DOI 10.29042/2019-4883-4889

Helix Vol. 9 (2): 4883- 4889

Regularization of Inverse Signal Recovery Problems

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Received: 25th February 2019, Accepted: 01st April 2019, Published: 30th April 2019

Abstract
The compensation for the distortions that occur during their registration is called the recovery of signals. The main problem studied in the literature review of this paper is the instability of the calculated estimations of input actions regarding the effects of feedback registration errors. Therefore, various techniques have been developed for regularizing the original equations on the basis of converting them into another equation, whose solution is calculated stably. The most famous technique is the regularization method developed by A. N. Tikhonov.
At the same time, this paper shows that the response may lack some information about the input effect, that is, even in the absence of measurement errors, the resulting solution will be approximate. A method is proposed for estimating the non-recoverable distortions caused by recording system operator itself, which can be used at the stage of its synthesis.
A linear form of representation of an impact component accessible to recovery through an impulse response is obtained so that the recovery task is reduced to the calculation of its coefficients. A method of regularization of systems of linear algebraic equations arising ,in this case, is proposed on the basis of adaptive estimation of registration error levels directly from the registered response.

Keywords
Signal Recovery, Distortion Compensation, Recording System Operator, Regularization Method.

Introduction
In the framework of this work, a signal is a function of time, the parameters of which contain some information about real processes or phenomena. Such signals are, in particular, channel signals of information transmission systems, responses to probing effects in radar, input influences in information-measuring systems, etc. Real signals arrive at the inputs of some systems, and manifest themselves in the form of responses at their outputs. In many cases, an integral model of signal conversion in recording systems is the integral relation of the convolution type [1-3].

$$u(t) = \int_0^{T_f} r(t-\tau) f(\tau) d\tau \,, \tag{1}$$

where - T_f is the duration of the signal (input) f ; $r(z)$ is the core integral relations (hardware function of the system), satisfying the condition of physical realizability.

$$r(z) \equiv 0, \ \forall z < 0 \tag{2}$$

In the following, it is assumed that the domain of the response $u(t)$ $0 \le t \le T_u$ is no less than the duration of the input, i.e. there is an inequality.

$$T_f \le T_u \,. \tag{3}$$

In addition, we assume that all the functions in (1) are continuous, real, and have a bounded Euclidean norm.

$$\|f\|^2 = \int_0^{T_f} f^2(t)dt < \infty \,; \|u\|^2 = \int_0^{T_u} u^2(t)dt < \infty \,; \| r_t \|^2 = \int_0 r^2(t-\tau)d\tau < \infty \,. \tag{4}$$

Recovery of signals is usually called estimation with a known core of input actions, based on processing the results of recording responses (empirical data).

Main Part of Research
Materials and Methods
The functioning of real recording systems is associated with discretization of the response domain, and the presence of distortions due to the uncontrolled effects of extraneous factors (interference, equipment noise) [2,4]. In accordance with this model, the registration of signals takes the form of:

$$w_i = u(i\Delta t) + \varepsilon(i\Delta t) = \int_0^{T_f} \phi_i(\tau) f(\tau) d\tau + \varepsilon_i, \ i = 1,..., N \,, \tag{5}$$

where Δt is response domain sampling interval.

$$\Delta t = T_u / (N-1) \,; \tag{6}$$

$$\phi_i(\tau) = r(i\Delta t - \tau),\ i = 1,...,N. \qquad (7)$$

Signal recovery from response logging results is an inverse problem [6,7]. One of the problems of its solution arises in view of the fact that in response, there may be a lack of information about the desired signal.

Indeed, it is known [4] that any function from L_2 may be uniquely represented as the sum of two orthogonal components:

$$f(\tau) = f_1(\tau) + f_2(\tau), \qquad (8)$$

where f_1 is lineal element:

$$f_1(\tau) = \sum_{i=1}^{N} \alpha_i \phi_i(\tau), \qquad (9)$$

and the second component is orthogonal to all functions of the form (7).

$$(f_2, \phi_i) = \int_0^{T_f} f_2(\tau)\phi_i(\tau)d\tau = 0, i = 1,...,N. \qquad (10)$$

Thus, the second component of the desired signal (8) does not manifest itself in the response. In other words, in general, only a component of the form (9) is available for recovery. Consider the possibility of a priori analysis of properties of components available for reconstruction when the core of integral relation (1), the discretization step of the response definition area and the model of the input action (desired signal) are specified.

