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The rotation of the non-rigid Earth at the second order II. The Poincaré model: non–singular complex canonical variables and Poisson terms

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ABSTRACT

We develop a Hamiltonian analytical theory for the rotation of a Poincaré Earth model (rigid mantle and liquid core) at the second order with respect to the lunisolar potential and moving ecliptic term. Since the Andoyer variables considered in the first order solution present virtual singularities, i.e., vanishing divisors, we introduce a set of non-singular complex canonical variables. This choice allows applying the Hori canonical perturbation method in a standard way. We derive analytical expressions for the first and second order solution of the precession and nutation of the angular momentum axis (Poisson terms).

Contrary to first order theories, there is a part of the Poisson terms that does depend on the Earth structure. The resulting numerical amplitudes, not incorporated in

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the International Astronomical Union nutation standard, are not negligible considering current accuracies. They are at the microarcsecond level for a few terms, with a very significant contribution in obliquity of about forty microarcseconds for the nutation argument with period −6798.38 days.

The structure dependent amplitudes present a large amplification with respect to the rigid model due to the fluid core resonance. The features of such resonance, however, are different from those found in first order solutions. The most prominent is that it does not depend on the second order nutation argument directly, but on the combination of first order arguments generating it. It entails that some first order approaches, like those based on the transfer function, cannot be applied to obtain the second order contributions.

Keywords: Earth — ephemeris — reference systems — methods: analytical — celestial mechanics

1. INTRODUCTION

Present applications of, among others, astronomy, geodesy, and navigation require an increase in the accuracy of the transformation between terrestrial and celestial reference systems. It entails that the models of the rotation of the Earth must incorporate theories of precession and nutation at the second order. As a matter of fact, the need of a prompt improvement Earth rotation theory has been reported recently by the International Astronomical Union (IAU)/International Association of Geodesy (IAG) Joint Working Group (JWG) on Theory of Earth Rotation and Validation (Ferrándiz et al. 2020).

Among the different kind of second order effects to be formulated (Escapa et al. 2020), the most challenging are those related to obtaining a more precise solution of the equations of the rotational motion. Namely, since it is not possible to obtain an exact solution of those equations, one tries to determine an approximate formal solution in powers of an small parameter ε. Roughly speaking, this parameter reflects the disturbing actions, like that of the external perturbers, on the torque-free motion (or a leading part of it) of the Earth.

This procedure leads to a sequence of first, second, etc. order solutions, accordingly to the degree of the polynomial in ε employed in the approximation. In Celestial Mechanics, it belongs to the realm of Perturbation Theories (e.g., Ferraz-Mello 2007), early developed by Delaunay, Bohlin, Lindstedt, Poincaré, Von Zeipel, etc. and improved around the mid-twentieth century with the introduction of Lie series (e.g., Hori 1966).

In the context of Earth rotation studies, the paradigmatic application of those techniques is found in the Hamiltonian theories pioneered by Kinoshita (1977) for a rigid model. In his theory there are already incorporated some second order contributions in the above sense of perturbation theories (later referred to as spin-spin coupling; nutation-nutation coupling; crossed-nutation effect, etc.). This task was refined in Kinoshita & Souchay (1990) and updated in the rigid model REN2000 (Souchay et al. 1999), with a targeted level of truncature of 0.1 μas (microarcsecond).

The extension of the rigid theory at the second order was finally completed in Getino et al. (2010), who removed the main simplification assumed in REN2000, restricted just to the angular momentum axis, and incorporated the effects of the Earth structure on the formulation. As a consequence some new second order contributions emerged above the 0.1 μas level threshold (Getino et al. 2010, Tables 5 and 7). They had a double origin: some contributions are related to the motion of the figure axis relative to the angular momentum axis (Oppolzer terms), entirely due to the Earth structure, hence absent in REN2000. The other ones affect to the angular momentum axis (Poisson terms) and must be added to the structure independent part computed in REN2000. From a qualitative point of view these last terms are very relevant because they limit the scope of the widespread affirmation that the motion of the angular momentum axis is independent of the Earth structure (e.g., Moritz & Mueller 1987, Chapter 3). That affirmation must be just circumscribed to the first order, as recognized firstly in Ferrándiz et al. (2004) for the precession of a non-rigid Earth.

The situation for non-rigid Earth models is more primitive to our knowledge. There is no complete theory at the second order analogous, for example, to that of Getino et al. (2010) for the rigid

Earth. That is an indication of the huge complication involved in extending a theory of the rotation of the non-rigid Earth at the second order. Indeed, there have been some preliminary and partial computations like those presented in Getino & Ferrándiz (2000); or comprehensive developments for the second order precession of a Poincaré and two-layer Earth models (Ferrándiz et al. 2004, Baenas et al. 2017); but no study, up to now, has developed a full analytical theory.

The case of the current IAU nutation standard (IAU 2000A nutation), based on the non-rigid Earth model MHB2000 (Mathews et al. 2002), requires further examination. That model developed a series for nutation containing 1365 terms of lunisolar and planetary origins. Their amplitudes were obtained applying the transfer function by Mathews et al. (2002) to each amplitude of the spectral component decomposition of the figure axis nutation, as derived from REN2000 rigid series [1] (Souchay et al. 1999). That transfer function was derived in the framework developed by Sasao et al. (1980), conveniently extended to tackle a more complex Earth model (three-layer, electromagnetic and viscous couplings, mantle anelasticity, etc.). In regard to the second order contributions, MHB2000 (Mathews et al. 2002) has no other second order terms, in the sense considered in this investigation, than those inherited from the REN2000 rigid model (Souchay et al. 1999).

The limitations of IAU 2000A nutation relative to the second order terms were recognized from the times of its adoption in Getino & Ferrándiz (2000) —see also Getino et al. (2010, Introduction)—, and have been presented in the final reports of the sub-working groups on Precession/Nutation of the IAU/IAG JWG on Theory of Earth Rotation (e.g., Escapa & Getino 2015) and of the IAU/IAG JWG on Theory of Earth Rotation and Validation (e.g., Escapa et al. 2019). Those difficulties have been raised neatly in Escapa et al. (2020), showing that IAU 2000A nutation (Mathews et al. 2002) modeled the second order effects in an incomplete and inconsistent way. The main points are:

[1] The origin of MHB2000 series is no clear at all. In Herring et al. (2002), it is stated that the transfer function was applied to 678 lunisolar terms and 687 planetary terms from REN2000 (Souchay et al. 1999). However, Souchay et al. (2007, Tables 2 to 5) pointed out the existence of some arguments present in MHB2000 and not in REN2000, and *viceversa*. Recently, Ferrándiz et al. (2018) found that the transfer function was not applied to the planetary terms, limiting in this way the accuracy of IAU 2000A nutation.

(1) REN2000 (Souchay et al. 1999) did not considered the influence of the Earth structure at the second order. Hence, that rigid theory lacks the part of the Poisson terms dependent on the structure and all the Oppolzer terms. Therefore, those contributions are absent in MHB2000 (Mathews et al. 2002), simply because they are not present in REN2000 (Souchay et al. 1999). Their magnitude can be relevant for non-rigid models, at the level of tens μas, due to the amplification of the fluid core.

(2) The MHB2000 transfer function was applied to the second order Poisson terms derived by REN2000 (Souchay et al. 1999). However, since those terms are independent of the structure except for a factor proportional to the squared dynamical ellipticity (H_d^2), that application is misleading. The right procedure to account for the change from a rigid to a non-rigid model in this case is to perform a rescaling of the amplitudes considering the ratio between the non-rigid and rigid squared dynamical ellipticities. This fact can lead to numerical differences at the level of μas.

(3) Even considering the structure dependent Poisson and Oppolzer second order terms for a rigid model (Getino et al. 2010, Section 4.1), the application of the MHB2000 transfer function (Mathews et al. 2002) does not provide the right second order non-rigid amplitudes. In other words, in its current formulation MHB2000 transfer function cannot be extended at the second order. Moreover, due to the intrinsic linearity of the transfer function, it is not evident that this approach can be generalized beyond first order models.

The program presented in Getino et al. (2010, Introduction) tries to fill these gaps. It aims at developing a complete analytical second order theory of the rotation of the non-rigid Earth. The difficulties of this task are enormous, mainly due to the increase in the dimensions of the phase space of the non-rigid Earth (for example, from 6 dimensions in the rigid case to 12 ones in two-layer models), and the need of generalizing first order frameworks. It requires a lot of cumbersome calculations that, even with the use of computer algebra systems (CAS), are difficult to manage.

In this research we continue that program by extending the rigid Earth model considered in Getino et al. (2010) to a classical non-rigid Earth model: the Poincaré one. The Poincaré model is the simplest one providing insight in the role played by the fluid core (e.g., Moritz & Mueller, 1987, Chapter 3). In this sense, once established the general second order framework for this model, its extension to other more realistic models (e.g., that of Sasao et al. 1980) is more feasible[2].

A similar strategy was employed when determining the second order precession of the non-rigid Earth that was first derived for a Poincaré model (Ferrándiz et al. 2004) and then extended to a two-layer one (Baenas et al. 2017). However, in contrast to the precession case, when including the nutation, the formulation of the problem in terms of Andoyer variables (Getino 1995a) becomes impracticable. So, firstly, it is necessary to build a new set of variables (the nonsingular complex canonical variables) that facilitates the second order computations. This circumstance made us to restrict the scope of this research to the nutations of the angular momentum axis (Poisson terms).

Namely, we will focus on developing the canonical framework necessary to extend the theory of the rotation of the Poincaré model to the second order; deriving the analytical expressions of the Poisson terms and quantifying them numerically; and discussing the effect of the fluid core on their amplitudes. In a forthcoming communication, we will present the second order expressions for the nutations of the figure axis by providing the second order Oppolzer terms, whose derivation is much more cumbersome than that of the Poisson ones (e.g., see Getino et al. 2010, Sections 3 and 4, in the rigid case).

The structure of the paper is as follows. In Section 2 we recall the Hamiltonian formulation of the Poincaré model in Andoyer variables under the same general premises as those considered in Getino et al. (2010) for the rigid Earth. The main difficulties of those variables for the Poincaré model, related to their ill definition in the equilibrium configuration (virtual singularities), is the absence of an

[2] From a numerical point of view the friction-generated effects in the second order solution would be, in principle, negligible considering the different magnitudes of first order in-phase and out-of-phase amplitudes (IERS Conventions 2010). However, the possible existence of related unexpected effects of small magnitude but theoretical interest makes this issue to be worthy of further research. Such study is far from being direct because the dissipation requires considering general, non-canonical, perturbation methods or doubling the dimension of the original phase space to preserve the Hamiltonian structure (Getino et al. 2010, Introduction).

explicit solution of the unperturbed problem, or auxiliary system, in terms of elementary functions. It hinders the direct application of the Hori perturbation method (Hori 1966). A new canonical set solving those difficulties is introduced in Section 3: the non-singular complex canonical variables ($NSCCV$). We proof their canonical character and formulate the Hamiltonian of the Poincaré model in terms of them. Those variables are defined at the equilibrium configuration and allows choosing a Hori kernel whose auxiliary system leads to an explicit analytical solution in a simple form. It makes easier the formulation of the Hori method that, in this way, runs parallel to that employed in Getino et al. (2010, Section 2.4), facilitating the comparisons with the rigid case. In Section 4, the practical implementation of the Hori's perturbation method is performed. We also provide the analytical expressions for the first and second order solutions of the motion of the angular momentum axis, comprising both precession and nutation. Those expressions turn out to be equivalent to that of Getino et al. (2010, Section 4.1), when reduced to the rigid case, and to Ferrándiz et al. (2004) and Baenas et al. (2017) for the second order precession of the Poincaré model.

The dependence of the derived amplitudes with the interior of the Earth is discussed in Section 5. We show that, in contrast to first order solutions, Poisson terms do depend on the Earth structure. Their numerical representation is also considered in that section, emerging some amplitudes above the 1 μas level. Specifically, it is the case of the terms with periods -6798.38, -3399.19, and 182.62 days, with a very significant contribution in obliquity for the term with period -6798.38 days, of about forty μas. Hence, considering current accuracies demands, this kind of terms can be no more ignored and must be incorporated in IAU Earth rotation models. We also compare those structure dependent amplitudes with their rigid counterparts, observing a noticeable amplification. We discuss their origin, concluding that it is due to the fluid core resonance. However, its features are different from that encountered in the first order nutations of the figure axis (Oppolzer terms) —at the first order Poisson terms are not amplified. The most relevant fact is that the amplification is not a function of the second order nutation frequency itself, but of the combinations of the original orbital frequencies generating it. This is one of the facts that prevents the use of MHB2000 transfer function (Mathews et al. 2002) to obtain the second order contribution, because that transfer function depends directly on the nutation frequency of each spectral component of the figure axis nutation (Mathews et al. 2002, Equation 7).

Finally, in Section 6 we summarize the main results of this investigation and draw some final conclussions. The paper is completed with four Appendixes A, B, C, and D were we include some lateral, but necessary, material for the development of our research.

2. HAMILTONIAN OF THE POINCARÉ MODEL IN ANDOYER CANONICAL VARIABLES

2.1. *Rotational dynamics*

For the development of our model, it is necessary to sketch the way in which the rotation of the Poincaré model of the Earth can be described by Hamiltonian methods. We limit to the main important points necessary to understand the construction of the Hamiltonian second order solution, referring the reader to the existing literature to obtain further details. For example, one can find first and second order solutions for the rigid Earth in Kinoshita (1977) and Getino et al. (2010), and first order solutions for the Poincaré model in Getino (1995a, 1995b). Other valuables references for the rigid model are Kinoshita & Souchay (1990), Escapa et al. (2002), Efroimsky & Escapa (2007), and Souchay et al. (1999). For the non rigid ones, the reader can consult Getino & Ferrándiz (2001), Escapa et al. (2001), Ferrándiz et al. (2004), Escapa (2011), Baenas et al. (2017), and Escapa et al. (2017).

The Poincaré model of the Earth[3] consists of a rigid mantle enclosing a liquid core. The mantle is assumed to be a symmetric ellipsoidal shell whose cavity is completely filled by the liquid. Attached to the mantle we consider a principal system of reference $Oxyz$, with associated basis vectors $\vec{e}_x$, $\vec{e}_y$, and $\vec{e}_z$, where O is the Earth barycenter; Oz the revolution axis of the ellipsoid, or the Earth figure axis $\vec{e}_z$; and xy the plane perpendicular to $\vec{e}_z$ (equatorial plane).

[3] The reader is referred to Melchior (2000), where an historical sketch of the first studies considering the Poincaré model can be found.

In the $Oxyz$ system, the tensors of inertia of the mantle, the core, and the whole Earth have the expressions

$$\mathbf{\Pi}_m = \begin{pmatrix} A_m & 0 & 0 \\ 0 & A_m & 0 \\ 0 & 0 & C_m \end{pmatrix}, \mathbf{\Pi}_c = \begin{pmatrix} A_c & 0 & 0 \\ 0 & A_c & 0 \\ 0 & 0 & C_c \end{pmatrix}, \mathbf{\Pi} = \mathbf{\Pi}_m + \mathbf{\Pi}_c = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{pmatrix}. \tag{1}$$

Because the mantle is rigid, its motion around the common barycenter O is described by the angular velocity $\vec{\omega}$ relative to the ecliptic of the epoch $OXYZ$ (basis vectors $\vec{e}_X$, $\vec{e}_Y$, and $\vec{e}_Z$). In the case of the core, its flow is dominated by the angular velocity by $\vec{\omega}_c$=$\vec{\omega} + \delta\vec{\omega}$, defined through a proper selection of a core system $Ox_cy_cz_c$ (Moritz & Mueller 1987, Chapters 3 and 4) with vectors $\vec{e}_{x_c}$, $\vec{e}_{y_c}$, and $\vec{e}_{z_c}$.

The rotation of the model is derived from the time evolution of the angular momentum vectors

$$\vec{L}_m = \mathbf{\Pi}_m\vec{\omega},\ \vec{L}_c = \mathbf{\Pi}_c(\vec{\omega} + \delta\vec{\omega}). \tag{2}$$

together with the corresponding rotation matrix that links the systems $OXYZ$ and $Oxyz$. Alternatively, instead of $\vec{L}_m$ one can consider the total angular momentum in Equation (2)

$$\vec{L} = \vec{L}_m + \vec{L}_c = \mathbf{\Pi}\vec{\omega} + \mathbf{\Pi}_c\delta\vec{\omega}. \tag{3}$$

A convenient method to determine that evolution of the angular momentum vectors and rotation matrix is by Hamiltonian mechanics, running a parallel way as that of the rigid Earth. Since in this case our dynamical system has 6 degrees of freedom, we need three pairs of canonical variables (p, q). Once selected them and constructed the Hamiltonian of the system, $\mathcal{H}(p, q)$, the equations of motion are given by

$$\frac{dp}{dt} = -\frac{\partial\mathcal{H}}{\partial q}, \frac{dq}{dt} = \frac{\partial\mathcal{H}}{\partial p}, \tag{4}$$

with the proper initial conditions at time $t = t_0$. So, the temporal evolution of any smooth function defined in the phase space $f(p, q)$ obeys to

$$\frac{df}{dt} = \{f; \mathcal{H}\}, \tag{5}$$

where the Poisson bracket in the canonical variables (p, q) has been represented by $\{-; -\}$.

In the case of the Poincaré model a suitable canonical set can be constructed from the Andoyer variables. In particular, a canonical set (M, μ); (N, ν); (Λ, λ) analogous to that used for the rigid Earth is extended (Getino 1995a) with (M_c, μ_c); (N_c, ν_c); (Λ, λ_c). This extension is necessary to account for the rotation of the core with respect to the mantle. We will refer to this set simply as Andoyer canonical variables for the Poincaré model of the Earth.

The Andoyer set has clear dynamical and geometrical meanings. The canonical momenta M, N, Λ and M_c, N_c, Λ_c are related to the Earth and core angular momentum, respectively. We have

$$\begin{aligned} M &= L, & M_c &= L_c, \\ N &= M\cos\sigma, & N_c &= M_c\cos\sigma_c, \\ \Lambda &= M\cos I, & \Lambda_c &= M_c\cos I_c. \end{aligned} \tag{6}$$

In these expressions the auxiliary variable σ is the angle between $\vec{e}_z$ and $\vec{e}_L = \vec{L}/L$; I between $\vec{e}_L$ and $\vec{e}_Z$; σ_c between $\vec{e}_{L_c} = \vec{L}_c/L_c$ and $\vec{e}_z$; and I_c between $\vec{e}_{L_c}$ and $\vec{e}_{z_c}$. The canonical coordinates λ, μ, and ν are defined geometrically when introducing two lines of nodes (Efroimsky & Escapa 2007, Figure 3 —noted as $\vec{i}$ and $\vec{j}$) defined by the vectors

$$\vec{n}_1 = \frac{\vec{e}_Z \times \vec{e}_L}{|\vec{e}_Z \times \vec{e}_L|}, \; \vec{n}_2 = \frac{\vec{e}_L \times \vec{e}_z}{|\vec{e}_L \times \vec{e}_z|}. \tag{7}$$

In this way λ is the (oriented) angle between $\vec{e}_X$ and $\vec{n}_1$; μ between $\vec{n}_1$ and $\vec{n}_2$; and ν between $\vec{n}_2$ and $\vec{e}_x$. It allows constructing the rotation matrix that transforms $OXYZ$ into $Oxyz$ by means of the sequence of rotations

$$\mathbf{R}_3(\nu)\mathbf{R}_1(\sigma)\mathbf{R}_3(\mu)\mathbf{R}_1(I)\mathbf{R}_3(\lambda), \tag{8}$$

where $\mathbf{R}_i$ denotes an elemental rotation matrix. A similar construction can be made for the coordinates μ_c, ν_c, λ_c, which brings $Ox_cy_cz_c$ into $Oxyz$ with

$$\mathbf{R}_3(\nu_c)\mathbf{R}_1(\sigma_c)\mathbf{R}_3(\mu_c)\mathbf{R}_1(I_c)\mathbf{R}_3(\lambda_c). \tag{9}$$

From a practical point of view, this matrix has less interest than that given in Equation (8).

The Earth is a fast rotator and in its motion departs slightly from the equilibrium configuration. The equilibrium configuration is defined as the dynamical state of the torque-free motion where the vectors $\vec{e}_L$ and $\vec{e}_{L_c}$ are parallel to the figure axis $\vec{e}_z$, the mantle and the core rotating with the same angular rate. Hence, in this situation the angular velocity vectors are also parallel to $\vec{e}_z$ with $\vec{\omega} = \omega_E \vec{e}_z$, $\vec{\omega}_c = \omega_E \vec{e}_z$, i.e., $\delta\vec{\omega} = \vec{0}$. In those circumstances (Equations 6), the angles σ and σ_c are nil ($M = N$, $M_c = N_c$) and the lines of nodes $\vec{n}_2$ and $\vec{n}_{2_c}$ are not defined. It entails that it is also the case for the angles ν and ν_c. Therefore, in the equilibrium configuration the Andoyer set is no well-defined, giving raise to the so-called virtual singularities (Henrard 2006, Lara 2018). However, the angles between $\vec{n}_2$ and $\vec{e}_x$ and $\vec{n}_{2_c}$ and $\vec{e}_x$ can be computed, i.e., both $\mu + \nu$ and $\mu_c + \nu_c$ remain perfectly defined.

