11-5-2022

Hydrodynamics

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Scholarly Commons Citation
Ludu, A. (2022). Hydrodynamics. In: Nonlinear Waves and Solitons on Contours and Closed Surfaces. Springer Series in Synergetics. Springer, Cham. https://doi.org/10.1007/978-3-031-14641-1_10

Springer Series in Synergetics

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Andrei Ludu

Nonlinear Waves and Solitons on Contours and Closed Surfaces

Third Edition

Springer

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Nonlinear Waves and Solitons on Contours and Closed Surfaces

Third Edition

Springer

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ISSN 0172-7389 ISSN 2198-333X (electronic)
Springer Series in Synergetics
ISBN 978-3-031-14640-4 ISBN 978-3-031-14641-1 (eBook)
https://doi.org/10.1007/978-3-031-14641-1

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

To my family, the most important presence.

Foreword

The story of solitary waves traces back to John Scott Russel. Approaching 200 years ago he wrote:

> *I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.*

Russel went on to conduct experiments and published his findings in 1845 (check this). Initially, major figures such as Stokes and Airy denied the existence of what we would now call a traveling wave on the surface of water in a channel. In the second half of the nineteenth century, one sees in the correspondence between Stokes and Raleigh that Stokes had changed his mind and this fact even appears in published work. In the period of this correspondence, Rayleigh found an approximate relation between the amplitude and speed of a solitary wave in a channel. However, it was left to Boussinesq in the 1870s to write down evolution equations that approximated the motion of disturbances on the surface of water and which featured exact solitary-wave solutions. One of these was the celebrated Korteweg-de Vries equation of water wave theory that was rederived by Joseph Korteweg and his student Gustav de Vries in 1895. The issue of existence of these so-called solitary waves having been settled, at least as far as the nineteenth century hydrodynamicists were concerned, the subject went moribund.

It came back to life, though in disguise, in work of Fermi, Pasta, Ulam and Tsingou on a lattice and spring model for heat conduction in the 1950s. Later, by taking an appropriate continuum limit of this mass and spring model, Kruskal and Zabusky came again to the Korteweg-de Vries equation. This time, however, the subject did not die. In 1967, the inverse scattering theory for this equation was discovered by Gardner, Greene, Miura and Kruskal. Peter Lax took the first step in putting this formalism into a very imaginative mathematical structure. Since then, the subject rapidly achieved industrial proportions, with tens of thousands of journal pages and with many, many applications of the theory.

As Andrei Ludu, the author of the present monograph writes in his introduction, considering the large literature on solitary waves, why yet another book? There are several things that set this text apart from others in the field. First is the overall focus upon solitary waves defined on compact spaces. Of course, one thinks initially of the classical cnoidal-wave solutions of the Korteweg-de Vries equation, but as Ludu ably shows, this is the tip of a very large iceberg. Another aspect of the text that strikes a new chord is the differential geometric perspective; the view that solitary waves can be realized as the motion of a planar or three-dimensional curve under particular flow conditions and with suitable initial conditions. This is not original to the text in question, but an overall assessment of these ideas and a comprehensive review of its applications is not to be found elsewhere in the literature. And, speaking of applications, the text ends with a large number of very diverse and interesting applications.

The text breaks into four parts. Parts I and II, which comprise the first eight chapters, contain a sketch of the relevant topology and especially the differential geometry of curves and surfaces in two and three spatial dimensions. It should be acknowledged that this material is not for beginners. Someone without prior knowledge of at least portions of this material will not find it easy going. However, as a reminder to those with some knowledge, and a focus on exactly what is needed from differential geometry in what follows, it is very helpful. Especially the material in Chapter 6 will be useful even for the cognoscenti.

Chapter 7 works out the connection between the motion of curves in two and three dimensions and integrable systems. Chapter 8 does the same thing for the motion of surfaces. Technically, this is the heart of the script. This will be new material to many readers; indeed, it is a developing subject in the mathematical firmament.

Ludu's exposition in Parts I and II is technically sound, but it makes much of its headway by way of appealing to our intuition. Not every theorem is proved in detail, which is quite okay given the overall goal of the text.

In Parts III and IV, the text becomes more concrete. It begins with a more or less standard discussion of the kinematics of fluid motion in Chapter 9. Knowledgeable readers may well skip this, but for folks a little rusty, it is helpful. Some of the notation is laid out in this chapter as well.

Chapters 10 and 11 find us deriving the Euler and Navier-Stokes equations. This includes a very detailed discussion of surface tension from a geometrical perspective. He goes on to derive many of our favorite approximate models, such as the Korteweg-de Vries equation, the modified Korteweg-de Vries equation, the Boussinesq equation and the cubic Schrodinger equation. He examines the well-known solitary-wave solutions of these equations by way of the mathematical structure developed in Part I. He also derives what he terms the GKdV equation (Generalized Korteweg-de Vries equation) that results from carrying out the formal asymptotics in the shallow water parameter and the nonlinear parameter to higher order. This equation specializes to the various more familiar equations. Again, what is distinctly non-standard is his concentration upon solitary waves defined on compact spaces that can be obtained via the motion of curves whose theory was developed in Part II. This part is also not for a beginner. Without prior background in these sorts of derivations, it will be hard going. Hard going, but worth the effort.

Chapters 12–15 might well have been lumped into Part II of the text. While they enlarge upon the theory, they emerge from physical considerations. Chapters 12–14 are concerned with the fascinating shape oscillations of liquid drops in two and three space dimensions. Chapter 15 presents another quite different point of view that yields some of the same fascinating shapes that appeared earlier in droplets.

In the fourth portion of the text, Ludu shows his scientific upbringing. He started life as a physicist and throughout his career he has been closely tied to real-world phenomena. He admirably shows off his breadth in Part IV of the text. Here we find him dealing with a whole stable of solitons that arise in some unlikely places. There are solitons on filaments of various sorts, solitons on stiff chains, solitons on the boundaries of microscopic structures, solitons at stellar scales.

The text finishes with a mathematical annex that includes some interesting remarks that didn't fit anywhere else in the text.

This book is not to be read in an armchair. As Ludu states in his opening remarks, it is meant to be studied with pencil and paper at hand and with an algebraic manipulation program up on the screen of a computer. It is a text dense with ideas and methods, both mathematical and scientific, and a serious addition to the literature. The fact that it is going into a third edition attests to its impact.

Chicago, USA

Hongqiu Chen
Jerry Bona

Preface to the Third Edition

In order to offer as much content as possible from all chapters of the book to readers with various prerequisites in mathematics, we present below a reader's map that can help readers to navigate through the book without being stuck in sections with denser mathematical content. Pretty much like on a skiing course, we introduce three possible paths to meet the interest of all our readers:

- No * asterisk is the path that doesn't request special prerequisites in mathematics, except calculus and first level course in mathematical physics. For these readers, we recommend the following path:

$$Introduction \to 2.1 \to 3.1 \to 3.2 \to 3.3 \to 3.12 \to 3.13 \to 4.1 \to 4.2 \to$$

$$5 \to 6.1 \to 6.5 \to 7.1 \to 7.3 \to 9.1 \to 9.3 \to 9.5 \to 9.6.1 \to 10.1 \to 10.2 \to$$

$$10.3 \to 10.4.1 \to 10.5 \to 10.6.1 \to 11.1 \to 12.1 \to 12.6 \to$$

$$13.1 \to 14.1 \to 14.2 \to 14.3 \to 17.3 \to 18.2 \to 18.3 \to 19$$

- Sections labeled with one asterisk * request some previous knowledge in real analysis, differential systems and elements of geometry. For these readers we recommend in addition to the "No * asterisk path" to add the following sections:

$$3.4 \to 3.5 \to 3.6 \to 3.7 \to 3.10 \to 6.3 \to 6.4 \to 7.5 \to 9.4 \to 9.6.2 \to$$

$$10.4.3 \to 10.6.2 \to 11.2 \to 11.5 \to 12.2 \to 12.3 \to 12.4 \to 12.5 \to 12.6 \to$$

$$13 \to 14.4 \to 14.5 \to 15.2 \to 18.1 \to 18.4.$$

- Sections labeled with two asterisks ** address to mathematicians or theoretical physicists, or anyone who finds useful de dedicate some time practicing a higher level of mathematics, like algebraic topology, differential geometry, or nonlinear differential systems. For these readers, we recommend in addition to the "* asterisk path" to add the following sections:

$$2.2 \to 2.3 \to 3.8 \to 3.9 \to 3.11 \to 6.2 \to 7.2 \to 7.4 \to 7.6 \to 8 \to 9.2$$

$$\to 10.4.2 \to 10.4.4 \to 10.4.5 \to 10.4.6 \to 10.4.7 \to 10.6.3 \to 11.3 \to$$

$$11.4 \to 15.1 \to 15.3 \to 16 \to 17.$$

Besides corrections made in the previous editions, the goal of this third edition is to implement latest results on solitons traveling on closed, compact surfaces or curves. We cover again mathematical and physical problems ranging from nuclear to astrophysical scales. The third edition provides additional examples of systems and models where the interaction between nonlinearities and the compact boundaries is essential for the existence and the dynamics of solitons.

The first historic mention of what we call today *soliton* was made in 1834 by John Scott Russell following his discovery of a new type of *waves of translation* [1]. The mathematical model for such waves, the Korteweg-de Vries (KdV) equation, was first introduced by Boussinesq in 1877, and it was rediscovered in 1895 by Diederik Korteweg and Gustav de Vries [2]. Relations between nonlinear differential equations and differential geometry, without any reference yet to solitons, were first discovered by Edmond Bour in 1862 in the course of the study of surfaces of constant negative curvature, like the Gauss—Codazzi equation for surfaces of curvature -1 in $\mathbf{R}^3$. This is the first mention of the sine-Gordon equation $u - \sin u = 0$. The equation was rediscovered by Frenkel and Kontorova in 1939 in their study of crystal dislocations [3]. Only starting with 1970, this equation attracted a lot of attention due to the presence of soliton solutions and its mathematical connection with differential geometry. Therefore, it is the main goal of this book to focus on such interesting and/or recent aspects of relations between nonlinear integrable systems with their soliton solutions and differential geometry, mainly defined on compact manifolds.

The book consists of 19 chapters organized in four parts, a mathematical annex, and a bibliography. The first part contains the fundamentals of topology, differential geometry, and analysis approaches. To render this book accessible to students in all STEM disciplines, Chap. 2 recalls some basic elements of topology with emphasis on the concept of being compact. Chapter 3 introduces the reader to calculus on differentiable manifolds, vector fields, differentiable forms, and various types of derivatives. We take the reader from the definition of the differential manifold all the way to the Poincaré lemma. Next, in this chapter, we introduce different types of fiber bundles, the Cartan theory of frames, and the theory of connection and mixed covariant derivatives. Without always presenting the proofs, we tried though to keep a high level of rigorousness (relying on classical mathematical textbooks) all across the text while we still introduce intuitive comments for each definition or affirmation. In Chap. 4, we review various representation formulas for various dimensions. These formulas justify how the information about the evolution of smooth physical fields inside a bounded region can be recovered only from the information on the region's boundary.

The second part of the book is devoted to applications of differential geometry in the theory of curves and surfaces. Chapter 5 lays the basis for the differential geometry of curves in spaces with three dimensions. We introduce special sections for the theory of closed curves and curves lying on surfaces. Complementary to these, in Chap. 6, we introduce elements of the geometry of surfaces with applications to the action of differential operators on these surfaces. In Chap. 7, we derive the theory of motion of curves in two and three dimensions, and we emphasize the relationships between theory of motion of curves and solitons. We devoted a section on the axiomatic deduction of the theory of curve motions based on differentiable forms and Cartan connection theory. We describe the relationships between some special motions of curves and solitons. We describe nonlinear integrable systems that can be represented by such motions. In Chap. 8, we discuss the theory of motion of surfaces and again relate such motions to nonlinear integrable systems and solitons.

The third part of the book is dedicated to applications of soliton theory, especially solitons on closed curves and surfaces, in fluid dynamics. The working frame of hydrodynamics, which is also the main content of Part III, is presented in Chap. 9. In Chap. 10, we discuss problems related to liquid surface tension effects and the associated representation theories for fluid dynamics models. Chapter 11 describes one-dimensional integrable systems on compact intervals, together with their periodic solutions. In this chapter, we introduce the most common and most used subject in nonlinear waves, the Korteweg-de Vries equation and system. In Chaps. 12–14, we approach the same type of problems except in higher dimensions. We describe and analyze nonlinear shape excitations for two and three-dimensional compact fluid systems, like liquid drops, liquid shells, etc.

Chapter 15 is devoted to other applications of soliton theory on compact surfaces in one to three dimensions like nonlinear shapes of layered liquid drops, compact supported solitons, or the relationship between solitons and collective motions of nonlinear dynamical systems with boundary.

In the fourth part of the book, as a closure for the first three parts, we present novel and interesting physical (and even biological) applications of the theory of nonlinear systems and their soliton solutions. We describe several physical systems at different space-time-energy scales. In Chap. 16, we study the vortex filaments and other one-dimensional flows. In Chap. 17, we describe microscopic applications of solitons and instantons in the theory of elementary particles and quantum fields, in description of exotic shapes of heavy nuclei, the phenomenon of exotic radioactivity and relationships between solitons on closed curves and quantum Hall drops.

Chapter 18 contains macroscopic scale applications of compact supported solitons in magnetohydrodynamic, plasma systems, elastic solids with surface, nonlinear surface diffusion, and neutron stars.

The book is completed by a mathematical annex, including an original section on the theory and applications of nonlinear dispersion relations, and their use for the qualitative description of the soliton solutions of nonlinear partial differential equations.

A legitimate question of the potential reader would be: "Why one more book on solitons?" First of all, we have to acknowledge the importance of the interactions between compact boundary manifolds and the dynamics of particles and fields in mathematical in physical models. Historically, the solitons are observed in sort of "infinite" systems like infinite long lines or curves, planes or open surfaces, or unbounded space. However, there is more and more evidence of the existence solitons or of localized patterns (like vortices) in compact lower dimensional spaces, like closed curves and/or surfaces. As examples, we can mention the unprecedented information technology advances in optical communication (light bullets and ultra-short optical pulses), solid-state spectroscopy, ultra-cold atom studies, soliton molecules, spinning solitons, quantum computers, spintronics and mass memory systems, femtosecond laser pulses, mesoscopic superconductivity, etc. Consequently, the reasons for writing this book are generated by a constantly increasing number of new challenges, vivid topics and hundreds of published articles. As one last comment, we mention that this book is not devoted to the teaching of general theory of solitons, or the Inverse Scattering Transform, and other traditional methods to obtain nonlinear solutions. This book opens a new direction in the field of nonlinear system, namely about nonlinear waves and solitons evolving in compact spaces, like closed curves, contours, and closed surfaces, etc.

If a substantial percentage of users of this book feel that it helped them to enlarge their outlook in the intersection between the fascinating worlds of nonlinear waves and compact surfaces and closed curves, its purpose has been fulfilled.

During the production of this third edition of this book, I have received the best support and uninterrupted encouragement from my family. I have also greatly benefited from discussions with my colleagues, and I am particularly grateful to Adrian S. Carstea and Denys Dutykh who provided valuable help in the elaboration of this edition.

Daytona Beach, FL, USA Andrei Ludu

Preface to the Second Edition

Nonlinear phenomena represent intriguing and captivating manifestations of nature. The nonlinear behavior is responsible for the existence of complex systems, catastrophes, vortex structures, cyclic reactions, bifurcations, spontaneous phenomena, phase transitions, localized patterns and signals, and many others. The importance of studying nonlinearities has increased over the decades and has found more and more fields of application ranging from elementary particles, nuclear physics, biology, wave dynamics at any scale, fluids, plasma to astrophysics. The soliton is the central character of this 178-year-old story. A soliton is a localized pulse traveling without spreading and having particle-like properties plus an infinite number of conservation laws associated to its dynamics. In general, solitons arise as exact solutions of approximate models. There are different explanation, at different levels, for the existence of solitons. From the experimenter's point of view, solitons can be created if the propagation configuration is long enough, narrow enough (like long and shallow channels, fiber optics, electric lines, etc.), and the surrounding medium has an appropriate nonlinear response providing a certain type of balance between nonlinearity and dispersion. From the numerical calculations point of view, solitons are localized structures with very high stability, even against collisions between themselves. From the theory of differential equations point of view, solitons are cross-sections in the jet bundle associated to a bi-Hamiltonian evolution equation (here Hamiltonian pairs are requested in connection to the existence of an infinite collection of conservation laws in involution). From the geometry point of view, soliton equations are compatibility conditions for the existence of a Lie group. From the physicist point of view, solitons are solutions of an exactly solvable model having isospectral properties carrying out an infinite number of non-obvious and counter-intuitive constants of motion.

The progress in the theory of solitons and integrable systems has allowed the study of many nonlinear problems in mathematics and physics: non-local interactions, collective excitations in heavy nuclei, Bose—Einstein condensates in atomic physics, propagation of nervous pulses, swimming of motile cells, nonlinear oscillations of liquid drops, bubbles, and shells, vortices in plasma and in atmosphere, tides in neutron stars, only to enumerate few of possible applications. A number of other applications of soliton theory also lead to the study of the dynamics of boundaries.

In that, the last three decades have seen the completion of the foundation for what today we call nonlinear *contour dynamics*. The subsequent stage of development along this topic was connected with the consideration of an almost *incompressible* systems, where the boundary (contour or surface) plays the major role.

Many of the integrable nonlinear systems have equivalent representations in terms of differential geometry of curves and surfaces in space. Such geometric realizations provide new insight into the structure of integrable equations, as well as new physical interpretations. That is why the theory of motions of curves and surfaces, including here filaments and vortices, represents an important emerging field for mathematics, engineering and physics.

