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MATHEMATICS

On Generalized Bessel Potentials and Perfect Functional Completions

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Abstract—The class of generalized Bessel potentials is the main object of study in this paper. The generalized Bessel potential is a negative real power of the operator $(I - \Delta_\gamma)$, where $\Delta_\gamma = \sum_{k=1}^n \frac{1}{x_k^{\gamma_k}} \frac{\partial}{\partial x_k} x_k^{\gamma_k} \frac{\partial}{\partial x_k}$ is the Laplace–Bessel operator and $\gamma = (\gamma_1, ..., \gamma_n)$ is a multi-index consisting of positive fixed real numbers. To solve various problems for differential equations, prove embedding theorems for some classes of functions, and invert integral operators, there is a need to consider functions up to some small (from the point of view of the problem under consideration) set. As such a small set, a set of Lebesgue measure zero is often taken. However, for many problems, sets of Lebesgue measure zero turn out to be too large to be disregarded. For example, when a boundary problem is solved, the behavior of the solution at the boundary is essential. In this regard, there arose the need to construct complete classes of admissible functions suitable for solving specific problems. Two stages of constructing a functional completion were presented by N. Aronszajn and K.T. Smith. The first of these stages consists in finding a suitable class of exceptional sets. The second is to find functions defined modulo of these exceptional classes that need to be joined to get a complete functional class. It turns out that there can be infinitely many suitable exceptional classes in a particular problem, but each of them corresponds in fact to one functional completion. It is clear that the most suitable functional completion is the one whose exceptional class is the smallest, since the functions will then be defined with the best possible accuracy. Whenever such a minimal exceptional class exists, the corresponding functional completion is called a perfect completion. In this paper, perfect completions are constructed using the norm associated with the kernel of the generalized Bessel potential.

Keywords: generalized Bessel potential, perfect completion of spaces

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1. INTRODUCTION

Classical Bessel potentials are the realizations of real negative powers of the differential operator $(I - \Delta)$ in the form:

$$(G^\alpha f)(x) = \int_{\mathbb{R}^n} \mathcal{G}_\alpha(xy) f(y) dy, \qquad \alpha > 0,$$

where $\mathcal{G}_\alpha(x) = \frac{2^{\frac{2-n-\alpha}{2}}}{\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)} \frac{K_{\frac{n-\alpha}{2}}(|x|)}{|x|^{\frac{n-\alpha}{2}}}$ and K_ν are modified Bessel functions of the second kind. The operators G^α are a powerful technical tool for harmonic analysis and its applications. The general theory of functional completions developed by Aronszajn and Smith in [1] is applied to spaces of Bessel potentials in [2]. The completions thus obtained are the most important classes of functions for studying differential problems, especially those of elliptic type. Classical Bessel potentials were also studied by Flett, Goldman, and others (see [2–4] for additional information and references).

In this paper, we deal with a singular differential Bessel operator of the form

$$(B_\gamma)_t = \frac{\partial^2}{\partial t^2} + \frac{\gamma}{t}\frac{\partial}{\partial t} = \frac{1}{t^\gamma}\frac{\partial}{\partial t} t^\gamma \frac{\partial}{\partial t}, \quad t > 0, \quad \gamma \in \mathbb{R}, \tag{1}$$

and consider the generalized Bessel potential, which is a negative real power of the operator $(I - \Delta_\gamma)$, where $\Delta_\gamma = \sum_{k=1}^n (B_{\gamma_k})_{x_k}$ is the Laplace–Bessel operator. The theory of such potentials was developed in [5–9]. Here we consider the functional class of generalized Bessel potentials and use a perfect functional completion to construct the space $\overset{\circ}{C}{}^\infty_{ev}$ with the norm associated with the generalized Bessel potential.

2. GENERALIZED BESSEL POTENTIAL AND ITS PROPERTIES

Let $\mathbb{R}^n$ be an n-dimensional Euclidean space,

$$\mathbb{R}^n_+ = \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \ \ x_1 > 0, \ldots, x_n > 0\right\},$$

$$\overline{\mathbb{R}}^n_+ = \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \ \ x_1 \geq 0, \ldots, x_n \geq 0\right\},$$

and let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a multi-index consisting of positive fixed real numbers γ_i, $i = 1, \ldots, n$, and $|\gamma| = \gamma_1 + \ldots + \gamma_n$.