Observationally, the departure from that equilibrium manifests in the smallness of the angles σ and σ_c, of the order of 10^{-6} radians, i. e., the vectors $\vec{e}_z$ and $\vec{L}$ keeping close. It makes useful to decompose the evolution of $\vec{e}_z$ relative to OZ or axis $\vec{e}_Z$ in two parts: the evolution of $\vec{L}$ relative to $\vec{e}_Z$ and the evolution of $\vec{e}_z$ with respect to $\vec{L}$. Abusing terminology, we refer to the first one as Poisson terms and to the second one as Oppolzer terms.

According to Equations (6) and (8), the Poisson terms are described by I and λ: I corresponds to the obliquity and λ to the longitude of the plane perpendicular to $\vec{L}$ that is named as Andoyer plane. The long-term part, or secular, of the time evolution of I and λ is the precession and the remaining part the nutation.

2.2. *Hamiltonian in Andoyer variables*

Once the canonical set to describe the rotational motion has been selected, the Hamiltonian of the system has to be written. It has the form

$$\mathcal{H} = T + V + E. \tag{10}$$

In this expression, T is the kinetic energy of the model; V the perturbing potential; and E an additional, or complementary, term. It appears when, instead of using the ecliptic of epoch, the motion is referred to the ecliptic of date that is a non–inertial system (Kinoshita 1977).

2.2.1. *Kinetic energy*

The kinetic energy is the sum of the rotational kinetic energy of the mantle and the core, i.e., depends on the interior structure of the Earth. According to the description of the Poincaré model (Equations 2 and 3) we have

$$T = T_m + T_c = \frac{1}{2}(\vec{L} - \vec{L}_c)^t \mathbf{\Pi}_m^{-1}(\vec{L} - \vec{L}_c) + \frac{1}{2}\vec{L}_c^t \mathbf{\Pi}_c^{-1}\vec{L}_c, \tag{11}$$

the superscript t denoting the transpose. From Equations (6, 8, and 9), the components of $\vec{L}$ and $\vec{L}_c$ in the $Oxyz$ system are

$$\vec{L} = \begin{pmatrix} \sqrt{M^2 - N^2}\sin\nu \\ \sqrt{M^2 - N^2}\cos\nu \\ N \end{pmatrix}, \vec{L}_c = \begin{pmatrix} \sqrt{M_c^2 - N_c^2}\sin\nu_c \\ -\sqrt{M_c^2 - N_c^2}\cos\nu_c \\ N_c \end{pmatrix}. \tag{12}$$

Thus, taking into account the expression of the tensors of inertia given in Equations (1), we can obtain

$$\begin{aligned} T = \ & \frac{1}{2A_m}\Bigg[\left(M^2 - N^2\right) + \frac{A}{A_c}\left(M_c^2 - N_c^2\right) + 2\sqrt{M^2 - N^2}\sqrt{M_c^2 - N_c^2}\cos(\nu + \nu_c) + \\ & + \frac{1}{2C_m}\left(N^2 - 2N\ N_c + \frac{C}{C_c}N_c^2\right)\Bigg]. \end{aligned} \tag{13}$$

This formula is model dependent and more complicated than in the rigid case. In particular, the kinetic energy for the Poincaré model depends on the coordinates ν and ν_c. Hence, in contrast to the symmetrical rigid case the Andoyer variables are not action-angle variables in the torque–free motion, what complicates the integration of the problem.

2.2.2. *Perturbing potential and moving ecliptic term*

Since in this research we are focused on determining the influence of Earth structure on some second order effects, we will reduce the perturbing potential just to the main contribution affecting the rotational dynamics. It is given by

$$V = \sum_{p=S,M} \frac{\kappa^2 m_p'}{r^3}(C - A)P_2(\sin\delta) = \sum_{p=S,M} k_p'\left(\frac{a_p}{r}\right)^3 P_2(\sin\delta), \tag{14}$$

where κ^2 is the universal constant of gravitation; r the distance of the perturbing body p (the Moon=M or the Sun=S) to O; m'_p its mass; and δ its latitude relative to $Oxyz$. This expression is formally the same as that appearing when considering the rigid Earth, the parameter k'_p characterizing the order of magnitude.

Therefore, we can directly borrow the process of writing Equation (14) in terms of the Andoyer variables from Getino et al. (2010, Section 2.3). We have

$$\begin{aligned} V = \sum_{p=S,M} k'_p \sum_i \Bigg[& \frac{1}{2}(3\cos^2\sigma - 1) B_i \cos\Theta_i - \frac{1}{2}\sin 2\sigma \sum_{\tau=\pm 1} C_{i,\tau}\cos(\mu - \tau\Theta_i) + \\ & + \frac{1}{4}\sin^2\sigma \sum_{\tau=\pm 1} D_{i,\tau}\cos(2\mu - \tau\Theta_i)\Bigg]. \end{aligned} \tag{15}$$

The auxiliary angle $I = \arccos(N/M)$ is implicitly contained in the orbital functions B_i, $C_{i,\tau}$, and $D_{i,\tau}$, which depend on the orbital motions of the Moon and the Sun[4] (Appendix A). It is also the case of λ through the arguments Θ_i. They are linear combinations of the Delaunay variables of the Moon and the Sun of the form

$$\Theta_i = m_{1i} l_M + m_{2i} l_S + m_{3i} F + m_{4i} D + m_{5i}(\Omega_0 - \lambda). \tag{16}$$

The values of the five integers m_{ji}, $j = 1, ..., 5$ characterize each index i and are obtained from a Fourier decomposition of the orbital motions of the perturbers, given by some ephemeris (Appendix A). When Θ_i must be considered as a known function of time, we take its time rate n_i, or orbital frequency, constant with

$$\Theta_i = n_i t + \Theta_{i0}. \tag{17}$$

In the development of the rotational theory, it is necessary to identify those arguments Θ_i leading to nil or very small, in absolute value, time rates n_i. The most important is the element i in the Fourier decomposition with the integer values $m_{ji} = 0$, $j = 1, ..., 5$. It will be denoted by $i = 0$. So, for this term we have $\Theta_0 = 0$ and $n_0 = 0$.

[4] We will also assume that the orbital functions do not depend explicitly on time (Escapa et al. 2017).

The additional term E depends on the angular velocity of the ecliptic of date with respect to the ecliptic of epoch and the total angular momentum of the Earth. Therefore, its expression is independent of the Earth model. So, we can take advantage of the derivations provided in Getino et al. (2010, Section 2.3). Then, we have

$$E = \Lambda e_1 + M \sin I (e_2 \cos\lambda + e_3 \sin\lambda), \tag{18}$$

where e_i are assumed to be constant and give an indication of the magnitude of E.

2.2.3. *Hamiltonian and equations of motion*

Equations (13), (15), and (18) provide the Hamiltonian $\mathcal{H}$ of the Poincaré model in Andoyer canonical variables. It has the functional dependencies

$$\mathcal{H} = T(M, N, \nu; M_c, N_c, \nu_c) + V(M, N, \Lambda, \mu, \lambda; t) + E(M, \Lambda, \lambda). \tag{19}$$

Its rotational dynamics is given by the equations of motion, derived with the aid of Equations (4).

The analytical resolution of the resulting system of non-linear differential equations is not practicable. However, the different relative magnitudes of T, V, and E make possible to construct an approximate analytical solution by perturbation methods. Indeed, the fast spin of the Earth rotation entails that V and E are much smaller than T, of the order of 10^{-7} times (Getino et al. 2010, Equation 29). So, they can be properly considered as perturbations and the Hamiltonian split as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 = T + (V + E), \tag{20}$$

where $\mathcal{H}_1$ is proportional to a small parameter ε and $\mathcal{H}_0$, equal to T, denotes the unperturbed Hamiltonian, leading to the torque–free motion. For V and E the parameter ε is proportional to the constants k'_p and e_i, respectively.

This decomposition seems formally similar to that used in Getino et al. (2010, Section 3) for the rigid case, but there is an important difference[5]. In the rigid case the Andoyer set is an action-angle

[5] Another difference appears when considering the expressions of the angular velocities components, not of the Euler angles, in terms of the canonical variables (Getino et al. 2010, Equation 19). Their derivation must consider the loss of osculation suffered by the Andoyer variables (Efroimsky & Escapa 2007, Escapa 2011).

variable set for $\mathcal{H}_0$, what provides a direct solution of the unperturbed problem. It is not the case for the Poincaré model, because of the dependence of T on ν and ν_c (Equation 13). Hence, the unperturbed problem might not have a direct solution in those canonical variables, as it is the case.

This difficulty can be circumvented by splitting the kinetic energy into two parts. One dominant close to the equilibrium configuration (where the Andoyer variables are not defined) and the torque–free motion dynamics with a direct solvable unperturbed Hamiltonian $\mathcal{H}_0$; and another residual part that would rise to a high order perturbation $\mathcal{H}_j$, proportional to ε^j, $j \geq 3$. We will return to this relevant point in Section 3.

2.3. *Second order integration*

2.3.1. *Hori's method*

To construct a second order solution of the equations related to Hamiltonian $\mathcal{H}$ (Equation 20) we will follow the Hori's method (Hori 1966), which allows the use of unspecified canonical variables combined with averaging. Its application is detailed in Getino et al. (2010, Section 2.4) and references therein. A very comprehensive exposition can be found in the monograph by Ferraz–Mello (2007, Chapter 6). The method consists on finding a generating transformation $\mathcal{W}$ which leads to a transformed Hamiltonian $\mathcal{H}^*$ easier to integrate in a new canonical set (p^*, q^*).

Specifically, we have for the generating function[6] up to the second order $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2$

$$\begin{aligned} \mathcal{W}_1 &= \int_{UP} \mathcal{H}_{1per}\, dt, \\ \mathcal{W}_2 &= \int_{UP} \mathcal{H}_{2per}\, dt + \frac{1}{2}\int_{UP} \{\mathcal{H}_1 + \mathcal{H}_{1sec}; \mathcal{W}_1\}_{per}\, dt, \end{aligned} \tag{21}$$

and for the transformed Hamiltonian $\mathcal{H}^* = \mathcal{H}_0^* + \mathcal{H}_1^* + \mathcal{H}_2^*$

$$\begin{aligned} \mathcal{H}_0^* &= \mathcal{H}_0, \\ \mathcal{H}_1^* &= \mathcal{H}_{1sec}, \\ \mathcal{H}_2^* &= \mathcal{H}_{2sec} + \frac{1}{2}\{\mathcal{H}_1 + \mathcal{H}_{1sec}; \mathcal{W}_1\}_{sec}. \end{aligned} \tag{22}$$

[6] There is a typo in Equation 23 of Getino et al. (2010). In the definition of $\mathcal{W}_2$, the element dt must be at the end of the line, i.e., the whole expression has to be integrated.

The generating functions are computed over the trajectories given by UP (unperturbed problem), that is to say, over the solutions of the system with the unperturbed Hamiltonian $\mathcal{H}_0^*$

$$\frac{dp^*}{dt} = -\frac{\partial \mathcal{H}_0^*}{\partial q^*}, \frac{dq^*}{dt} = \frac{\partial \mathcal{H}_0^*}{\partial p^*}. \tag{23}$$

This system is more properly named as auxiliary system and the Hamiltonian $\mathcal{H}_0^*$ as Hori kernel, since generally speaking they may differ from the unperturbed situation. Both play a fundamental role in the theory of perturbations (e.g., Hori 1973; Ferraz–Mello 2007; Baenas et al. 2017).

The per and sec subindexes in Equations (21 and 22) refer to the periodic[7] and secular parts of the corresponding functions computed from the auxiliary system. Basically, the periodic part of any function is that involving short period terms which stem from any presence of the variables μ, ν, μ_c, and ν_c. When those canonical variables are not present, the periodic part is due to any linear combination of one or several orbital arguments Θ_i, Θ_j, etc. with time rate of the linear combination different from 0 (or larger than a pre-fixed very small value).

The secular part of a function is the part that is not periodic. For example, if we considered $\mathcal{H}_1$ as given by Equations (15) and (18), we have that

$$\mathcal{H}_1^* = \mathcal{H}_{1sec} = \sum_{p=S,M} \frac{k_p'}{2}(3\cos^2\sigma^* - 1)B_0^* + \Lambda^* e_1 + M^* \sin I^*(e_2 \cos\lambda^* + e_3 \sin\lambda^*), \tag{24}$$

the remaining part of $\mathcal{H}_1$ being periodic, according to the former considerations.

This scheme allows determining the evolution of any function of the phase space $f(p,q)$ at the second order, once known the solution of the new canonical variables (p^*, q^*) from the canonical equations with Hamiltonian $\mathcal{H}^*$. In particular, we have

$$f(p,q) = f^*(p^*, q^*) + \Delta f(p^*, q^*), \tag{25}$$

with $\Delta f = \Delta_1 f + \Delta_2 f + \Delta_3 f$ given by

$$\begin{aligned} \Delta_1 f &= \{f^*; \mathcal{W}_1\}, \\ \Delta_2 f &= \{f^*; \mathcal{W}_2\}, \\ \Delta_3 f &= \frac{1}{2}\{\{f^*; \mathcal{W}_1\}; \mathcal{W}_1\}. \end{aligned} \tag{26}$$

[7] Strictly, the functions are usually quasi-periodic.

2.3.2. *Solution of the auxiliary system in Andoyer variables*

The practical implementation of the Hori's method in Andoyer variables to obtain a second order solution in the small parameter ε, i.e., in k_p' and e_i, requires the solution of the auxiliary system. If we take as the Hori kernel of the perturbation the torque–free motion Hamiltonian $\mathcal{H}_0^* = \mathcal{H}_0 = T$, the auxiliary system is given by the equations

$$\begin{aligned}
\dot{\mu} =& \ \frac{M}{A_m}\left[1 + \frac{\sqrt{M_c^2 - N_c^2}}{\sqrt{M^2 - N^2}}\cos(\nu + \nu_c)\right], \\
\dot{\nu} =& \ -\frac{N}{A_m}\left[1 + \frac{\sqrt{M_c^2 - N_c^2}}{\sqrt{M^2 - N^2}}\cos(\nu + \nu_c)\right] + \frac{N - N_c}{C_m}, \\
\dot{N} =& \ \frac{1}{A_m}\sqrt{M^2 - N^2}\sqrt{M_c^2 - N_c^2}\sin(\nu + \nu_c), \\
\dot{\Lambda} =& \ 0,\ \dot{M} = 0,\ \dot{\lambda} = 0,
\end{aligned} \tag{27}$$

for the total Earth related variables, where for simplicity we have omitted the asterisk in the canonical variables. For those of the core, it is obtained

$$\begin{aligned}
\dot{\mu}_c =& \ \frac{M_c}{A_m}\left[\frac{A}{A_c} + \frac{\sqrt{M^2 - N^2}}{\sqrt{M_c^2 - N_c^2}}\cos(\nu + \nu_c)\right], \\
\dot{\nu}_c =& \ -\frac{N_c}{A_m}\left[\frac{A}{A_c} + \frac{\sqrt{M^2 - N^2}}{\sqrt{M_c^2 - N_c^2}}\cos(\nu + \nu_c)\right] - \frac{1}{C_m}\left(N - \frac{C}{C_c}N_c\right), \\
\dot{N}_c =& \ \frac{1}{A_m}\sqrt{M^2 - N^2}\sqrt{M_c^2 - N_c^2}\sin(\nu + \nu_c), \\
\dot{\Lambda}_c =& \ 0,\ \dot{M}_c = 0,\ \dot{\lambda}_c = 0.
\end{aligned} \tag{28}$$

Although this system of differential equations has particular constants of motion, like $N - N_c$, its general solution is not given in terms of elementary functions. The difficulty arise from the non-linear character of the time evolution of the pairs (N, ν) and (N_c, ν_c), which avoids solving them by quadratures in a standard way. In turn, that evolution would determine that of the variables μ and μ_c. Well–know strategies like expanding the dynamics around the equilibrium configuration (Arnold 1989, e.g., Chapter 5) fail in this case, because the virtual singularities of the Andoyer set render the factors

$$\frac{\sqrt{M_c^2 - N_c^2}}{\sqrt{M^2 - N^2}} = \frac{M_c \sin\sigma_c}{M \sin\sigma}, \ \frac{\sqrt{M^2 - N^2}}{\sqrt{M_c^2 - N_c^2}} = \frac{M \sin\sigma}{M_c \sin\sigma_c} \tag{29}$$

in Equations (27) and (28) not defined in such configuration (they tend to 0/0).

The way that was envisaged to circumvent this problem is due to Getino (1995b). He introduced non-canonical variables to compute the integrals over the unperturbed solution, keeping in this way the advantages of the Andoyer set. From a systematic perspective, that procedure can be identified with making a non-canonical change of variables to

$$\begin{gathered}\sqrt{M^2 - N^2}\cos\nu, \ \sqrt{M^2 - N^2}\sin\nu, \ \mu + \nu, \\ \sqrt{M_c^2 - N_c^2}\cos\nu_c, \ \sqrt{M_c^2 - N_c^2}\sin\nu_c, \ \mu_c + \nu_c,\end{gathered} \tag{30}$$

instead of working with the canonical ones N, μ, ν and N_c, μ_c, ν_c. In those new variables the equilibrium configuration is well-defined. The corresponding equations of motion, equivalent to those of the original auxiliary system, can be approximately integrated by neglecting the quadratic monomials in σ and σ_c.

As a matter of fact in Getino (1995b) —and subsequent works, e.g., Getino & Ferrándiz (1997, 2001)— the integrals over the unperturbed problem are evaluated through the approximated equations of motion themselves, not requiring the explicit solution of all the transformed variables except for $\mu + \nu$. Then, such integrals are re-expressed in terms of the original Andoyer canonical set and the first order integration is completed according to Hori's method. This process is legitimate and can be extended to a second order integration as it has been done for studying the precession motion (Ferrándiz et al. 2004, Baenas et al. 2017).

However, the introduction of the non-canonical variables given by Equations (30) destroys, to some extent, the advantages of a canonical formulation. For example, it translates into a more cumbersome computation of averaging and generating functions than if we had available the explicit solution of the auxiliary system in a canonical set. This is especially important when constructing the nutations at the second order, since, in contrast to precession, its evaluation requires the calculation of two generating functions (Equations 21).

Besides, this method departs from the developed standard Hamiltonian procedure to get a second order solution for the rigid Earth (Getino et al. 2010, Section 2.4). It makes more difficult to compare the different features of rigid and non-rigid models in the process of obtaining that approximate analytical second order solution (e.g., Hori kernel, auxiliary system solution, etc.). Therefore, it would be very expedient to construct a canonical set for Poincaré model of the Earth that, if possible, skips the former drawbacks of the Andoyer canonical and non-canonical sets. This objective is accomplished in the next section. At any rate, the Andoyer canonical variables still plays a role, because of their clear geometric and dynamical meaning. Hence, they will act as a proxy to connect more abstract canonical sets with the physics of the rotation of the Earth.

3. HAMILTONIAN OF THE POINCARÉ MODEL IN NON-SINGULAR COMPLEX CANONICAL VARIABLES

3.1. *Non-singular complex canonical set*

3.1.1. *Non-singular canonical variables*

The difficulties referred to in formulating the Poincaré model in Andoyer variables are related to the virtual singularities that they present in the equilibrium configuration. In fact, a similar problem arises in the orbital motion when Delaunay variables face to zero inclination and eccentricity. This type of singularities are avoided by constructing a new canonical set named Poincaré variables in its different variations (Brower & Clemence 1961, Chapter XVII).

In the case of the non-rigid Earth modeling, similar sets were introduced in Getino et al. (2000) for a two-layer Earth model and modified by Escapa et al. (2001) for a three-layer Earth model. They were denominated as non-singular canonical variables. Since in the case of the Poincaré model the angle I_c does not enter into the Hamiltonian (Equations 13, 15, and 18), we will follow Getino et al. (2000). It avoids the use of the more involved definitions by Escapa et al. (2001), which are necessary when considering an inner core in the Earth rotation modeling.