The first problem about such compact systems is that shape solitons, which usually exist in infinite long and shallow propagation media, cannot survive on a circle or sphere. That is because such compact manifolds cannot offer the requested type of environment (long and narrow), even by the introduction of shallow layers and rigid cores. However, there is another basic idea that supports, in a natural way, the existence of nonlinear solutions on compact spaces. Because of its high localization, a soliton is not a unique solution for the partial differential system. Its position in space is undetermined because, far away from its center, the excitation is practically zero. On the other hand, all linear equations provide uniqueness properties for their solutions. It results that strongly localized solutions and almost compact supported solutions can be generated only within nonlinear equations. There is an exception here: the finite difference equations with their compact supported wavelet solutions, but in some sense, a finite difference equation is similar to a nonlinear differential one.

Despite the many applications and publications on nonlinear equations on compact domains, there are still no books introducing this theory, except for several sets of lecture notes. One reason for this may be that the field is still undergoing a major development and has not yet reached the perfection of a systematic theory. Another reason is that a fairly deep knowledge of integrable systems on compact manifolds has been required for the understanding of solitons on closed curves and compact surfaces.

The goal of the second edition of this book is to analyze the existence and describe the behavior of solitons traveling on closed, compact surfaces or curves. The approach of the physical problems ranging from nuclear to astrophysical scales is made in the language of differential geometry. The text is rather intended to be an introduction to the physics of solitons on compact systems like filaments, loops, drops, etc., for students, mathematicians, physicists, and engineers. The author assumes that the reader has some previous knowledge about solitons and nonlinearity in general. The book provides the reader examples of systems and models where the interaction between nonlinearities and the compact boundaries is essential for the existence and the dynamics of solitons.

We focused on interesting and recent aspects of relations between integrable systems and their solutions and differential geometry, mainly on compact manifolds. The book consists of 17 chapters, a mathematical annex, and a bibliography. First part contains the fundamental differential geometry and analysis approach. To render this book accessible to students in science and engineering, Chapter 2 recalls some basic elements of topology with emphasis on the concept of being compact. In Chapter 3, we review the representation formulas for different dimensions. The formulas express how a lot of information about the evolution of differentiable forms and fields inside a compact domain can be recovered only from its boundary. Chapter 4 introduces the reader to the calculus on differentiable manifolds, vector fields, forms, and various types of derivatives. We take the reader from map all the way to the Poincaré lemma. Next we introduce different types of fiber bundles, including the Cartan theory of frames, and the theory of connection and mixed covariant derivative (for immersions). Without always presenting the proofs, we tried though to keep a high level of rigorousness (relying on classical mathematical textbooks) all across the text while we still introduce intuitive comments for each definition or affirmation. Chapter 5 lays the basis for the differential geometry of curves in $\mathbf{R}_3$ We devote here special sections to closed curves and curves lying on surfaces. Complementary, in Chapter 7, we introduce the elements of differential geometry of the surfaces with applications to the action of differential operators on surfaces. In Chapter 6, we derive the theory of motion of curves, both in two dimensions, and in the general case. We devoted a section on the axiomatic deduction of the theory of motions based on differentiable forms and Cartan connection theory. We relate these motions with soliton solutions and find the nonlinear integrable systems that can be represented by such motions of curves. In Chapter 8, we discuss the theory of motion of surfaces, and we also relate it to integrable systems.

The second part of the monograph contains an exposition of the basic branches of nonlinear hydrodynamics. The working frame of hydrodynamics is the main content of the first part of the monograph, namely Chapter 9. In Chapter 10, we discuss the problems on surface tension effects and representation theorems for fluid dynamics models. Chapter 11 concentrates with one-dimensional integrable systems on compact intervals, and their periodic solutions. Chapters 12 and 13 deal with nonlinear shape excitations of two-dimensional and three-dimensional liquid drops and bubbles. Chapter 14 is devoted to various applications of three-dimensional nonlinear drops and also to compact supported solitons.

In the third part of the book, as a final goal for the first two parts, we present additional physical applications of nonlinear systems and their soliton solutions on various systems of different scales. In Chapter 15, we study the vortex filaments and other one-dimensional flows. In Chapter 16, we describe microscopic applications like elementary particles as solitons, instantons, exotic shapes in heavy nuclei, exotic radioactivity and quantum Hall drops. Chapter 17 deals with macroscopic applications like magnetohydrodynamic plasma systems, elastic spheres, nonlinear surface diffusion, and neutron stars.

The book is closed by a mathematical annex including a section on nonlinear dispersion relations and their use for nonlinear systems of partial differential equations.

A legitimate question of the potential reader would be: "Why one more book on solitons?" First of all, we have to acknowledge the importance of the interactions between compact boundary manifolds and the dynamics of particles and fields in mathematical in physical models. Historically, the solitons are observed in sort of “infinite” systems like infinite long lines or curves, planes or open surfaces, or unbounded space. However, there is more and more evidence of the existence solitons or of localized patterns (like vortices) in compact lower dimensional spaces, like closed curves and/or surfaces. As examples, we can mention the unprecedented information technology advances in optical communication (light bullets and ultra-short optical pulses), solid-state spectroscopy, ultra-cold atom studies, soliton molecules, spinning solitons, quantum computers, spintronics and mass memory systems, femtosecond laser pulses, mesoscopic superconductivity, etc. Consequently, the reasons for writing this book are generated by a constantly increasing number of new challenges, vivid topics and hundreds of published articles.

If a substantial percentage of users of this book feel that it helped them to enlarge their outlook in the intersection between the fascinating worlds of nonlinear waves and compact surfaces and closed curves, its purpose has been fulfilled.

While writing the second edition of this book, I have greatly benefited from discussions with my colleagues. I am particularly grateful to Ivailo Mladenov, Thiab Taha, Annalisa Calini, Adrian Stefan Carstea who provided an inspirational and valuable help in the elaboration of this second edition. For the best advices and uninterrupted encouragement, I am indebted to my family.

May 2011

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Preface to the First Edition

Everything the Power of the World
does is done in a circle. The sky is
round and I have heard that the earth
is round like a ball and so are all the stars.
The wind, in its greatest power, whirls.
Birds make their nests in circles,
for theirs is the same religion as ours.
The sun comes forth and goes down
again in a circle. The moon does the
same and both are round. Even the
seasons form a great circle in their
changing and always come back again
to where they were. The life of a man
is a circle from childhood to childhood.
And so it is everything where power moves.
Black Elk (1863-1950)

Nonlinearity is a captivating manifestation of the observable Universe, whose importance has increased over the decades, and has found more and more fields of application ranging from elementary particles, nuclear physics, biology, wave dynamics at any scale, fluids, plasmas to astrophysics. The central character of this 172-year-old story is the soliton. Namely, a localized pulse traveling without spreading and having particle-like properties plus an infinite number of conservation laws associated to its dynamics. In general, solitons arise as exact solutions of approximative models. There are different explanations, at different levels, for the existence of solitons. From the experimentalist point of view, solitons can be created if the propagation configuration is long enough, narrow enough (like long and shallow channels, fiber optics, electric lines, etc), and the surrounding medium has an appropriate nonlinear response providing a certain type of balance between nonlinearity and dispersion. From the numerical calculations point of view, solitons are localized structures with very high stability, even against collisions between themselves. From the theory of differential equations point of view, solitons are cross-sections in the jet bundle associated to a bi-Hamiltonian evolution equation (here Hamiltonian pairs are requested in connection to the existence of an infinite collection of conservation laws in involution). From the geometry point of view, soliton equations are compatibility conditions for the existence of a Lie group. From the physicist point of view, solitons are solutions of an exactly solvable model having isospectral properties carrying out an infinite number of non-obvious and counter-intuitive constants of motion.

The progress in the theory of solitons and integrable systems has allowed the study of many nonlinear problems in mathematics and physics: elementary particle non-local interactions, collective excitations in heavy nuclei, Bose-Einstein condensates in atomic physics, propagation of nervous influxes, nonlinear oscillations of liquid drops, bubbles, and shells, vortexes in plasma and in atmosphere, tides in neutron stars, etc., only to enumerate few of possible applications. A number of other applications of soliton theory also lead to the study of the dynamics of boundaries. In that, the last three decades have seen the completion of the foundation for what today we call nonlinear *contour dynamics*. The subsequent stage of development along this topic was connected with the consideration of a almost *incompressible* systems, where the boundary (contour or surface) plays the major role.

The first problem about such compact systems is that shape solitons, that usually exist in infinite long and shallow propagation media, can not survive on a circle or sphere. That is because such compact manifolds can not offer the requested type of environment (long and narrow), even by the introduction of shallow layers and rigid cores. However, there is another basic idea that supports, in a natural way, the existence of nonlinear solutions on compact spaces. Because of its high localization, a soliton (or a compacton) is not a unique solution for the partial differential system. Its position in space is undetermined because, far away from its center, the excitation is practically zero. On the other hand, all linear equations provide uniqueness properties for their solutions. It results that strongly localized solutions and almost-compact supported solutions can be generated only within nonlinear equations. There is an exception here: the finite difference equations with their compact supported wavelet solutions, but in some sense, a finite-difference equation is similar to a nonlinear differential one.

Despite the many applications and publications on nonlinear equations on compact domains, there are still no books introducing this theory, except for several sets of lecture notes. One reason for this may be that the field is still undergoing a major development and has not yet reached the perfection of a systematic theory. Another reason is that a fairly deep knowledge of integrable systems on compact manifolds has been required for the understanding of solitons on closed curves and compact surfaces.

The main aim of this book is to present models of nonlinear phenomena that occur mainly on closed, compact surfaces or curves, especially where solitons and solitary waves are involved. The approach of the physical problems ranging from nuclear to astrophysical scales is made in the language of differential geometry. The text is rather intended to be an introduction to the physics of solitons on compact systems like filaments, loops, drops, etc., for students, mathematicians, physicists, and engineers. However, the book does not elaborate on the general theory of solitons, or the inverse scattering problem, for example. The author assumes that the reader has some previous knowledge about solitons, integrable systems and nonlinearity in general. The book furnishes the reader with models related to compact boundaries and their nonlinear dynamics, and, if available, with soliton-like solutions. This is a book to be read with pencil, paper, and a symbolic computer program at hand. Our intention is to furnish readers with enough knowledge to be able to identify, understand, and model such nonlinear systems.

This text is still far from being a comprehensive study on the topic of solitons on compact systems. It consists of 18 chapters, an appendix, and a bibliography. First part contains the fundamental differential geometry and analysis approach. To render this book accessible to students in science and engineering, Chapter 2 recalls some basic elements of topology. In Chapter 3, we review some representation formulas for different dimensions, as expressions of the comprehensive information contained in the boundaries. Chapter 4 introduces the reader in the calculus on differentiable manifolds, vector fields, forms, and various type of derivatives. Chapter 5 lays the basis for the differential geometry of curves in $\mathbf{R}_3$. In Chapter 6, we derive the theory of motion of curves, and we relate these motions with soliton solutions. In Chapter 7, we recall some elements of differential geometry of the surfaces, with applications on the action of differential operators on surfaces. In Chapter 8, we discuss the theory of motion of surfaces.

The second part of the monograph contains an exposition of the basic branches of nonlinear hydrodynamics. The working frame of hydrodynamics is the main content of the first part of the monograph, namely, Chapter 9. In Chapter 10, we discuss the problems on surface tension effects and representation theorems for fluid dynamics models. Chapter 11 concentrates with one-dimensional integrable systems on compact intervals, and their periodic solutions. Chapters 12 and 13 deal with nonlinear shape excitations of two-dimensional, and three-dimensional liquid drops and bubbles. Chapter 14 is devoted for various applications of three-dimensional nonlinear drops, and also to compact supported solitons.

In the third part of the book, as a final goal for the first two parts, we present additional physical applications of nonlinear systems and their soliton solutions on various systems of different scales. In Chapter 15, we study the vortex filaments and other one-dimensional flows. In Chapter 16, we describe microscopic applications like exotic shapes in heavy nuclei, exotic radioactivity, and quantum Hall drops. Chapter 17 deals with macroscopic applications like magnetohydrodynamic plasma systems, elastic spheres, neutron stars, etc.

The book is closed by a mathematical annex including a section on nonlinear dispersion relations and their use for nonlinear systems of partial differential equations.

The last comment of this preface would be: Why one more book on solitons, and why on compact spaces? A first answer is that there are already a large number of application on these vivid topics and hundreds of published articles. On the other hand, there is the importance of compact manifolds themselves in physics.

If a substantial percentage of users of this book feel that it helped them to enlarge their outlook in the intersection between the fascinating worlds of nonlinear waves and compact surfaces and closed curves, its purpose has been fulfilled.

I have greatly benefited from discussions with my colleagues and students, and I am particularly grateful to Thiab Taha for his sedulous and constant effort to provide the frame for such discussions through his nonlinear waves meetings. I should like to thank to whom gave me help and support to write this book: Randall J. Webb, Austin L. Temple, and the National Science Foundation (through the grant PHYS-0140274). For interesting and helpful conversations, I am indebted to many friends. For discussions and constant encouragement, I am indebted to my family. During the completion of the manuscript, Bob Odom has given me valuable suggestions. The last but not at all the least I am thankful to the Watson Library and the group working with the Illiad interlibrary borrowing who offered me the chance to cover all the necessary references.

June 2009

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Contents

Symbols

In general the spaces ($\mathbb{R}^3$ for example), the vectors ($\boldsymbol{v}$), and the matrices are denoted with bold letters, and the dimension is represented as a subscript.

$A \lhd B$	$A \subset B$ and A has the same structure as B (it is a sub-structure)
A, dA	Element of area, surface element
$\boldsymbol{a}, \boldsymbol{A}, \boldsymbol{v}$	Vector field
$\alpha, \beta, \delta, \ldots$ (not γ)	Unspecified labels in general (or labels from 1 to 2)
$\boldsymbol{b}$	Binormal in the Serret–Frenet frame
$C^k(M)$	Class of differentiable functions of order k defined on M
$C^\infty(M)$	Class of infinite-differentiable functions defined on M, also called *smooth* in this book
$D_{\boldsymbol{x}}$	Directional derivative
$Diff(A, B)$, $Hom(A, B)$, etc.	Diffeomorphisms, homeomorphisms, etc. from A to B.
▼	Covariant differential
∂M	Boundary of the domain M
E, F, G, e, f, g	Second fundamental form coefficients
f, df, f^*	Mapping, its differential, and the pull-back
g	Metrics
γ, Γ	Parametrized curve, or
Γ, ω	Connection, connection form
H	Mean curvature
$i, j, k, l, \ldots$	Labels in general (or labels in the $1, 2, \ldots, n \geq 3$)
$i = \sqrt{-1}$	If specified in the context
I	First fundamental form
K	Gaussian curvature
κ	Curvature of a curve
$\kappa_{1,2}$	Principal curvatures

κ_n	Normal curvature
κ_g	Geodesic curvature
M, X, Y	Manifold
ν	Viscosity
$\nabla_v, \nabla_{\alpha'}$	Covariant derivative
$\nabla_\Sigma, \nabla_\Sigma \cdot, \nabla_\Sigma \times, \Delta_\Sigma$	Surface gradient, surface divergence, surface curl, surface Laplacean
$\boldsymbol{n}, \boldsymbol{N}$	Normal to a curve in the Serret–Frenet frame, normal to a surface
ODE, PDE	Ordinary or partial (system of) differential equation(s)
Ω, ω	Differential form
Π	Second fundamental form of a surface
Σ	Surface
s, ds	Arc-length
$TM, TX, TY, \ldots$	Tangent space
t	Time
t	Unit tangent to a curve
$\boldsymbol{t}, \boldsymbol{N}, \boldsymbol{t}^\perp = \boldsymbol{N} \times \boldsymbol{t}$	Darboux frame associated to a given curve lying on a surface
τ	Torsion
$\hat{A}$	Tensor in general
τ_g	Geodesic torsion
u, v	Surface parameters
u, t	Curve parameters
v	Volume, only if results from context
$dv, d^n x, d^3 x$	Element of volume
$\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{V}, \boldsymbol{U}, \boldsymbol{W}$	Velocities or vorticities
$\boldsymbol{v}(\boldsymbol{w})$	Lie derivative of $\boldsymbol{w}$ with respect to (or along) $\boldsymbol{v}$
w.s.	Without summation
$x^i = \{x, y, z\}$	Specific three-dim coordinate notation
$x^\sigma = \{u, v\}$	Specific two-dim coordinate notation.

In general, the notations below represent the same expression

$$\int^x f(x)dx = \int^x f(x')dx' = \int_0^x f(x)dx + \mathcal{C}.$$

Chapter 10
Hydrodynamics

The mathematical description of the states of a fluid is based on the study of three fields defined on the domain occupied by the fluid: the velocity field $\boldsymbol{V}$, the density ρ, and the pressure field P. These three "unknowns" are determined by integrating other five scalar equations, namely the mass conservation (continuity equation), the three components of the equation of momentum balance (Euler or Navier–Stokes), and the energy balance. This last equation needs in addition information about the thermodynamics of the fluid, so it may need to be supplied with some equation of state. In addition to these five equations, we request regularity, asymptotic and, if it is the case, boundary conditions, to provide a unique solution. When we study the dynamics of the fluid confined in a compact domain with free boundaries, the system is slightly more complicated, and we have to add the kinematical equation of the free surface, as well as equations of momentum balance at the surface. If we take into account the nonlinear terms in the dynamical equations, and in the associated curved geometry, some interesting solutions occur. Special nonlinear effects related to fluids on compact domains with free surface could be Gibbs–Marangoni effect, dividing the flow in cells (Bènard effect), couplings between different modes, collective effects, separation of flow in layer (boundary layer, turbulence), standing traveling surface waves, etc. In this chapter, we introduce some elements of general hydrodynamics which we will use later on in the book, boundary conditions especially at free surfaces, surface pressure theory, and representation theorems.