It is obvious that such an analysis is related to modeling of the direct problem, and the formation of a response, for which in particular it is necessary to calculate a set of integrals of the form (10). As a quadrature formula, we use the formula of rectangles (the lid on the top means an estimate of the response).

$$\widehat{u}_i = \Delta\tau \sum_{k=1}^{M} \phi_{ik} f_k, \qquad (11)$$

Where $\Delta\tau$ is the sampling interval of the region of integration, determined by analogy with the relation (6);

$\widehat{u}_i \approx u(i\Delta t)$; $\phi_{ik} = \phi_i(k\Delta\tau)$; $f_k = f(k\Delta\tau), k = 1,...,M$.

Further implementation of the following inequality is assumed:

$$N \le M \qquad (12)$$

Set

$$\Phi = \{\Delta\tau\phi_{ik}\}, i = 1,..., N : k = 1,..., M, \qquad (13)$$

Therefore, the set of the relations (11) is approximated by approximate matrix equality.

$$\vec{u} = (u_1,...,u_N)' \approx \Phi\vec{f},\ \vec{f} = (f_1,..., f_M)'. \qquad (14)$$

In turn, for an approximation of the component (9) it is natural to use the vector representation

$$\vec{f}_1 = \Phi'\vec{\alpha}. \qquad (15)$$

The second of the components (8) is determined by the ratio

$$\vec{f}_2 = \vec{f} - \Phi'\vec{\alpha}, \qquad (16)$$

So according to (10) the orthogonality relation must hold

$$\Phi\vec{f}_2 = \Phi\vec{f} - \Phi\Phi'\vec{\alpha} = \vec{0}. \qquad (17)$$

Since relation (15) determines the orthogonal projection of the vector of samples of the desired signal, component (16) should have the minimum Euclidean norm. Therefore, the vector of coefficients must satisfy the following variational condition:

$$\mathrm{F}(\vec{\alpha}) = \|\vec{f} - \Phi'\vec{\alpha}\|^2 = \min F(\vec{\beta}) = \min \|\vec{f} - \Phi'\vec{\beta}\|^2, \forall\vec{\beta} \in R^N, \qquad (18)$$

It is of interest to obtain a method for calculating the projection (15) for a given vector and matrix, which makes it possible to carry out a priori analysis of the components available for reconstruction based on the simulation.

It is known [5] that the matrices of the form (12) can be represented as a singular decomposition:

$$\Phi = QL^{1/2}G', \qquad (19)$$

where the prime means the transpose operation;

$$L = diag(\lambda_1,..,\lambda_N)\ ;\ \lambda_1 \ge \lambda_2 \ge ... \ge \lambda_N \ge 0; \qquad (20)$$

Q is orthogonal matrix of eigenvectors of a symmetric non-negatively defined matrix.

$$A = \Phi\Phi', \qquad (21)$$

$$AQ = QL; \qquad (22)$$

$Q = \{\vec{q}_1 ... \vec{q}_N\}, \vec{q}_i = (q_{1i},...,q_{Ni})', i = 1,...,N$; (23)

$Q'Q = diag(1,...,1)$; (24)

If the rank of the matrix (13) is equal, and

$K < N$, (25)

then there will be equality:

$\lambda_{K+1} = \lambda_{K+2} = ... = \lambda_N = 0$. (26)

In this case, you can exclude the columns of the matrix corresponding to zero eigenvalues.

In turn G –in general, is an orthogonal matrix of dimension $M * N$, columns of which are eigenvectors of a symmetric non-negatively defined matrix:

$B = \Phi'\Phi$, (27)

$BG = GC$; $C = diag(\lambda_1,...,\lambda_N)$; (28)

$G'G = diag(1,...,1)$. (29)

The following is true.