The non-singular canonical set is composed of the pairs $\left(\widehat{Y}_1, \widehat{y}_1\right)$; $\left(\widehat{Y}_2, \widehat{y}_2\right)$; $\left(\widehat{Y}_3, \widehat{y}_3\right)$ for the Earth and $\left(\widehat{Y}_{1c}, \widehat{y}_{1c}\right)$; $\left(\widehat{Y}_{2c}, \widehat{y}_{2c}\right)$; $\left(\widehat{Y}_{3c}, \widehat{y}_{3c}\right)$ for the core. They are linked with the Andoyer variables by means of

$$\begin{aligned} \widehat{Y}_1 &= M, & \widehat{y}_1 &= \mu + \nu, \\ \widehat{Y}_2 &= \sqrt{2(M-N)}\cos\nu, & \widehat{y}_2 &= -\sqrt{2(M-N)}\sin\nu, \\ \widehat{Y}_3 &= \Lambda, & \widehat{y}_3 &= \lambda, \end{aligned} \tag{31}$$

with analogous relationships for the core variables. The transformation is canonical and the new Hamiltonian $\widehat{\mathcal{H}}$ is obtained by expressing the Andoyer variables in terms of the non-singular set $\left(\widehat{Y}, \widehat{y}\right)$ in $\mathcal{H}$.

The advantage of this set is that, as expected, removes the virtual singularities, since it contains the combination $\mu + \nu$ and the factor (Equation 6)

$$\sqrt{2(M-N)} = \sqrt{2M(1-\cos\sigma)} = 2\sqrt{M}\sin\frac{\sigma}{2}. \tag{32}$$

As we pointed out, the combination $\mu + \nu$ is well-defined in the equilibrium configuration. It is not the case of the angle ν. However, it enters in Equations (31) through the bounded functions sine and cosine that are multiplied by $\sqrt{2(M-N)}$; but, since this factor is zero in the equilibrium configuration, the virtual singularities disappear with $\widehat{Y}_2 = \widehat{y}_2 = 0$. A similar argument is valid for the core variables leading to $\widehat{Y}_{2c} = \widehat{y}_{2c} = 0$ in that configuration. The former Equation (32) also entails that the pairs $\left(\widehat{Y}_2, \widehat{y}_2\right)$ and $\left(\widehat{Y}_{2c}, \widehat{y}_{2c}\right)$ are of the order of σ and σ_c, respectively.

3.1.2. *Complexification*

The non-singular set as defined formerly would be completely useful for our purposes. Nevertheless, we can obtain a further simplification considering the dynamical symmetry of the Poincaré model, i.e., the equality of the equatorial moments of inertia. This strategy was partially employed when using the non-canonical variables of Equation (30), as in Getino & Ferrándiz (1997), and it also appears in some texts of Mechanics (e.g., Arnold 1989, Appendix 7).

It starts from combining the pair $\left(\widehat{Y}_2, \widehat{y}_2\right)$ in a complex valued pair (Y_2, y_2), in such a way that the variables Y_2 and y_2 become complex conjugates with each other. The same procedure is applied to $\left(\widehat{Y}_{2c}, \widehat{y}_{2c}\right)$. The transformation is completed for the other variables in order to obtain a new canonical set (Y, y). We will refer to this set as non-singular complex canonical variables ($NSCCV$).

The process of determining explicitly the form of the transformation relies in the necessary and sufficient conditions of canonicity. In its more general form (Witner 1941, Chapter 1), it is required that

$$\mathbf{M}^t\mathbf{I}\mathbf{M} = \upsilon\mathbf{I}, \text{ with } \mathbf{I} = \begin{pmatrix} \mathbf{0}_6 & \mathbf{1}_6 \\ -\mathbf{1}_6 & \mathbf{0}_6 \end{pmatrix}. \tag{33}$$

The symbols $\mathbf{0}_6$ and $\mathbf{1}_6$ represent the zero and unit sixth dimension matrices, respectively, leading to the matrix $\mathbf{I}$ of dimension 12 —the symplectic matrix. The scalar υ is the multiplier of the transformation[8] and $\mathbf{M}$ its Jacobian matrix

$$\mathbf{M} = \frac{\partial(Y, y)}{\partial(\widehat{Y}, \widehat{y})}. \tag{34}$$

The transformed Hamiltonian $\mathcal{H}'$ is given by

$$\mathcal{H}' = \upsilon\widehat{\mathcal{H}} + \widehat{\mathcal{R}}, \tag{35}$$

where $\widehat{\mathcal{R}}$ is the remainder function, which appears for time-dependent transformations as it was the case of the additional term E (Equation 18).

To determine the explicit form of a simple transformation fulfilling those conditions of canonicity, we analyzed different linear transformations. One of the possible solutions that we found consists on keeping some original variables unaltered or scaled by $-\mathrm{i}$, with $\mathrm{i} = \sqrt{-1}$,

$$Y_1 = -\mathrm{i}\widehat{Y}_1, Y_3 = -\mathrm{i}\widehat{Y}_3, y_1 = \widehat{y}_1, y_3 = \widehat{y}_3, \tag{36}$$

and the remaining ones combined to get a complex conjugate pair

$$Y_2 = \frac{1}{\sqrt{2}}(\widehat{y}_2 - \mathrm{i}\widehat{Y}_2),\ y_2 = \frac{1}{\sqrt{2}}(\widehat{y}_2 + \mathrm{i}\widehat{Y}_2), \tag{37}$$

[8] In the literature there are different definitions of canonical transformations (e.g., Goldstein et al. 2001, Arnold 1989, Ferraz–Mello 2007, etc.). Nowadays the most extended is that taking $\upsilon = 1$, although the general case with $\upsilon \neq 1$ as considered here is useful in solving some problems. In some references, the scalar υ is also named as valence (e.g., Gantmacher 1975) and the symplectic matrix is defined with the opposite sign (e.g., Arnold 1989).

with similar relations for the core variables. The multiplier of the transformation turned out to be $\upsilon = -\mathrm{i}$, with $\widehat{\mathcal{R}} = 0$, since the process is time independent. So, the transformed Hamiltonian is given by $\mathcal{H}' = -\mathrm{i}\widehat{\mathcal{H}}$.

The Equations (36) and (37), and the respective ones for the core, define the canonical transformation giving raise to the *NSCCV*. Moreover, since the canonical transformations form a group (Witner 1941, Chapter 1), we can relate the *NSCCV* directly to Andoyer variables with the use of Equation (31), avoiding in this way the use of the intermediate set $\left(\widehat{Y}, \widehat{y}\right)$.

Therefore, we have

$$\begin{aligned} Y_1 &= -\mathrm{i}M,\ Y_2 = -\mathrm{i}\sqrt{M-N}e^{-\mathrm{i}\nu},\ Y_3 = -\mathrm{i}\Lambda, \\ y_1 &= \mu + \nu,\ y_2 = \mathrm{i}\sqrt{M-N}e^{\mathrm{i}\nu},\ y_3 = \lambda. \end{aligned} \tag{38}$$

Those relationships are supplemented with similar formulae for the core variables. The transformed Hamiltonian is given by

$$\mathcal{H}' = -\mathrm{i}\mathcal{H}. \tag{39}$$

3.2. *Hamiltonian in the NSCCV*

3.2.1. *Perturbing potential and complementary term*

The Hamiltonian in the *NSCCV* is given by Equation (39). From the formulae derived in Section 2 and given in Equations (19), (13), (15), and (18), the transformed Hamiltonian is

$$\mathcal{H}' = -\mathrm{i}\mathcal{H} = -\mathrm{i}(T' + V' + E'), \tag{40}$$

where the terms T', V', and E' are now expressed in terms of the *NSCCV*.

This process can be done with the aid of the inverse relations of Equations (38). Specifically, the Andoyer momenta can be written as

$$M = \mathrm{i}Y_1,\ N = \mathrm{i}Y_1 - Y_2 y_2,\ \Lambda = \mathrm{i}Y_3. \tag{41}$$

Accordingly, the auxiliary angle I is given by

$$I = \arccos\left(\frac{Y_3}{Y_1}\right), \tag{42}$$

With respect to the canonical coordinates, first we consider the substitutions

$$\mu = y_1 - \nu,\ \lambda = y_3, \tag{43}$$

and then we perform the transformations

$$\cos\nu = \frac{\mathrm{i}}{2}\frac{Y_2 - y_2}{\sqrt{Y_2 y_2}},\ \ \sin\nu = -\frac{1}{2}\frac{Y_2 + y_2}{\sqrt{Y_2 y_2}}. \tag{44}$$

The same procedure is applied for the Andoyer variables related to the core.

In this way the functional dependence of the Hamiltonian $\mathcal{H}'$ appears as

$$\mathcal{H}' = -\mathrm{i}[T'(Y_1, Y_2, y_2; Y_{1c}, Y_{2c}, y_{2c}) + V'(Y_1, Y_2, Y_3, y_1, y_2, y_3; t) + E'(Y_1, Y_3, y_3)]. \tag{45}$$

In particular, the literal expression of the perturbing potential turns out to be

$$\begin{aligned} V' = \sum_{p=S,M} k'_p \sum_i \sum_{\tau=\pm1} \Bigg[& \frac{1}{2} B_i e^{\mathrm{i}\tau\Theta_i}\left(1 + 3\mathrm{i}\frac{Y_2 y_2}{Y_1}\right) - \\ & \mathrm{i}\frac{\sqrt{2}}{2}\frac{C_{i,\tau}}{\sqrt{\mathrm{i}Y_1}}\left(Y_2 e^{\mathrm{i}(y_1 - \tau\Theta_i)} - y_2 e^{-\mathrm{i}(y_1 - \tau\Theta_i)}\right) + \\ & \mathrm{i}\frac{D_{i,\tau}}{4Y_1}\left(Y_2^2 e^{\mathrm{i}(2y_1 - \tau\Theta_i)} + y_2^2 e^{-\mathrm{i}(2y_1 - \tau\Theta_i)}\right)\Bigg], \end{aligned} \tag{46}$$

where, as inherited from Andoyer variables, the orbital functions B_i, $C_{i,\tau}$, and $D_{i,\tau}$ depends on I and the arguments Θ_i on y_3. In a similar way, the additional term due to the motion of the ecliptic of date is

$$E' = \mathrm{i}[e_1 Y_3 + Y_1 \sin I(e_2 \cos y_3 + e_3 \sin y_2)]. \tag{47}$$

The expression of the kinetic energy of the Poincaré model can be constructed in the same way. However, it is simpler to write $\vec{L}$ and $\vec{L}_c$ (Equation 12) in terms of the *NSCCV*

$$\vec{L}' = \begin{pmatrix} -\dfrac{1}{2}\sqrt{\mathrm{i}2Y_1 - Y_2 y_2}(Y_2 + y_2) \\ \\ \dfrac{\mathrm{i}}{2}\sqrt{\mathrm{i}2Y_1 - Y_2 y_2}(Y_2 - y_2) \\ \\ \mathrm{i}Y_1 - Y_2 y_2 \end{pmatrix},\ \vec{L}'_c = \begin{pmatrix} -\dfrac{1}{2}\sqrt{\mathrm{i}2Y_{1c} - Y_{2c} y_{2c}}(Y_{2c} + y_{2c}) \\ \\ -\dfrac{\mathrm{i}}{2}\sqrt{\mathrm{i}2Y_{1c} - Y_{2c} y_{2c}}(Y_{2c} - y_{2c}) \\ \\ \mathrm{i}Y_{1c} - Y_{2c} y_{2c} \end{pmatrix}, \tag{48}$$

and then apply Equation (11) to obtain T'. Since the derived expression is quite lengthy, we omit its writing, returning to the relevant part of it when determining the Hori kernel of the problem. From now on, the prime on the functions depending on the *NSCCV* is omitted to lighten the notation.

3.2.2. *Hori kernel*

Once formulated the Poincaré model in the *NSCCV*, it is necessary to select the unperturbed Hamiltonian $\mathcal{H}_0$, i.e., the Hori kernel within the perturbation procedure. If we took the whole expression of the kinetic energy, $-\mathrm{i}T$, we would face to similar problems to that of the Andoyer variables regarding the no availability of a direct complete solution for the generated auxiliary system.

However, the non-singular character of the new variables allows employing a common technique in Mechanics (e.g., Arnold 1989, Chapter 5), this is the expansion of $\mathcal{H}_0$ around the equilibrium configuration. Indeed, this procedure has been used for studying the synchronous rotation of some celestial bodies (e.g., Henrard 2006). It consists of developing the Hamiltonian in powers of the differences of the canonical variables with respect to their equilibrium values and keeping just the quadratic terms.

That decomposition leads to a linear auxiliary system, so solvable. Besides, this choice has the virtue that the Hori kernel is representative of the perturbed dynamics (Ferraz–Mello 2007, Chapter 6) and that the remaining neglected part is indeed very small.

In the equilibrium configuration, the variables appearing in T takes the following values (Section 2 and Equations 38)

$$y_2 = Y_2 = y_{2c} = Y_{2c} = 0,\ Y_1 = -\mathrm{i}C\omega_E, Y_{1c} = -\mathrm{i}C_c\omega_E. \tag{49}$$

The quadratic expansion of T, which is denoted as T_0, would lead to a second degree polynomial in $Y_1 + \mathrm{i}C\omega_E$, Y_2, y_2, $Y_{1c} + \mathrm{i}C_c\omega_E$, Y_{2c}, y_{2c} with no linear terms —the expansion point is an equilibrium solution. The resulting expression of T_0 can be written in the form

$$\begin{aligned} T_0 = \ & \omega_E\left[\frac{C - A_m}{A_m} y_2 Y_2 + \frac{A}{A_c}\frac{C_c}{A_m} y_{2c} Y_{2c} - \frac{\sqrt{CC_c}}{A_m}(y_2 y_{2c} + Y_2 Y_{2c})\right] - \\ & -\frac{1}{2C_m}\left[(Y_1 - Y_{1c})^2 + \frac{C_m}{C_c} Y_{1c}^2\right]. \end{aligned} \tag{50}$$

By so doing, we have decomposed the kinetic energy in the *NSCCV* as

$$T = T_0 + \Delta T, \tag{51}$$

where ΔT contains the terms of T not included in T_0. It is about 10^{-20} times smaller than T_0, and hence completely negligible in our context[9]. Therefore, a convenient choice for the Hori kernel of the present problem is given by $\mathcal{H}_0 = -\mathrm{i}T_0$.

3.3. *Solution of the auxiliary system in the NSCCV*

3.3.1. *First solutions*

The auxiliary system is formed from Equations (23) with $\mathcal{H}_0^* = \mathcal{H}_0 = -\mathrm{i}T_0$. Considering the functional form of T_0 (Equation 51), the evolution of the canonical variables in the unperturbed problem are of three different kinds.

First, there is a group of variables that keep constant, since their respective coordinate or momentum are absent in the T_0 expression. Namely,

$$\begin{aligned} &Y_1 = Y_{10},\ Y_3 = Y_{30},\ y_3 = y_{30}, \\ &Y_{1c} = Y_{1c0},\ Y_{3c} = Y_{3c0},\ y_{1c} = y_{1c0},\ y_{3c} = y_{3c0}. \end{aligned} \tag{52}$$

Some of them will enter in the numerical evaluation of our formulae, with the expressions

$$Y_{10} = -\mathrm{i}\ C\omega_E, Y_{1c0} = -\mathrm{i}C_c\omega_E, Y_{30} = -\mathrm{i}\ C\omega_E\cos I,\ y_{30} = \lambda, \tag{53}$$

where it must be understood that all the former values are referred to the epoch J2000 (Table 1).

We also have a variable that evolves linearly with time, since its time rate is constant according to Equations (52) and (53)

$$\frac{dy_1}{dt} = \mathrm{i}\frac{Y_1 - Y_{1c}}{C_m} \Rightarrow y_1 = \omega_E t + y_{10}. \tag{54}$$

[9] The order of magnitude of the variables Y_2, y_2, Y_{2c}, y_{2c} can be estimated with the help of Equation (32), with a value of about 10^{-6} radians for σ and σ_c. With respect to Y_1, we can write $Y_1 \simeq -\ \mathrm{i}C\omega_E(1+m_3)$. Here, m_3 is the variation of the z component of $\vec{\omega}$, $\omega_z = \omega_E(1+m_3)$, with $m_3 \sim 10^{-8}$ (Gross 2015). We have employed a similar expression for the core. There are other possibilities to estimate Y_1 like, for example, that of Williams (1994), with no significant change in the obtained numerical order of magnitude.

Finally, the variables Y_2, y_2, Y_{2c}, y_{2c} have coupled their dynamics, since their evolution obeys to the linear differential system

$$\frac{d}{dt}\begin{pmatrix} y_2 \\ Y_{2c} \\ Y_2 \\ y_{2c} \end{pmatrix} = \mathrm{i}\begin{pmatrix} \mathbf{R} & \mathbf{0}_2 \\ \mathbf{0}_2 & -\mathbf{R} \end{pmatrix}\begin{pmatrix} y_2 \\ Y_{2c} \\ Y_2 \\ y_{2c} \end{pmatrix}, \tag{55}$$

where the matrix $\mathbf{R}$ is given by

$$\mathbf{R} = \begin{pmatrix} r_1 & r_2 \\ -r_2 & r_3 \end{pmatrix} = \frac{\omega_E}{A_m}\begin{pmatrix} A_m - C & \sqrt{CC_c} \\ -\sqrt{CC_c} & AC_c/A_c \end{pmatrix}. \tag{56}$$

In Equation (55) we have ordered the matrix column in such a way that the resulting system is further simplified. Indeed, due to the block structure of the matrix of the differential system given in Equation (55), the four dimensional system can be described by two decoupled two dimensional systems as

$$\frac{d}{dt}\begin{pmatrix} y_2 \\ Y_{2c} \end{pmatrix} = \mathrm{i}\mathbf{R}\begin{pmatrix} y_2 \\ Y_{2c} \end{pmatrix}, \; \frac{d}{dt}\begin{pmatrix} Y_2 \\ y_{2c} \end{pmatrix} = -\mathrm{i}\mathbf{R}\begin{pmatrix} Y_2 \\ y_{2c} \end{pmatrix}. \tag{57}$$

This is a significant simplification stemming from the complex character of the $NSCCV$ and will facilitate the second order integration. Moreover, since the involved pairs of the canonical variables are complex conjugate, once computed the solution of the first system, with the initial conditions y_{20} and Y_{2c0},

$$\begin{pmatrix} y_2 \\ Y_{2c} \end{pmatrix} = e^{\mathrm{i}\mathbf{R}t}\begin{pmatrix} y_{20} \\ Y_{2c0} \end{pmatrix}, \tag{58}$$

the second one is automatically derived from

$$Y_2 = \bar{y}_2, \; y_{2c} = \bar{Y}_{2c}, \tag{59}$$

where $\bar{z}$ denotes the complex conjugate of z.

3.3.2. *Literal expression of the exponential matrix $e^{\mathrm{i}\mathbf{R}t}$*

To finish the integration of the auxiliary problem, it is necessary to compute explicitly the matrix $e^{\mathrm{i}\mathbf{R}t}$. The availability of such a solution (Hori 1973) will simplify the calculations of the generating functions $\mathcal{W}_1$ and $\mathcal{W}_2$ (Equations 21). The procedure that we follow is described in Gantmacher (1959, Chapter 5) —se also Apostol (1969, Chapter 7)— and relies on determining the eigenvalues of the matrix $\mathbf{R}$.

First, we introduce the adimensional parameters describing the Poincaré model of the Earth

$$e = \frac{(C-A)}{A},\; e_c = \frac{(C_c - A_c)}{A_c},\; r_{cm} = \frac{A}{A_m}. \tag{60}$$

The parameters e and e_c are referred to as ellipticities[10] with values of about 10^{-3}. In terms of them, the elements of the matrix $\mathbf{R}$ (Equation 56) are

$$\begin{aligned} r_1 &= -\omega_E[r_{cm} + e(1+r_{cm})], \\ r_2 &= \omega_E\sqrt{r_{cm}(1+r_{cm})(1+e)(1+e_c)}, \\ r_3 &= \omega_E(1+r_{cm})(1+e_c). \end{aligned} \tag{61}$$

So, the characteristic equation of $\mathbf{R}$ has the form

$$m^2 - \omega_E[1-(1+r_{cm})(e-e_c)]m - \omega_E^2(1+r_{cm})e(1+e_c) = 0. \tag{62}$$

Its solutions can be written as

$$m_1 = \omega_E\left(1 + \frac{1}{P_{FCN}}\right),\; m_2 = -\frac{\omega_E}{P_{CW}}, \tag{63}$$

where P_{CW} and P_{FCN} denote the periods related to the well-known normal modes for a Poincaré model (Moritz & Mueller 1987, Chapter 3), i.e., the Chandler Wobble (CW) and the Free Core Nutation (FCN)[11]. It is customarily to express them at the first order in e and e_c. Then, Equation (62) leads to the familiar approximated expressions

$$\frac{1}{P_{FCN}} = (1+r_{cm})e_c,\; \frac{1}{P_{CW}} = (1+r_{cm})e. \tag{64}$$

[10] The dynamical ellipticity is given by $H_d = (C-A)/C = e/(1+e)$.