A. Ludu, *Nonlinear Waves and Solitons on Contours and Closed Surfaces*,
Springer Series in Synergetics,
https://doi.org/10.1007/978-3-031-14641-1_10

10.1 Momentum Conservation: Euler and Navier–Stokes Equations

The continuity equation for fluid dynamics (9.32) was derived in Sect. 9.6 and it has the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0, \tag{10.1}$$

where $\boldsymbol{V} = (V_i)$ is the Lagrangian or material velocity of the fluid particle, and ρ is the fluid density. Because we study the fluid in the three-dimensional Euclidean space of flat metric, there is no difference between covariant and contravariant character of the Euclidean vectors, so we will place the label as subscripts as a rule in this section. The momentum of the unit of fluid volume is given by

$$p_i \equiv \frac{\partial}{\partial t}(\rho V_i) = f_i = \frac{F_i}{V}, \tag{10.2}$$

where $\boldsymbol{f} = (f_i)$ is the volume force density, derived for the total force field in the fluid $\boldsymbol{F}$. From (10.1) and (10.2), we have

$$\frac{\partial V_i}{\partial t} + V_k\left(\frac{\partial}{\partial x_k} V_i\right) = -\frac{\partial}{\partial x_j}(P\delta_{ij} + \rho V_i V_j) \equiv -\frac{\partial}{\partial x_j}\Theta_{ij}, \tag{10.3}$$

where P is the pressure, and we define the fluid symmetric *momentum flux tensor* as $\hat{\Theta}$. In the inviscid case, where we have no loss of momentum in viscosity and internal frictions, this tensor has the property

$$f_i = \frac{\partial p_i}{\partial t} = \frac{\partial}{\partial t}(\rho V_i) = -\frac{\partial}{\partial x_i}\Theta_{ij}^{inviscid}. \tag{10.4}$$

If we draw an imaginary smooth surface with unit normal $\boldsymbol{N}$, (10.4) can be written in the form

$$\hat{\Pi}^{inviscid} \cdot \boldsymbol{N} = P\boldsymbol{N} + \rho \boldsymbol{V}(\boldsymbol{V} \cdot \boldsymbol{N}), \tag{10.5}$$

which represents the balance of *reversible* momentum. The LHS term represents how much momentum is transferred per unit of time and cross-section area in the direction $\boldsymbol{N}$, the first term on the RHS is the change of momentum by molecular motion and interaction, and the last term is the change of momentum by bulk flow only.

If we consider the viscosity, η, we have to extend the momentum flux tensor with an extra term, namely

$$\Theta_{ij}^{inviscid} \to \Theta_{ij} = P\delta_{ij} + \rho V_i V_j - \sigma_{ij}^{'}. \tag{10.6}$$

In literature [1–5, 14, 27], authors use another tensor, namely the fluid *stress tensor* $\hat{\sigma}$, inspired from the study of elasticity, representing the total momentum transferred by molecular motion both reversible and irreversible, and defined by

$$\sigma_{ij} = -P\delta_{ij} + \sigma^{'}_{ij}, \tag{10.7}$$

so that

$$\Theta_{ij} = -\sigma_{ij} + \rho V_i V_j. \tag{10.8}$$

So far we took for granted that these stress tensors are symmetric. The proof is based on the judgment that the total torque, $dM_i = \epsilon_{ijk} x_j \partial\Theta_{kl}/\partial x_l dV$, produced by fluid forces in an infinitesimal domain depends only on the surface of the domain, because inside forces between different elements cancel each other in action–reaction pairs. From the Green theorem applied on this domain, we obtain that $\epsilon_{ijk}\Theta_{jk} = 0$, where from $\Theta_{ij} = \Theta_{ji}, \sigma_{ij} = \sigma_{ji}$.

To have an expression for the stress tensor, we need to use the *Newtonian fluid* hypothesis, namely the part of the momentum flux tensor which results from frictional interaction of the fluid in relative motion (represented by the viscous stress tensor σ') depends only on the instantaneous gradient of fluid velocity. In addition, this dependence is approximated to be linear. If we keep the general dependence on the gradient, the fluid is called *Stokesian fluid*, but the hypothesis need to be supplemented by requiring smoothness, isotropy, and homogeneity [6, 7]. So, we can write

$$\sigma^{'}_{ij} = C_{ijkl}\frac{\partial V_k}{\partial V_l}. \tag{10.9}$$

To determine the tensor C, we note that a global rotation of the fluid should not introduce any stress, so we have $C_{ijkl} = C_{ijlk}$. In addition we require C to be an *isotropic tensor*, namely invariant to any rotation. We know that the only rotational invariant tensors of rank 0 is a scalar, of rank 1 there is none, of rank 2 is the Kronecker symbol δ_{ij}, and of rank 3 is the Levi–Civita tensor ϵ_{ijk}. The number of linear independent isotropic tensors of rank k is given by the Motzkin recursion formula

$$N_k = \frac{k-1}{k+1}(2N_{k-1} + 3N_{k-2}), \quad k = 1, 2, \ldots,$$

from where it results $N_4 = 3$ [8]. To obtain the general formula for the C tensor, we can use a theorem from elasticity [9, 10]. This theorem states that a rank 2 symmetric tensor (i.e., $\hat{\sigma}^{'}$) generated by all possible linear combinations between another rank 2 tensor $\nabla \boldsymbol{V}$ and a rank 4 isotropic tensor $\hat{C}$ with the above listed properties is a linear combination of the symmetric part of $\nabla \boldsymbol{V}$ and the Kronecker tensor times the trace of $\nabla \boldsymbol{V}$. That is

$$\hat{\sigma} = -P\hat{I} + \eta(\nabla \boldsymbol{V} + (\nabla \boldsymbol{V})^t) + \lambda \text{Tr}(\nabla \boldsymbol{V})\hat{I}, \tag{10.10}$$

where $\hat{(I)} = \delta_{ij}$, and where the second term on the RHS is the symmetric part of $\nabla \boldsymbol{V}$ (containing the transpose), also called the *rate of deformation* (or *rate of strain*), and $\mathrm{Tr}(\nabla \boldsymbol{V}) = \nabla \cdot \boldsymbol{V}$ is called *rate of expansion* [3, 5]. The last assumption on the stress tensor (Stokes' assumption) namely $\hat{\sigma}'$ makes no contributions to the mean normal stress, so we have $\lambda = -2/3$ from here. It results

$$\hat{\sigma} = -P\hat{I} + \eta\left[(\nabla\hat{\boldsymbol{V}} + (\nabla \boldsymbol{V})^t) - \frac{2}{3}\mathrm{Tr}(\nabla \boldsymbol{V})\right]\hat{I}$$

$$= -P\delta_{ij} + \eta\left[\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} - \frac{2}{3}\frac{\partial V_k}{\partial x_k}\delta_{ij}\right]. \tag{10.11}$$

If we neglect the Stokesian assumption, and we also consider the contribution of a *dilatational viscosity*, we correct (10.11) into

$$\sigma_{ij} = -P\delta_{ij} + \eta\left[\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} - \frac{2}{3}\frac{\partial V_k}{\partial x_k}\delta_{ij}\right] + \zeta\frac{\partial V_k}{\partial x_k}\delta_{ij}, \tag{10.12}$$

where ζ is the *coefficient of dilatational viscosity*. In the non-Newtonian fluid, we have $\eta, \zeta = f(\partial v_i/\partial x_k)$.

We can rewrite (10.12) in a vectorial form, such that the dynamical equation for a viscous fluid reads

$$\rho\left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\nabla)\boldsymbol{V}\right] = -\nabla P + \rho\boldsymbol{f} + \eta\triangle\boldsymbol{V} + \left(\zeta + \frac{\eta}{3}\right)\nabla(\nabla\cdot\boldsymbol{V}), \tag{10.13}$$

which is the famous Navier–Stokes equation of a fluid in the presence of a volume density force $\boldsymbol{f}$. In the case of incompressible fluid, (10.14) becomes

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\nabla)\boldsymbol{V} = -\frac{1}{\rho}\nabla P + \boldsymbol{f} + \frac{\eta}{\rho}\triangle\boldsymbol{V}, \tag{10.14}$$

which reduces to the Euler equation in absence of viscosity

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\nabla)\boldsymbol{V} = -\frac{1}{\rho}\nabla P + \boldsymbol{f}. \tag{10.15}$$

10.2 Boundary Conditions

Boundary conditions at the surface of a fluid Σ can be of three types: separation between two fluids (fluid interface), free surface of a fluid in a rarefacted gaseous atmosphere (or vacuum), and contact with rigid surfaces. The expressions of the conditions of continuity in each case depend if the fluid (fluids) is viscous or inviscid.

Basically, we can write a general continuity condition for the separation of two fluids (say fluids 1 and 2), and this condition can be modified for the other two cases.

The continuity of the velocity at the interface is a relation strongly dependent on the model (viscous or not, slipping interface or not, etc.), so we will use it for every situation in particular. Nevertheless, we can write a provisional continuity condition in the form $\boldsymbol{V}_1|_\Sigma = \boldsymbol{V}_2|_\Sigma$ or

$$\boldsymbol{V}_{n,1}|_\Sigma = \boldsymbol{V}_{n,2}|_\Sigma, \quad \boldsymbol{V}_{\shortparallel,1}|_\Sigma = \boldsymbol{V}_{\shortparallel,2}|_\Sigma, \tag{10.16}$$

where the two components are the normal and the parallel one to the surface. In many models, it is more practical to rewrite the continuity conditions (10.16) in another form,

$$\begin{aligned} \boldsymbol{V}_{n,1}|_\Sigma &= \boldsymbol{V}_{n,2}|_\Sigma, \\ \boldsymbol{N} \cdot (\nabla_\Sigma \cdot \boldsymbol{V}_1|_\Sigma) &= \boldsymbol{N} \cdot (\nabla_\Sigma \cdot \boldsymbol{V}_2|_\Sigma), \\ \boldsymbol{N} \cdot (\nabla_\Sigma \times \boldsymbol{V}_1|_\Sigma) &= \boldsymbol{N} \cdot (\nabla_\Sigma \times \boldsymbol{V}_2|_\Sigma), \end{aligned} \tag{10.17}$$

namely the continuity of the normal components of the velocity, of the divergence and the curl of the velocity. The last one is nothing but the continuity of the normal component of the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{V}$. The operator ∇_Σ is the surface gradient. Basically, it represents the gradient expressed in surface curvilinear coordinates, acting on vectors in the tangent plane to Σ. Its rigorous definition and properties are described in Sect. 6.5. Equation (10.17) represent mixed Dirichlet and von Neumann boundary conditions, and guarantee the uniqueness of the solution of the (elliptic type partial differential equations) Euler or Navier–Stokes equations (see (10.13) and (10.15)).

In the case of rigid surface in contact with the fluid, because of the cohesive forces, we ask $\boldsymbol{V}|_\Sigma = 0$. Such a relation cannot be fulfilled by the Euler equation (it would generate zero solutions all over the space), but it can be fulfilled at least for the normal components in the case of inviscid fluids (or actually the normal component of fluid velocity should be equal to the local velocity of the rigid surface), while $\boldsymbol{V}_\shortparallel \neq 0$ for ideal fluids. Consequently, the separation between the fluid and the rigid boundary is a special zone, so-called "vortex-sheet" or "boundary layer" where we model the discontinuity for the tangent velocity. In the boundary layer the vorticity is nonzero, but because the equation for vorticity in the viscous case is a diffusion type of equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nu \Delta \boldsymbol{\omega},$$

where we eliminate the volume forces for simplification, we expect the vorticity to decay toward the bulk of the fluid, away from the boundary layer. This also implies that out of the boundary layer the velocity is almost potential.

The balance of the momentum across the surface is

$$\boldsymbol{F}_1|_\Sigma = \boldsymbol{F}_2|_\Sigma \rightarrow N_i \sigma^1_{ik}|_\Sigma = N_i \sigma^2_{ik}|_\Sigma \tag{10.18}$$

or in tensor form

$$(\hat{\sigma}^1 - \hat{\sigma}^2) \cdot \boldsymbol{N} = 0, \quad \text{on } \Sigma. \tag{10.19}$$

For a free surface, (10.18) reduces to

$$N_i \sigma'^1_{ik}|_\Sigma = P|_\Sigma N_k. \tag{10.20}$$

In tensor form the continuity condition across a free surface reads

$$(\hat{\sigma}' \cdot \boldsymbol{N})_\Sigma = P|_\Sigma \cdot \boldsymbol{N} = 2\sigma H,$$

$$(\hat{\sigma}' \cdot \boldsymbol{t}_{a,b})_\Sigma = 0, \tag{10.21}$$

where $\boldsymbol{t}_{a,b}$ form a basis in the tangent space of the surface, σ is the *coefficient of surface tension*, and H is the mean curvature of the surface. These equations will be elaborated in detail in Sect. 10.4. In this case of an isolated droplet, the driving force (the surface tension) acts always perpendicularly to the free surface. Therefore, the tangential stress on the surface vanishes, and the normal stress is the driving force. In Chap. 8, we have noticed that there are a lot of other interactions at the interface between two fluids, especially if the surface is material and it is moving.

If the surface of separation carries some material properties, for example it has mass distribution, internal viscoelastic forces, etc. (in this case the separation is called an interface), the continuity equations for the stress (10.19) and (10.21) change correspondingly

$$(\hat{\sigma}^1 - \hat{\sigma}^2) \cdot \boldsymbol{N}|_\Sigma = \boldsymbol{F}_{\text{net},\Sigma}, \tag{10.22}$$

where the RHS is the net force per unit of surface area acting upon the physical surface, sometimes denoted $\boldsymbol{\sigma}_\Sigma$. This surface density force, $\boldsymbol{F}_{\text{net}} = F_n \boldsymbol{N} + \boldsymbol{F}_{\shortparallel}$, contains the surface tension and many other terms related to the existence of surface elasticity, viscosity, shear, surfactants, mass transfer, etc. Its expression is obtained on differential geometry grounds in Sect. 8.4 (see (8.41) and (8.56)).

10.3 Circulation Theorem

This subject was initially investigated by Thomson [11] and Helmholtz [12]. Some different proofs of the theorems on vortex motion were given later by Lord Kelvin [13]. The circulation theorem states that:

Theorem 24 (Kelvin Circulation Theorem) *The line integral of the fluid velocity* $\boldsymbol{v}$ *along a closed circuit* Γ *(the circulation of the velocity) which moves together with the fluid is constant in time if the fluid is perfect*

$$C_{\boldsymbol{v},\Gamma} = \oint_{\Gamma} \boldsymbol{v} \cdot \boldsymbol{t} ds = const. \tag{10.23}$$

Here $\boldsymbol{v}$ is calculated in the Lagrangian frame and $\boldsymbol{t}$ is unit tangent to Γ.

By perfect fluid we understand here inviscid isentropic flow, governed by Euler equation (10.15) in the presence of only potential external forces

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla P - \nabla U, \tag{10.24}$$

where $\boldsymbol{a}$ is the Lagrangian acceleration and U is the potential of external forces acting on the fluid. This result is important both for vortex motion and potential motion. However, in spite of the fact that the concept of closed circuit *moving with the fluid* is intuitive, and it is based on the Lagrangian point of view, this concept is not quite rigorously defined geometrically. In the following, we give two proofs for the circulation theorem differing in the degree of rigorousness and geometry involved [1, 3, 14].

Proof 1 (*Equation of State Approach*) The rate of change of the circulation is

$$\frac{dC_{\boldsymbol{v},\Gamma}}{dt} = \oint_{\Gamma} \boldsymbol{a} \cdot \boldsymbol{t} ds + \oint_{\Gamma} \boldsymbol{v} \cdot d\left(\frac{d\boldsymbol{r}}{dt}\right). \tag{10.25}$$

The second integral on the RHS is a total differential ($\boldsymbol{v} d\boldsymbol{v}$) and it provides zero contribution on the closed circuit. According to the hypotheses, the acceleration is given by the Euler equation (10.15). If the flow is isentropic, the Lagrangian variation of the entropy of the unit of mass of the fluid is zero, $d\left(\frac{S}{m}\right) = ds = 0$. Consequently, we can write the variation of the enthalpy of the unit of mass

$$dh = \frac{VdP + TdS}{m} = \frac{1}{\rho}dP, \tag{10.26}$$

where P is the pressure. In this way the acceleration becomes a gradient $\boldsymbol{a} = -\nabla(h + U)$, and the first integral in (10.25) is also zero. The circulation of velocity on any closed circuit moving with the fluid is indeed constant. □

In other approaches (for example [1]) Theorem 24 is formulated with a different hypothesis. It is stated that in the inviscid fluid the density is either constant or function of pressure only (barotropic flow). The equivalence of the two formulations is obvious: if the fluid is isentropic, then the constancy of entropy provides an equation of state in terms of density and pressure only, $s = s(p, \rho)$, from where the requested dependence [14].

It is interesting to observe that, for inviscid fluids which are not isentropic (not barotropic fluids) and for which the circulation is not conserved, the acceleration has the property

$$\nabla \times \boldsymbol{a} = \nabla P \times \nabla \frac{1}{\rho}. \tag{10.27}$$

This means that the rate of change of circulation can be expressed through the Stokes theorem in the form

$$\frac{dC_{\boldsymbol{v},\Gamma}}{dt} = \int_{\Sigma} \left(\nabla P \times \nabla \frac{1}{\rho} \right) \cdot \boldsymbol{N} dA, \tag{10.28}$$

where Σ is a surface bounded by the circuit Γ. That means that the average (over a small surface) rate of change of the circulation is directed along the intersection between isobaric surfaces and surfaces of constant density. A lot of convection effects, including for example the surface vs. bottom salted water current between the Black Sea and the Mediterranean Sea, are generated by this mechanism [1].