Let Ω be a finite or infinite open set in $\mathbb{R}^n$ symmetric with respect to each hyperplane $x_i = 0$, $i = 1, \ldots, n$. Let $\Omega_+ = \Omega \cap \mathbb{R}^n_+$ and $\overline{\Omega}_+ = \Omega \cap \overline{\mathbb{R}}^n_+$, where $\overline{\mathbb{R}}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n, x_1 \geq 0, \ldots, x_n \geq 0\}$. We will work with the class $C^m(\Omega_+)$ of functions that are m-times differentiable for Ω_+. By $C^m(\overline{\Omega}_+)$, we denote the subset of functions from $C^m(\Omega_+)$ such that all derivatives of these functions with respect to x_i extend continuously up to $x_i = 0$ for all $i = 1, \ldots, n$. Let the class $C^m_{ev}(\overline{\Omega}_+)$ consist of all functions from $C^m(\overline{\Omega}_+)$ such that $\left.\frac{\partial^{2k+1} f}{\partial x_i^{2k+1}}\right|_{x_i=0} = 0$ for all non-negative integers $k \leq \frac{m-1}{2}$. In what follows, we use the notation C^m_{ev} for $C^m_{ev}(\overline{\mathbb{R}}^n_+)$. We set $C^\infty_{ev}(\overline{\Omega}_+) = \bigcap C^m_{ev}(\overline{\Omega}_+)$, where the intersection is taken over all finite m, and $C^\infty_{ev}(\overline{\mathbb{R}}_+) = C^\infty_{ev}$.

Let $\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+)$ be the space of all finitely supported functions $f \in C^\infty_{ev}(\overline{\Omega}_+)$. We use the notation $\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+) = \mathcal{D}_+(\overline{\Omega}_+)$ and $\overset{\circ}{C}{}^\infty_{ev}(\overline{\mathbb{R}}_+) = \overset{\circ}{C}{}^\infty_{ev}$.

Let $L^\gamma_p(\mathbb{R}^n_+) = L^\gamma_p$, $1 \leq p < \infty$, consist of all measurable functions on $\mathbb{R}^n_+$ even for each of the variables x_i, $i = 1, \ldots, n$, such that

$$\|f\|_{L^\gamma_p(\mathbb{R}^n_+)} = \|f\|_{p,\gamma} = \left(\int_{\mathbb{R}^n_+} |f(x)|^p \, x^\gamma dx\right)^{1/p} < \infty, \quad x^\gamma = \prod_{i=1}^n x_i^{\gamma_i}.$$

The multidimensional Hankel transform of the function $f \in L^\gamma_1(\mathbb{R}^n_+)$ is defined as follows:

$$\mathbf{F}_\gamma[f](\xi) = \mathbf{F}_\gamma[f(x)](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n_+} f(x)\mathbf{j}_\gamma(x;\xi)x^\gamma dx, \quad \mathbf{j}_\gamma(x;\xi) = \prod_{i=1}^n j_{\frac{\gamma_i - 1}{2}}(x_i\xi_i).$$

The symbol j_ν is used to denote the normalized Bessel function of the first kind $j_\nu(x) = \frac{2^\nu \Gamma(\nu+1)}{x^\nu} J_\nu(x)$, where J_ν is the Bessel function of the first kind.

Let $f \in L_1^\gamma(\mathbb{R}_+)$ be a function of bounded variation in the vicinity of point x of its continuity. Then, for $\gamma > 0$, the inversion formula for the transform has the form

$$\mathbf{F}_\gamma^{-1}[\hat{f}(x)](x) = f(x) = \frac{2^{n-|\gamma|}}{\prod_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int_{\mathbb{R}_+^n}\mathbf{j}_\gamma(x,\xi)\hat{f}(\xi)\xi^\gamma d\xi.$$

The subspace of the space of rapidly decreasing (Schwarz) functions S_{ev} has the form

$$S_{ev} = \left\{f \in C_{ev}^\infty : \sup_{x\in\mathbb{R}_+^n}\left|x^\alpha D^\beta f(x)\right| < \infty \quad \forall\alpha, \beta \in \mathbb{Z}_+^n\right\},$$

where $\alpha = (\alpha_1, ..., \alpha_n)$, $\beta = (\beta_1, ..., \beta_n)$, $\alpha_1, ..., \alpha_n$, $\beta_1, ..., \beta_n$ are non-negative integers, $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}$, $D^\beta = D_{x_1}^{\beta_1}\dots D_{x_n}^{\beta_n}$, and $D_{x_j} = \frac{\partial}{\partial x_j}$.

