the covariant time derivative (9.12) and (9.13) has the following actions

$$\frac{d_c A}{dt} = \frac{dA}{dt} + \gamma^t A, \text{ on covariant vectors,}$$
$$\frac{d_c A}{dt} = \frac{dA}{dt} - \gamma A, \text{ on contravariant vectors,}$$
$$\frac{d_c \Omega}{dt} = \frac{d\Omega}{dt} - \gamma\Omega - \Omega\gamma^t, \text{ on (2.0) tensors,}$$
$$\frac{d_c \Omega}{dt} = \frac{d\Omega}{dt} + \gamma^t\Omega + \Omega\gamma, \text{ on (0.2) tensors.}$$

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