Statement 1. Given the vector $\vec{f}$ umatrix $\Phi\partial$ for components (15) satisfying condition (17), the following relation is valid:

$\vec{f}_1 = GG'\vec{f}$. (30)

Evidence.

$\vec{f}_2 = \vec{f} - GG'\vec{f}$ (31)

It is easy to show that the vector is orthogonal to all rows of the matrix (13).For this (31) on the left and on the right we multiply by this matrix. As a result, we have:

$\Phi\vec{f}_2 = (\Phi - \Phi GG')\vec{f} = 0\vec{f}$. (32)

The zero matrix in parentheses is obtained by substituting the decomposition (19) taking into account the property (29).

We now consider the variational condition (18), and represent its right-hand side in the following form:

$F(\vec{\beta}) = \| \vec{f} \|^2 - 2\vec{\beta}'\Phi\vec{f} + \vec{\beta}'\Phi\Phi'\vec{\beta}$

Differentiating by vector and using representation (19), and taking into account the property orthogonality (29), we obtain the equality satisfied by the optimal in the sense of (18) vector of coefficients:

$QL^{1/2}G'\vec{f} = QL^{1/2}L^{1/2}Q'\vec{\alpha}$.

The diagonality of the matrix allows us to obtain equality:

$L^{1/2}Q'\vec{\alpha} = G'\vec{f}$.

Multiplying the last relation from the left and the right by the matrix, and taking into account the decomposition (19), we obtain:

$GL^{1/2}Q'\vec{\alpha} = \Phi'\vec{\alpha} = GG'\vec{f}$, (33)

which completes the proof of the above statement.

As a consequence of justice (30), we obtain the relation for the square of the norm of the vector (31):

$\|\vec{f}_2\| = \| \vec{f} \|^2 - \| G'\vec{f} \|^2$. (34)

From here, and from (29) it follows that the equality of the orthogonal component to zero is achieved on lineal vectors:

$\vec{f} \equiv \vec{f}_1 = G\vec{\beta}$, (35)

where $\vec{\beta}$ is vector with arbitrary real components $\vec{\beta} = (\beta_1,...,\beta_N)'$.

In the general case, when developing a signal recovery method, it seems natural to be guided by the fact that only a component of the form (9) is accessible to the restoration, using its discrete approximations (15) or (35). Then calculations are reduced to determining the coefficients of these representations based on the recorded values of the responses.

To illustrate the importance of taking into account the distortion of information on input effects in the response, consider the following example. This row of the matrix (13) has the form $\vec{\phi}_i = (0,..,0,v(1),...,v(Mi),0,..,0)$, where the number of zeros at the beginning is $i-1$, $Mi = 150$. $v(k) = 0,5(1-\cos(2\pi k / Mi))$, $i = 1,...,N = 200$, $M = N + 149$. As a vector of readout samples, we use a rectangular "impulse" where the number of zeros at the beginning is 50, and the number of consecutive units is 20. This corresponds to the simulation of radar measurements in range.

Fig. 1 shows graphs of the components of the vector $\vec{f}$,and based on the ratio (30) of its components $\vec{f}_1$. These graphs clearly demonstrate that the remaining component may differ significantly from the exact impact, and false "bursts" (the second negative impulse) may appear, which, when reconstructed, will be perceived as real.

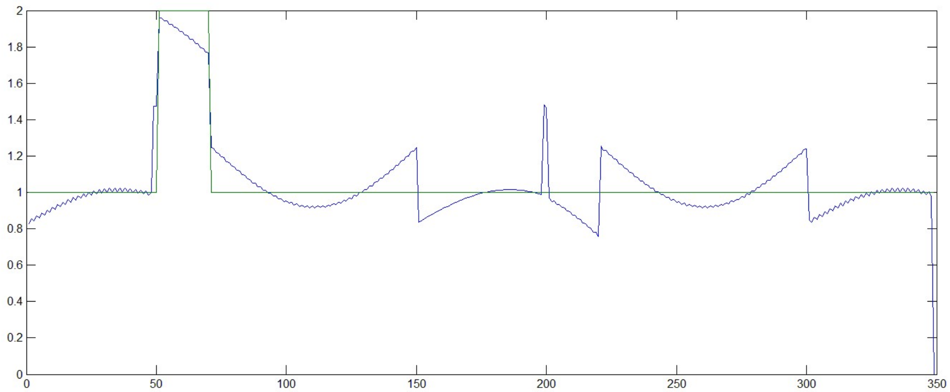

Fig.1. Graphs of Input Effects (Solid Line) and its Component (30)