On the other hand, the circulation Theorem 24 helps to understand the permanent character of the potential flow: once the curl of velocity is zero in some region and at some initial moment of time, the velocity will be irrotational in any region of the space and at any later moment, by circulation (zero in this case) conservation. The irrotational character of the flow is transported by physical fluid particles in all the flow region.

Proof 2 (*Free Surface Approach*) The physical hypotheses are the same: ideal inviscid isentropic fluid with potential external forces. We need to work with the concept of moving particle circuit, i.e., the closed curve of particles moving with the fluid. In other words a closed contour always consists of the same fluid particles. For a rigorous geometric definition of particle lines and circuits in terms of fiber bundles, the reader can return to the Sects. 9.2, 9.2.2, 9.2.4, and 9.2.5.

We prepare the proof of the Kelvin theorem by using traditional definitions of path lines and particle contours, like those introduced in Sects. 9.1.2, 9.2.3, and 9.3. Later on we reformulate the theorem in terms of differential geometry. Let us choose at $t = 0$ a compact, connected, and simply connected surface Σ made by fluid particles, and consider its boundary the closed curve $\Gamma = \partial \Sigma$. We call Γ a particle circuit. The existence and stability in time of such a curve are discussed in the above-mentioned sections. We parametrize this curve with the equation $\boldsymbol{r}_0(s)$, where s labels the fluid particles in the circuit. At a later moment of time, within some finite time interval $t \in [0, T]$, we construct a diffeomorphic deformation of Σ into Σ', i.e., the fluid flow. This mapping induces a diffeomorphic deformation of Γ into Γ', described by $\boldsymbol{r}_0(s) \to \boldsymbol{r}(t, s)$. The $\boldsymbol{r}(t, s)$ function represents the position of the s fluid particle at moment t. When time runs, the diffeomorphism generates a family of curves (particle circuits moving with the fluid) each one parameterized by the same label s. The set

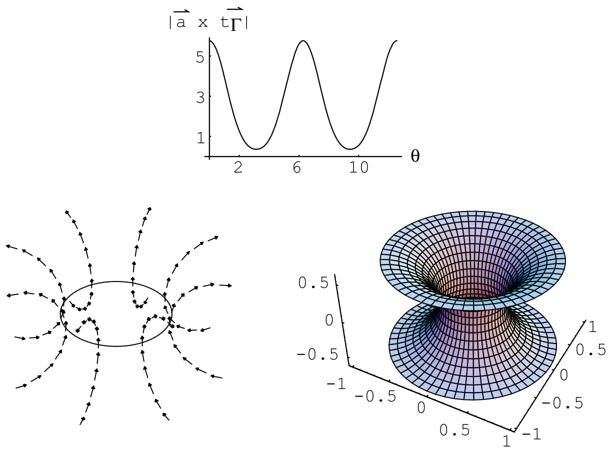

Fig. 10.1 *Left*: particle circuit Γ (*horizontal circle*) and corresponding particle paths (Γ_s, *arrows*). *Right*: resulting tube of flow $\Sigma^{\Gamma}_{[0,\sigma_{max}]}$. *Top*: the regularity condition in Theorem 25 is fulfilled, i.e., $\boldsymbol{a}(\boldsymbol{r}(\sigma, s)) \times \boldsymbol{r}'_{\Gamma}(s) \neq 0$

of these closed curves is called a *tube of flow* based on the particle sheets Σ and Σ'. The question is if this tube of flow described by the curves $\boldsymbol{r}(t, s)$ is a *regular surface*. The answer is given by Theorem 25.

Theorem 25 *Let $\boldsymbol{a}(\boldsymbol{r})$ be a differential vector field on an open domain $\mathcal{D} \subset \mathbb{R}^3$ and $\Gamma \subset \mathcal{D}$ be an arc-length parameterized regular simple closed curve of equation $\boldsymbol{r}_{\Gamma}(s)$ with $s \in [0, L_{\Gamma}]$ and $\boldsymbol{r}_{\Gamma}(0) = \boldsymbol{r}_{\Gamma}(L_{\Gamma})$. For every $s \in [0, L_{\Gamma}]$ we build a regular simple parameterized curve Γ_s of equation $\boldsymbol{r}(\sigma, s)$ with $\sigma \in [0, \sigma_{max}]$ as follows:*

1. *The equation $\boldsymbol{r}(\sigma, s) = \boldsymbol{r}_{\Gamma}(s)$ has one and only one solution $\sigma = 0$.*
2. *If $\boldsymbol{t}_{\Gamma_s}(\sigma)$ is the unit tangent for each Γ_s curve, then $\forall \sigma \in [0, \sigma_{max}]$*

$$\frac{\partial \boldsymbol{r}}{\partial \sigma}(\sigma, s) \equiv \boldsymbol{t}_{\Gamma_s}(\sigma) = \boldsymbol{a}(\boldsymbol{r}(\sigma, s)),$$

$$\boldsymbol{a}(\boldsymbol{r}(\sigma, s)) \times \frac{d\boldsymbol{r}_{\Gamma}}{ds}(s) \neq 0.$$

$\boldsymbol{r}(\sigma, s)$ is a regular parameterized surface $\Sigma^{\Gamma}_{[0,\sigma_{max}]}$ for $\sigma \in [0, \sigma_{max}]$, $s \in [0, L_{\Gamma}]$.

Proof See Fig. 10.1. Since the field $\boldsymbol{a}$ is differentiable, the curves Γ_s are its integral curves and depend smoothly on their natural arc-length parameter σ. Also, from the Frobenius existence and uniqueness theorem (Theorem 3), all these curves depend smoothly on their initial data, i.e., the s parameter (see also [15, Theorem 1, p. 176]). Consequently $\boldsymbol{r}(\sigma, s)$ is a differentiable function. From the hypotheses each integral curve intersects the contour only one time. The Jacobian matrix

$$\hat{J}\boldsymbol{r}(\sigma, s) = \left(\frac{\partial x^i}{\partial \sigma}, \frac{\partial x^i}{\partial x^j_\Gamma}\frac{dx^j_\Gamma}{ds}\right) = (a^i(\boldsymbol{r}(\sigma, s)), \delta_{ij}t^j_\Gamma(s)) \neq 0$$

is nonzero by hypothesis. The Jacobian has rank 2 and hence the tangent map $d\boldsymbol{r}$ is one-to-one. Consequently $\boldsymbol{r}(\sigma, s)$ is a regular parametrized surface. □

From Theorem 25 we know that moving particles arranged in a closed contour Γ generate a tube of flow $\boldsymbol{r}(t, s)$ based on Γ and Γ'. Now we can come back to the second proof of the Kelvin circulation theorem. We write (10.23) in the form

$$\oint_\Gamma \boldsymbol{v} \cdot \boldsymbol{t} ds = \oint_{\Gamma'} \boldsymbol{v} \cdot \boldsymbol{t} ds,$$

where Γ, Γ' represent the particle contour at two different moments of time.

The vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ has the property $\nabla \cdot \boldsymbol{\omega} = 0$ which means that, for any domain $\mathcal{D}$, we have

$$\int_{\mathcal{D}} \nabla \cdot \boldsymbol{\omega} dV = \oint_{\partial\mathcal{D}} \boldsymbol{\omega} \cdot \boldsymbol{N} dA = 0,$$

where dV, dA are the volume and area elements and $\boldsymbol{N}$ is the unit normal to Σ. We choose $\mathcal{D}$ to be the inside of a tube of flow bounded by Σ, Σ' and a side area described by the flows $\boldsymbol{r}(t, s)$, denoted in the following Σ_f. We have

$$0 = \oint_{\Sigma \cup \Sigma' \cup \Sigma_f} \boldsymbol{\omega} \cdot \boldsymbol{N} dA = \int_{\Sigma_f} \boldsymbol{\omega} \cdot \boldsymbol{N} dA + \int_{\Sigma \cup \Sigma'} \boldsymbol{\omega} \cdot \boldsymbol{N} dA. \tag{10.29}$$

Because Σ, Σ' are particle surfaces, we have

$$\boldsymbol{v}|_\Sigma \times \boldsymbol{N}_\Sigma = 0, \boldsymbol{v}|_{\Sigma'} \times \boldsymbol{N}_{\Sigma'} = 0, \tag{10.30}$$

and hence we have $\boldsymbol{v}|_\Gamma \cdot \boldsymbol{t}_\Gamma = \boldsymbol{v}|_{\Gamma'} \cdot \boldsymbol{t}_{\Gamma'} = 0.$[1] Consequently

$$0 = \oint_\Gamma \boldsymbol{v} \cdot \boldsymbol{t} ds = \int_\Sigma \boldsymbol{\omega} \cdot \boldsymbol{N} dA$$

$$0 = \oint_{\Gamma'} \boldsymbol{v} \cdot \boldsymbol{t}' ds = \int_{\Sigma'} \boldsymbol{\omega} \cdot \boldsymbol{N}' dA', \tag{10.31}$$

which cancel the second term on the RHS of (10.29). So, we have

$$\int_{\Sigma_f} \boldsymbol{\omega} \cdot \boldsymbol{N} dA = 0, \tag{10.32}$$

[1] For the proof of these relations, see Problem 5 at the end of this chapter.

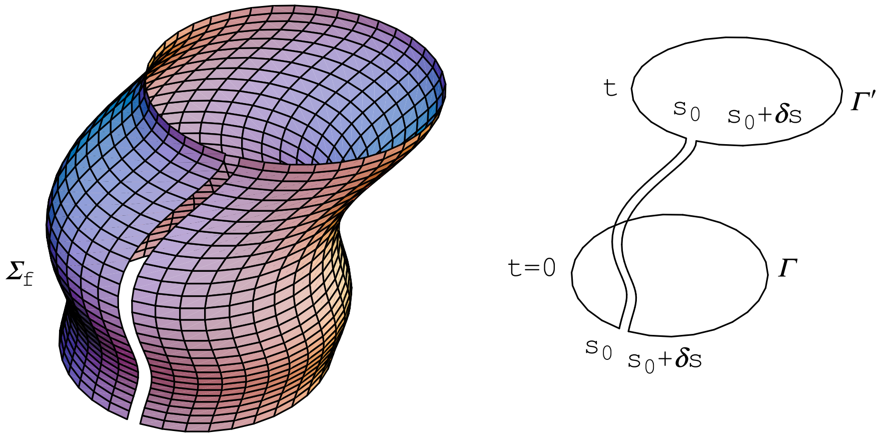

Fig. 10.2 Closed contour of integration on a tube of flow

i.e., the flux of vorticity through the side surface is zero.[2] Now we choose $t = 0$ and another moment of time t, and s_0, $s_0 + \delta s$ two close points on Γ and Γ'. We integrate $\boldsymbol{v}$ along a closed curve lying in Σ_f, composed by $\boldsymbol{r}_\Gamma|_{s\in[s_0+\delta s, s_0]}$, connected to $\boldsymbol{r}|_{[0,t]\times\{s_0\}}$, connected to $\boldsymbol{r}_{\Gamma'}|_{s\in[s_0,s_0+\delta s]}$, and finally connected to $\boldsymbol{r}|_{[t,0]\times\{s_0+\delta s\}}$, like in Fig. 10.2. We integrate $\boldsymbol{v}$ along the curve in Fig. 10.2 in the limit $\delta s \to 0$, and from (10.32) we have

$$\lim_{\delta s=0} \oint \boldsymbol{v}\cdot\boldsymbol{t}ds = \int_{\Sigma_f} \boldsymbol{\omega}\cdot\boldsymbol{N}dA = 0. \tag{10.33}$$

But

$$\oint \boldsymbol{v}\cdot\boldsymbol{t}ds = \int_\Gamma \boldsymbol{v}\cdot\boldsymbol{t}ds + \int_{\boldsymbol{r}(s_0,0)}^{\boldsymbol{r}(s_0,t)} \boldsymbol{v}\cdot\boldsymbol{t}ds - \int_{\Gamma'} \boldsymbol{v}\cdot\boldsymbol{t}ds + \int_{\boldsymbol{r}(s_0+\delta s,t)}^{\boldsymbol{r}(s_0+\delta s,0)} \boldsymbol{v}\cdot\boldsymbol{t}ds. \tag{10.34}$$

In the limit $\lim_{\delta s=0}$, the second and the fourth terms in the RHS of (10.34) cancel each other, and by using (10.33) we prove the Kelvin circulation theorem.

Traditional proofs of the same theorem can be found, for example, in Article 146 from [3], in Sect. 3.51 from [1], or in Sect. 8 from [14].

Proof 3 (*Comment*) There is a geometrical way to prove (10.32). Since we work only on the fluid particle surface, it is natural to use the surface differential operators instead of the full three-dimensional ones. We apply the surface divergence theorem (6.61), where we substitute $\boldsymbol{A} = \boldsymbol{v}\times\boldsymbol{N}$. From the formula (6.69) in the problems at the end of Chap. 6, we have $\nabla_{\Sigma_f}\cdot(\boldsymbol{v}\times\boldsymbol{N}) = \boldsymbol{N}\cdot(\nabla_{\Sigma_f}\times\boldsymbol{v}) - \boldsymbol{v}\cdot(\nabla_{\Sigma_f}\times\boldsymbol{N})$ and

[2] The fact that the flux of vorticity is zero on a tube of flow surface is an interesting result by itself. For more discussions, also see Problem 5 at the end of this chapter.

this reduces to $\boldsymbol{N} \cdot (\nabla_{\Sigma_f} \times \boldsymbol{v})$ because of the property of the normal from in (6.54). It results

$$\iint_{\Sigma_f} \nabla_{\Sigma_f} \cdot (\boldsymbol{v} \times \boldsymbol{N}) dA = \oint \boldsymbol{\omega} \cdot \boldsymbol{N} dA,$$

where the contour integral is taken along the curve in Fig. 10.2. Both RHS terms in the surface divergence theorem formula cancel. On one hand we have

$$\oint (\boldsymbol{v} \times \boldsymbol{N}) \cdot \boldsymbol{t}^{\perp} ds = \oint (\boldsymbol{t}^{\perp} \times \boldsymbol{v}) \cdot \boldsymbol{N} ds = 0,$$

because $\boldsymbol{v} \parallel \boldsymbol{t}^{\perp}$ by the definition of $\boldsymbol{\Sigma}_f$. The second term on the RHS of the divergence theorem formula cancels by construction

$$-2 \iint H (\boldsymbol{v} \times \boldsymbol{N}) \cdot \boldsymbol{N} dA = 0,$$

so it results (10.32). The reason we wanted to mention this geometric amendment is related to (10.30). In Proof 2, these equations are somehow postulated on physical grounds (i.e., particles contained in the surface move together with the surface), however in this comment they result automatically as a rigorous consequence.

10.4 Surface Tension

10.4.1 Physical Problem

In this section, we study certain phenomena that occur in the neighborhood of a closed surface of separation between two continuous media that do not mix. In reality, the two systems in contact are separated by a thin boundary layer having special properties. However, in the following, we neglect the internal structure of this transition layer, and we assimilate it with an infinite thin geometric surface. In the neighborhood of a curved surface of separation, the pressure in the two media is different, and we call this pressure difference *surface tension*. In Sect. 8.4 (see (8.32)), we introduce the same surface tension in another manner, starting from dynamical considerations. Here, we assume that the free energy of this state of tension (the stress between two adjacent elements of surface) depends only on the area of the common boundary, on the nature of the two media, and on temperature. The special case of additional electric, acoustic, etc., fields, or presence of surfactants will be discussed later in another chapter. For a more detailed discussion on the topic, see Article 265 in [3]. Although, the original first treatment of the problem belongs to Lagrange who first determined a *minimal* surface in 1760. A review on the topics of capillarity is presented in [16] and references herein.

In the stationary case $\boldsymbol{v} = 0$ for a fluid with free boundary S, the Euler equation reads

$$-\frac{1}{\rho}\nabla P + \boldsymbol{f} = 0, \tag{10.35}$$

where ρ is the fluid density, P is the pressure, and $\boldsymbol{f}$ is the mass density of the force field acting inside the fluid. If the force field is potential, $\boldsymbol{f} = -\triangle u$, the stationary Euler equation reduces to the simplest Bernoulli type of equation, namely $P = P_0 - \rho u$. However, this equation cannot predict the pressure infinitesimally close to the surface, where stronger nonlinear effects occur. To obtain the pressure next to the fluid surface, we have to use other approach [14].

The expression of surface tension can be obtained by using the equations of thermodynamic equilibrium. Let us assume that locally the surface of separation suffers a variation in the form of an infinitesimal displacement. The only displacement that counts physically is that one normal to the surface, because we neglect the internal structure of the surface, and we consider it to be homogenous from the physical point of view. Let us describe the surface of separation as a parameterized regular geometrical surface $\boldsymbol{r}(u, v) : U \to S$ (see Chap. 19) with unit normal $\boldsymbol{N}(u, v)$. We define the *normal variation* of the surface S as the function

$$\boldsymbol{r}^t(u, v, t) = \boldsymbol{r}(u, v) + t\, h(u, v)\boldsymbol{N}(u, v), \tag{10.36}$$

where $(u, v) \in U, t \in (-\varepsilon, \varepsilon)$ is a parameter, and $h(u, v)$ is a differential real function defined on U. For each t, the map $\boldsymbol{r}^t : U \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ is a regular parametrized surface (see Fig. 10.3). For $t = 0$, the normal variation reduces to the original surface.