[11] The mode corresponding to $-m_1$ is referred to as Nearly Diurnal Free Wobble —NDFW— (e.g., Smith 1980).

To construct the matrix $e^{\mathrm{i}\mathbf{R}t}$ the relevant fact is that the structure of $\mathbf{R}$ provides two distint eigenvalues. So, it is possible to apply the Lagrange interpolation polynomial that defines a function of a matrix (Gantmacher 1959, Chapter 5), in this case the exponential one. We can, however, get a more compact result if first we rewrite $\mathbf{R}$ in terms of its eigenvalues. There are different possibilities to do that, one of them is given by

$$\mathbf{R} = \begin{pmatrix} r_1 & r_2 \\ -(m_1 - r_1)(m_2 - r_1)/r_2 & m_1 + m_2 - r_1 \end{pmatrix}, \tag{65}$$

where the values of $m_1 m_2$ and $m_1 + m_2$ can be derived from Equation (62) —with no approximation—

$$m_1 m_2 = -\omega_E^2(1 + r_{cm})e(1 + e_c), \;\; m_1 + m_2 = \omega_E[1 - (1 + r_{cm})(e - e_c)]. \tag{66}$$

In this way, the expression that will be employed in the second order integration for $e^{\mathrm{i}\mathbf{R}t}$ is given by

$$e^{\mathrm{i}\mathbf{R}t} = \frac{1}{m_1 - m_2} \begin{pmatrix} (r_1 - m_2)e^{\mathrm{i}m_1 t} - (r_1 - m_1)e^{\mathrm{i}m_2 t} & r_2\big(e^{\mathrm{i}m_1 t} - e^{\mathrm{i}m_2 t}\big) \\ -r_2\big(e^{\mathrm{i}m_1 t} - e^{\mathrm{i}m_2 t}\big) & -(r_1 - m_1)e^{\mathrm{i}m_1 t} + (r_1 - m_2)e^{\mathrm{i}m_2 t} \end{pmatrix}. \tag{67}$$

4. SECOND ORDER MOTION OF THE ANDOYER PLANE: ANALYTICAL SOLUTION

4.1. *Practical application of the perturbation method*

The Hamiltonian framework in the *NSCCV* developed so far is totally general and allows constructing the second order solution of any quantity related to the rotation of the Poincaré model. Moreover, it keeps its utility for studying other two-layer Earth models that preserve the canonical structure, because their deviations with respect to the Poincaré model —like the mantle elasticity— can be considered as perturbations. Hence, the Hori kernel and the auxiliary system would have the same form as that introduced in Section 3, together with the derived explicit solution of the unperturbed Hamiltonian.

In the remaining sections of this article, we focus on determining the second order nutations of the Andoyer plane, i.e., the second order solution of the Poisson terms. The involved algebraic manipulations, even with the use of a CAS like Maple, are much cumbersome than in the rigid case due to the higher dimension of the present problem. However, they can be simplified to some extent by establishing a parallelism with the procedure developed to obtain the second order solution for the rigid Earth by Getino et al. (2010), which we will follow here[12]. This is possible thanks to the introduction of the *NSCCV*.

Specifically, the key point is that the role played by the monomials in σ in the rigid Earth model, now it is played by the monomials in Y_2, y_2, Y_{2c}, and y_{2c}. To lighten the notation, we represent such several variables monomials of degree[13] k by ζ^k. Therefore, the same arguments as those given by Getino et al. (2010, Section 5, Equations 74 and 75) make possible that we can consider truncated expansions in Y_2, y_2, Y_{2c}, and y_{2c} for the different functions involved in the construction of the second order solutions.

In particular, the nutations of the Andoyer plane are specified by obtaining the evolution of the functions (Equations 41 and 42)

$$\lambda = y_3, \; I = \arccos\left(\frac{Y_3}{Y_1}\right), \tag{68}$$

which are functions of degree 0 in ζ, as it is the case in σ for the rigid Earth model. Therefore, it is just necessary to keep first degree monomials ζ^1 in the perturbing potential V, the additional term E, and the first order generating function $\mathcal{W}_1$; and zero degree monomials ζ^0 in the second order generating function $\mathcal{W}_2$ and transformed Hamiltonian $\mathcal{H}^*$(see also Table 1 in Getino et al. 2010). Those truncations will make easier the computations.

We must underline that those practical simplifications are both dependent on the function whose solution is obtained and the order of perturbation. For example, for a first order solution of the Poisson terms, it is enough with keeping zero degree monomials ζ^0 in all the functions. If we maintain the same scheme for the second order, we would lose some contributions, precisely those depending on the Earth structure as we will show. Oppolzer terms at the second order would require keeping

[12] As far as possible we adopt the notations given in Getino et al. (2010), so their detailed explanation can be consulted in that work.

[13] For example, a monomial ζ^2 of the second degree has the form $Y_2^{k_1} y_2^{k_2} Y_{2c}^{k_3} y_{2c}^{k_4}$, where k_i are non-negative integers with $\sum_{i=1}^{4} k_i = 2$.

second degree monomials ζ^2 in some functions, what complicates significantly the computations even with respect to the second order Poisson terms.

Taking into account the former considerations it is possible to compute the first and second order functions entering in the Hori's method (Equations 21 and 22). The main steps in those calculations are presented in Appendix B.

4.2. *Precession*

4.2.1. *General form*

The equations of motion (Equations 4 and 5) determine the secular evolution of the longitude (λ) and obliquity (I) of the Andoyer plane that are functions of the new canonical variables (Equations 68). They stem from the transformed Hamiltonian $\mathcal{H}^*$ (Equations 50, B6, and B12), which has the following functional dependencies in the new variables (to lighten the notation we have omitted their asterisks)

$$\mathcal{H}^* = \mathcal{H}_0^*(Y_1, Y_2, y_2; Y_{1c}, Y_{2c}, y_{2c}) + \mathcal{H}_1^*(Y_1, Y_3, y_3) + \mathcal{H}_2^*(Y_1, Y_3). \tag{69}$$

Thus, we get

$$\begin{aligned} \frac{d\lambda}{dt} &= \{\lambda; \mathcal{H}^*\} = \frac{\mathrm{i}}{Y_1 \sin I} \frac{\partial(\mathcal{H}_1^* + \mathcal{H}_2^*)}{\partial I}, \\ \frac{dI}{dt} &= \{I; \mathcal{H}^*\} = -\frac{\mathrm{i}}{Y_1 \sin I} \frac{\partial \mathcal{H}_1^*}{\partial y_3}. \end{aligned} \tag{70}$$

The value of the variable Y_1 in the former equations is constant (y_1 is not present in $\mathcal{H}^*$) and taken equal to that of the unperturbed problem (Equation 53). Those relationships provide the precession of the Andoyer plane.

Their expressions are derived from Equations (B6, and (B12), resulting

$$\begin{aligned} \frac{d\lambda}{dt} &= S_E^L + \sum_{p=S,M} \frac{k_p}{\sin I} S_1^L + \sum_{p,q=S,M} \frac{k_p k_q}{\sin^2 I} \sum_{\tau,\rho=\pm 1} \left[\sum_{i,j\neq 0} S_{2a}^L + \sum_{i,j} S_{2b}^L \right]_{\tau\Theta_i=\rho\Theta_j}, \\ \frac{dI}{dt} &= S_E^O, \end{aligned} \tag{71}$$

where $k_{p,q} = k'_{p,q}/(C\omega_E)$. We have followed the same notation as that introduced in Getino et al. (2010, Section 4.1): the superscripts L and O denote the longitude and the obliquity, respectively; and the subscripts the origin of the contribution[14] (E for the complementary term, 1 for $\mathcal{H}_1^*$, and 2 for $\mathcal{H}_2^*$).

4.2.2. *Formulae*

The explicit formula of each function S_α^β is given by[15]

$$
\begin{aligned}
S_E^L = &\ e_1 - (e_2\cos\lambda + e_3\sin\lambda)\frac{\cos I}{\sin I}, \\
S_E^O = &\ e_3\cos\lambda - e_2\sin\lambda, \\
S_1^L = &\ -B_0', \\
S_{2a}^L = &\ \frac{1}{4}\frac{m_{5i}}{n_i}\left[B_i\left(B_j'' - \frac{\cos I}{\sin I}B_j'\right) + B_i'B_j'\right],
\end{aligned}
\tag{72}
$$

and

$$
S_{2b}^L = \frac{1}{2}\frac{\omega_E - \tau n_i - r_3}{\prod\limits_{k=1,2}(\omega_E - \tau n_i - m_k)}\sin I\left(C_{i,\tau}'C_{j,\rho} + C_{i,\tau}C_{j,\rho}'\right). \tag{73}
$$

The above formulae determine the second order solution of the secular motion of the Poisson terms. The second order part is characterized by the terms proportional to k_pk_q, i.e., ε^2. There is no equivalent terms of the form k_pe_i or e_ie_j, since the whole additional term E is secular (Equation B5). The moving ecliptic term E is the single responsible of the contribution to the precession in obliquity, because it is the only one providing a dependence in the variable y_3 (Equations B13). Such circumstances are similar in the rigid case.

The most interesting feature of the solution given by Equation (71) is the dependence on the Earth model through the term S_{2b}^L (Equation 73). It comes from $\mathcal{H}_2^*$ and involves an indirect influence of the

[14] There is a typo in Getino et al. (2010) in the fifth line after Equation 51: when referring to $\mathcal{W}_1$ and $\mathcal{W}_2$, it must appear $\mathcal{H}_1^*$ and $\mathcal{H}_2^*$, respectively.

[15] Let us recall that the prime in an orbital function denotes derivation with respect to I.

core due to r_3 and the normal modes of the Poincaré Earth model (Equations B12). So, in contrast to first order contribution S_1^L, some second order terms of the precession in longitude do depend on the Earth's interior. A result first pointed out in Ferrándiz et al. (2004) —see also Baenas et al. (2017).

In Appendix C we show the equivalence of our precession formulate when reducing the Poincaré model to a rigid one (Getino et al. 2010), and with those derived in Ferrándiz et al. (2004) and Baenas et al. (2017) for the second order precession of the Poincaré model employing the Andoyer variables in the modeling.

4.3. *Nutation*

4.3.1. *General form*

The quasi-periodic motion of λ and I, i.e., the nutations of the Andoyer plane, can be computed with the help of Equations (26) and then numerical evaluated through the solutions of the transformed Hamiltonian. They arise from $\mathcal{W}_1$, $\mathcal{W}_2$, and the $\mathcal{W}_1$ crossed terms ($\Delta_3 f$ in Equation 26)

$$\begin{aligned} \Delta\lambda = &\ \{\lambda;\mathcal{W}_1\} + \{\lambda;\mathcal{W}_2\} + \frac{1}{2}\{\{\lambda;\mathcal{W}_1\};\mathcal{W}_1\}, \\ \Delta I = &\ \{I;\mathcal{W}_1\} + \{I;\mathcal{W}_2\} + \frac{1}{2}\{\{I;\mathcal{W}_1\};\mathcal{W}_1\}. \end{aligned} \tag{74}$$

Those computation are made easier if the second order generating function (Appendix B) is split as

$$\mathcal{W}_2 = \mathcal{W}_{2s} + \mathcal{W}_{2p} = \int_{UP} \{\mathcal{H}_{1\mathrm{sec}};\mathcal{W}_1\}\ dt + \frac{1}{2}\int_{UP} \{\mathcal{H}_{1\mathrm{per}};\mathcal{W}_1\}\ dt, \tag{75}$$

In this way, the Poisson brackets can be computed as in Equations (70), but now considering the following functional dependencies (Equations B7, B13, and B14)

$$\mathcal{W}_1 = \mathcal{W}_1(Y_1, Y_2, Y_3, y_1, y_2, y_3; Y_{2c}, y_{2c}; t),\ \mathcal{W}_{2s} = \mathcal{W}_{2s}(Y_1, Y_3, y_3; t),\ \mathcal{W}_{2p} = \mathcal{W}_{2p}(Y_1, Y_3, y_3; t). \tag{76}$$

The evaluation of the direct Poisson bracket with $\mathcal{W}_1$ can be done disregarding the terms proportional to ζ^1, since the resulting expressions are considered at ζ^0 degree as we have pointed out previously. This is not the case of the crossed term involving $\mathcal{W}_1$. Here, it is necessary to compute the Poisson brackets considering the whole expression of $\mathcal{W}_1$, which, afterwards, can be truncated at ζ^0.

The resulting nutations can be ordered in a similar way and with the same notation as done in Getino et al. (2010, Section 4.1). Therefore, we can write the nutations in longitude as

$$\begin{aligned}\Delta\lambda = & \sum_{p=S,M}\frac{k_p}{\sin I}\sum_{i\neq 0}\Big[(\mathcal{L}_1+\mathcal{L}_E^{in})\sin\Theta_i+\mathcal{L}_E^{out}\cos\Theta_i\Big]+\\ & \sum_{p,q=S,M}\sum_{i\neq 0}\frac{k_pk_q}{\sin^2 I}\mathcal{L}_2^s\sin\Theta_i+\\ & \sum_{p,q=S,M}\frac{k_pk_q}{\sin^2 I}\sum_{\tau,\rho=\pm 1}\Bigg[\sum_{i,j\neq 0}(\mathcal{L}_2^{p1}+\mathcal{L}_3^1)\sin(\tau\Theta_i-\rho\Theta_j)+\\ & \sum_{i,j}(\mathcal{L}_2^{p2}+\mathcal{L}_3^2)\sin(\tau\Theta_i-\rho\Theta_j)\Bigg]_{\tau\Theta_i\neq\rho\Theta_j},\end{aligned} \tag{77}$$

and in obliquity

$$\begin{aligned}\Delta I = & \sum_{p=S,M}\frac{k_p}{\sin I}\sum_{i\neq 0}\Big[(\mathcal{O}_1+\mathcal{O}_E^{in})\cos\Theta_i+\mathcal{O}_E^{out}\sin\Theta_i\Big]+\\ & \sum_{p,q=S,M}\sum_{i\neq 0}\frac{k_pk_q}{\sin^2 I}\mathcal{O}_2^s\cos\Theta_i+\\ & \sum_{p,q=S,M}\frac{k_pk_q}{\sin^2 I}\sum_{\tau,\rho=\pm 1}\Bigg[\sum_{i,j\neq 0}(\mathcal{O}_2^{p1}+\mathcal{O}_3^1)\cos(\tau\Theta_i-\rho\Theta_j)+\\ & \sum_{i,j}(\mathcal{O}_2^{p2}+\mathcal{O}_3^2)\cos(\tau\Theta_i-\rho\Theta_j)\Bigg]_{\tau\Theta_i\neq\rho\Theta_j}.\end{aligned} \tag{78}$$

The amplitudes have been denoted by $\mathcal{L}_\alpha^\beta$ for the longitude and $\mathcal{O}_\alpha^\beta$ for the obliquity. As in the case of the precession, the subscripts reflect the first, second, or additional term origin of each amplitude. In the case of E terms, the superscripts denote the in-phase and out-of-phase contributions. For the remaining second order amplitudes, we have that the superscript s amplitudes come from $\mathcal{W}_{2s}$; $p1$ and $p2$ ones from $\mathcal{W}_{2p}$; and 1 and 2 from $\mathcal{W}_1$ crossed terms.

4.3.2. *Formulae*

In the case of the first order solution of the Poisson terms, the explicit expressions are (Equation B7)

$$\mathcal{L}_1 = -\frac{B'_i}{n_i}, \mathcal{O}_1 = -m_{5i}\frac{B_i}{n_i}, \tag{79}$$

which are independent of the Earth model. This fact is well-known in the literature (e.g., Moritz & Mueller 1987, Chapter 3), but, as we are emphasizing, that affirmation must be restricted just to the Poisson terms in a first order solution.

For the second order terms, first we consider those emerging from the ecliptic of date motion (Equations B13). They provide in-phase contributions

$$\begin{aligned}\mathcal{L}_E^{\rm in} &= -\frac{m_{5i}}{n_i^2}\left[e_1 B'_i + \frac{(e_2\cos\lambda + e_3\sin\lambda)}{\sin I}\left(\frac{B_i}{\sin I} - \cos I\ B'_i\right)\right],\\ \mathcal{O}_E^{\rm in} &= -\frac{1}{n_i^2}\left[m_{5i}^2 e_1 B_i + (e_2\cos\lambda + e_3\sin\lambda)\left(B'_i - m_{5i}^2\frac{\cos I}{\sin I}B_i\right)\right],\end{aligned} \tag{80}$$

and out-of-phase ones

$$\begin{aligned}\mathcal{L}_E^{\rm out} &= \frac{1}{n_i^2}(e_2\sin\lambda - e_3\cos\lambda)B''_i,\\ \mathcal{O}_E^{\rm out} &= \frac{m_{5i}}{n_i^2}(e_2\sin\lambda - e_3\cos\lambda)\left(\frac{\cos I}{\sin I}B_i - B'_i\right).\end{aligned} \tag{81}$$

Those terms are also model-independent and of the second order through $k_p e_i$. For the same reasons as those explained in the case of the precession, the additional term does not give raise to any second order term of the form $e_i e_j$.

This kind of second order terms arise because of our choice for the Hori kernel H_0^* is free from any contribution coming from E (Equation 50). Hence, its effects stem from the coupling with $\mathcal{W}_1$ (Equation B10), including some out-of-phase terms not related with any dissipative torque —absent in our model. Since the structure of Equations (80) and (81) is the same as in the rigid Earth model, we refer the reader to Getino et al. (2010, Section 5) to get further explanations about this question.

The remaining second order terms are proportional to $k_p k_q$. Those due to $\mathcal{W}_{2s}$ are given by

$$
\begin{aligned}
\mathcal{L}_2^s &= \frac{m_{5i}}{n_i^2}\left[B_i\left(B_0'' - \frac{\cos I}{\sin I}B_0'\right) + B_i'\ B_0'\right],\\
\mathcal{O}_2^s &= \frac{m_{5i}^2}{n_i^2} B_i\ B_0',
\end{aligned}
\tag{82}
$$

which do not depend on the Earth's interior, being also common with the rigid Earth solution developed in Getino et al. (2010).

As in the case of the terms proportional to $k_p e_i$ (Equations 80 and 81), other rigid Earth theories (e.g., Souchay et al. 1999) obtain these amplitudes as first order contributions. The reason is that they include the secular part of the perturbing potential V (Equation 14) in the Hori kernel of their problem (see Getino et al. 2010, Section 5).

The next amplitudes proportional to $k_p k_q$ come from $\mathcal{W}_{2p}$. We can distinguish a model independent part

$$
\begin{aligned}
\mathcal{L}_2^{p1} &= \frac{1}{8}\frac{1}{\tau n_i - \rho n_j}\left(\frac{1}{\tau n_i} + \frac{1}{\rho n_j}\right)\left[\tau m_{5i} B_i\left(B_j'' - \frac{\cos I}{\sin I}B_j'\right) + \rho m_{5j} B_i' B_j'\right],\\
\mathcal{O}_2^{p1} &= \frac{1}{8}\frac{\tau m_{5i}}{\tau n_i - \rho n_j}\left(\frac{1}{\tau n_i} + \frac{1}{\rho n_j}\right)\left(\tau m_{5i} B_i B_j' + \rho m_{5j} B_i' B_j\right),
\end{aligned}
\tag{83}
$$

and a dependent one given by

$$
\begin{aligned}
\mathcal{L}_2^{p2} &= \frac{\sin I}{2}\frac{1}{(\tau n_i - \rho n_j)}\frac{\omega_E - \tau n_i - r_3}{\prod\limits_{k=1,2}(\omega_E - \tau n_i - m_k)}\left(C_{i,\tau}' C_{j,\rho} + C_{i,\tau} C_{j,\rho}'\right),\\
\mathcal{O}_2^{p2} &= \frac{\sin I}{2}\frac{\tau m_{5i} - \rho m_{5j}}{(\tau n_i - \rho n_j)}\frac{\omega_E - \tau n_i - r_3}{\prod\limits_{k=1,2}(\omega_E - \tau n_i - m_k)}\ C_{i,\tau} C_{j,\rho}.
\end{aligned}
\tag{84}
$$

This set of terms arises from the summand in $\mathcal{C}_P$ (Equation B11), which depends on the fluid core and entails an indirect contribution (like in S_{2b}^L, Equation 73). We call this dependence indirect because the same form is kept for the rigid Earth, although with different values of r_3 and normal modes of the Poincaré model, m_1 and m_2, just reduced to the the Eulerian one.