Another problem in solving the signal recovery problem arises when some elements of the matrix are in the expansion (19) (the singular numbers of the matrix will not be large enough compared to measurement errors). In fact, referring to the representations (14) and (35), and also a decomposition of the form (19), taking into account the property (29), the model of real measurements of responses (5) can be approximated by the following relation:

$$\vec{w} = (w_1,\dots,w_N)' = QL^{1/2}\vec{\beta} + \vec{\varepsilon}\,. \qquad (36)$$

Here is the error vector $\vec{\varepsilon} = (\varepsilon_1,\dots,\varepsilon_N)'$, where it is advisable to include the approximation errors due to the use of quadratures, it is assumed to be unknown, so that to calculate the unknown coefficient vector $\vec{\beta}$ has to use the presentation:

$$\vec{w} = QL^{1/2}\vec{\beta}\,. \qquad (37)$$

Sustainable assessment of $\vec{\beta}$ is possible only when the elements of diagonal matrix are not very small, compared with the values of squares of the vector components $\vec{\varepsilon}$ at (36). Otherwise, it is necessary to use special techniques for constructing stable approximations to the desired signals. These techniques are called regularization.

Results and Discussion

Extensive literature is devoted to the problem of regularization of restoration problem. The main contradiction arises between the desire to achieve stability and at the same time ensure the reproduction of sufficiently subtle details of the input actions in the presence of unknown errors in response registration model.

Note that the known approaches to the construction of approximate solutions do not take into account the fact that, in response in the general case, there is no information about the second component of the decomposition (8).

The most general approach to the regularization of inverse problems was developed in the works of A. Tikhonov et al. [3,5,6]. It is based on the method of replacing the original equation with those "close" to it, whose solution is stable. This is implemented using the variational principle of minimizing the regularizing functional.

$$G(f,\alpha) = \int_0^{T_u} [w(t) - \int_0^{T_f} r(t-\tau)f(\tau)d\tau]^2\,dt + \alpha\Omega(f) = \min\ \forall f \in X\,, \qquad (38)$$

where it is assumed that the empirical data is a function of time; $\alpha \geq 0$ is regularization parameter; X is some functional space, for example the Sobolev space or L_2; $\Omega(f) \geq 0$ - stabilizing functional (stabilizer) defined on a given functional space.

The specificity of the convolution equation is the possibility of using the algebraic relation:

$$U(\omega) = R(\omega)F(\omega)\,, \qquad (39)$$

Between Fourier transformants (spectra, capital letters) of the functions included in the relation (1), the definition of which has the following form: $Z(\omega) = \int_{-\infty}^{\infty} z(y)\exp(-j\omega y)dy\,,$ (40)

where $\omega = 2\pi v$ is circular frequency; v is frequency; $j = (-1)^{1/2}$.

The conditions for the existence of such integrals are considered in many manuals, for example, in [7]. The existence of inverse transformations is also assumed.

$$z(y)=\int_{-\infty}^{\infty} Z(\omega)\exp(j\omega y)d\omega/2\pi . \quad (41)$$

In this regard, the principle of regularization (38) can be implemented on the basis of the model:

$$f_\alpha(\tau)=\int_{-\infty}^{\infty} W(\omega)R^*(\omega)\exp(j\omega\tau)d\omega/(|R(\omega)|^2+\alpha\omega^2)/2\pi , \quad (42)$$

where the asterisk denotes complex conjugation of the Fourier transform of the kernel, and Euclidean norm of the first derivative of the desired signal is used as a stabilizer.