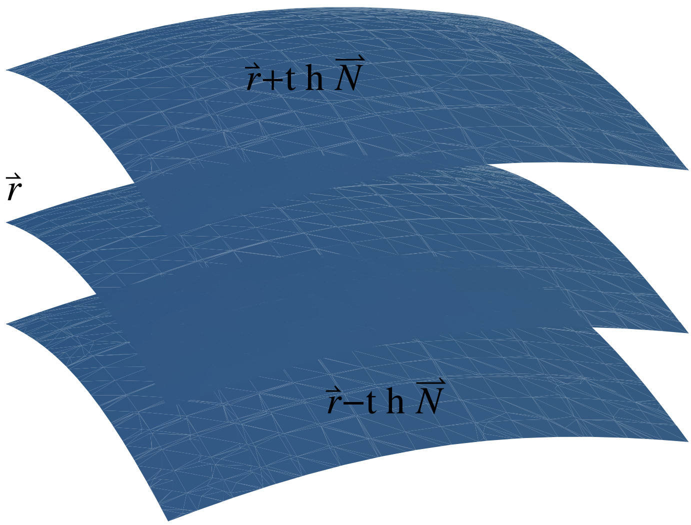

Fig. 10.3 A normal variation of $\boldsymbol{r}(U)$

We assume that the original surface suffered a normal variation determined by the $h(u, v)$ function, and it is not anymore in thermodynamic equilibrium. The elementary volume of an infinitesimal element of space bounded by the original surface and by the graphs of the function $\boldsymbol{r}^t$ is $t\, h(u, v) dA(u, v)$, where dA is the elementary area of the original surface, $dA = \sqrt{EG - F^2} du dv$ (from Definition 52). We denote by P_1 and P_2 the pressures in the medium 1 and medium 2, respectively, separated by S, in the neighborhood of the surface, and we choose the direction from 1 to 2 in the direction of the unit normal $\boldsymbol{N}$. The work produced by a compression upon this elementary volume, which is also the change in its free energy $\mathcal{F}$, is

$$W_{vol} = \delta \mathcal{F}_{vol} = t \iint_{\bar{U}} (P_2 - P_1) h \sqrt{EG - F^2} du dv. \tag{10.37}$$

The total change in the free energy of the system is given by δW_{vol} plus the work associated with the variation of the area of the separation surface, i.e., the superficial (or surface) energy. In a simple model, this second part of the free energy is given by the product between a constant σ and the variation of the area δA. The constant σ is called *surface tension coefficient* and depends on the nature of the two media, and on temperature. The total variation in the free energy becomes

$$\delta \mathcal{F} = t \iint_{\bar{U}} (P_2 - P_1) h \sqrt{EG - F^2} du dv + \sigma \delta A. \tag{10.38}$$

The equilibrium condition is $\delta \mathcal{F} = 0$, and from here we obtain the expression of the surface tension, $P|_S = P_2 - P_1$. We prove in Sect. 10.4.2 that the expression of the surface tension at a point $\boldsymbol{r}$ on the surface is

$$P_2 - P_1 = P_{\boldsymbol{r} \in S} = \sigma(\kappa_1 + \kappa_2),$$

where $\kappa_{1,2}$ are the two principal curvatures of the surface at p. In all our examples, we choose the orientation of the surfaces such that the normal is toward the convexity of the curve, and the direction from medium 1 to medium 2 is chosen along this normal. To check the correct sign of the surface pressure expression, we choose for the surface the graphics of a differential function $z = \eta(x)$. The profile depends only on x, and we have full symmetry along the other coordinate y. In this one-dimensional case, we have just one principal curvature nonzero, this $\kappa_1 = \kappa$ ($\kappa_2 = 0$) is called the curvature of the function η, and it has the expression $\kappa = \frac{\eta''}{(1+\eta'^2)^{\frac{3}{2}}}$. If we choose a convex function with $\eta'' < 0$, we have $\kappa < 0$ and consequently $P_1 > P_2$. That pressure P_1 inside the concavity is larger, as it should be. A more geometrical definition of the surface tension can be found in Sect. 8.4 or in [6, 17].

10.4.2 Minimal Surfaces

To find the explicit expression for the surface tension in the most general situation, we need to calculate the RHS term in (10.38). The coefficients of the first fundamental form of the modified surface $\boldsymbol{r}^t$ are

$$E^t = E + 2th\boldsymbol{r}_u \cdot \boldsymbol{N}_u + t^2h^2\boldsymbol{N}_u \cdot \boldsymbol{N}_u + t^2(h_u)^2,$$

$$F^t = F + th(\boldsymbol{r}_u \cdot \boldsymbol{N}_v + \boldsymbol{r}_v \cdot \boldsymbol{N}_u) + t^2h^2\boldsymbol{N}_u \cdot \boldsymbol{N}_v + t^2h_uh_v,$$

$$G^t = G + 2th\boldsymbol{r}_v \cdot \boldsymbol{N}_v + t^2h^2\boldsymbol{N}_v \cdot \boldsymbol{N}_v + t^2(h_v)^2. \tag{10.39}$$

By using the definition relations for the second fundamental form of the surface (see Chap. 19)

$$e = -\boldsymbol{r}_u \cdot \boldsymbol{N}_u, \quad f = -(\boldsymbol{r}_u \cdot \boldsymbol{N}_v + \boldsymbol{r}_v \cdot \boldsymbol{N}_u)/2, \quad g = -\boldsymbol{r}_v \cdot \boldsymbol{N}_v$$

and the definition of the mean curvature of a surface (6.10)

$$H = \frac{Eg - 2fF + Ge}{2(EG - F^2)}, \tag{10.40}$$

we obtain

$$E^tG^t - (F^t)^2 = EG - F^2 - 2th(Eg - 2fF + Ge) + \mathcal{O}(t)$$

$$= (EG - F^2)(1 - 4thH) + \mathcal{O}(t), \tag{10.41}$$

where $\mathcal{O}(t)$ is a term that approaches zero more rapidly than t when $t \to 0$. From (10.41), it results that, if ε is small enough, the surface $\boldsymbol{r}^t$ is a regular parameterized surface. Just now we can use $\boldsymbol{r}^t$ as the equation of a surface in the calculation of the free energy and surface tension. The area $A(t)$ of $\boldsymbol{r}^t(\bar{U})$ is given by

$$A(t) = \iint_{\bar{U}} \sqrt{E^tG^t - (F^t)^2}dudv$$

$$= \iint_{\bar{U}} \sqrt{1 - 4thH + \frac{\mathcal{O}(t)}{EG - F^2}}\sqrt{EG - F^2}dudv. \tag{10.42}$$

It follows that, in the limit of small ε, $A(t)$ is differentiable with respect to t, and its derivative at $t = 0$ is

$$\frac{dA}{dt}(0) = -2\iint_{\bar{U}} hH\sqrt{EG - F^2}dudv = -\int hHdA. \tag{10.43}$$

So, the variation of the area during this deformation parameterized by the parameter t is $\delta A = (dA/dt)dt$. At $t = 0$ we have

$$\delta A = -2 \iint_{\bar{U}} h\, H \sqrt{EG - F^2} du\, dv\, dt = -\int (\varkappa_1 + \varkappa_2) h dA dt, \qquad (10.44)$$

where $\varkappa_{1,2}$ are the principal curvatures of the surface at the point of coordinates (u, v) (see Chap. 19). Equation (10.44) can provide an interesting interpretation of the mean curvature, in terms of the minimal surfaces. We can define the *mean curvature vector* by $\boldsymbol{H} = H\boldsymbol{N}$, and by choosing $h = H$ in (10.44) we can write

$$\delta A = -2 \iint_{\bar{U}} \boldsymbol{H} \cdot \boldsymbol{H} \sqrt{EG - F^2} du\, dv\, dt. \qquad (10.45)$$

Equation (10.45) means that the area of the deformed surface $\boldsymbol{r}^t(U)$ always decreases if we deform it in every point toward the direction of the mean curvature vector. For a given surface, the mean curvature vector points toward the direction where this surface tends to become a minimal surface. For example, in the case of an infinitesimal normal variation of a spherical surface, the mean curvature is still negative (the corrections in the first order in ε are smaller than 1) and since the normal is directed outside the sphere and $H < 0$, the vector $\boldsymbol{H}$ points toward the center. This is indeed the direction along which the area of an elementary spherical surface would become smaller, by flattening toward a plane.

The unit normal field for S is a divergence-free vector field. This comes from the fact that the mean curvature is related to the normal direction of the surface by the equation

$$H = -\frac{1}{2} \nabla_S \cdot \boldsymbol{N},$$

from Proposition 5 (Sect. 6.5.2), where $\nabla_S \cdot$ is the surface divergence operator. From here it results

Proposition 9 *For a minimal surface the normal vector field is surface divergence free.*

Coming back to the dynamics of the surface, if we consider the variation of the original area from $t = 0$ to a certain small value of t, we have $dt = t$, and introducing (10.44) in (10.38), we have the condition of equilibrium in the form

$$\iint_{\bar{U}} (P_2 - P_1 - \sigma(\varkappa_1 + \varkappa_2)) t\, h \sqrt{EG - F^2} du dv = 0.$$

Since the function h is arbitrary, we have to fulfill

$$P_2 - P_1 = \sigma(\varkappa_1 + \varkappa_2) = 2\sigma H \qquad (10.46)$$

which determines the expression of the surface pressure (Laplace formula for capillarity). H is the mean curvature. For a more physical proof the reader can check (8.55). If, for example, the principal curvatures are positive, it results that $P_1 > P_2$, i.e., the pressure is larger in the medium located inside the concavity of the surface.

We end this section with a property of minimal surfaces which results as a consequence of the divergence integral theorem (6.61). From the relation

$$\nabla_S \times \boldsymbol{r} = 0,$$

where $\nabla_S\times$ is the surface curl and $\boldsymbol{r}$ is the position vector, we can write two integral conditions valid for any closed curve Γ on any minimal surface S

$$\oint_\Gamma \boldsymbol{t}^\perp ds = 0 \tag{10.47}$$

$$\oint_\Gamma \boldsymbol{r} \times \boldsymbol{t}^\perp ds = 0, \tag{10.48}$$

where $\boldsymbol{t}^\perp = \boldsymbol{N} \times \boldsymbol{t}$ with $\boldsymbol{t}$, $\boldsymbol{r}$ having their regular interpretation and s being the arc-length along Γ. These two equations can be regarded as the dynamical equilibrium conditions for the minimal surface. The first one represents force balance, and the second one represents the momentum balance of a domain of S surrounded by Γ.

10.4.3 Application

To have a better intuition of the direction of the surface tension gradient, we present in the following a simpler example. Let us choose a parameterized surface S as the graph of a differential function $z = h(x, y)$ and U is an open set of the xOy $\mathbb{R}^2$ plane. The parameterizations of the surface are $\boldsymbol{r} = (u, v, h(u, v))$ with $u = x$ and $v = y$. We have

$$\boldsymbol{N}(x, y) = \frac{(-h_x, -h_y, 1)}{(1 + h_x^2 + h_y^2)^{1/2}} \tag{10.49}$$

and

$$H = \frac{(1 + h_x^2)h_{yy} - 2h_x h_y h_{xy} + (1 + h_y^2)h_{xx}}{(1 + h_x^2 + h_y^2)^{1/2}}. \tag{10.50}$$

For a more concrete example, we consider the surface of a semicylinder having the axis along Ox and its points at $z = f(x, y) > 0$. If it rains from above, this cylinder will not keep the water. Close to the top of the cylinder, we have $\boldsymbol{N} \simeq (0, 0, 1)$, and the normal is oriented upward, toward positive z. It means medium 1 (we choose medium 1 to be liquid) is under the cylinder, inside its concavity, and medium 2 (we choose medium 2 to be air) is above the cylinder. We also assume that the cylinder radius R is large enough so we can neglect nonlinear terms in the expression of the mean curvature. At points close to the top of this cylinder ($x \simeq 0$, $z \simeq R$), we have, according to (10.46) and (10.50)

$$P_2 - P_1 = \sigma(\varkappa_1 + \varkappa_2) \simeq h_{yy}, \tag{10.51}$$

and because at this points $h_{yy} < 0$ it results $P_2 < P_1$, so the liquid is under more pressure than the ambient atmosphere, which is in agreement with the Laplace law of capillarity.

We can use the condition (10.46) to find the equilibrium free surface S for $P_1 = P_2 =$ constant. This is a system subjected to the same internal and external pressure in all its points, i.e., a system consisting only in free surfaces, like soap films in microgravity. The total free energy of this system is proportional to the area of the surface, and attains its minimum when the area is minimal. The surface equation $\boldsymbol{r}$ is a *minimal* surface (i.e., $H = 0$) if and only if $\delta A = 0$, i.e., when $A'(t = 0) = 0$, for all normal variations of the surface S. Indeed, if the surface is minimal, $H = 0$ and according to (10.43), $A' = 0$. Conversely, let us assume that $A' = 0$ but let us make the hypothesis that $H \neq 0$, at least in a certain open subset of U. Then, we can always choose $h = H$ in that open set, and zero elsewhere, and it results that $A' < 0$ which contradicts the hypothesis.

To understand the role of surface tension in the geometry of the free surface, we analyze a region of fluid, in the stationary case, and in absence of any external (bulk) forces. The Euler equation reduces to $\nabla P = 0$, so the pressure is the same everywhere inside the fluid (Pascal principle). Because the pressure outside of the liquid P_0 is also considered to be the same, we find the equilibrium condition

$$P - P_0 = (P - P_0)_S = -2\sigma H = -2\sigma(\kappa_1 + \kappa_2) = \text{const.} \tag{10.52}$$

Consequently, the free boundary of a stationary, isolated (no external forces) drop of liquid should have the mean curvature constant all over it. If the mean curvature is constant and there are no other superficial constraints, the surface is spherical. The $H =$ const. condition is not dependent on the compressibility of the fluid, as far as the forces are absent. However, if the free surface is supported by a fixed curve, the shape is much more complicated (see for example Fig. 10.4). In the case of rigid boundaries for the free surface, the parameterized surface is not anymore regular. In the general case there will be singularities along the rigid boundaries. This problem was first formulated in the following form: for any given closed curve $\alpha \in \mathbb{R}^3$, there is a surface S of minimum area with α as boundary. There is a special case when this problem becomes simpler, namely when the liquid forms itself one or more very thin layers, like the above-mentioned soap films, suspended by some closed rigid curves, and exposed to the same external pressure P_0 in every point. Actually, no matter how thin the films are, there are always three-dimensional regions of liquid bounded by these surfaces. Because the liquid region is very thin compared to its overall dimensions, we can describe the liquid film as being bounded by two identical surfaces, separated by a very small distance along the common unit normal.

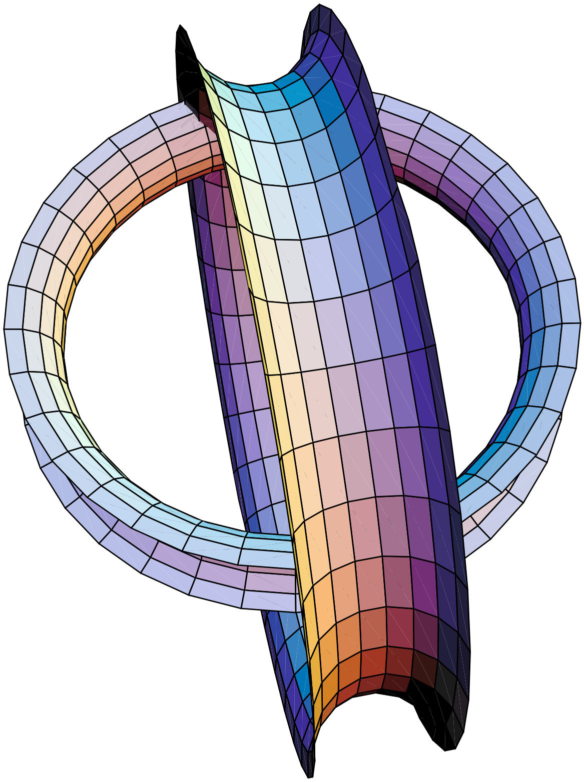

Fig. 10.4 Simulation of an experimental minimal surface produced by dipping a 4-circles wire frame into a soap solution

We consider locally these two surfaces as two identical copies of the same surface, separated by a very small normal displacement. By local we mean here any open domain of the surfaces which do not intersect the boundary curves. On every such open domain, the unit normals $H_{1,2}$ of these two surfaces have the same support, except they point in opposite directions (Fig. 10.5). Any point inside the fluid is infinitesimally close to any of these two identical surfaces, so we can write the surface tension condition as

$$P - P_0 = -2\sigma H_1 = -2\sigma H_2 = 2\sigma H_1. \tag{10.53}$$

It results that the only possibility is to have zero mean curvature in all points. In conclusion, in the absence of forces and in the stationary case, the surface tension and the mean curvature of the free surface are either constant for a free regular surface surrounding the liquid or zero for a thin liquid film. When $H = 0$ we call these surface *minimal*, because they have indeed the minimum area under given constraints. Some of the properties of the minimal surfaces also apply to surfaces of constant mean curvature [18].

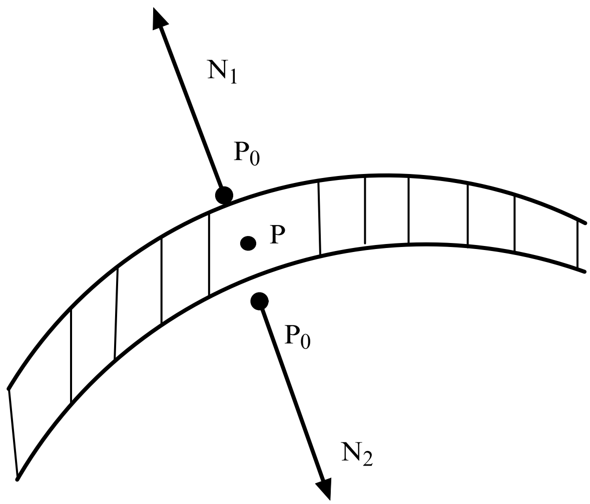

Fig. 10.5 The pressure inside a thin liquid film

10.4.4 Isothermal Parametrization

According to (10.40) and (10.44), the local criterium for the existence of minimal surfaces is played by the PDE $H = Eg - 2fF + Ge = 0$. The structure of this equation simplifies considerably if the coordinate system on the surface S is orthogonal, namely $F = \boldsymbol{r}_u \cdot \boldsymbol{r}_v = 0$. It is always possible to choose such an *orthogonal parametrization* (also called orthogonal curvilinear system of coordinates) for a regular surface. Indeed, for any point $p \in S$ there is a parametrization $\boldsymbol{r}(u, v)$ in a neighborhood of p, $\mathcal{V}(p)$, with the property that the curves $u =$ const. and $v =$ const. are perpendicular. For example, if we choose two differentiable vector fields on S defined by $\boldsymbol{w}_1 = \boldsymbol{r}_u$ and $\boldsymbol{w}_2 = -\frac{F}{E}\boldsymbol{r}_u + \boldsymbol{r}_v$. Moreover, if the vectors of the local basis have equal norms, $E = G$, then the minimal surface local condition reduces to a Laplace equation.