Finally, the $\mathcal{W}_1$ crossed terms also provide second order contributions that can be split in the same way. One part does not depend on the Earth model

$$\mathcal{L}_3^1 = \frac{1}{8}\frac{1}{\tau n_i\,\rho n_j}\left[\tau m_{5i} B_i' B_j' + \rho m_{5j} B_j \left(B_i'' - \frac{\cos I}{\sin I} B_i'\right)\right],$$
$$\mathcal{O}_3^1 = \frac{1}{8}\frac{\tau m_{5i}}{\tau n_i \rho n_j}\left[\tau m_{5i} B_i B_j' + \rho m_{5j} B_j \left(B_i' - \frac{\cos I}{\sin I} B_i\right)\right], \tag{85}$$

and another one does depend on it

$$\mathcal{L}_3^2 = \frac{\sin I}{2}\frac{(\omega_E - \tau n_i - r_3)(\omega_E - \rho n_j - r_3) - r_2^2}{\prod\limits_{k=1,2}[(\omega_E - \tau n_i - m_k)(\omega_E - \rho n_j - m_k)]} C_{i,\tau} C'_{j,\rho},$$
$$\mathcal{O}_3^2 = \frac{\sin I}{2}(\cos I - \tau m_{5i})\frac{(\omega_E - \tau n_i - r_3)(\omega_E - \rho n_j - r_3) - r_2^2}{\prod\limits_{k=1,2}[(\omega_E - \tau n_i - m_k)(\omega_E - \rho n_j - m_k)]} C_{i,\tau} C_{j,\rho}. \tag{86}$$

In this case the core contribution to the second order solution is twofold. There is an indirect contribution and a direct one that comes from r_2^2, which is linked to the core. It would totally disappear for the rigid Earth model since there are no core parameters, i.e., $r_2 = 0$ (Equation 61). For further comparisons it is convenient to separate them, as it was done for studying the precession motion at the second order (Baenas et al. 2017). The indirect parts are given by

$$\mathcal{L}_{3\text{-id}}^2 = \frac{\sin I}{2}\frac{(\omega_E - \tau n_i - r_3)(\omega_E - \rho n_j - r_3)}{\prod\limits_{k=1,2}[(\omega_E - \tau n_i - m_k)(\omega_E - \rho n_j - m_k)]} C_{i,\tau} C'_{j,\rho},$$
$$\mathcal{O}_{3\text{-id}}^2 = \frac{\sin I}{2}(\cos I - \tau m_{5i})\frac{(\omega_E - \tau n_i - r_3)(\omega_E - \rho n_j - r_3)}{\prod\limits_{k=1,2}[(\omega_E - \tau n_i - m_k)(\omega_E - \rho n_j - m_k)]} C_{i,\tau} C_{j,\rho}. \tag{87}$$

and the direct ones by

$$\mathcal{L}_{3\text{-d}}^2 = -\frac{\sin I}{2}\frac{r_2^2}{\prod\limits_{k=1,2}[(\omega_E - \tau n_i - m_k)(\omega_E - \rho n_j - m_k)]} C_{i,\tau} C'_{j,\rho},$$
$$\mathcal{O}_{3\text{-d}}^2 = -\frac{\sin I}{2}(\cos I - \tau m_{5i})\frac{r_2^2}{\prod\limits_{k=1,2}[(\omega_E - \tau n_i - m_k)(\omega_E - \rho n_j - m_k)]} C_{i,\tau} C_{j,\rho}. \tag{88}$$

The last amplitudes are the single ones that lead to a direct contribution from the core within our model. In Appendix C we proof the equivalence of our nutation formulate when reducing the Poincaré model to a rigid one (Getino et al. 2010).

5. NUMERICAL RESULTS AND DISCUSSION

5.1. *Dependencies of the amplitudes on Earth structure*

The determined analytical solutions for the Poisson terms (Equations 72 and 73; and Equations from 79 to 86) depend on orbital (motions of the Moon and the Sun) and Earth model parameters. As we have explained in Section 4, there are contributions that are independent of the Earth model. They correspond to the terms

$$S_E^L, S_E^O, S_1^L, S_{2a}^L \tag{89}$$

in the case of precession (Equations 72), and

$$\begin{gathered}\mathcal{L}_1, \mathcal{L}_E^{\text{in}}, \mathcal{L}_E^{\text{out}}, \mathcal{L}_2^s, \mathcal{L}_2^{p1}, \mathcal{L}_3^1. \\ \mathcal{O}_1, \mathcal{O}_E^{\text{in}}, \mathcal{O}_E^{\text{out}}, \mathcal{O}_2^s, \mathcal{O}_2^{p1}, \mathcal{O}_3^1\end{gathered} \tag{90}$$

for nutation (Equations 79, 80, 81, 82, 83, and 85).

It is necessary, however, to make a precision: although those contributions are indeed independent of the Earth model features, the particular model enters in the corresponding solution through the parameters k_p or $k_p k_q$, which are proportional to the Earth dynamical ellipticity H_d and H_d^2, respectively. That is the only way in which this kind of Poisson terms are affected by the Earth model (Escapa et al. 2020). Since H_d is a global parameter of the Earth, we could have different interior configurations leading to the same value of H_d. Hence, when analyzing the influence of the Earth's interior in the motion, it is convenient to freeze the value of this parameter in order to isolate the contributions of the structure.

There is a second group of terms that do depend on the Earth interior (they also keep the dependence in $k_p k_q$)

$$S_{2b}^L;\ \mathcal{L}_2^{p2},\ \mathcal{L}_3^2;\ \mathcal{O}_2^{p2},\ \mathcal{O}_3^2, \tag{91}$$

appearing both in precession and nutation (Equations 73, 84, and 86). They are part of the second order solution and show that Poisson terms are affected by the Earth structure. All of them come from the terms of the perturbing potential proportional to the orbital function $C_{i,\tau}$ (Equation B4), i.e., the terms of degree one in ζ, which are combined in different ways to provide the second order contributions as shown.

The particular influence of the Earth model is neatly appreciated in the dependence of the former contributions with the model normal modes. In turn, those are determined by the Earth layered structure —one, two, or three layers— and tuned by features like elasticity, dissipation, etc. In our case, there appear the two characteristic normal modes of the Poincaré model: the Chandler Wobble (CW) and the Nearly Diurnal Free Wobble (NDFW), which induces the Free Core Nutation (FCN). The last one is specially important because it is resonant for some orbital frequencies (e.g., Moritz & Mueller 1987, Chapter 3, Ferrándiz et al. 2004), what can amplify some amplitudes that are negligible in the rigid case.

We can separate the terms given in Equation (91) according to its dependence. We have contributions that are present for a rigid Earth, but affected by the model and its normal modes. Those are the indirect contributions. Among them, we have

$$S_{2b}^{L};\ \mathcal{L}_2^{p2};\ \mathcal{O}_2^{p2}. \tag{92}$$

The direct ones are those disappearing in the rigid case. So, they are exclusively due to the presence of the core and induced by the Y_{2c} and y_{2c} terms in $\mathcal{W}_1$ (Equation B7). The amplitudes $\mathcal{L}_3^2$ and $\mathcal{O}_3^2$, coming from the $\mathcal{W}_1$ crossed terms, have both indirect and direct parts. They are the only ones that provide direct terms, proportional to r_2^2 in this case. Besides to their theoretical interest, the dependencies on the Earth model (Equations 91) are not numerically negligible in the second order solution of the Poisson terms, as we show below.

5.2. *Second-order numerical amplitudes of the Poisson terms*

We determine the numerical contributions of the second order terms in longitude and obliquity of Poisson terms with a twofold objective. First, it will allow ascertaining whether the particular magnitude of the contributions reach or not the threshold of nowadays accuracy targets of the numerical standards of Earth rotation, established about the μas level for the nutation amplitudes (e.g., Ferrándiz et al. 2020). This fact is specially important, because of current IAU nutation model IAU 2000A nutation, based on MHB2000 (Mathews et al. 2002), modeled the second order effects in a inconsistent and incomplete way (Escapa et al. 2020). In the affirmative, it would entail the need of incorporating this kind of second order effects in the next Earth rotation models to be considered by IAU.

The second objective will provide a quantitative information about the influence of the Earth's structure features on Poisson terms, showing its relevance specially when compared with the rigid case due to the fluid core amplification, as it is the case for first order Oppolzer terms (e.g., Moritz & Mueller 1987, Chapter 4, or Getino 1995b).

5.2.1. *Numerical results*

The evaluation of the analytical solutions of the Poisson terms (Equations 72 and 73; and Equations from 79 to 86) is performed by considering a first group of variables and parameters related to the ecliptic motion, some initial conditions of the rotational motion, etc. at the epoch J2000. They are displayed in Table 1.

It is also necessary to provide the values of the arguments Θ_i, time rates n_i, etc. coming from the orbital motion of the Moon and the Sun. They are taken from Getino et al. (2010, Tables 8 and 9) and, for the sake of convenience, are reproduced in Appendix A. They consist of the eleven main arguments Θ_i, whose ratio n_i/ω_E runs, in module, from 0 (infinite period term, i.e., Θ_0) to about 0.11 (nine days period term).

When constructing the second order terms, those orbital arguments combine as $\tau\Theta_i - \rho\Theta_j$, providing second order nutations with arguments Θ_k. Therefore, the initial orbital list of eleven terms is considerably increased, although not all the combinations lead to significant nutations (Table 3). We will denote each combination giving Θ_k, positive or negative, as $(\Theta_i, \Theta_j, \tau, \rho)_{\varepsilon\Theta_k}$with $\varepsilon = \pm 1$. So, the associated frequency n_k of a second order nutation can be generated by different constituents τn_i and ρn_j (see Appendix A for an example). Those constituents are one of the key elements to explain

Table 1. Numerical parameters independent of the Earth model

Parameter	Value
I	-0.4090928041 rd
$\omega_E\,(\simeq \dot{\Phi})$	230121.67526278 rd cy^{-1}
e_1	0 arcsec cy^{-1}
e_2	5.341 arcsec cy^{-1}
e_3	46.82 arcsec cy^{-1}
λ	0 rd

NOTE—Extracted from Table 10 in Getino et al. (2010). The values of the parameters are referred to J2000 and are common for the different numerical computations performed in this work.

the second order amplitudes, more than Θ_k itself. This is a fundamental difference with respect to first order nutations.

As we have explained in Section 4, there is a second group of parameters that depend on the Earth model, necessary to compute numerically the amplitudes at the second order. Commonly, some of them —the basic Earth parameters (BEP)— are determined by a process of data fitting (e.g., Getino & Ferrándiz 2001). Hence, their values will emerge after adjusting the whole theory of the rotation of the Earth, i.e., including all the theoretical contributions to its motion, to the available observations.

Since such a numerical process is out of the scope of this research, the choice of their particular values for a fixed Earth model is conventional to some extent. Nonetheless, it allows providing the order of magnitude of the new second order contributions derived in this work. As we have pointed out, giving precise values would require a re-fitting of a complete theory of the rotation of the Earth. Having in mind those considerations, we have selected the relevant parameters for the Poincaré model of the Earth from some of the BEP fitted in Getino & Ferrándiz (2001) for a two-layer Earth model.

Specifically, we have taken the values given in Table 2. From those values it is possible to obtain m_1 and m_2 directly with the aid of Equations (63). Then, the associated ellipticities of the Earth e and the core e_c are derived by solving numerically the Equations (66), which gives raise to r_1, r_2,

Table 2. Poincaré model parameters

Parameter	Value	Source
k_M	7567.870647 arcsec	Getino & Ferrándiz (2001)
k_S	3474.613747 arcsec	Getino & Ferrándiz (2001)
r_{cm}	0.123234	Getino & Ferrándiz (2001)
P_{CW}	401.80 (sidereal days)	Getino & Ferrándiz (2001)
P_{FCN}	434.13 (sidereal days)	Getino & Ferrándiz (2001)
m_1	230651.750759 rd cy^{-1}	Derived
m_2	$-$572.726917 rd cy^{-1}	Derived
r_1	$-$28931.685571 rd cy^{-1}	Derived
r_2	85799.277026 rd cy^{-1}	Derived
r_3	259010.709413 rd cy^{-1}	Derived

Note—The first five rows are extracted from Table 1 in Getino & Ferrándiz (2001). They were obtained by fitting the nutation amplitudes of a two–layer Earth model to the observations. In that work P_{CW} and P_{FCN} were given in mean solar days and k_S as $k_S = k_M\, k_{S/M}$. The derived values displayed in this table were obtained through Eqs. (63), (66), and (61) and are required to evaluate the nutation amplitudes. See the main text for a discussion.

and r_3 through Equations (61). In this way, the Poincaré model is completely characterized for our purposes. The computations can be done for other different numerical sets (see Appendix D), but there is no essential difference in the order of magnitude of the obtained contributions.

Once fixed the values of the orbital and Earth model parameters, it is possible to compute numerically the second order contributions to Poisson terms. We exclude from our analysis both the second order precession and the second order nutations arising from the amplitudes $\mathcal{L}_E^{\text{in}}$, $\mathcal{L}_E^{\text{out}}$, $\mathcal{L}_2^s$ and $\mathcal{O}_E^{\text{in}}$, $\mathcal{O}_E^{\text{out}}$, $\mathcal{O}_2^s$ (Equations 80, 81, and 82). They were comprehensively discussed in Ferrándiz et al. (2004) and Baenas et al. (2017), and in Getino et al. (2010, Section 5), respectively, and, as we have pointed out, our analytical results are consistent with theirs. Hence, we focus on the remaining second order nutations whose amplitudes are presented in Table 3.

In columns (7) and (12) we have displayed the second order amplitudes $\mathcal{L}_2^{p1} + \mathcal{L}_3^1$ and $\mathcal{O}_2^{p1} + \mathcal{O}_3^1$ (Equations 83 and 85) that are independent of the Earth model, except for the factor $k_p k_q$. As we have pointed out, the corresponding analytical expressions are the same as in Getino et al. (2010,

Table 3. Second order nutations of the Andoyer plane for the Poincaré model (unit: μas)

Argument					Period	$\Delta\lambda$(sin)					ΔI(cos)				
l_M	l_S	F	D	Ω	Days	$\mathcal{L}_2^{p1}+\mathcal{L}_3^1$	$\mathcal{L}_2^{p2}$	$\mathcal{L}_{3-\mathrm{id}}^2$	$\mathcal{L}_{3-\mathrm{d}}^2$	Total	$\mathcal{O}_2^{p1}+\mathcal{O}_3^1$	$\mathcal{O}_2^{p2}$	$\mathcal{O}_{3-\mathrm{id}}^2$	$\mathcal{O}_{3-\mathrm{d}}^2$	Total
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
+0	+0	+0	+0	+1	−6798.38	−29.97	+6.65	+0.54	−4.73	−27.53	+28.92	+43.17	−0.09	+0.83	+72.82
+0	+0	+0	+0	+2	−3399.19	−1226.11	+3.52	−0.02	+0.17	−1222.45	+239.1	−1.96	*	−0.05	+237.09
+0	+0	+0	+0	+3	−2266.13	+21.54	−0.06	*	*	+21.48	−3.79	+0.03	*	*	−3.76
+0	+1	+0	+0	+1	+386.00	+0.99	−0.04	−0.03	+0.23	+1.15	+0.16	+0.05	*	−0.06	+0.15
+0	+1	−2	+2	−3	−385.96	−1.96	*	*	*	−1.95	−0.28	*	*	*	−0.28
+0	+1	+0	+0	+0	+365.26	+1.02	−0.65	*	*	+0.38	−0.11	*	+0.10	−0.86	−0.88
+0	−1	+2	−2	+2	+365.22	−1.47	+0.06	*	−0.07	−1.47	+0.67	−0.04	*	+0.05	+0.68
+0	+1	+0	+0	−1	+346.64	+1.43	+0.05	*	+0.04	+1.51	+0.14	+0.02	*	+0.03	+0.19
+0	+1	−2	+2	−1	−346.60	+1.54	*	*	*	+1.53	+1.21	*	*	*	+1.20
+0	+0	+2	−2	+4	+192.99	+1.40	*	*	*	+1.40	−0.28	*	*	*	−0.28
+0	+0	+2	−2	+3	+187.66	−117.95	+0.28	*	+0.10	−117.58	+17.32	−0.13	*	−0.03	+17.16
+0	+0	+2	−2	+2	+182.62	−0.05	−7.77	−0.04	+0.38	−7.48	+0.02	+3.97	+0.03	−0.28	+3.73
+0	+0	+2	−2	+1	+177.84	+93.11	−0.26	+0.07	−0.57	+92.34	−73.34	+0.28	−0.03	+0.21	−72.88
+0	+0	+2	−2	+0	+173.31	−1.04	+0.03	*	*	−1.02	+0.83	*	*	*	+0.83
+0	+1	+2	−2	+3	+123.97	−4.61	+0.01	*	*	−4.60	+0.68	*	*	*	+0.68
+0	+1	+2	−2	+1	+119.61	+3.64	−0.01	*	−0.02	+3.61	−2.88	+0.01	*	*	−2.86
+0	+0	+4	−4	+4	+91.31	−4.27	+0.04	*	−0.06	−4.29	+0.85	−0.02	*	+0.02	+0.85
+1	+0	+0	+0	+1	+27.67	+0.66	*	*	+0.02	+0.68	*	*	*	*	−0.01
+1	+0	+0	+0	−1	+27.44	+0.67	*	*	+0.02	+0.69	*	*	*	*	+0.02
+0	+0	+0	+2	+0	+14.77	+1.33	+0.13	*	*	+1.46	−0.82	*	+0.01	−0.06	−0.87
+0	+0	+2	+0	+3	+13.69	−19.12	+0.04	*	*	−19.08	+2.85	−0.02	*	*	+2.83
+0	+0	+2	+0	+2	+13.66	−4.88	−0.93	*	*	−5.80	+0.95	+0.49	*	*	+1.45
+0	+0	+2	+0	+1	+13.63	+15.19	−0.03	*	+0.05	+15.20	−12.03	+0.08	*	−0.02	−11.96
+0	+0	+2	+0	+0	+13.61	−2.16	+0.02	*	*	−2.14	−0.46	*	*	*	−0.46
+0	+0	+4	−2	+4	+12.71	−1.38	+0.01	*	−0.02	−1.39	+0.27	*	*	*	+0.28
+1	+0	+2	+0	+3	+9.14	−2.45	*	*	*	−2.45	+0.37	*	*	*	+0.36
+1	+0	+2	+0	+1	+9.12	+1.95	*	*	−0.02	+1.93	−1.54	*	*	*	−1.53

NOTE—The terms whose total amplitude in longitude or obliquity is, in absolute value, equal or greater than 0.5 μas have been displayed. The symbol "*" designs amplitudes whose absolute value is below 0.01 μas, accordingly the internal accuracy used in the computations. The same conventions will be followed for similar tables in this work. The amplitudes result from the parameters displayed in Table 1 and Table 2, and the nutation formulae given in Eqs. (83, 84, 85, 87, and 88). Columns (7) and (12) contain second order amplitudes independent of the Earth model. Columns (8), (9) and (10), and (13), (14), and (15) represent the dependent parts. The indirect effects of the fluid core are given in columns (8) and (9), and (13) and (14); whereas the direct ones are displayed in columns (10) and (15). See the main text for a discussion.

Appendix D). So, their numerical contributions are similar to those appearing in Table 3 by Getino et al. (2010). The slight differences are due to the k_M and k_S values employed in that work, which were borrowed from the rigid Earth theory REN2000 (Souchay et al. 1999) in contrast to the values employed here (Table 2). Generally speaking, the larger second order contributions arise from combinations $(\Theta_i, \Theta_j, \tau, \rho)_{\varepsilon\Theta_k}$, with small values of n_i and n_j; large orbital functions (Appendix A); and not nil integers m_{5i} and m_{5j} (Equations 83 and 85).

The remaining columns (8) to (10) and (13) to (15) are affected by the presence of the fluid core, both in indirect and direct ways. Specifically, the indirect contributions are given by $\mathcal{L}_2^{p2}$ (Equation 84) and $\mathcal{L}^2_{3\text{-id}}$, columns (8) and (9), and $\mathcal{O}_2^{p2}$ (Equation 84) and $\mathcal{O}^2_{3\text{-id}}$, columns (13) and (12); the direct ones by $\mathcal{L}^2_{3\text{-d}}$, column (10), and $\mathcal{O}^2_{3\text{-d}}$.

Although the magnitudes of that structure dependent amplitudes are usually below 1 μas, some terms contribute in a significant way considering nowadays accuracies and cannot be neglected. Hence, in addition to its theoretical interest, the structure dependent part of the second order Poisson terms is numerically relevant. It is clear the case for the terms with periods −6798.38, −3399.19, and 182.62 days, with a very significant contribution in obliquity for the term with period −6798.38 days, of about forty μas.