Thus, the implementation of (42) involves the definition of the Fourier transform of the response and the calculation of the inverse transform at a certain value of the regularization parameter. It is interesting to analyze this procedure.

First, we note that from discrete data only the response spectrum estimate can be calculated.

$$W_d(\omega)=\Delta t\sum_{k=1}^{N} w_k \exp(-j\omega\Delta t(k-1)), \quad (43)$$

which is periodic

$$W_d(\omega+2\pi mP_w)=W_d(\omega), \quad (44)$$

with a period

$$P_w=2\pi/\Delta t . \quad (45)$$

Therefore, the integral in (42) should be considered in the frequency domain no wider:

$$-\pi/\Delta t\le\omega\le\pi/\Delta t . \quad (46)$$

At the same time, the following relationship is valid [9]:

$$W_d(\omega)=W(\omega)+\sum_{k=1}^{\infty} (W(\omega+2\pi kP_w)+W(\omega-2\pi kP_w)), \quad (47)$$

From which it follows that the response spectrum can be significantly distorted compared with the spectrum of continuous implementation, the frequency components near the boundaries of the interval (46) will be particularly distorted.

In reality, the spectrum (43) is also calculated in a discrete set of points,

$$\omega_i=\pi i/(\Delta t(I-1)),\ -I\le i\le I , \quad (48)$$

in which the spectrum of the nucleus must also be calculated. Therefore, (42) is approximated by the integral sum.

$$f_\alpha(k)=\Delta\tau\Delta\omega\sum_{i=-I}^{I} W(\omega_i)R^*(\omega_i)\exp(ji\Delta\omega(k-1))/(|R(\omega_i)|^2+\alpha\omega_i^2),\ \Delta\omega=\pi/(I-1) . \quad (49)$$

The spectral discretization step is often chosen in accordance with the principle of the discrete Fourier transform (DFT) (this allows the use of the Fast Fourier Transform (FFT) algorithm) [9].

$$\Delta\omega=2\pi/(\Delta tN) . \quad (50)$$

Thus, the spectrum of the evaluation of the desired signal will be limited to the interval (46), which can lead to errors. In addition, since

$$\Delta t=\Delta\tau . \quad (51)$$

But if it is necessary to preserve fine details of the input signal profile, for example, to ensure the resolution of closely spaced extremes, then one should appropriately select the sampling rate of the response domain. In particular, the Parseval equality [8] implies the requirements:

$$\int_{-\infty}^{\infty} r^2(t)dt=\int_{-\infty}^{\infty} |R(\omega)|^2 d\omega/2\pi\approx\int_{-\pi/\Delta t}^{\pi/\Delta t} |R(\omega)|^2 d\omega/2\pi , \quad (52)$$

$$\int_{-\infty}^{\infty} f^2(t)dt=\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega/2\pi\approx\int_{-\pi/\Delta t}^{\pi/\Delta t} |F(\omega)|^2 d\omega/2\pi . \quad (53)$$

It is clear that the fulfillment of the latter condition cannot be confirmed. Thus, it seems appropriate to apply such a regularization method, when the sampling intervals are not connected by a strict condition of the form (51), which can be impracticable when using empirical data.

The choice of the value of the regularization parameter is one of the main problems, which is characterized by a contradiction between the desire to ensure the sustainability of the evaluation of the input signal and the influence of inaccuracy of a priori knowledge about the properties of errors, along with the need to identify subtle details. The solution to this problem is based on the assumptions about the level of errors, for example, in the sense of its dispersion.