We call *isothermal* [15], a parameterized surface $\boldsymbol{r}(u, v)$ fulfilling the conditions

$$\boldsymbol{r}_u \cdot \boldsymbol{r}_u = \boldsymbol{r}_v \cdot \boldsymbol{r}_v, \quad \boldsymbol{r}_u \cdot \boldsymbol{r}_v = 0, \tag{10.54}$$

which basically means $E = G$ and $F = 0$. Isothermal parameterized surfaces are endowed with orthogonal, yet not normalized, curvilinear coordinates. Orthonormality would imply $E = G =$ const. In the isothermal case the norms of the local basis vectors are equal, but not constant on the surface. It is not easy to parameterize surfaces with isothermal or orthonormal coordinates. For example, the graphs of a differentiable function as a parameterized surface in the independent variable parametrization, $(u, v, f(u, v))$, can never be an isothermal surfaces, because, by using (10.50), we would need $f_u = f_v = 0$ (the only isothermal surface emerging from a graphics is the plane). However, we can provide the following result.

Theorem 26 *Given a parameterized surface* $\boldsymbol{r}(u, v)$*, we can change the parametrization* $(u, v) \to (\alpha, \beta)$ *by the map* $(u, v) = \Phi(\alpha, \beta) : W \subset \mathbb{R}^2 \to U \subset \mathbb{R}^2$ *such that* $(\tilde{\boldsymbol{r}} \circ \Phi)(\alpha, \beta)$ *is isothermal.*

Proof We have $\Phi(u(\alpha, \beta), v(\alpha, \beta))$ and

$$\tilde{\boldsymbol{r}}_\alpha = \tilde{\boldsymbol{r}}_u u_\alpha + \tilde{\boldsymbol{r}}_v v_\alpha, \quad \tilde{\boldsymbol{r}}_\beta = \tilde{\boldsymbol{r}}_u u_\beta + \tilde{\boldsymbol{r}}_v v_\beta,$$

and we request $\tilde{\boldsymbol{r}}_\alpha \cdot \tilde{\boldsymbol{r}}_\beta = 0$ and $\tilde{\boldsymbol{r}}_\alpha \cdot \tilde{\boldsymbol{r}}_\alpha = \tilde{\boldsymbol{r}}_\beta \cdot \tilde{\boldsymbol{r}}_\beta$. These conditions are equivalent with the following system of two nonlinear PDE

$$\begin{cases} E u_\alpha u_\beta + F(u_\alpha v_\beta + u_\beta v_\alpha) + G v_\alpha v_\beta = 0 \\ E u_\alpha^2 + 2F u_\alpha u_\beta + G u_\beta^2 = E v_\alpha^2 + 2F v_\alpha v_\beta + G v_\beta^2 \end{cases} \tag{10.55}$$

The two solutions of this PD system of equations $u(\alpha, \beta)$, $v(\alpha, \beta)$ should also fulfill the compatibility conditions $u_{\alpha,\beta} = u_{\beta,\alpha}$, $v_{\alpha,\beta} = v_{\beta,\alpha}$. By using the theorem of existence and uniqueness from Sect. 3.3, we can always find solutions for (10.55) defined in a neighborhood, under Cauchy arbitrary conditions. Consequently, we can always provide the given parameterized surface with new isothermal curvilinear coordinates. □

For example, if $S = \{(x, y, z) \in S_2 \subset \mathbb{R}^3 | z > 0\}, x = u, y = v$, originally parameterized as the graphics of the function $z = f(u, v) = \sqrt{1 - u^2 - v^2}$, we have $\boldsymbol{r} = (u, v, f(u, v))$

$$\boldsymbol{r}_u = (1, 0, f_u), \quad \boldsymbol{r}_v = (0, 1, f_v), \quad \boldsymbol{N} = \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}},$$

and $E = 1 + f_u^2$, $G = 1 + f_v^2$, and $F = f_u f_v$. Obviously this surface is not isothermal, but if we map u, v into spherical coordinates θ, φ we have $\tilde{\boldsymbol{r}} = (\sin(\theta)\cos(\varphi), \sin(\theta)\sin\varphi, \cos(\theta))$. The new first fundamental form reads $\tilde{E} = 1$, $\tilde{F} = 0$, and $\tilde{G} = \sin^2\theta$. We need to map these new coordinates into a new set of curvilinear coordinates, α, β, which have to fulfill again the isothermal conditions (10.54), i.e.,

$$\begin{cases} \theta_\alpha \theta_\beta + \sin^2\theta \varphi_\alpha \varphi_\beta = 0 \\ \theta_\alpha^2 + \sin^2\theta \theta_\beta^2 = \varphi_\alpha^2 + \sin^2\theta \varphi_\beta^2 \end{cases}$$

A possible solution of the above system is provided by $\varphi = \beta$ and $\theta(\alpha) = 2\arctan C_0 e^{\pm\alpha}$, with arbitrary constant C_0. In Fig. 10.6 we present a subset of the surface S in all these three parameterizations.

The main result of this section can be expressed by the following affirmation regarding minimal isothermal surfaces.

Theorem 27 *If the parameterized surface* $\boldsymbol{r}(u, v)$ *is isothermal, we can write*

$$\boldsymbol{H} = H\boldsymbol{N} = \frac{1}{2E}\triangle\boldsymbol{r}, \tag{10.56}$$

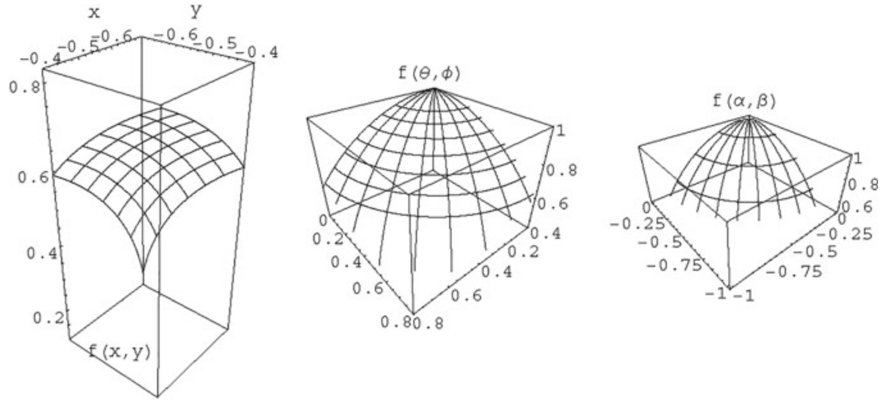

Fig. 10.6 From left to right: a domain of a sphere represented in cartesian coordinates, in spherical coordinates, and in the α, β coordinates

where $\triangle = \partial_{uu} + \partial_{vv}$ *is the Laplace operator in the surface curvilinear coordinates, and we introduce the mean curvature vector* $\boldsymbol{H}$.

Proof By differentiating $\boldsymbol{r}_u \cdot \boldsymbol{r}_v = 0$ and $\boldsymbol{r}_u \cdot \boldsymbol{r}_u = \boldsymbol{r}_v \cdot \boldsymbol{r}_v$ with respect to u and v, we obtain $\boldsymbol{r}_v \cdot \triangle \boldsymbol{r} = \boldsymbol{r}_u \cdot \triangle \boldsymbol{r}$, so $\triangle \boldsymbol{r}$ is parallel to $\boldsymbol{N}$. On the other side, we have $H = (e+g)/(2E) = \boldsymbol{N} \cdot \triangle \boldsymbol{r}/(2E)$ so $\boldsymbol{H} = \boldsymbol{N}(\boldsymbol{N} \cdot \triangle \boldsymbol{r})/(2E)$. □

Theorem 27 has a different expression if instead of the full three-dimensional Laplace operator we use the surface Laplace operator $\triangle_S$ defined in Sect. 6.5.3. In the surface differential operator case, we have

Proposition 10 *On a surface* Σ *parameterized with orthogonal coordinates, we have*

$$\triangle_S \boldsymbol{r} = 2H\boldsymbol{N},$$

and the Laplacian of the position vector is zero for minimal surfaces.

The proof follows from (6.47). In case of orthogonal coordinates ($F = 0$) this relation becomes (6.48). Even more interesting, in the case of a minimal surface, the normal component of the position vector of the surface $r_n = \boldsymbol{r} \cdot \boldsymbol{N}$ is given by (6.51), namely $\triangle_S(r_n) = 2r_n K$.

As a direct consequence of Theorem 27, an isothermal parameterized surface is minimal if and only if its parametrization function is harmonic (i.e., $\triangle \boldsymbol{r} = (\triangle x(u,v), \triangle y(u,v), \triangle z(u,v)) = 0$). Theorem 27 provides an invaluable tool to find minimal surfaces through a very well-studied PDE. For example, if we identify the parameter space with the complex plane by setting $z = u + iv \in \mathbb{C}$, $(u,v) \in U \subset \mathbb{R}^2$ and if we express the regular parameterized surface $\boldsymbol{r}$ through the equations $\varphi_j = \frac{\partial x_j}{\partial u} - i\frac{\partial x_j}{\partial v}$, $j = 1, 2, 3$, then, the parameterized surface $\boldsymbol{r}$ is isothermal if and only if $\varphi_1 + \varphi_2 + \varphi_3 = 0$ and this surface is minimal if and only if the three complex functions φ_j are analytic. Indeed, analyticity implies harmonicity of the coordinate functions by the Cauchy–Riemann conditions. In Fig. 10.7 we present some traditional examples of minimal surfaces. The Scherk's surface [15] is such an example of complex surface.

In addition to their simplification over the minimal surfaces equation, the isothermal surfaces ($E = G$, $F = 0$) have another interesting property related to the Laplace operator. The Gaussian curvature is $K = \frac{1}{2E} \triangle \log E$ [19].

10.4.5 Topological Properties of Minimal Surfaces

Minimal surfaces have a lot of interesting topological properties. The zeros of the Gaussian of a minimal surface are isolated, meaning that if a minimal surface has planar or parabolic points, they are isolated. In other words, there is no straight escaping line along a minimal surfaces, they are really "very twisted." Also, there are no compact minimal surfaces. This is easy to prove, because all the points of a regular minimal surface are hyperbolic. If a minimal surface S is compact (bounded and closed), we can find an S_2 sphere of radius R containing S. We can choose R such that $S_2 \cap S = \varnothing$. Then, we decrease R continuously until the intersection between S and the sphere becomes nonempty. If the intersection is an open set for the first time, this set should be homeomorphic to an open part of S_2, having all its points elliptic points, which is forbidden by $H = 0$. If the intersection consists in only isolated points $q \in S \cap S_2$, we can find neighborhoods of these points $\mathcal{V}(q) \subset S$ lying both inside and outside S_2, contradicting hence the hypothesis. So, all (regular) minimal surfaces are unbounded, hence noncompact. We remember here that compact regular surfaces have at least one elliptic ($K > 0$) point.

If S is a regular closed minimal surface which is not a plane, the image of the Gauss map is dense in the sphere S_2. When a point moves along the surface, the normal $\boldsymbol{N}$ takes "almost" all possible orientations in $\mathbb{R}^3$. That is, for every arbitrary direction $\boldsymbol{N}_0$, there are open sets of points on S, such that the corresponding normal of these points approaches the given direction as close as we want.

We also mention another property of the minimal surfaces. If S is minimal and has no planar points ($K \neq 0$ on S), then the angle of intersection of any two curves on S and the angle of intersection of their spherical images (images through the tangent map of the Gauss map) are equal up to a sign. In terms of equation this fact reads $\forall p \in S, \forall \boldsymbol{v}, \boldsymbol{w} \in T_p S, d\boldsymbol{N}_p(\boldsymbol{v}) \cdot d\boldsymbol{N}_p(\boldsymbol{w}) = -K_p \boldsymbol{v} \cdot \boldsymbol{w}$. In terms of thin layers of fluid, this behavior of the free minimal surface means that the two variations of the gradient of pressure, when we move toward two perpendicular directions of the tangent plane, are perpendicular.

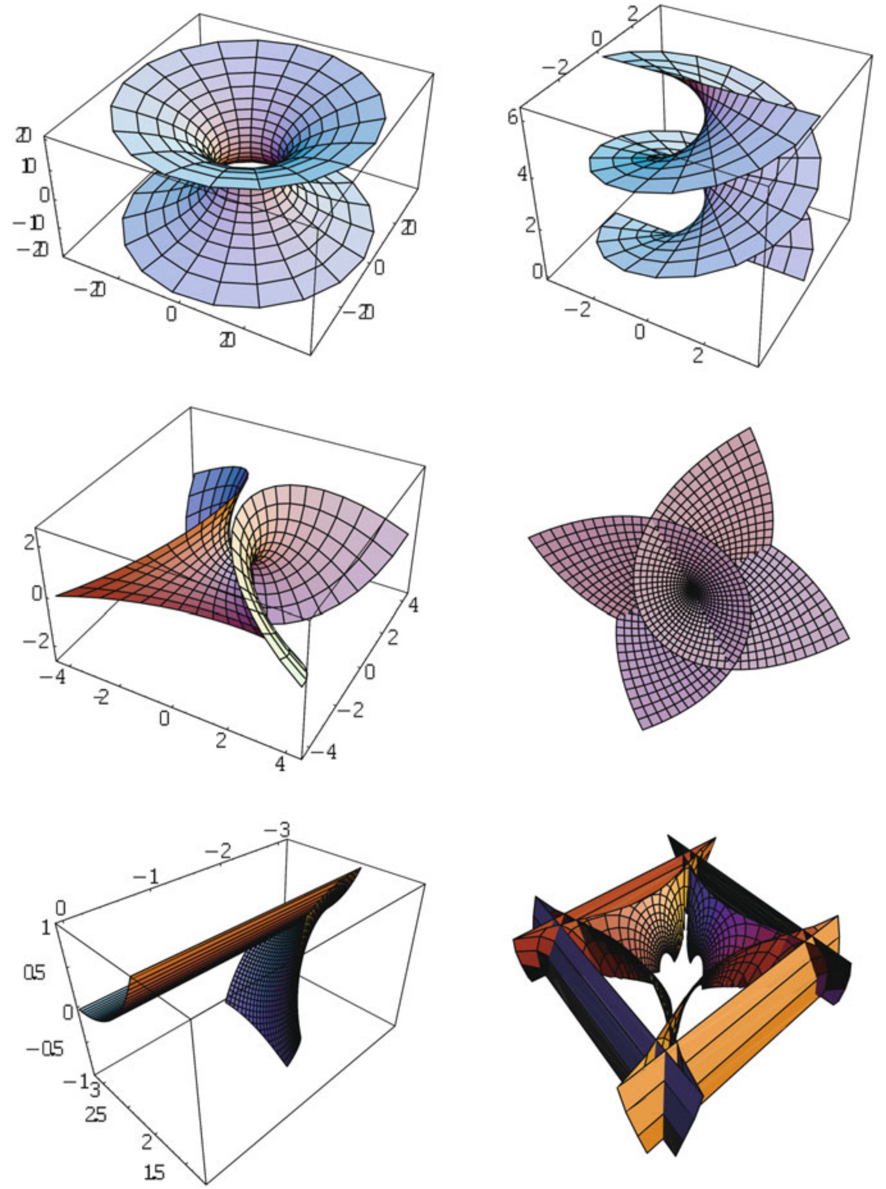

Fig. 10.7 Examples of minimal surfaces. *Upper line*: catenoid and helicoid. *Middle line*: Enneper's polynomial surface. *Lower line*: Scherk's periodical surface from complex analysis

10.4.6 General Condition for Minimal Surfaces

In the following we want to provide a general expression for the local condition $H = 0$ for a minimal surface, expressed in different systems of curvilinear coordinates. In such systems we use for the surface parameters two of the three curvilinear coordinates, and one free function (the *shape function*) depending on these two coordinates. In the cartesian case $(u, v) = (x, y)$, we have $\boldsymbol{r} = u, v, h(u, v)$ where $h(u, v)$ is the shape function. The mean curvature is

$$H = \frac{h_{uu} + h_{vv} + h_v^2 h_{uu} - 2h_u h_v h_{uv} + h_u^2 h_{vv}}{(1 + h_u^2 + h_v^2)^{\frac{3}{2}}}. \tag{10.57}$$

In cylindrical symmetry, the surface can be parameterized in cylindrical coordinates $((u, v) = (\varphi, z))$ in the form $\boldsymbol{r} = (\rho(u)\cos u, \rho(u)\sin u, v)$ with shape function $\rho(u)$. The mean curvature is

$$H = \frac{\rho^3 + 2\rho\rho_u^2 - \rho^2\rho_{uu}}{(\rho^2 + \rho_u^2)^2}. \tag{10.58}$$

In spherical symmetry $(u, v) = (\theta, \varphi)$, the surface becomes $\boldsymbol{r} = ((R + \rho(\theta, \varphi))\sin\theta\cos\varphi, (R + \rho(\theta, \varphi))\sin\theta\sin\varphi, (R + \rho(\theta, \varphi))\cos\theta)$ and, in terms of the shape function $\rho(u, v)$, the mean curvature is

$$H = \frac{\mathcal{B} - ((R + \rho)^2 + \rho_\theta^2)\rho_\theta \sin\theta\cos\theta + \mathcal{C}\sin^2\theta}{2\left((R + \rho)^2 + \rho_\theta^2 + \frac{\rho_\varphi^2}{\sin^2\theta}\right)^2 \sin^2\theta}, \tag{10.59}$$

where

$$\begin{aligned}
\mathcal{B} &= 3R\rho_\varphi^2 + 3\rho\rho_\varphi^2 - R^2\rho_{\varphi\varphi} - 2R\rho\rho_{\varphi\varphi} - \rho^2\rho_{\varphi\varphi} - \rho_\theta^2\rho_{\varphi\varphi} \\
&\quad + 2\rho_\theta\rho_\varphi\rho_{\theta\varphi} - \rho_{\theta\theta}\rho_\varphi^2 - 2\rho_\theta\rho_\varphi^2\cot\theta, \\
\mathcal{C} &= (R + \rho)(2(R + \rho)^2 + 3\rho_\theta^2 - R\rho_{\theta\theta} - \rho\rho_{\theta\theta}).
\end{aligned}$$

If the shape function is small compared to the radius, $\rho \ll R$, we have the following hierarchy of orders of smallness in ρ/R for H

$$\begin{aligned}
\mathcal{O}(0) &= -\frac{1}{R}, \\
\mathcal{O}(1) &= \frac{\rho}{R^2} + \frac{1}{2R^2}\triangle_\Omega\rho, \\
\mathcal{O}(2) &= -\frac{\rho^2}{R^3} + \frac{\rho_\theta^2}{2R^3} - \frac{\rho\rho_\theta}{R^3} + \frac{\rho_\varphi^2}{2R^3\sin^2\theta} - \frac{\rho\rho_{\varphi\varphi}}{R^3\sin^2\theta} - \frac{\rho\rho_\theta\cot\theta}{R^3},
\end{aligned} \tag{10.60}$$

where

$$\triangle_{\Omega} = \rho_{\theta\theta} + \cot\theta \rho_{\theta} + \frac{\rho_{\varphi\varphi}}{\sin^2\theta}$$

is the angular part of the Laplace operator in spherical coordinates.