The multidimensional generalized translation is defined by the equality

$$({}^\gamma\mathbf{T}_x^y f)(x) = {}^\gamma\mathbf{T}_x^y f(x) = ({}^{\gamma_1}T_{x_1}^{y_1}\dots{}^{\gamma_n}T_{x_n}^{y_n} f)(x), \tag{2}$$

where each one-dimensional generalized translation ${}^{\gamma_i}T_{x_i}^{y_i}$, $i = 1, ..., n$, acts according to the formula

$$({}^{\gamma_i}T_{x_i}^{y_i} f)(x) = c(\gamma_i)\int_0^\pi f(x_1,\dots,x_{i-1},\sqrt{x_i^2+y_i^2-2x_iy_i\cos\varphi_i},x_{i+1},\dots,x_n)\sin^{\gamma_i-1}\varphi_i d\varphi_i,$$

where $c(\gamma_i) = \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma_i}{2}\right)}$. In what follows, we will use the notation $C(\gamma) = \pi^{-\frac{n}{2}}\prod_{i=1}^n\frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}$.

The generalized convolution generated by the multidimensional generalized translation ${}^\gamma\mathbf{T}_x^y$ has the form

$$(f*g)_\gamma(x) = (f*g)_\gamma = \int_{\mathbb{R}_+^n} f(y)({}^\gamma\mathbf{T}_x^y g)(x)y^\gamma dy. \tag{3}$$

The generalized Bessel potential is given by the relation (see [5–7])

$$u = (\mathbf{G}_\gamma^\alpha\varphi)(x) = \int_{\mathbb{R}_+^n} G_\alpha^\gamma(y)({}^\gamma\mathbf{T}_x^y\varphi(x))y^\gamma dy, \tag{4}$$

where

$$G_\alpha^\gamma(x) = \mathbf{F}_\gamma^{-1}\left[\left(1+|\xi|^2\right)^{-\alpha/2}\right](x) \tag{5}$$

is the generalized Bessel kernel. In [5], the space $\mathbf{B}_\gamma^\alpha(L_p^\gamma) = \{u{:}u = \mathbf{G}_\gamma^\alpha\varphi,\ \varphi \in L_p^\gamma\}$ with the norm $\|u\|_{\mathbf{B}_\gamma^\alpha(L_p^\gamma)} = \|\varphi\|_{L_p^\gamma(\mathbb{R}_+^n)}$ was introduced using B-hypersingular integrals. In a large review in [6], in particular, the connection between the Sobolev–Liouville space and the Bessel potential space is discussed. It was shown in [7] that

$$G_\alpha^\gamma(x) = \frac{2^{\frac{n-|\gamma|-\alpha}{2}+1}}{|x|^{\frac{n+|\gamma|-\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)\prod_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}K_{\frac{n+|\gamma|-\alpha}{2}}(|x|), \tag{6}$$

where $K_{\frac{n+|\gamma|-\alpha}{2}}$ is the modified Bessel function of the second kind.

Since G_α^γ is integrable with weight the x^γ (see [6, 7]), its Hankel transform exists for each ξ. The kernel G_α^γ is analytical for $\alpha > 0$ as a function of α. From (5) by analytical continuation, we find that the Hankel transform of the generalized Bessel kernel for $\alpha > 0$ is $\mathbf{F}_\gamma[G_\alpha^\gamma](\xi) = (1 + |\xi|^2)^{-\alpha/2}$.

Let us introduce the function

$$\omega_{\alpha,\gamma}(|x|) = \frac{2^{\frac{n-|\gamma|-\alpha}{2}+1}}{\Gamma\left(\frac{\alpha}{2}\right)\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}|x|^{\frac{n+|\gamma|-\alpha}{2}}K_{\frac{n+|\gamma|-\alpha}{2}}(|x|). \tag{7}$$

The operator $\mathbf{G}_\gamma^\alpha$ can be represented in terms of generalized convolution (3) with $\omega_{\alpha,\gamma}(|x|)$:

$$\mathbf{G}_\gamma^\alpha\varphi = \left(\frac{\omega_{\alpha,\gamma}(|x|)}{|x|^{n+|\gamma|-\alpha}} * \varphi\right)_\gamma, \quad \alpha > 0.$$

The generalized Bessel potential has the following main properties:

1. the semi-group property $\mathbf{G}_\gamma^\alpha\mathbf{G}_\gamma^\beta\varphi = \mathbf{G}_\gamma^{\alpha+\beta}\varphi$, $\varphi \in L_p^\gamma$,
2. $\mathbf{G}_\gamma^0\varphi = \varphi$, $\varphi \in L_p^\gamma$,
3. $\mathbf{F}_\gamma(\mathbf{G}_\gamma^\alpha\varphi)(x) = \left(1+|\xi|^2\right)^{-\alpha/2}\mathbf{F}_\gamma[\varphi](x)$, $\varphi \in S_{ev}$,
4. $\mathbf{G}_\gamma^{\alpha+2k}(I-\Delta_\gamma)^k\varphi = \mathbf{G}_\gamma^\alpha\varphi$, $\varphi \in S_{ev}$.