5.2.2. *Numerical differences with the rigid case*

There is a very significant difference among the values of $\mathcal{L}_2^{p2}$, $\mathcal{L}_3^2$, $\mathcal{O}_2^{p2}$, and $\mathcal{O}_3^2$ previously displayed and those ones of the rigid case, which can be obtained by evaluating our analytical formulae reducing the Poincaré model of the Earth to a rigid one. To this end, the particularized rigid parameters are derived following the same guidelines as in Section C.1, i.e., with $A_c = 0$ and keeping $e_c = 0$, alternatively taking $P_{FCN} \to +\infty$ (Equation 64), in the Poincaré model. With those values it is possible to obtain r_1, r_2, r_3, m_1 and m_2 (Equations C15, C16, and C17) needed to compute the Poisson terms. The values of k_M, k_S, and P_{CW} are the same as in Tables 1 and 2.

By doing so, the obtained differences can be attributed just to the influence of the fluid core. It is also possible to obtain the rigid amplitudes from the formulae given in Getino et al. (2010, Appendix D), by considering those values and $n_\mu = \omega_E\big(1 + P_{CW}^{-1}\big) = 230694.40$ rd/cy (Equations 63 and C18) .

Table 4. Poincaré and rigid Earth models: Structure dependent part of the second order Poisson terms (unit: μas)

Argument					Period	$\Delta\lambda$(sin)			ΔI(cos)		
l_M	l_S	F	D	Ω	Days	Poi.	Rig.	Dif.	Poi.	Rig.	Dif.
+0	+0	+0	+0	+1	−6798.38	+2.46	−0.45	+2.91	+43.91	+1.39	+42.52
+0	+0	+0	+0	+2	−3399.19	+3.67	+0.06	+3.61	−2.01	−0.04	−1.97
+0	+1	+0	+0	+0	+365.26	−0.65	+0.03	−0.68	−0.76	+0.01	−0.77
+0	+0	+2	−2	+2	+182.62	−7.43	−0.20	−7.23	+3.72	+0.11	+3.61
+0	+0	+2	−2	+1	+177.84	−0.76	*	−0.76	+0.46	*	+0.46
+0	+0	+2	+0	+2	+13.66	−0.93	−0.04	−0.89	+0.49	+0.02	+0.47

Note—The displayed amplitudes are the structure dependent part of the second order Poisson terms, i.e., $\mathcal{L}_2^{p2}$, $\mathcal{L}_3^2$, $\mathcal{O}_2^{p2}$, and $\mathcal{O}_3^2$. The Poincaré ones correspond with the sums in Table 3 of columns (8), (9), and (10) for longitude; and (13), (14), and (15) for obliquity. The rigid amplitudes are computed with the parameters stemming from reducing the Poincaré model to a rigid one. See the main text for a discussion.

802 The rigid model values shown in Table 4 for both longitude and obliquity (columns denoted as Rig.) are very close to those ones computed in Getino et al. (2010, Table 5). The small differences, less than +0.05 μas in modulus, can be attributed to the different values used for k_M, k_S, and n_μ —in the rigid case (Equation C18) we can write $n_\mu = \omega_E\big(1 + P_{CW}^{-1}\big) = \omega_E(1 - H_d)^{-1}$. Their magnitude is very small, the largest contributions arising from the term of period −6798.38 days with values of about −0.5 and +1.4 μas in longitude and obliquity. They stem from the amplitudes $\mathcal{L}_2^{p2}$ and $\mathcal{O}_2^{p2}$ with no significant contribution from $\mathcal{L}_3^2$ and $\mathcal{O}_3^2$ (Getino et al. 2010, Table 5).

809 In contrast, for the Poincaré model (columns denoted as Poi. in Table 4) the amplitudes depending on the Earth structure are noticeably amplified. Next, we will find out the source of such an amplification by discussing the contributions of the fluid core to those amplitudes in comparison with their rigid counterparts.

813 ### 5.3. *Influence of the fluid core on the second order amplitudes*

814 The analytical character of our theory makes possible to understand also qualitatively the origin the fluid core amplification. With this aim, we will develop asymptotic estimates that describe the

role played by the fluid core in the structure dependent amplitudes $\mathcal{L}_2^{p2}$, $\mathcal{L}_3^2$, $\mathcal{O}_2^{p2}$, and $\mathcal{O}_3^2$ (Equations 84 and 86).

Those estimates, however, cannot always be given in a simple, neat, and direct way. The reason is that their values depend more on the particular constituents $(\Theta_i, \Theta_j, \tau, \rho)_{\varepsilon\Theta_k}$ than on the final argument Θ_k, hence a multiplicity of situations arises. Nevertheless, it is interesting to perform that kind of analysis both to validate the derived second order amplitudes and to show their intricate features.

5.3.1. *Amplitudes $\mathcal{L}_2^{p2}$ and $\mathcal{O}_2^{p2}$*

We consider the situation for $\mathcal{L}_2^{p2}$ and $\mathcal{O}_2^{p2}$. From Table 3, columns (8) and (13), those amplitudes provide the largest second order contributions dependent on the Earth structure for most terms. The ratio η_2^{p2} between the Poincaré model and rigid amplitudes is the same for longitude and obliquity. It can be expressed as (Equations 84 and C19)

$$\eta_2^{p2} = \frac{\mathcal{L}_2^{p2}}{\mathcal{L}_{2\mathrm{R}}^{p2}} = \frac{\mathcal{O}_2^{p2}}{\mathcal{O}_{2\mathrm{R}}^{p2}} = \frac{\omega_E - \tau n_i - r_3}{\prod\limits_{k=1,2} (\omega_E - \tau n_i - m_k)} (n_\mu - \tau n_i). \tag{93}$$

We can establish a simple asymptotic estimate[16] for η_2^{p2}. With this objective, we observe from Table 2 and Equations (64) that e, e_c are of the order of 2×10^{-3}. In addition, $|n_i/\omega_E|$ (Appendix A) belong to the interval $[0, 0.11]$. Hence, neglecting all those parameters with respect to 1, we can write Equations (61) and (C18) as

$$\begin{aligned} r_3 &= \omega_E(1 + r_{cm})(1 + e_c) \sim \omega_E(1 + r_{cm}), \\ n_\mu - \tau n_i &= \omega_E(1 + e) - \tau n_i \sim \omega_E. \end{aligned} \tag{94}$$

Analogously, considering that the eigenvalue m_2 is proportional to e (Equations 64); the expression of m_1 (Equations 62); and the above approximations, we have

$$\begin{aligned} \omega_E - \tau n_i - m_1 &= -\left(\tau n_i + P_{FCN}^{-1}\right), \\ (\omega_E - \tau n_i - m_2) &\sim \omega_E. \end{aligned} \tag{95}$$

[16] This ratio also appears when considering the precession function S_{2b}^L (Equation 73).

Then, the ratio given in Equation (93) can be estimated as

$$\eta_2^{p2} \sim \frac{\tau n_i/\omega_E + r_{cm}}{\tau n_i/\omega_E + P_{FCN}^{-1}} = \frac{1 + \dfrac{r_{cm}}{\tau n_i/\omega_E}}{1 + \dfrac{P_{FCN}^{-1}}{\tau n_i/\omega_E}}. \tag{96}$$

This expression can be further simplified considering that $P_{FCN}^{-1} \sim 2 \times 10^{-3}$ (Table 2), leading to the following asymptotic approximations

$$\eta_2^{p2} \sim \begin{cases} 1 + r_{cm}/(\tau n_i/\omega_E), \ P_{FCN}^{-1} \ll |n_i/\omega_E| \\ r_{cm} P_{FCN}, \ P_{FCN}^{-1} \gg |n_i/\omega_E| \end{cases}. \tag{97}$$

Depending on the particular value of τn_i, this estimate runs in a range from about -22 to 24 in the first case and about 50 in the the second one, which is related to the terms with the larger periods.

The condition $P_{FCN}^{-1} \sim |n_i/\omega_E|$ requires further consideration. In our case, it affects the orbital arguments with annual periods, so $P_{FCN}^{-1} \sim n_i/\omega_E$. If for those n_i terms we split P_{FCN}^{-1} as

$$P_{FCN}^{-1} = (n_i/\omega_E)(1 - \delta_i), \tag{98}$$

we get that δ_i is about 0.1563 and 0.1564 for the arguments with periods 365.26 and 365.22 days, respectively. Hence, we obtain

$$\eta_2^{p2} \sim \begin{cases} -r_{cm} P_{FCN}/\delta_i, \tau = -1 \\ r_{cm} P_{FCN}/2, \ \tau = 1 \end{cases}, \tag{99}$$

leading to the exact fluid resonance when $\delta_i = 0$, i.e., if there were some orbital argument with frequency $n_i = P_{FCN}^{-1}\omega_E$, corresponding to about 433 (mean solar) days, and $\tau = -1$. In our case, just the arguments with periods 365.26 and 365.22 days are relatively close to P_{FCN}, providing a value of η_2^{p2} about -250, taking $\delta_i \sim 0.2$. That ratio is replaced by about 25 when $\tau = 1$.

The features of η_2^{p2} are the same as those encountered in the development of the first order theory of the nutations of the Poincaré model[17]. The fluid core resonance, also referred to as fluid core

[17] Commonly, first order theories do not represent the ratio η_2^{p2} itself, but the ratio of the whole nutations of the figure axis, i.e., both Poisson and Oppolzer terms, i.e., the so called transfer function. This is the case of Figure 4 in Smith (1980), Figure 1 in Sasao et al. (1977), or Figure 3.8 in Moritz & Mueller (1987). The definition of the orbital frequencies in Moritz & Mueller (1987) has the opposite sign to that usually employed as can be checked from comparing their Table 3.1 and Table 2 in Sasao et al. (1980).

amplification, appears in the first order amplitudes of the Oppolzer terms and it is inherited by the nutations of the figure axis. That resonance is due to the to the NDFW mode of the Poincaré model (Equation 63) and is located in the retrograde diurnal band in the terrestrial system, corresponding to an "inertial" period of about 433 days with $\tau = -1$, and it has been extensively recognized in the literature (e.g., Sasao et al. 1977; Smith 1980; or Moritz & Mueller 1987, Chapter 4).

Moreover, Equation (93) is analytically equivalent to Equation 68 in Getino (1995b). Therefore, the fluid resonance of the ratio of the first order Oppolzer terms also appears in the structure dependent part of the second order Poisson terms given by the amplitudes $\mathcal{L}_2^{p2}$ and $\mathcal{O}_2^{p2}$. Nevertheless, the whole first and second order amplitudes themselves are quite different because of the dependencies on the orbital arguments and functions (i.e., compare Equations 77, 78, and 84 with Equation 64 in Getino 1995b).

The reason of that equivalence arises from the form of $\mathcal{W}_{2p}$ (Equation B14), which gives rise to $\mathcal{L}_2^{p2}$ and $\mathcal{O}_2^{p2}$. As we stated in Appendix B, the determination of that function can be carried out just keeping zero degree monomials ζ^0. It entails that in the process of constructing $\mathcal{W}_{2p}$ the only integration depending on the Earth model is that proportional to first degree monomials ζ^1 in $\mathcal{W}_1$ (Equation B7), just as in the case of first order Oppolzer terms (Getino 1995b).

There is, however, an important difference. Whereas the ratio η_2^{p2} of the first order Oppolzer terms is a direct function of the nutation argument Θ_k, $\eta_2^{p2} = \eta_2^{p2}(\varepsilon n_k)$; it is not the case of η_2^{p2} for the second order Poisson terms. In that case, as for other second order amplitudes, that dependence is with Θ_i but not with Θ_k, i.e., $\eta_2^{p2} = \eta_2^{p2}(\tau n_i)$. Therefore, the fluid core amplification appears in a different way for each of the constituents $(\Theta_i, \Theta_j, \tau, \rho)_{\varepsilon\Theta_k}$ of the nutation argument Θ_k.

Hence, the total ratio of the non-rigid and rigid second order amplitudes of Θ_k is a result of the multiple individual ratios for every combination $(\Theta_i, \Theta_j, \tau, \rho)_{\varepsilon\Theta_k}$. This fact prevents, in a general situation, the derivation of a direct analytical expression for that total ratio. In turn, it entails that it is not possible to apply the MHB2000 transfer function (Mathews et al. 2002) to derive the second order amplitudes of the non-rigid Earth. The reason is that this transfer function depends directly on the nutation frequency of each nutation argument (Mathews et al. 2002, Equation 7, in their notation σ plays the role of εn_k in the terrestrial system), regardless it comes from a first order contribution or a second order one. In this way, it is not possible to recover the ratio associated to each constituent of the argument Θ_k.

For example, the second order term with period 365.26 days arises from the proper combinations among the orbital arguments with periods 365.26, 365.22, 182.62, and 121.75 days, and the secular one. Each of them has an associated ratio η_2^{p2} that runs from a factor of about -290 for the 365.26 days term, with $\tau = -1$, to 55 for the secular one, considering the exact expression given in Equation (93). However, the value of the total ratio for the whole second order 365.26 days amplitude in $\mathcal{L}_2^{p2}$ and $\mathcal{O}_2^{p2}$ reaches a factor of about -23.

The former considerations and the analytical estimates given by Equations (97 and 99) explain the numerical amplifications of $\mathcal{L}_2^{p2}$ and $\mathcal{O}_2^{p2}$ for the Poincaré model (Table 4). First, it is necessary that the constituents of the second order nutation argument Θ_k come from some combination involving an orbital period of 365.26 or 365.22 days, since for them, with $\tau = -1$, the amplification is maximum in absolute value (about 250). Another favorable situation arises for the terms with larger orbital periods, typically the secular one and those with periods -6798.38 and -3399.19 days, which produce a factor amplification of about 50. In contrast, if the constituents involve shorter periods, or the annual ones with $\tau = +1$, the amplification is quite modest.

Second, the considered amplification is applied to the rigid amplitudes $\mathcal{L}_2^{p2}$ and $\mathcal{O}_2^{p2}$. So, to obtain a significant second order contribution for the Poincaré model, it is needed that the rigid counterpart amplitude also reaches an appreciable value. As explained in Getino et al. (2010, Equations D5), it is just the case for those constituents of Θ_k that combine large orbital periods and coefficients (see also Appendix A).

A paradigmatic example is given by the secular orbital term Θ_0. It is a constituent of all the second order arguments Θ_k that are present in the original set of the orbital terms (Appendix A), since, for all τ and ρ, the combinations $(\Theta_0, \Theta_j, \tau, \rho)$ and $(\Theta_i, \Theta_0, \tau, \rho)$ give raise to the same argument $\Theta_k = \Theta_j = \Theta_i$ (with the proper positive or negative signs). The term Θ_0, with $n_0 = 0$, produces a significant factor of amplification of about 55 and has a relative large value of its orbital coefficients

(Appendix A). As a result, it contributes very significatively to the amplitudes $\mathcal{L}_2^{p2}$ and $\mathcal{O}_2^{p2}$ of the Poincaré model.

As a matter of fact if the secular term is excluded from the combinations, those amplitudes are considerably reduced for most terms, even below the 0.01 μas threshold. For example, the amplitudes of the term with period -6798.38 days would change its values to -1.43 and $+1.52$ μas in longitude and obliquity, respectively —compare with columns (8) and (13) in Table 3.

Indeed, if the annual terms, with $\tau = -1$, enter into the set of constituents of Θ_k the amplification is the greatest one. However, since for those terms the rigid amplitude is very small, the resulting contribution is still reduced and does not contribute to the total amplitude significatively. As an illustration, the second order term with period 182.62 days is generated from the suitable combinations of the orbital arguments with periods 365.26, 365.22, 182.62, and 121.75 days, and the secular one (Appendix A). The ratio η_2^{p2} reaches the largest value, in modulus, of about -290 when the constituents involve the annual terms, with $\tau = -1$. Nevertheless, in those situations the rigid amplitudes, e.g., in longitude, are about 10^{-5} μas, or smaller, thus their final contribution is clearly below 0.01 μas. In other cases there are some contributions above that level, but still very small because the tiny magnitude of the original rigid amplitudes.

5.3.2. *Amplitudes $\mathcal{L}_3^2$ and $\mathcal{O}_3^2$*

With respect to the amplitudes $\mathcal{L}_3^2$ and $\mathcal{O}_3^2$ (Equations 86), also affected by the core, we explained in Section 4 that they split in indirect and direct parts (Equations 87 and 88).

The direct parts $\mathcal{L}_{3\text{-d}}^2$ and $\mathcal{O}_{3\text{-d}}^2$ are proportional to r_{cm}, through r_2^2, accordingly absent in the rigid case. They are about one order of magnitude larger than the indirect ones —see columns (9) and (10), and (14) and (15) in Table 3. The indirect parts $\mathcal{L}_{3\text{-in}}^2$ and $\mathcal{O}_{3\text{-in}}^2$ show a very significant amplification with respect to their rigid counterparts that are below the threshold of 0.01 μas for all the terms (Getino et al. 2010, Table 5). Both facts are consistent with the analytical amplitudes expressions (Equations 87 and 88) as we show below.

5.3.2.1. *Direct and indirect contributions*—To understand the dominant role of the direct part of the amplitudes $\mathcal{L}_3^2$ and $\mathcal{O}_3^2$ relative to the indirect one, let us consider the expression of the ratio between the direct and indirect parts

$$\alpha^2_{\text{3-d-in}} = \frac{\mathcal{L}^2_{\text{3-d}}}{\mathcal{L}^2_{\text{3-in}}} = \frac{\mathcal{O}^2_{\text{3-d}}}{\mathcal{O}^2_{\text{3-in}}} = -\frac{r_2^2}{(\omega_E - \tau n_i - r_3)(\omega_E - \rho n_j - r_3)}. \tag{100}$$

We can now proceed in an analogous way as in Section 5.3.1. From Equations (61) and neglecting e and e_c relative to r_{cm} (Table 2), we have

$$\alpha^2_{\text{3-d-in}} \sim -\frac{r_{cm}(1 + r_{cm})}{\left(\tau \frac{n_i}{\omega_E} + r_{cm}\right)\left(\rho \frac{n_j}{\omega_E} + r_{cm}\right)}. \tag{101}$$

We can also ignore $|n_i/\omega_E|$ and $|n_j/\omega_E|$ relative to r_{cm}, which holds for most orbital terms (the larger ones). Therefore, we get

$$\alpha^2_{\text{3-d-in}} \sim -\frac{(1 + r_{cm})}{r_{cm}} \sim -9, \tag{102}$$

what explains the different magnitudes of columns (9) and (10), and (14) and (15) in Table 3.

It is difficult to get more precise estimates for the particular value of the ratio for each Θ_k than $\alpha^2_{\text{3-d-in}} \sim -9$. As we have pointed out, those difficulties are inherent to second order terms, since the argument Θ_k arises from multiple combinations $(\Theta_i, \Theta_j, \tau, \rho)_{\varepsilon\Theta_k}$ that provide a variety of values η_3^2, as indicated from Equation (101). In this case, $\alpha^2_{\text{3-d-in}}$ is a function of τn_i and ρn_j, $\alpha^2_{\text{3-d-in}} = \alpha^2_{\text{3-d-in}}(\tau n_i, \rho n_j)$, but not directly of εn_k unless we use a rough estimation as that given in Equation (102). For example, that argument Θ_k with period -6798.38 days has a global ratio of about -9 (Table 3), as a result of mixing its constituent ratios. Their values run from about -50 to -4 depending on the particular combination of Θ_i, Θ_j, τ, and ρ. The ratio taking the smallest or the largest values comes from the terms with the lower orbital periods, below 14 days, where $\tau n_i/\omega_E$ or $\rho n_i/\omega_E$ are close to r_{cm} or $-r_{cm}$, respectively.

5.3.2.2. *Amplification of the indirect contributions*—To understand why the presence of the fluid produces a large amplification of the indirect parts $\mathcal{L}^2_{\text{3-in}}$ and $\mathcal{O}^2_{\text{3-in}}$, we can perform a similar analysis as that done in the case of $\mathcal{L}^{p2}_2$ and $\mathcal{O}^{p2}_2$.