In all these examples, the expression of H is very close to the Laplacian of the free function describing the surface in the corresponding curvilinear coordinates. If the curvilinear coordinates are isothermal, the mean curvature equation is precisely the Laplace equation, and this behavior is natural in view of (10.56). It is interesting to check how does the Laplacian of $\triangle\boldsymbol{r}$ reduce to the Laplacian of the shape scalar function, $\triangle h$ or $\triangle\rho$, like in the examples above. In general, orthogonal curvilinear coordinates are not isothermal, so we expect H to contain in addition to the Laplacian of the free function, also some other terms. The question is: to what extent, in some given curvilinear coordinates, we can approximate the minimal surface equation $H = 0$ and the surface pressure expression, with the Laplace equation of the curvilinear coordinates? It would be of practical application to find the approximate expression of the surface tension for surfaces that are small deviation from an isothermal, or at least orthogonally parameterized surface.

10.4.7 Surface Tension for Almost Isothermal Parametrization

We consider a thin liquid surface S, initially in "equilibrium," parameterized by isothermal coordinates, $\boldsymbol{r}_0(u, v)$ defined in an open set $(u, v) \in U$, with $E = G$, $F = 0$. Next to this surface, the pressure is the surface tension and it has the expression provided by (10.52) and (10.56)

$$P = \frac{2\sigma}{2E}|\triangle\boldsymbol{r}|.$$

We consider that some external interaction occurs (like the presence of a force field or a nonuniform change in temperature) and produces a deformation of this surface. This deformation, or variation, is defined as a new parameterized surface $\boldsymbol{r}(u, v) = \boldsymbol{r}_0(u, v) + \epsilon\boldsymbol{\rho}(u, v)$. We consider this new surface to be a small variation of the original isothermal one if $\epsilon \max_{(u,v)\in U}\{|\boldsymbol{\rho}|\} \ll |\boldsymbol{r}_0|$. In the following we denote any quantity that refers to the original isothermal surface with a zero label, like for example $\boldsymbol{r}_{0u} \cdot \boldsymbol{r}_{0u} = \boldsymbol{r}_{0v} \cdot \boldsymbol{r}_{0v} = E_0 = G_0$ and $\boldsymbol{r}_{0u} \cdot \boldsymbol{r}_{0v} = F_0 = 0$. The surface tension expression

$$P(u, v, \epsilon, \rho(u, v)) = \sigma\frac{Eg - 2fF + Ge}{(EG - F^2)} \tag{10.61}$$

reduces in the limit $\lim_{\epsilon=0} P = P_0 = 2\sigma H_0 = \sigma(g_0 + e_0)/E_0$. For small variations we work in the first linear approximation of ϵ and we neglect $\mathcal{O}(\epsilon^2)$.

In the following we choose a *normal variation* $\boldsymbol{\rho} = \rho(u, v)\boldsymbol{N}_0(u, v)$. There is no loss of generality in this choice, because any arbitrary deformation can be reduced to a normal one by a reparameterization. Besides, in the case of orthogonal curvilinear coordinates, the deformed surface is always normal, since the deformation occurs along the orthogonal parameter. For example in the spherical case, $\boldsymbol{r}_0 = (R \sin u \cos v, R \sin u \sin v, R \cos u)$ with $R = \text{const.}$, the usual variation of the coordinate surface has the form $\boldsymbol{\rho} = \epsilon\rho(u, v)(\sin u \cos v, \sin u \sin v, \cos u)$, which means $\boldsymbol{r}_0 \perp \boldsymbol{\rho}$, and consequently the variation is normal.

Since we are interested in surfaces close to the isothermal one, we follow the calculations just in the first order in ϵ. From the definition of the normal variation, and from $E_0 = G_0$, $F_0 = 0$, we obtain

$$\begin{aligned} \boldsymbol{r}_u &= \boldsymbol{r}_{0u} + \epsilon\rho_u \boldsymbol{N}_0 + \epsilon\rho\boldsymbol{N}_{0u}, \\ \boldsymbol{r}_v &= \boldsymbol{r}_{0v} + \epsilon\rho_v \boldsymbol{N}_0 + \epsilon\rho\boldsymbol{N}_{0v}, \end{aligned}$$

and consequently we have the coefficients of the first fundamental form of the deformed surface in the first order in ϵ

$$E = E_0 - 2\epsilon\rho e_0, \quad G = E_0 - 2\epsilon\rho g_0, \quad F = -2\epsilon\rho f_0. \tag{10.62}$$

We notice that it is impossible to have, in general, a surface and its infinitesimal normal variation, simultaneously isothermal, $F_0 = F = 0$. This is possible in the linear approximation only if $f_0 = 0$. The unit normal has the form

$$\boldsymbol{N} = \boldsymbol{N}_0 - \frac{\epsilon}{E_0}(\rho_u \boldsymbol{r}_{0u} + \rho_v \boldsymbol{r}_{0v}) + \mathcal{O}(\epsilon^2).$$

The second fundamental form has the coefficients

$$\begin{aligned} e &= e_0 + \epsilon\left(\rho_{uu} - \tfrac{1}{2E_0}(\rho_u E_{0u} - \rho_v E_{0v}) - \tfrac{\rho}{E_0}(e_0^2 + f_0^2)\right), \\ g &= g_0 + \epsilon\left(\rho_{vv} - \tfrac{1}{2E_0}(\rho_v E_{0v} - \rho_u E_{0u}) - \tfrac{\rho}{E_0}(g_0^2 + f_0^2)\right), \\ f &= f_0 + \epsilon\left(\rho_{uv} - \tfrac{1}{2E_0}(\rho_u E_{0v} + \rho_v E_{0u}) - \tfrac{\rho f_0}{E_0}(e_0 + g_0)\right). \end{aligned}$$

By introducing all these coefficients in (10.40), we obtain

$$H = \frac{e_0 + g_0}{2E_0} + \epsilon\frac{\rho(e_0^2 + g_0^2)}{2E_0^2} + \epsilon\frac{\Delta\rho}{2E_0} + \mathcal{O}(\epsilon^2), \tag{10.63}$$

which describes the mean curvature of the infinitesimal normal variation of an isothermal surface in the linear approximation. This form is a linear operator in ρ with variable coefficients, and the surface tension may be written as

$$P_S = -2\sigma(\mathcal{A} + \epsilon\mathcal{B}\rho + \epsilon\mathcal{C}\triangle\rho) + \mathcal{O}(\epsilon^2), \tag{10.64}$$

where the three variable coefficients $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ can be identified from (10.63).

Such a simple form as (10.63) for the surface pressure is not always available. In practical situations one uses orthogonal curvilinear coordinates which are not necessarily isothermal, mainly because $E_0 \neq G_0$. In the following we obtain a similar first-order approximation of the mean curvature for a normal deviation starting from an orthogonal parameterized surface.

Definition 62 Three families of smooth (of rank 3) surfaces are a *triply orthogonal* system in an open $U \subset \mathbb{R}^3$ if one unique surface of each family passes through any point $P \in U$, and if the three surfaces that pass through each point $p \in U$ are pairwise orthogonal.

The second constraint means that $\boldsymbol{r}_u$, $\boldsymbol{r}_v$, and $\boldsymbol{r}_w$ are always orthogonal. The curves of intersection of any pair of surfaces from different system are lines of curvature in each of the respective surfaces, i.e., the intersection lines are principal directions. The traditional 12 systems of curvilinear coordinates are the examples (cartesian, cylindric, spherical, elliptic, parabolic, bowls, etc.). In the case of orthogonal parametrization, the coefficients of the first fundamental form are similar to (10.62). The normal is different

$$\boldsymbol{N} = \boldsymbol{N}_0 - \epsilon\begin{pmatrix} \rho_u \boldsymbol{r}_{0u} \\ \rho_v \boldsymbol{r}_{0v} \end{pmatrix}.$$

The coefficients of the second fundamental form are different

$$e = e_0 + \epsilon\left(\rho_{uu} - \frac{1}{2E_0G_0}(\rho_u E_{0u} G_0 - \rho_v E_{0v} E_0) - \rho\frac{e_0^2 G_0 + f_0^2 E_0}{E_0 G_0}\right) + \mathcal{O}(\epsilon^2),$$

$$g = g_0 + \epsilon\left(\rho_{vv} - \frac{1}{2E_0G_0}(\rho_v G_{0v} E_0 - \rho_u G_{0u} G_0) - \rho\frac{f_0^2 G_0 + g_0^2 E_0}{E_0 G_0}\right) + \mathcal{O}(\epsilon^2),$$

$$f = f_0 + \epsilon\left(\rho_{uv} - \frac{1}{2E_0G_0}(\rho_v G_{0u} E_0 + \rho_u E_{0v} G_0) - \rho f_0\frac{e_0 G_0 + g_0 E_0}{E_0 G_0}\right) + \mathcal{O}(\epsilon^2).$$

In the end, the form for the mean curvature of the deformed surface in the first order of approximation is

$$H = \frac{e_0 G - 0 + g_0 E_0}{2E_0G_0} + \epsilon\left(\frac{G_0\rho_{uu} + E_0\rho_{vv}}{2E_0G_0} - \frac{\rho_u E_{0u}}{4E_0^2} - \frac{\rho_v G_{0v}}{4G_0^2}\right.$$
$$\left. + \frac{\rho_u G_{0u} + \rho_v E_{0v}}{4E_0G_0} + \frac{\rho g_0^2}{2G_0^2} + \frac{\rho e_0^2}{2E_0^2} + \frac{3\rho f_0^2}{2E_0G_0}\right) + \mathcal{O}(\epsilon^2).$$

It is easy to check that (10.65) reduces to the particular cases discussed above for spherical, cartesian, etc., coordinates. Still this expression is a linear second-order differential operator acting on ρ with variable coefficients.

10.5 Special Fluids

There are important differences between Newtonian (traditional or small molecule) fluids obeying Newtonian fluid dynamics and "polymeric" (macromolecular) fluids. The features of the macromolecular architecture influence the flow behavior. Polymeric fluids have molecular weights several orders of magnitude higher than normal fluids, and besides, this molecular weight is not uniformly distributed in the mass of the fluid. In addition, the polymers have a huge number of metastable configurations at equilibrium, and consequently the flow is altered in time and space by the local stretching and alignment of macromolecules. In high concentration polymers (melts), the macromolecules can form entanglement networks, and the number of entanglement junctions can change with the flow conditions. In [20] there is a detailed discussion of such types of flow. The most important property of macromolecular fluids is the non-Newtonian viscosity, i.e., the fact that the viscosity of the fluid changes with the shear rate. In *viscoplastic* (or *dilatant*) fluids, there is present the phenomenon of shear thickening, namely the viscosity of the fluid increases with the shear rate. Such fluids will not flow at all unless acted on by at least some critical shear stress, called *yield stress*. In some other polymeric fluids, we have the phenomenon of elasticity and memory of the flow, called the *viscoelastic* property. After the external pressure is removed, the fluid begins retreating in the direction from which it came. The fluid, however, does not return all the way to its original position (like an ideal rubber band for example), since its temporary entanglement junctions have a finite lifetime, and they are continuously being created and destroyed by the flow. Such a viscoelastic fluid behaves like having a *fading* memory.

10.6 Representation Theorems in Fluid Dynamics

10.6.1 Helmholtz Decomposition Theorem in $\mathbb{R}^3$

Theorem 28 (Helmholtz Theorem for the Whole Space) *Any single-valued continuous vector field* $\boldsymbol{v}(\boldsymbol{r}) : \mathbb{R}^3 \to \mathbb{R}^3$ *satisfying*

$$\nabla \cdot \boldsymbol{v} \to 0, \quad \nabla \times \boldsymbol{v} \to 0, \quad \textit{when } r \to \infty,$$
$$\exists \epsilon > 0, |\boldsymbol{v}| < \frac{1}{r^{1+\epsilon}}, \quad \textit{when } \ r \to \infty,$$

may be written as the sum of an irrotational (or conservative or lamellar) part and a solenoidal part

$$\boldsymbol{v} = \nabla \Phi + \nabla \times \boldsymbol{A},$$

such that

$$\Phi(\boldsymbol{r}) = \frac{1}{4\pi}\iiint_{\mathbb{R}^3} \frac{\nabla' \cdot \boldsymbol{v}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} d^3\boldsymbol{r}'$$
$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{4\pi}\iiint_{\mathbb{R}^3} \frac{\nabla' \times \boldsymbol{v}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} d^3\boldsymbol{r}' \;\; \textit{and} \;\; \nabla \cdot \boldsymbol{A} = 0.$$

For a proof of the theorem see [5, 6, 21].

Usually, the Helmholtz theorem is formulated as "source plus condition at infinity" problem. Given the source fields $\rho(\boldsymbol{r})$, $\boldsymbol{j}(\boldsymbol{r})$ defined on $\mathbb{R}^3$ with the regularity propriety at $|\boldsymbol{r}| \to \infty$, ρ, $\boldsymbol{j} \to 0$, and the vector field equation

$$\nabla \cdot \boldsymbol{v} = \rho, \;\; \nabla \times \boldsymbol{v} = \boldsymbol{j},$$

there is a unique solution for the unknown vector field $\boldsymbol{v} = \nabla\Phi + \nabla \times \boldsymbol{A}$, with the *potentials*$\rho$, $\boldsymbol{A}$ solutions of the equations

$$\triangle\Phi = \rho, \;\; \triangle\boldsymbol{A} = \boldsymbol{j}, \;\; \nabla \cdot \boldsymbol{A} = 0.$$

Also, the potentials are not uniquely determined up to their gauge transformations. Namely, Φ is defined modulo addition of an arbitrary harmonic function $\Phi \to \Phi + f(\boldsymbol{r})$, $\triangle f = 0$, and $\boldsymbol{A}$ is defined modulo addition of the gradient of an arbitrary function $\boldsymbol{A} \to \boldsymbol{A} + \nabla g(\boldsymbol{r})$.

The Helmholtz theorem (Theorem 28) can be extended by using a Neumann–Debye decomposition [21]. Instead of using one scalar Φ and one vector function $\boldsymbol{A}$ plus the divergence constraint (i.e., $1 + 3 - 1 = 3$ degrees of freedom), we can use three scalar functions. If the field $\boldsymbol{v}$ is continuous and single-valued, and it fulfills the same regularity conditions at ∞ as in the Helmholtz theorem, we have the following decomposition

$$\boldsymbol{v} = \nabla\Phi + \nabla \times (\boldsymbol{r}\Psi) + \nabla \times (\nabla \times \boldsymbol{r}\chi) = \nabla\Phi + \boldsymbol{L}\Psi + \boldsymbol{Q}\chi, \tag{10.65}$$

where the operators are $\boldsymbol{L} = -\boldsymbol{r} \times \nabla$ (angular momentum) and $\boldsymbol{Q} = \nabla \times \boldsymbol{L}$. The functions Ψ, χ are the so-called *Debye potentials* and are related to the operators by the equations

$$\Phi(\boldsymbol{r}) = \frac{1}{4\pi}\iiint_{\mathbb{R}^3} \frac{\nabla' \cdot \boldsymbol{v} d^3\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}$$
$$\Psi(\boldsymbol{r}) = \frac{1}{4\pi}\iiint_{\mathbb{R}^3} \boldsymbol{r}' \cdot (\nabla' \times \boldsymbol{v}) \ln(1 - \boldsymbol{r}' \cdot \boldsymbol{r}) d^3\boldsymbol{r}'$$
$$\chi(\boldsymbol{r}) = \frac{1}{16\pi^2}\iiint_{\mathbb{R}^3} d^3\boldsymbol{r}' \ln(1 - \boldsymbol{r} \cdot \boldsymbol{r}')(\boldsymbol{r}' \cdot \nabla') \iiint_{\mathbb{R}^3} \frac{\nabla'' \cdot \boldsymbol{v}(\boldsymbol{r}'')}{|\boldsymbol{r}'-\boldsymbol{r}''|} d^3\boldsymbol{r}''$$
$$- \frac{1}{4\pi}\iiint_{\mathbb{R}^3} \ln(1 - \boldsymbol{r} \cdot \boldsymbol{r}')\boldsymbol{r}' \cdot \boldsymbol{v} d^3\boldsymbol{r}'.$$

The operators involved in this generalized Helmholtz theorem fulfill interesting algebraic relations. The angular momentum operator is closed under commutation relation and spans the $su(1,1)$ Lie algebra by $[L_i, L_j] = \mathcal{E}_{ijk} L_k$. The operator $\boldsymbol{Q}$ is a left ideal of this algebra $[L_i, Q_J] = \mathcal{E}_{ijk} Q_k$, and the Laplace operator is the Casimir element of this algebra $[\boldsymbol{L}, \triangle] = [\boldsymbol{Q}, \triangle] = 0$.