It seems expedient to develop such a method of recovering the input signal from empirical data, which takes into account only the first component in the additive mixture (8), and allows us to estimate the level of errors in recording response directly from the available data.
Since the second component of the sum (8) is lost, it is natural to use the representation as the source vector of values of the signal being restored.

$$\vec{f}_1 = G\vec{\beta}, \tag{54}$$

Taking into account the property (29) it is easy to get the ratio:

$$\|\vec{f}_1\|^2 = \|\vec{\beta}\|^2 . \tag{55}$$

whereas for the norm of the signal part of the response (14) in mind (19), and (24) takes place:

$$\|\vec{u}\|^2 = \sum_{k=1}^{N} \lambda_k \beta_k^2 . \tag{56}$$

Set

$$\vec{\gamma} = Q^{'}\vec{w} = Q^{'}\vec{u} + Q^{'}\vec{\varepsilon} . \tag{57}$$

In view of (36), we have:

$$\vec{y} = Q^{'}\vec{u} = (y_1, ..., y_N)^{'} = L^{1/2}\vec{\beta}, \tag{58}$$

$$y_k = \lambda_k^{1/2}\beta_k, \; k = 1, ..., N . \tag{59}$$

Thus, the fulfillment of equalities of the form (36) entails

$$y_k = 0, \; k = K+1, ..., N . \tag{60}$$

Therefore, the regularizing functional of A. N. Tikhonov formed on the basis of (55), (57) and (58) must consider only nonzero singular numbers:

$$G(\vec{\beta}, \alpha) = \sum_{k=1}^{K} (\lambda_k^{1/2}\beta_k - \gamma_k)^2 + \alpha \sum_{k=1}^{K} \beta_k^2 . \tag{61}$$

Minimization of this function with a fixed regularization parameter is achieved on a vector with components of the form:

$$\beta_k = \lambda_k^{1/2}\gamma_k / (\alpha + \lambda_k), \; k = 1, ..., K . \tag{62}$$

In accordance with (60), the remaining components of the coefficient vector in (54) should be given zero values.
When substituting this representation into the first sum of the right-hand side of (61), it is easy to obtain the ratio for the square of the norm of the vector of deviations from the empirical data:

$$H(\alpha) = \alpha^2 \sum_{k=1}^{K} \gamma_k^2 / (\alpha + \lambda_k)^2 . \tag{63}$$

If we now set its value, then the equation for the regularization parameter will be determined:

$$\alpha^2 \sum_{k=1}^{K} \gamma_k^2 / (\alpha + \lambda_k)^2 = s^2 . \tag{64}$$

It is easy to get the ratio:

$$dH(\alpha) / d\alpha = \alpha \sum_{k=1}^{K} \lambda_k \gamma_k^2 / (\alpha + \lambda_k)^2 ,$$

which shows that the right-hand side of (63) does not monotonously decrease, and, therefore, the root of equation (64) will be unique, and it should be borne in mind

$$s^2 < \sum_{k=1}^{K} \gamma_k^2 / \lambda_k^2 , \tag{65}$$

so that the root is non-negative and limited. Use equality:

$$s^2 = 0 \tag{66}$$

gives a zero value of the regularization parameter; therefore, it should be applied in cases when the error vector norm in (36) is small compared to the information vector norm (37), in particular, when the eigenvalues of matrix (21) are sufficiently large. Otherwise, it seems natural to use the estimate of the expectation for close to zero eigenvalues (see (60)).

$$s^2 = K \sum_{k=K+1}^{N} \gamma_k^2 / (N - K) . \tag{67}$$

  Helix E-ISSN: 2319-5592; P-ISSN: 2277-3495

We also note that to find the root of equation (64) which should use the method of successive approximations based on the representation

$$\alpha = s(\sum_{k=1}^{K} \gamma_k^2 / (\alpha + \lambda_k)^2)^{-1/2}, \tag{68}$$

which converges in fulfilling inequality (65).

Conclusion

In the present paper, the problem of sustainable signal recovery in linear systems with constant parameters is considered. It is shown that in the response of the system, some of the information about the input effect may be missing, and a method has been developed for the priori analysis of the component available for recovery. A projection method for the stable restoration of signals based on singular decomposition of matrices is proposed. The basic relations have been obtained, which make it possible to regularize recovery problem and calculate the regularization parameter directly from the measurement data.

Acknowledgment:

The present work was done with the financial support of the Ministry of Education and Science in framework of state assignment of the Belarusian State University (project # 8.2201.2017 / 4.6).

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