The ratio $\eta^2_{\text{3-in}}$ between the Poincaré model and the rigid amplitudes is identical for longitude and obliquity. With the help of Equations (C19) and (87), it is got that

$$\eta^2_{\text{3-in}} = \frac{\mathcal{L}^2_{\text{3-in}}}{\mathcal{L}^2_{\text{3R}}} = \frac{\mathcal{O}^2_{\text{3-in}}}{\mathcal{O}^2_{\text{3R}}} = \frac{(\omega_E - \tau n_i - r_3)(\omega_E - \rho n_j - r_3)}{\prod\limits_{k=1,2} [(\omega_E - \tau n_i - m_k)(\omega_E - \rho n_j - m_k)]}(n_\mu - \tau n_i)(n_\mu - \rho n_j). \tag{103}$$

If we compare this equation with Equation (93), we can write

$$\eta^2_{3\text{-in}} = \left[\frac{\omega_E - \tau n_i - r_3}{\prod\limits_{k=1,2}(\omega_E - \tau n_i - m_k)}(n_\mu - \tau n_i)\right]\left[\frac{\omega_E - \rho n_j - r_3}{\prod\limits_{k=1,2}(\omega_E - \rho n_j - m_k)}(n_\mu - \rho n_j)\right] = \eta_2^{p2}(\tau n_i)\eta_2^{p2}(\rho n_j). \tag{104}$$

Hence, we have for $\eta^2_{3\text{-in}}$ the estimate (Equation 96)

$$\eta^2_{3\text{-in}} \sim \left(\frac{1 + \dfrac{r_{cm}}{\tau n_i/\omega_E}}{1 + \dfrac{P_{FCN}^{-1}}{\tau n_i/\omega_E}}\right)\left(\frac{1 + \dfrac{r_{cm}}{\rho n_j/\omega_E}}{1 + \dfrac{P_{FCN}^{-1}}{\rho n_j/\omega_E}}\right). \tag{105}$$

Therefore, the asymptotic behavior of $\eta^2_{3\text{-in}}$ can be extracted directly from that of η_2^{p2} expressed in Equations (97) and (99) but now considering the product $\eta_2^{p2}(\tau n_i)\eta_2^{p2}(\rho n_j)$. So, the ratio $\eta^2_{3\text{-in}}$ presents a double fluid resonance[18], a circumstance that has no parallel in the first order theory of the Poincaré model nor in no first order theory of the non-rigid Earth. As in the former section, this fact entails again that it is not possible to apply MHB2000 transfer function (Mathews et al. 2002) to obtain these second order amplitudes.

The double resonance in $\eta^2_{3\text{-in}}$ arises from the second order contributions to longitude and obliquity due to the $\mathcal{W}_1$ crossed terms (Equations 26 and 74). It involves two times the term proportional to first degree monomials ζ^1 in $\mathcal{W}_1$ (Equation B7), which depends on the Earth model (one for Θ_i, n_i, and τ, and other for Θ_j, n_j, and ρ), providing in this way the found structure of $\eta^2_{3\text{-in}}$. It leads to very significant amplifications that, in modulus, could reach about a factor up to 1.3×10^4, if we combine annual terms, with $\tau = \rho = -1$, with the largest period ones[19]. One more time, that double resonance depends on the constituents $(\Theta_i, \Theta_j, \tau, \rho)_{\varepsilon\Theta_k}$ of the nutation argument Θ_k, and not on Θ_k itself. However, in spite of the very large amplification of the ratio $\eta^2_{3\text{-in}}$ for some combinations, the associated second order terms are very small —columns (9) and (14) in Table 3.

The reason is the same as that explained for η_2^{p2} but intensified in this case, since the rigid amplitudes $\mathcal{L}^2_{3\mathrm{R}}$ and $\mathcal{O}^2_{3\mathrm{R}}$ are below the threshold of 0.01 μas for all the terms (Getino et al. 2010, Table 5).

[18] This double resonance is different from the *double résonance* introduced in Poincaré (1910).

[19] The situation in which both terms have the same annual period $\Theta_i = \Theta_j$, with $\tau = \rho = -1$, is excluded by the summation conditions of Equations (77) and (78). It is also the case for close annual terms like those with periods 365.26 days and 365.22 days, which lead to a second order term with a very large period (larger than 10 000 y).

For example, if we analyze the second order term with period 365.26 days formerly considered, we find a global amplification factor of about 5200 (computed from Equation 93). The largest value in modulus for the ratio $\eta^2_{3\text{-in}}$ is about 16 000, derived for the constituents involving the secular term and the 365.26 days one term (with $\tau = \rho = -1$). However, the corresponding rigid amplitude $\mathcal{L}^2_{3\mathrm{R}}$ for those constituents reaches a maximum value in modulus of about 4×10^{-6} μas. Hence, even with the strong amplification of those terms, the resulting non-rigid amplitude is far from the 0.5 μas truncation level that we have considered in this research.

6. SUMMARY AND CONCLUSIONS

We have constructed a Hamiltonian framework to derive in a systematic way the analytical solutions of the rotation of the Poincaré model of the Earth at the second order, in the sense of perturbation theories (or spin-spin coupling), extending the rigid Earth solution by Getino et al. (2010). It has allowed determining and analyzing the contributions of the second order effects to the precession and nutation of the angular momentum axis (Poisson terms).

To develop that process we have had to abandon the first order canonical formulation of the problem (Getino 1995a), since it is not suitable to tackle second order effects. The reason is related to the virtual singularities that Andoyer variables have in the equilibrium configuration (Henrard 2006). They prevent from constructing a Hori kernel with an auxiliary system having an explicit and simple analytical solution. Because of that the application of the Hori perturbation method (Hori 1966) deviates from its standard implementation, what makes unworkable its extension up to the second order of perturbation.

That difficulty has been overcome with the introduction of a set of non-singular canonical variables (e.g., Getino et al. 2000 or Escapa et al. 2001) of Poincaré kind. Due to the axial symmetry of the Poincaré model, the complexification of those variables simplifies further the computations. It has lead to the final definition of the non-singular complex canonical variables ($NSCCV$) considered in this work (Equations 38). Since those variables are free from virtual singularities in the equilibrium configuration, it has been possible to define a Hori kernel in a rigorous way. That definition follows a technique common in Mechanics (e.g., Arnold 1989, Chapter 5) and provides an unperturbed

Hamiltonian quadratic in powers of the differences of the canonical variables with respect to their equilibrium values.

This procedure offers several advantages. First, the solution of the auxiliary system can be computed explicitly, so the Hori method can be applied in a standard way what facilitates the comparison with the rigid model solution (Getino et al. 2010). Second, some $NSCCV$ are zero in the equilibrium configuration and keep small in the rotational evolution. Hence, the functions entering in the construction of the approximate analytical can be expanded in terms of their monomials of degree k, ζ^k. By doing so, it is possible to consider truncated expansions up to the proper powers of ζ, what simplifies considerably the second order computations as it is done with the angle σ in the rigid Earth model (Getino et al. 2010).

The Hori method (Equations 21 and 22) was implemented through the $NSCCV$ and the selected Hori kernel, allowing the computation of the first and second order transformed Hamiltonians and generating functions. From them, we have derived the second order analytical solutions for the precession (Equations 71) and nutation (Equation 77) of the angular momentum axis.

The obtained second order amplitudes can be divided into two groups. The first one is independent from the Earth model (S^L_{2a} in Equations 72; $\mathcal{L}^s_2$, $\mathcal{L}^{p1}_2$, $\mathcal{O}^s_2$, and $\mathcal{O}^{p1}_2$ in Equations 82 and 83) with the exception of a global factor proportional to the squared dynamical ellipticity H^2_d (equivalently to $k_p k_q$). Their expressions turned to be equal to those previously determined by Getino et al. (2010) for the rigid Earth using the classical Andoyer variables, what can be viewed as a first validation of the approach built in this investigation.

The second group depends on the Earth interior (S^L_{2b} in Equations 73; $\mathcal{L}^{p2}_2$, $\mathcal{L}^2_3$, $\mathcal{O}^{p2}_2$, and $\mathcal{O}^2_3$ in Equations 84 and 86). This is one of the most important conclusion derived in this study and generalizes the result firstly pointed out in Ferrándiz et al. (2004) just for the precession (S^L_{2b} term). So that, in contrast to first order Poisson terms, there is a part of second order Poisson terms affected by the structure of the Earth, hence different for rigid, elastic, two-layer, etc. models. It limits to the first order solutions the spread affirmation that the rotational motion of the angular momentum axis is independent of the internal constitution of the Earth (Moritz & Mueller 1987, Chapter 3).

We have also shown that those amplitudes are the same as in Getino et al. (2010) when the Poincaré model is reduced to a rigid one. It is also the case of S_{2b}^{L} when compared with the formulae given in Ferrándiz et al. (2004) or Baenas et al. (2017). This is a second validation of our approach.

The presence of the fluid core affects the structure dependent Poisson amplitudes and makes them very different from their rigid counterparts. Such differences depend on the particular amplitudes. We found three distinct situations. There is a part of the amplitudes $\mathcal{L}_3^2$ and $\mathcal{O}_3^2$ that depend directly on the fluid core, that is to say, they are not present in the rigid case.

The amplitudes $\mathcal{L}_2^{p2}$, $\mathcal{O}_2^{p2}$ and S_{2b}^{L} provide an indirect contribution from the fluid core, showing a significant amplification with respect to the rigid Earth. As derived from the performed asymptotic estimates (Equations 98, and 97), that amplification is driven by one of the normal mode of the Poincaré model: the Free Core Nutation (FCN). Specifically, there appears a fluid resonance in the amplitudes, since some orbital arguments with annual periods are close to that of the FCN, which is about 433 days for our model. A similar situation was found for the first order Oppolzer terms of the Poincaré model (Getino 1995b). However, there is an essential difference. In the first order Oppolzer terms the resonance is related directly with the nutation argument Θ_k. In contrast, the resonance of this part of the second order Poisson terms depends not on Θ_k but on the constituents $(\Theta_i, \Theta_j, \tau, \rho)_{\varepsilon\Theta_k}$ whose combination produces Θ_k.

Lastly, the amplitudes $\mathcal{L}_3^2$ and $\mathcal{O}_3^2$ also give an indirect contribution. Our analysis (Equations 104 and 105) have shown that those amplitudes present a double fluid resonance, a circumstance that has no equivalent in the first order models of the non-rigid Earth and has been found here for the first time. It entails a large amplification that can reach a value of about 16 000 in the most favorable case. As in the former case, that resonance depends on the constituents $(\Theta_i, \Theta_j, \tau, \rho)_{\varepsilon\Theta_k}$ and not on the nutation argument Θ_k.

Numerically, the structure dependent Poisson amplitudes have provided contributions that cannot be neglected considering nowadays accuracy demands (Ferrándiz et al. 2020). Namely, the arguments with periods -3399.19, and 182.62 days have amplitudes about a few μas, whereas the term with period −6798.38 days gives values of about three and forty μas in longitude and obliquity, respectively (Table 3). Those contributions are absent in current IAU nutation model (Mathews et al. 2002).

To conclude, the analytical and numerical results obtained in this investigation allow us to draw some final conclusions in two different levels. On the one hand, because of their numerical contributions, second order spin-spin coupling contributions can no longer be ignored in Earth rotation studies. So, models superseding IAU 2000A (Mathews et al. 2002) must include that kind of effects. It is expected that this conclusion will be reinforced when the second order Oppolzer terms be determined. As we have pointed out, their calculation is challenging since it is necessary to increase the truncation order in ζ. We will present the computations and their numerical contributions in a forthcoming communication.

On the other hand, we have unveiled the dependencies of the second order Poisson terms with the Earth structure, particularly with the fluid core. Indeed, since they do depend on the Earth interior they are different from first order Poisson terms (independent from the Earth model). But also from first order Oppolzer terms, due to the more complex role played by the fluid resonance. It entails that it is not possible to employ current first order formulations, like that based on the MHB transfer function (Mathews et al. 2002), to capture the second order contributions. In this sense, the Hamiltonian approach as that developed in this work provides a suitable framework to extend the rotation of the non-rigid Earth models at the second order.

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APPENDIX

Table 5. List of the main orbital arguments Θ_i used in this work (taken from Getino et al. 2010)

Argument	Period	Moon (10^{-7} rad)			Sun (10^{-7} rad)	
l_M l_S F D Ω	Days	$A^{(0)}$	$A^{(1)}$	$A^{(2)}$	$A^{(0)}$	$A^{(2)}$
+0 +0 +0 +0 +0	$+\infty$	4963035.3	0	0	5002105.4	0
+0 +0 +0 +0 +1	−6798.36	0	448720.5	0	0	0
+0 +0 +0 +0 +2	−3399.18	0	0	40433.0	0	0
+0 +1 +0 +0 +0	365.26	−1559.1	0	0	250710.3	0
+0 −1 +2 −2 +2	365.22	0	0	−100.0	0	−83543.3
+0 +0 +2 −2 +2	182.62	0	0	7880.7	0	9993042.1
+0 +1 +2 −2 +2	121.75	0	0	338.0	0	584450.7
+1 +0 +0 +0 +0	27.55	811948.6	0	0	0	0
+0 +0 +2 +0 +2	13.66	0	0	9880171.3	0	0
+0 +0 +2 +0 +1	13.63	0	−443830.4	0	0	0
+1 +0 +2 +0 +2	9.13	0	0	1891661.7	0	0

A. ORBITAL ARGUMENTS, COEFFICIENTS, AND FUNCTIONS

For the sake of convenience, we include in Table 5 the list of the eleven main orbital arguments Θ_i derived from a Fourier decomposition of the orbital motions of the Moon and the Sun as given in Getino et al. (2010, Table 8). It contains the orbital coefficients $A_i^{(0,1,2)}$ necessary to compute the orbital functions B_i, $C_{i,\tau}$, and $D_{i,\tau}$ appearing in the precession and nutation amplitudes. Their expressions were provided by Kinoshita (1977) as[20]

$$\begin{aligned}
B_i =& \ \frac{1}{6}\left(3\cos^2 I - 1\right)\ A_i^{(0)} - \frac{1}{2}\sin 2I\ A_i^{(1)} - \frac{1}{4}\sin^2 I\ A_i^{(2)}, \\
C_{i,\tau} =& -\frac{1}{4}\sin 2I\ A_i^{(0)} + \frac{1}{2}(1+\tau\cos I)(-1+2\tau\cos I)\ A_i^{(1)} \\
& +\frac{1}{4}\tau\sin I(1+\tau\cos I)\ A_i^{(2)}, \\
D_{i,\tau} =& -\frac{1}{2}\sin^2 I\ A_i^{(0)} + \tau\sin I(1+\tau\cos I)\ A_i^{(1)} \\
& -\frac{1}{4}(1+\tau\cos I)^2 A_i^{(2)}.
\end{aligned} \tag{A1}$$

[20] There is a typo in Equation (6) by Getino et al. (2010). In the definition of $C_{i,\tau}$ the orbital coefficients $A_i^{1)}$ and $A_i^{2)}$ must be swapped.

The value of the orbital argument Θ_i is constructed from Equation (16)

$$\Theta_i = m_{1i} l_M + m_{2i} l_S + m_{3i} F + m_{4i} D + m_{5i}(\Omega_0 - \lambda) \tag{A2}$$

and Table 5. That expression also allows writing Θ_i as a function of time and computing its rate[21] n_i (Equation 17). To this end, it is necessary to know the time evolution of a combination of the Delaunay variables of the Moon and the Sun as appearing in Table 9 in Getino et al. (2010).

As explained in the main part of the text, the original eleven orbital arguments Θ_i combine as $\tau\Theta_i - \rho\Theta_j$ to generate the second order arguments $\varepsilon\Theta_k$. Hence, there appears a multiplicity of terms not present in Table 5 (e.g., see Table 3). Not all the possible combinations are present finally, since many of them provide contributions to the second order nutation or precession below the established numerical threshold.

In obtaining those second order contributions, one must take into account that the corresponding amplitudes are computed from the combinations, or constituents, leading to $\varepsilon\Theta_k$ that are denoted as $(\Theta_i, \Theta_j, \tau, \rho)_{\varepsilon\Theta_k}$. For example, the constituents of the second order term with period 182.62 days are

$$\begin{gathered}
(+\infty, 182.62, -, -)_+;\; (182.62, +\infty, +, -)_+;\; (365.26, 365.22, +, -)_+;\; (365.26, 121.75, -, -)_+;\\
(+\infty, 182.62, +, -)_+;\; (182.62, +\infty, +, +)_+;\; (365.22, 365.26, +, -)_+;\; (121.75, 365.26, +, +)_+;\\
(+\infty, 182.62, -, +)_-;\; (182.62, +\infty, -, -)_-;\; (365.22, 365.26, -, +)_-;\; (365.26, 121.75, +, +)_-;\\
(+\infty, 182.62, +, +)_-;\; (182.62, +\infty, -, +)_-;\; (365.26, 365.22, -, +)_-;\; (121.75, 365.26, -, -)_-;
\end{gathered} \tag{A3}$$

where to simplify the notation we have identified the corresponding arguments Θ_i and Θ_j by their respective periods as indicated in Table 5; displayed just the signs of τ and ρ; and denoted the subscript $\varepsilon\Theta_k$ by the corresponding sign of ε.

[21] Analogously to Getino et al. (2010, Section 5), our perturbation scheme entails that n_i is computed using Ω_0 whereas Θ_i involves $\Omega = (\Omega_0 - \lambda)$.

B. COMPUTATION OF THE FIRST AND SECOND ORDER FUNCTIONS

We sketch the principal guidelines in the computation of the first and second order functions entering in the Hori's method (Equations 21 and 22). As indicated in Section 4, this process is simplified by considering the proper truncation expansions in ζ.

B.1. *First order functions*

The first order stage of the Hori's perturbation method (Section 2) are implemented via the generating function $\mathcal{W}_1$ and transformed Hamiltonian $\mathcal{H}_1^*$. They come from the periodic and secular parts of the perturbing Hamiltonian in *NSCCV* given by Equations (46) and (47).

The argument Θ_i, with $i \neq 0$, and the variable y_1 (Equation 54) evolve fast, so the periodic part is given by

$$\mathcal{H}_{1\mathrm{per}} = \sum_{p=M,S} k'_p \sum_{\tau=\pm 1} \Bigg\{ -\sum_{i\neq 0} \frac{\mathrm{i}}{2} B_i e^{\mathrm{i}\tau\Theta_i} + \sum_i \frac{\sqrt{2}}{2} \frac{C_{i,\tau}}{\sqrt{\mathrm{i}Y_1}} \left(y_2 e^{-\mathrm{i}(y_1-\tau\Theta_i)} - Y_2 e^{\mathrm{i}(y_1-\tau\Theta_i)} \right) \Bigg\}, \tag{B4}$$

and the secular one by

$$\mathcal{H}_{1\mathrm{sec}} = -\sum_{p=M,S} \mathrm{i}k'_p B_0 + e_1 Y_3 + Y_1 \sin I (e_2 \cos y_3 + e_3 \sin y_3). \tag{B5}$$

We have truncated $\mathcal{H}_{1\mathrm{per}}$ at ζ^1 degree, since it gives raise to $\mathcal{W}_1$; and $\mathcal{H}_{1\mathrm{sec}}$ at ζ^0 because it appears directly in the transformed Hamiltonian $\mathcal{H}^*$.

The first order transformed Hamiltonian is $\mathcal{H}_{1\mathrm{sec}}$, but expressed in the variables (Y^*, y^*)

$$\mathcal{H}_1^* = -\sum_{p=M,S} \mathrm{i}k'_p B_0^* + e_1 Y_3^* + Y_1^* \sin I^* (e_2 \cos y_3^* + e_3 \sin y_3^*). \tag{B6}$$

With regard to the generating function, it is computed by integrating $\mathcal{H}_{1\mathrm{per}}$ over the solutions of the auxiliary system. Considering the Equations (17), (52), and (54) for the evolution of Θ_i, I, y_3, and

y_1; and the Equations (58), (59), and (67) for that of y_2 and Y_2, and after a little algebra, we get

$$\begin{aligned}\mathcal{W}_1 = \; & \mathrm{i} \sum_{p=M,S} k'_p \sum_{\tau=\pm1} \sum_{i\neq0} \frac{\mathrm{i}}{2} \frac{B^*_i}{\tau n_i} e^{\mathrm{i}\tau\Theta^*_i} + \\ & \mathrm{i} \sum_{p=M,S} k'_p \sum_{\tau=\pm1} \sum_{i} \frac{1}{\sqrt{2\mathrm{i}Y^*_1}} \frac{C^*_{i,\tau}}{\prod\limits_{k=1,2} (\omega_E - \tau n_i - m_k)} \times \\ & \big\{ e^{-\mathrm{i}(y^*_1 - \tau\Theta^*_i)} [(\omega_E - \tau n_i - r_3) y^*_2 + r_2 Y^*_{2C}] + \\ & e^{\mathrm{i}(y^*_1 - \tau\Theta^*_i)} [(\omega_E - \tau n_i - r_3) Y^*_2 + r_2 y^*_{2C}] \big\}.\end{aligned} \tag{B7}$$

The former expression is equivalent to that derived in Getino (1995b), or Getino & Ferrándiz (2001), in terms of Andoyer variables when reducing their model to the Poincaré one. Nevertheless, as we have mentioned and can be checked in those references, the procedure followed there to obtain $\mathcal{W}_1$ is more cumbersome and non–systematic due to the need of using non-canonical variables in the computation of the generating function.