A very useful version of the Neumann–Debye (10.65) is related to the linear Navier–Stokes fluid dynamics equation in absence of external forces

$$\frac{\partial \boldsymbol{V}}{\partial t} = -\frac{1}{\rho} \nabla P - \nu \nabla \times (\nabla \times \boldsymbol{V}), \tag{10.66}$$

where the fluid velocity field $\boldsymbol{V}(\boldsymbol{r}, t)$ is a smooth nonsingular time-dependent (Euclidean) vector field defined on a domain $D \subset \mathbb{R}^3$ with values in $T\mathbb{R}^3$; ρ and ν are positive constants, density and viscosity, respectively, and $P(\boldsymbol{r}, t)$ is the pressure scalar field, also defined on $D \subset \mathbb{R}^3$. If we ask for the velocity field to be divergence free on D, i.e., to have no net sources of fluid,

$$\nabla \cdot \boldsymbol{V} = 0, \tag{10.67}$$

it is possible to apply the representation theorem (10.65) for solutions of (10.67) in D. We have

Theorem 29 *Let us define a vector field*

$$\boldsymbol{V} = \nabla \times (\boldsymbol{Q}\beta) + \nabla \times \nabla \times (\boldsymbol{Q}b) + \nabla c$$

and the scalar field

$$P = -\rho \frac{\partial c}{\partial t},$$

where $\boldsymbol{Q}(\boldsymbol{r}, t)$ is an arbitrary smooth vector field on $D \times \mathbb{R}$, and β, b, c are arbitrary smooth scalar fields depending on $(\boldsymbol{r}, t) \in D \times \mathbb{R}$. Then $\boldsymbol{V}$, P defined above are solutions for the Navier–Stokes equations (10.66) *in the divergence-free condition* (10.67) *if the following conditions are fulfilled on $D \times \mathbb{R}$*

$$\nu \triangle \beta = \frac{\partial \beta}{\partial t}, \quad \nu \triangle b = \frac{\partial b}{\partial t}, \quad \triangle c = 0, \quad \boldsymbol{Q} = C_0 \boldsymbol{r},$$

with C_0 an arbitrary constant.

The proof of the theorem is by direct calculation. Details and applications can be found in [22, 23].

An interesting version of the Helmholtz theorem in a domain $\mathcal{D}$ with boundary $\partial\mathcal{D} \neq \emptyset$ is presented in Chorin and Marsden's book [24], under the name of Helmholtz–Hodge theorem. In this formalism, a vector field $\boldsymbol{v}$ is decomposed into a potential field $\nabla\Phi$ and an incompressible vector field $\boldsymbol{u}$, $div\,\boldsymbol{u} = 0$ which is parallel to the boundary of $\mathcal{D}$, $(\boldsymbol{u} \cdot \boldsymbol{N})_{\partial\mathcal{D}} = 0$. The existence of the Helmholtz–Hodge decomposition is guaranteed by the existence of a solution to the Neumann-associated problem for Φ. Uniqueness is guarantied by the fact that the two terms of the decomposition are orthogonal in an average taken through an integration over $\mathcal{D}$. Indeed, $\int_{\mathcal{D}} \boldsymbol{u} \cdot grad\Phi = 0$ through Gauss formula and because of the properties of $\boldsymbol{u}$. Consequently, any two distinct Helmholtz–Hodge decompositions must have same $\boldsymbol{u}$ and same Φ, up to an additive constant. In this form the theorem is more adapted to hydrodynamics problems where one has incompressible fluid in a bounded region. Because the velocity is divergence free and vanishes on the boundary, the Navier–Stokes equation can be projected into a divergence-free component which does not contain the pressure, i.e., the gradient term.

Hydrodynamics is perhaps one of the best-studied fields of application of nonlinear equations, waves, and their solutions, and we have barely touched the subject. A very comprehensive and extended treatment of hydrodynamics in general, toward the nonlinear problems open at the time when the book was written, is [3]. The book is dense in solved examples and problems in almost any field of basic hydrodynamics. The book goes hand in hand with mathematical physics text books like [25, 26] or in the same style. The calculations are detailed and comprehensive, very much relying on expansions in series of functions and independent mode analysis. A book which complements Lamb's book on hydrodynamics and is written in the same *grand* style is [5], especially for magnetohydrodynamics and fluid and plasma stability problems. Another comprehensive book on hydrodynamics, where very special problems are solved in very original ways, is [14]. If the reader is more concerned about mathematical rigorousness, toward functional analysis and operator approach in hydrodynamics, a good lecture would be [4]. More restrictive topics, yet presented on a fundamental basis and mathematical rigorous, are approached in [2, 6, 27]. In this last mentioned spectrum, more oriented toward mathematics is the attractive and clear book of Chorin and Marsden [24], or more toward applied mathematics [11]. For specific topics on waves in general and nonlinear waves in fluids, the reader may consider to consult [28, 29].

10.6.2 Decomposition Formula for Transversal Isotropic Vector Fields

This special decomposition works for axially and/or translational symmetric vector fields. It is particularly useful in convective hydrodynamics stability calculations, and in general in physical systems exhibiting transport and transformation processes. It is also useful in the dynamics of viscous drops submerged in viscous fluids [30]. This decomposition formula was introduced for a particular axisymmetric field in [5, Sect. 61], and later, for spherical surfaces and even for more general situations in [31]. The big advantage of this decomposition consists in the fact that the vector field $\boldsymbol{v}$ can be expressed as function of the radial component v_r, the divergence $div\boldsymbol{v}$ and the radial component of the vorticity, ω_r, where $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$. When the flow is incompressible, and the velocity field has spherical symmetry, this decomposition becomes very useful because of its simplicity. Moreover, for solenoidal fields, like vorticity, this divergence term is also canceled and the vector field can be constructed from the radial components only.

In general, the formula works for any curvilinear orthogonal system of coordinates of the form (r, q_1, q_2) with a local basis $\{\boldsymbol{e}_r, \boldsymbol{q}_1, \boldsymbol{q}_2\}$, where $r =$ const. describes closed coordinate surface homotopic to the sphere S_2. At the same time, we can expand any vector field $\boldsymbol{v}(r, q_1, q_2)$ in an orthogonal basis of functions defined on the compact surface $r =$ const. This surface S, being homotopic to S_2, allows the existence of an $L_2(S)$ Hilbert space with countable basis of harmonic polynomials defined on S_2. In the case of spherical coordinates, these are the spherical harmonics $Y_{l,m}$. In the following we introduce this vector decomposition in spherical coordinates (r, θ, φ). For the calculation of components and operator action, we refer to Sect. 19.3.

Any vector field, like for example the velocity field $\boldsymbol{v}$, can be decomposed in its normal (radial for spherical) and parallel components

$$\boldsymbol{v} = v_r \boldsymbol{e}_r + \boldsymbol{v}_\parallel, \tag{10.68}$$

and also the gradient and Laplace operators can be decomposed in a similar way

$$\nabla_\parallel = \nabla - \boldsymbol{e}_r(\boldsymbol{e}_r \cdot \nabla) = \nabla - \boldsymbol{e}_r \frac{\partial}{\partial r}, \quad \triangle = \triangle_r(r, \partial/\partial r) + \triangle_\parallel(\theta, \partial/\partial\theta, \varphi, \partial/\partial\varphi,). \tag{10.69}$$

From vector analysis we have the formula

$$\begin{aligned} \triangle \boldsymbol{v}_\parallel &= \nabla(\nabla \cdot \boldsymbol{v}_\parallel) - \nabla \times (\nabla \times \boldsymbol{v}_\parallel) \\ &= \nabla_\parallel(\nabla \cdot \boldsymbol{v}_\parallel) - [\nabla \times (\nabla \times \boldsymbol{v}_\parallel)]_\parallel, \end{aligned} \tag{10.70}$$

where we retain on the RHS only the parallel terms (the normal terms cancel each other), because the LHS in (10.70) contains by definition only parallel terms. We have

$$\triangle \boldsymbol{v}_\parallel = \nabla_\parallel(\nabla \cdot \boldsymbol{v}) - \nabla_\parallel \mathcal{D} v_r - [\nabla \times (\nabla \times \boldsymbol{v}_\parallel)]_\parallel, \tag{10.71}$$

where $\mathcal{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2)$, i.e., the radial part of the div operator in the curvilinear coordinates.

We can expand the vector field $\boldsymbol{v}(r, \theta, \varphi)$ in spherical harmonics. We have

$$\boldsymbol{v} = v_r \boldsymbol{e}_r + \boldsymbol{v}_\parallel = \sum_{l,m} \boldsymbol{v}_{l,m}(r, t) Y_{l,m}(\theta, \varphi), \tag{10.72}$$

where $\boldsymbol{v}_{l,m} = \boldsymbol{e}_r v_{r,lm} + \boldsymbol{v}_{\parallel,lm}$. With these notations we obtain

$$\triangle \boldsymbol{v}_\parallel = \triangle_r \boldsymbol{v}_\parallel + \triangle_\parallel \boldsymbol{v}_\parallel = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} \boldsymbol{v}_\parallel\right) + \triangle_\Omega \boldsymbol{v}_\parallel, \tag{10.73}$$

where Ω is the angular (parallel) part of the Laplace operator (see Sect. 19.3). For any l, m component we can write

$$\triangle \boldsymbol{v}_{\parallel,lm} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\boldsymbol{v}_{\parallel,lm}\right) - \frac{l(l+1)}{r^2}\boldsymbol{v}_{\parallel,lm}, \tag{10.74}$$

accordingly to the action of the angular Laplacian operator on spherical harmonics. It results

$$\boldsymbol{v}_{\parallel,lm} = \frac{r^2}{l(l+1)}(\triangle_r \boldsymbol{v}_\parallel - \triangle \boldsymbol{v}_\parallel). \tag{10.75}$$

In the following equations, we skip the labels l, m, but we refer to the l, m component, unless otherwise stated. From (10.68), (10.71), and (10.75), we have the following preliminary form for the decomposition

$$\boldsymbol{v} = v_r \boldsymbol{e}_r + \frac{r^2}{l(l+1)}\left(\triangle_r \boldsymbol{v}_\parallel + \nabla_\parallel \mathcal{D} v_r - \nabla_\parallel(\nabla \cdot \boldsymbol{v}) + [\nabla \times (\nabla \times \boldsymbol{v}_\parallel)]_\parallel\right). \tag{10.76}$$

In the following, we focus on the first and fourth term in the RHS parenthesis in (10.76). We have

$$\triangle_r \boldsymbol{v}_\parallel + [\nabla \times (\nabla \times \boldsymbol{v}_\parallel)]_\parallel = \triangle_r \boldsymbol{v}_\parallel + (\nabla \times \boldsymbol{\omega})_\parallel - [\nabla \times (\nabla \times u_r \boldsymbol{e}_r)]_\parallel, \tag{10.77}$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ is the vorticity field. We also notice that $\omega_r = \boldsymbol{e}_r \cdot (\nabla_\parallel \times \boldsymbol{v}_\parallel)$. This is possible because of the relation

$$\nabla \times \boldsymbol{v} = \nabla_\parallel \times \boldsymbol{v}_\parallel + \boldsymbol{e}_r(\boldsymbol{e}_r \cdot \nabla) \times \boldsymbol{v}_\parallel - \nabla_\parallel \times \boldsymbol{e}_r v_r - \boldsymbol{e}_r(\boldsymbol{e}_r \cdot \nabla) \times \boldsymbol{e}_r v_r,$$

where all the last three terms are perpendicular on $\boldsymbol{e}_r$, hence they have only parallel components. The only normal component in the RHS of the equation above is contained the first term. We also notice the identity [31, (H1.12)]

$$\nabla \times \boldsymbol{v} = \boldsymbol{e}_r \omega_r + \boldsymbol{e}_r \times \left(\frac{1}{r}\frac{\partial}{\partial r}(r\boldsymbol{v}_\parallel) - \nabla_\parallel u_r\right). \tag{10.78}$$

From (10.77) and (10.78), we have

$$\begin{aligned} &\triangle_r \boldsymbol{v}_\parallel + (\boldsymbol{e}_r(\boldsymbol{e}_r \cdot (\nabla_\parallel \times \boldsymbol{\omega}_\parallel))) + \boldsymbol{e}_r\left[\tfrac{1}{r}\tfrac{\partial}{\partial r}(r\boldsymbol{\omega}_\parallel) - \nabla_\parallel \omega_r\right]_\parallel - [\nabla \times (\nabla \times v_r \boldsymbol{e}_r)]_\parallel \\ &= \triangle_r \boldsymbol{v}_\parallel + \boldsymbol{e}_r \times \left[\tfrac{1}{r}\tfrac{\partial}{\partial r}(r\boldsymbol{\omega}_\parallel)\right] - [\nabla \times (\nabla \times v_r \boldsymbol{e}_r)]_\parallel - \boldsymbol{e}_r \times \nabla_\parallel \omega_r \\ &= -\boldsymbol{e}_r \times \nabla_\parallel \omega_r. \end{aligned} \tag{10.79}$$

The last equality holds because the first three terms in the second line of (10.79) cancel each other, as one can check by direct calculations in spherical coordinates components. Consequently we have

$$\triangle_r \boldsymbol{v}_\parallel + [\nabla \times (\nabla \times \boldsymbol{v}_\parallel)]_\parallel = -\boldsymbol{e}_r \times \nabla_\parallel \omega_r. \tag{10.80}$$

From (10.76) to (10.80), we can write the final decomposition formula

$$\boldsymbol{v} = v_r \boldsymbol{e}_r + \frac{r^2}{l(l+1)}\left[\nabla_\parallel \mathcal{D} v_r - \nabla_\parallel(\nabla \cdot \boldsymbol{v}) - \boldsymbol{e}_r \times \nabla_\parallel \omega_r\right]. \tag{10.81}$$

That is, we can express the velocity field function of its radial component, and function of the radial component of the vorticity and the divergence of velocity.

10.6.3 Solenoidal–Toroidal Decomposition Formulas

Another version of the above decomposition formula can be obtained for an axisymmetric solenoidal vector field. We use a cylindrical system of coordinates (r_c, φ, z), and the axis of symmetry is taken in the z-direction. In this case the field can be expressed as a superposition of a poloidal and toroidal field in terms of two azimuth-independent scalar functions $U(r_c, z)$ and $V(r_c, z)$ [5]

$$\boldsymbol{v} = -r_c \frac{\partial U}{\partial z} \boldsymbol{e}_{r_c} + r_c V \boldsymbol{e}_\varphi + \frac{1}{r_c} \frac{\partial}{\partial r_c}(r_c^2 U)\boldsymbol{e}_z. \tag{10.82}$$

An equivalent and unified way of writing (10.82) and the curl of velocity is

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{e}_z \times \boldsymbol{r} V + \nabla \times (\boldsymbol{e}_z \times \boldsymbol{r} U), \quad \text{and} \\ \nabla \times \boldsymbol{u} &= -\boldsymbol{e}_z \times \boldsymbol{r} \triangle_5 U + \nabla \times (\boldsymbol{e}_z \times \boldsymbol{r} V), \end{aligned} \tag{10.83}$$

where $\triangle_5$ is the Laplacian operator in a five-dimensional Euclidean space in cylindrical coordinates. According to Chandrasekhar [5, Sect. 61], there is a particular advantage of this representation in that no matter of how many times one applies curl operator to the velocity and vorticity fields, the representations in (10.83) have the same type of expression.

In spherical coordinates, the Chandrasekhar poloidal–toroidal decomposition of an axisymmetric solenoidal field has the form

$$\boldsymbol{u} = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta U)\boldsymbol{e}_r - \frac{\sin\theta}{r}\frac{\partial}{\partial r}(r^2 U)\boldsymbol{e}_\theta + r\sin\theta V \boldsymbol{e}_\varphi. \tag{10.84}$$

The interpretation of the scalars U, V is straightforward. Since fields derived only from the scalar U have components only in the meridional planes, it results that the U field is nothing but the Stokes' stream function for motions in these planes (meridional motions). The field V defines motions which are entirely rotational. Another advantage is this types of representations reciprocity: a poloidal field has toroidal vorticity and, conversely, a toroidal field has poloidal vorticity.

10.7 Problems

1. In Sect. 10.3 we conjecture (10.30) and (10.32) by using the physical intuition that particles contained in particle surface move together with the surface, and never tangent to it. Prove this affirmation on a more geometrical background. Hint: use the integral formulas in Sect. 6.5.
2. *Monge's potential representation*: show that an arbitrary differentiable vector field $\boldsymbol{v}$ can be always represented as

$$\boldsymbol{v} = \nabla\varphi + \psi\nabla\chi,$$

where the first term on the RHS is irrotational field, and the second term has the property of being perpendicular to its curl, $(\psi\nabla\chi)\cdot(\nabla\times\psi\nabla\chi)$. Such fields are called *complex lamellar* fields [6].

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