In contrast, the computation developed here runs parallel to that of the rigid Earth (Getino et al. 2010) and the standard method of canonical perturbations, although even with the introduction of *NSCCV* the difficulties increases due to the higher dimension of the phase space.

The structure of $\mathcal{W}_1$ in Equation (B7) reflects both the form of $\mathcal{H}_{1\mathrm{per}}$ and the solution of the auxiliary system. The features of the Earth model enter through the part proportional to the orbital coefficients $C_{i,\tau}$ in $\mathcal{H}_{1\mathrm{per}}$ that depend on y_2 and Y_2.

Although those variables refer to the whole Earth, after performing the integration, the variables related to the core y^*_{2C} and Y^*_{2C} also appear. This is a consequence of the coupled dynamics among them generated by the Hori kernel (Equations 55). Since that dynamics is governed by a linear system, the degree of the monomials in ζ is conserved when computing the integrals through the unperturbed problem. That is to say, monomials ζ^k transform into monomials ζ^{*k}, although its particular decomposition can be altered. This property allows employing the same truncating degree for $\mathcal{H}_{1\mathrm{per}}$ and $\mathcal{W}_1$.

The influence of the Earth interior with respect to the rigid model can be decomposed in two groups of terms. The first one involves y_2^* and Y_2^* and depends indirectly on the core by means of the matrix element r_3 and the eigenvalues m_1 and m_2. The second group contains a direct contribution of the core due to the variables y_{2C}^* and Y_{2C}^*, alternatively to the matrix element r_2, which are linked to the core. It would totally disappear in the rigid Earth model.

Similar observations to those pointed out for $\mathcal{W}_1$ can be extended to other functions entering the the construction of the second order solutions of the Poisson terms. To lighten the notation, in the following we will omit the asterisks in the transformed canonical variables unless there is risk of confusion.

B.2. *Second order functions*

The second order functions appearing in Hori's perturbation method (Section 2) give raise to the generating function $\mathcal{W}_2$ and the transformed Hamiltonian $\mathcal{H}_2^*$. They can be managed (Getino et al. 2010, Section 3.3) by computing the Poisson brackets

$$\mathcal{C}_S = \{\mathcal{H}_{1\mathrm{sec}}; \mathcal{W}_1\}\ ,\ \mathcal{C}_P = \frac{1}{2}\{\mathcal{H}_{1\mathrm{per}}; \mathcal{W}_1\}, \tag{B8}$$

leading to (Equations 21 and 22)

$$\begin{aligned} \mathcal{H}_2^* &= \mathcal{C}_{P\mathrm{sec}}, \\ \mathcal{W}_2 &= \mathcal{W}_{2s} + \mathcal{W}_{2p} = \int_{UP} \mathcal{C}_S\ dt + \int_{UP} \mathcal{C}_{P\mathrm{per}}\ \ dt, \end{aligned} \tag{B9}$$

since for our Hamiltonian $\mathcal{H}_2 = 0$ (Equation 20).

Once calculated[22] $\mathcal{C}_S$ and $\mathcal{C}_P$, the computations are alleviated, since, as we have previously indicated, both $\mathcal{H}_2^*$ and $\mathcal{W}_2$ can be truncated at ζ^0 degree in the construction of the second order solution of the Poisson terms. So, after computing the Poisson brackets of Equations (B8) we can skip all the

[22] It is not necessary to introduce the functions $\mathcal{C}_\alpha^\beta$ considered in Getino et al. (2010, Section 3.3) when developing the computations up to σ^2, since they only were needed to calculate the Oppolzer terms at the second order.

terms of degree one in ζ. In this way, it is obtained

$$
\begin{aligned}
\mathcal{C}_S = \; & \frac{\mathrm{i}}{2} \sum_{p=M,S} k'_p \sum_{\tau=\pm 1} \sum_{i\neq 0} \frac{e^{\mathrm{i}\tau\Theta_i}}{\tau n_i} \times \\
& \left\{ \tau m_{5i} B_i \left[\frac{\cos I}{\sin I} (e_2 \cos y_3 + e_3 \sin y_3) - e_1 \right] + \right. \\
& \mathrm{i} B'_i \, (e_2 \sin y_3 - e_3 \cos y_3) - \\
& \left. \sum_{q=M,S} \frac{\mathrm{i} k'_q}{Y_1 \sin I} \tau m_{5i} B_i B'_0 \right\}
\end{aligned}
\tag{B10}
$$

and

$$
\begin{aligned}
\mathcal{C}_P = & \sum_{p=M,S} \sum_{q=M,S} \frac{k'_p k'_q}{Y_1 \sin I} \sum_{\tau,\rho=\pm 1} \times \\
& \left\{ \sum_{i\neq 0} \sum_{j\neq 0} \frac{1}{8} \frac{e^{\mathrm{i}(\tau\Theta_i - \rho\Theta_j)}}{\tau n_i} \big(\tau m_{5i} B_i B'_j + \rho m_{5j} B'_i B_j \big) + \right. \\
& \left. \sum_i \sum_j \frac{\sin I}{2} \frac{\omega_E - \tau n_i - r_3}{\prod\limits_{k=1,2} (\omega_E - \tau n_i - m_k)} C_{i,\tau} C_{j,\rho} \cos(\tau\Theta_i - \rho\Theta_j) \right\}.
\end{aligned}
\tag{B11}
$$

Previous formulae show that the whole $\mathcal{C}_S$ is independent of the Earth model, whereas $\mathcal{C}_P$ does depend on it through the last summand.

The expression of the transformed Hamiltonian $\mathcal{H}_2^*$ arises from the secular part of $\mathcal{C}_P$. It is given by the combinations $\tau\Theta_i - \rho\Theta_j$ equal to zero, since they provide a nil time rate. Hence, we have

$$
\begin{aligned}
\mathcal{H}_2^* = & \sum_{p,q=M,S} \frac{k'_p k'_q}{Y_1 \sin I} \sum_{\tau,\rho=\pm 1} \left\{ \sum_{i\neq 0} \sum_{j\neq 0} \frac{1}{8} \frac{1}{\tau n_i} \big(\tau m_{5i} B_i B'_j + \rho m_{5j} B'_i B_j \big) + \right. \\
& \left. \sum_i \sum_j \frac{\sin I}{2} \frac{\omega_E - \tau n_i - r_3}{\prod\limits_{k=1,2} (\omega_E - \tau n_i - m_k)} C_{i,\tau} C_{j,\rho} \right\}_{\tau\Theta_i = \rho\Theta_j}.
\end{aligned}
\tag{B12}
$$

With respect to the generating functions $\mathcal{W}_{2s}$ and $\mathcal{W}_{2p}$, their computation is direct in this case, since they do not depend on Y_2, y_2, Y_{2c}, and y_{2c}. The evolution of the variables appearing in them are given by Equations (58), (59), so we obtain

$$\begin{aligned}\mathcal{W}_{2s} = \ & \frac{1}{2}\sum_{p=M,S} k_p' \sum_{\tau=\pm1}\sum_{i\neq 0}\frac{e^{\mathrm{i}\tau\Theta_i}}{n_i^2}\times \\ & \left\{\tau m_{5i}B_i\left[\frac{\cos I}{\sin I}(e_2\cos y_3 + e_3\sin y_3) - e_1\right] + \right. \\ & \mathrm{i}B_i'\,(e_2\sin y_3 - e_3\cos y_3) - \\ & \left.\sum_{q=M,S}\frac{\mathrm{i}k_q'}{Y_1\sin I}\tau m_{5i}B_iB_0'\right\}.\end{aligned} \tag{B13}$$

and

$$\begin{aligned}\mathcal{W}_{2p} = \ & \sum_{p=M,S}\sum_{q=M,S}\frac{k_p'k_q'}{Y_1\sin I}\sum_{\tau,\rho=\pm1}\times \\ & \left\{-\sum_{i\neq0}\sum_{j\neq0}\frac{1}{8}\frac{\mathrm{i}\ e^{\mathrm{i}(\tau\Theta_i-\rho\Theta_j)}}{\tau n_i(\tau n_i-\rho n_j)}\left(\tau m_{5i}B_iB_j' + \rho m_{5j}B_i'B_j\right) + \right. \\ & \left.\sum_i\sum_j\frac{\sin I}{2}\frac{\omega_E-\tau n_i - r_3}{\prod_{k=1,2}(\omega_E-\tau n_i-m_k)}C_{i,\tau}C_{j,\rho}\frac{\sin(\tau\Theta_i-\rho\Theta_j)}{\tau n_i-\rho n_j}\right\}_{\tau\Theta_i\neq\rho\Theta_j}.\end{aligned} \tag{B14}$$

The dependence on the Earth model of $\mathcal{H}_2^*$, $\mathcal{W}_{2s}$, and $\mathcal{W}_{2p}$ is inherited from that of $\mathcal{C}_S$ and $\mathcal{C}_P$. Therefore, $\mathcal{W}_{2s}$ is independent of the Earth model, whereas $\mathcal{H}_2^*$ and $\mathcal{W}_{2p}$ depend on it through the terms proportional to $C_{i,\tau}C_{j,\rho}$. That dependence is of the indirect kind, i.e., there is no r_2 factor.

C. PARTIAL COMPARISONS OF THE SECOND ORDER FORMULAE FOR THE POINCARÉ MODEL

The procedure to obtain the second order solution of the Poisson terms is systematic, but cumbersome. It makes desirable to establish comparisons, at least in a partial way, with other second order results existing in the literature, checking the correctness of some of the expressions derived in this section. Considering the nature of our second order theory, we will just focus on other analytical investigations.

C.1. *Second order solution of the rigid Earth model (Getino et al. 2010)*

The first and obligated comparison is with the second order solution of the rigid Earth (Getino et al. 2010). In spite of the fact that we have followed the same guidelines as in that work for constructing our solution, it is worth to compare with the rigid model. The main reason is that its solution was derived directly with Andoyer variables —what is feasible for a rigid model— whereas we have introduced the *NSCCV*.

With respect to the terms that are independent of the Earth interior (Equations 89 and 90), their analytical expressions are the same as those displayed in Getino et al. (2010, Appendixes C and D)[23] The contributions depending on the Earth model (Equation 91) are obviously different. However, we can test the consistency of our model if, when reducing it to the rigid case, we recover the corresponding formulae by Getino et al. (2010, Appendixes C and D). To do that we have to take $A_c = 0$ and keeping $e_c = 0$ in the Poincaré model, what leads to a rigid Earth model with ellipticity e and equatorial moment of inertia $A = A_m$.

If we consider such reductions in Equations (61), we have

$$r_{1\mathrm{R}} = -\omega_E e,\ r_{2\mathrm{R}} = 0, r_{3\mathrm{R}} = \omega_E, \tag{C15}$$

where the subscript R refers to the rigid particularization. The characteristic equation (Equation 62) has now the form

$$m_{\mathrm{R}}^2 - \omega_E(1-e)m_{\mathrm{R}} - \omega_E^2 e = 0, \tag{C16}$$

entailing, with no approximation, the solutions

$$m_{1\mathrm{R}} = \omega_E, m_{2\mathrm{R}} = -\omega_E e = \omega_E - n_\mu, \tag{C17}$$

[23] We have detected two missprints in Appendix D by Getino et al. (2010). The second B_j' in the expression of $\mathcal{O}_2^{p1}$ must appear as B_j (no prime). The term B_i' in $\mathcal{O}_3^1$ must be $B_i' - B_i \cos I/\sin I$.

where n_μ is the mean motion of the Andoyer variable μ of the rigid Earth (Getino et al. 2010)

$$n_\mu = \omega_E(1+e). \tag{C18}$$

The eigenvalue $m_{2\mathrm{R}}$ provides the opposite to the characteristic Eulerian frequency of the rigid Earth, which is the single proper mode of this model within our context.

With the former simplifications, the Earth parameters dependencies of the contributions given in Equations (91) reduce to

$$\begin{aligned} \frac{\omega_E - \tau n_i - r_{3\mathrm{R}}}{\prod\limits_{k=1,2}(\omega_E - \tau n_i - m_{k\mathrm{R}})} &= \frac{1}{n_\mu - \tau n_i}, \\ \frac{(\omega_E - \tau n_i - r_{3\mathrm{R}})(\omega_E - \rho n_j - r_{3\mathrm{R}}) - r_{2\mathrm{R}}^2}{\prod\limits_{k=1,2}[(\omega_E - \tau n_i - m_{k\mathrm{R}})(\omega_E - \rho n_j - m_{k\mathrm{R}})]} &= \frac{1}{n_\mu - \tau n_i}\frac{1}{n_\mu - \rho n_j}. \end{aligned} \tag{C19}$$

They lead to the same expressions derived in Getino et al. (2010, Appendixes C and D) for the rigid model. The analytical equivalences shown above have been also confirmed numerically (Section 5.2.2), the rigid Earth model derived from the Poincaré one with $r_{cm} = 0$ and $e_c = 0$ provides the same numerical results for the Poisson terms as those derived with the formulae by Getino et al. (2010).

C.2. *Second order precession solution of a two-layer Earth model (Ferrándiz et al. 2004, Baenas et al. 2017)*

There is also available a partial second order solution for the non-rigid Earth. It considers the contributions of the Earth structure to the precession in longitude. Indeed, to our knowledge, the second order precession in longitude of a Poincaré model by Ferrándiz et al. (2004) was the first study where it was recognized that the core affects the precessional motion in a non-negligible amount. Later, that work was extended to incorporate the effects of the elasticity of the mantle in Baenas et al. (2017).

Therefore, it is possible to compare their results with those obtained here. In particular, we will consider Equations 24 in Baenas et al. (2017), which correspond to Equations 18 in Ferrándiz et al. (2004). Both refer to a Poincaré model and provide the second order contribution to the precession in longitude δp that is the opposite to $d\lambda/dt$. As we have pointed out, the way of obtaining δp was formally different from that developed here, because the Andoyer variables set of the Poincaré model was employed in those works.

Considering Equation (71), the contribution to δp is given by

$$\delta p = -\sum_{p,q=S,M} \frac{k_p k_q}{\sin^2 I} \sum_{\tau,\rho=\pm 1} \left[\sum_{i,j\neq 0} S^L_{2a} + \sum_{i,j} S^L_{2b} \right]_{\tau\Theta_i=\rho\Theta_j}. \tag{C20}$$

That expression can be expanded, taking into account those values of i, j, τ, and ρ for which $\tau\Theta_i = \rho\Theta_j$. In particular, when $\Theta_i = \Theta_j$ the combination $\tau\Theta_i - \rho\Theta_j$ is zero only for $\tau = \rho$. If $\Theta_i = \Theta_j = 0$, any value of τ and ρ is possible to nill $\tau\Theta_i - \rho\Theta_j$.

In this way, we have for S^L_{2a} (Equation 72)

$$\begin{aligned}
-\sum_{p,q=S,M} \frac{k_p k_q}{\sin^2 I} \sum_{\tau,\rho=\pm 1} \sum_{i,j\neq 0} S^L_{2a} = & -\sum_{p,q=S,M} \frac{k_p k_q}{\sin^2 I} \sum_{\tau=\rho=\pm 1} \sum_{i=j\neq 0} S^L_{2a} = \\
& -\sum_{p,q=S,M} \frac{k_p k_q}{\sin^2 I} \sum_{\tau=\pm 1} \sum_{i\neq 0} \frac{1}{4} \frac{m_{5i}}{n_i} \left[B_i \left(B''_i - \frac{\cos I}{\sin I} B'_i \right) + B'_i B'_i \right] = \\
& -\frac{1}{2} \sum_{i\neq 0} \sum_{p,q=S,M} \frac{k_p k_q}{\sin^2 I} \frac{m_{5i}}{n_i} \left[B_i \left(B''_i - \frac{\cos I}{\sin I} B'_i \right) + B'_i B'_i \right],
\end{aligned} \tag{C21}$$

since the orbital functions B_i are independent of τ. This expression is the same as δp^{00}_P in Baenas et al. (2017).

Similar arguments can be applied to get the expanded form of the term S^L_{2b} (Equation 73) —in this case $C_{i,\tau}$ functions do depend on τ. It is also necessary to rewrite the function that contain the Earth parameters as

$$\frac{\omega_E - \tau n_i - r_3}{\prod\limits_{k=1,2} (\omega_E - \tau n_i - m_k)} = \frac{-\omega_E[(1 + r_{cm}) + (1 + r_{cm})e_c] + (\omega_E - \tau n_i)}{(-m_2 + \omega_E - \tau n_i)(-m_1 + \omega_E - \tau n_i)} = \frac{r_{4\mathrm{B}} + n_{h_{i,\tau}\mathrm{B}}}{f_{1;i,\tau\mathrm{B}} f_{2;i,\tau\mathrm{B}}} = F^{1a}_{i,\tau\mathrm{B}}, \tag{C22}$$

where the subscript B refers to the notation employed in Baenas et al. (2017). The resulting formulae are identical to δp^{10}_P and δp^{11}_P reported by Baenas et al. (2017). Such equivalences have also been corroborated numerically.

Table 6. Differences of the second order Poisson terms: Poincaré minus PREM models (unit: μas)

Argument					Period	$\Delta\lambda$(sin)	ΔI(cos)
l_M	l_S	F	D	Ω	Days	Dif.	Dif.
+0	+0	+0	+0	+1	−6798.38	−0.47	+7.27
+0	+0	+0	+0	+2	−3399.19	+0.65	−0.35
+0	+1	+0	+0	+1	+386.00	+0.53	+0.03
+0	+1	+0	+0	+0	+365.26	−5.78	−2.89
+0	+0	+2	−2	+2	+182.62	−0.91	+0.53
+0	+1	+2	−2	+2	+121.75	+0.69	−0.35

NOTE—The displayed amplitudes are the differences of columns (10) and (14) in Table 3 and the total second order terms computed from the parameters of the PREM Earth model. See the main text for a discussion.

D. EFFECT OF THE PARTICULAR POINCARÉ MODEL ON THE NUMERICAL AMPLITUDES

The numerical magnitude of the amplitudes of the second order Poisson terms (Table 3) will depend on the particular values of the employed Poincaré model of the Earth (Table 2). As we pointed out in Section 5.2.1, our choice just aimed at providing the order of magnitude of the new second order contributions. We can corroborate that point by calculating those second order amplitudes for a different parameterizations of the Poincaré model.

With that objective, we have considered a Poincaré model characterized with the parameters corresponding to the Preliminary Earth Model (PREM, Gilbert & Dziewonski 1981) as given in Mathews et al. (1991), a model quite far from that considered previously. It is defined by $e = 3.247 \times 10^{-3}$, $e_c = 2.547 \times 10^{-3}$, and $r_{cm} = 0.128407$, from which it is possible to obtain the values of r_1, r_2, r_3, m_1 and m_2 (Equations 61 and 62) necessary to evaluate the Poisson terms. As in the previous cases the constants k_M and k_S are the same as in Table 1.

In Table 6 we have displayed the differences among the amplitudes of the Poincaré model (Table 3) and those of the PREM one. There are just a few second order nutation arguments showing numerical differences larger than 0.5 μas, so both models provide close second order results. However, there exist some variations at the μas level, especially in the case of the terms with periods -6798.38 and $+365.26$ days. Basically, they can be attributed to the different values of the FCN of each model as explained below.

For the Poincaré model we have that $P_{FCN} = 434.13$ (sidereal days, Table 2), whereas for the PREM model we get (Equation 62) $P_{FCN} = 348.09$ (sidereal terms). Those differences, together with the small ones related to the values of r_{cm}, make, for example, that the product of $r_{cm}P_{FCN}$ (Equations 97 and 99) changes from about 54 to 47, decreasing the amplitude for some constituents like the obliquity of the term with period -6798.38 days. Another important deviation is due to the fluid resonance itself, since for the PREM model corresponds to about 347 (mean solar) days, and $\tau = -1$, much closer to the orbital annual periods than in the Poincaré model (about 433 mean solar days).

Indeed, the associated values δ_i (Equation 98) changes from 0.1563 and 0.1564 to -0.05221 and -0.05220 for the terms with periods $+365.26$ and $+365.22$ days, respectively. It entails that the fluid core resonance for the annual terms, with $\tau = -1$, is more profound, multiplying by a factor of about -2.5 the combination of parameters $-r_{cm}P_{FCN}/\delta_i$ (Equation 99) in the Poincaré model. This fact is neatly appreciated by the magnitude of the differences in the term $+365.26$ days in Table 4, as a result of the large amplification of the constituents involving that annual argument itself and the secular one, and the large values of their respective orbital coefficients (Appendix A).

In consequence, even considering a very different Poincaré model like the PREM one, the order of magnitude of the structure dependent part of second order Poisson terms is kept. There are differences for some nutation amplitudes at the order of a few μas, as shown, but they do not alter the global picture of the derived contributions. Therefore, as we pointed out before, these kind of second order contributions can no longer be ignored considering nowadays accuracies and must be incorporated to the nutation series.

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