It follows from property 4, in particular, that the function $f(x) = \mathbf{G}_\gamma^{2k}\varphi(x)$, $x \in \mathbb{R}_+^n$ is the solution of the equation $(I - \Delta_\gamma)^k f(x) = \varphi(x)$, $k = 1, 2, \ldots$.

Example. The solution of the problem

$$f(x) - \Delta_\gamma f(x) = \mathbf{j}_\gamma(x,\xi), \quad f(0) = \frac{1}{1+|\xi|^2}$$

has the form $f(x) = (\mathbf{G}_\gamma^2)_x\mathbf{j}_\gamma(x,\xi) = \dfrac{\mathbf{j}_\gamma(x,\xi)}{1+|\xi|^2}$.

In [8], it was shown that, for $\alpha > 0$, $1 \le p \le \infty$, and $\varphi \in L_p^\gamma$, the generalized Bessel potential $\mathbf{G}_\gamma^\alpha\varphi$ coincides with the integral

$$\mathbf{G}_\gamma^\alpha\varphi(x) = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\int_0^\infty t^{\frac{\alpha}{2}-1}e^{-t}u_\gamma(x,t)dt \tag{8}$$

almost for all x, where $u_\gamma(x, t)$ is the generalized Gauss–Weierstrass integral of the form

$$u_\gamma(x,t) = \int_{\mathbb{R}_+^n} W_\gamma(y,t)({}^\gamma\mathbf{T}_x^y\varphi(x))y^\gamma dy, \quad W_\gamma(x,t) = C_{n,\gamma}\frac{e^{-\frac{|x|^2}{4t}}}{t^{\frac{n+|\gamma|}{2}}}, \quad C_{n,\gamma} = \frac{2^{-|\gamma|}}{\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}.$$

In this case, the function $u_\gamma(x, t)$ is the solution of the Cauchy problem $u_t = \Delta_\gamma u$, $u = u(x, t)$, $u(x, 0) = \varphi(x)$, on $S = \mathbb{R}_+^n \times (0, b)$ for $\varphi \in C_{ev}^0(S)$.

3. FUNCTIONAL CLASS OF GENERALIZED BESSEL POTENTIALS

The Banach space of integrated or generalized functions on the n-dimensional Euclidean space $\mathbb{R}^n$, which generalizes the usual Sobolev space of functions, whose derivatives belong to L_p classes, is the space of Bessel potentials. If Δ denotes the Laplace operator, then the Bessel potential space $\mathbf{B}^\alpha$ can be defined as the space of functions (or distributions) f such that $(I - \Delta)^{\alpha/2}f$ belongs to the Lebesgue space L_p normalized by the related Lebesgue norm. The operator $(I - \Delta)^{\alpha/2} = \mathbf{G}^{-\alpha}$ is, for $\alpha > 0$, a kind of fractional differentiation and is defined using some regularization (see [6]). The space of Bessel potentials was studied in [2].

We consider the space of generalized Bessel potentials

$$\mathbf{B}_\gamma^\alpha = \{u : u = \mathbf{G}_\gamma^\alpha \varphi, \varphi \in L_p^\gamma\}$$

with the norm $\|u\|_{\mathbf{B}_\gamma^\alpha(L_p^\gamma)} = \|\varphi\|_{L_p^\gamma(\mathbb{R}_+^n)}$. The space $\mathbf{B}_\gamma^\alpha$ of generalized Bessel potentials was first introduced by Lyakhov in [5] using the Stein–Lizorkin approach. In [5], the B-hypersingular integrals introduced earlier by Lyakhov and the Riesz B potentials were used to construct the norm in $\mathbf{B}_\gamma^\alpha$. We use another approach for introducing the norm in $\mathbf{B}_\gamma^\alpha$ based on studies of Aronszajn and Smith [2].

The spaces of generalized Bessel potentials of arbitrary order α are needed to determine the classes of solutions to a boundary-value problem of the form

$$Au = f \text{ in } D, \quad B_i u = 0 \text{ on } \partial D,$$

where A is an elliptic operator that contains Bessel differential operators (1); in particular, A can be the Laplace–Bessel operator Δ_γ.

The simplest norm on $\mathbf{B}_\gamma^\alpha$ is given by the weighted Dirichlet integral of order α by the formula

$$d_{\alpha,\gamma}(u) = \int_{\mathbb{R}_+^n} |\xi|^{4\alpha} \left|\mathbf{F}_\gamma[u](\xi)\right|^2 \xi^\gamma d\xi. \tag{9}$$

It was shown in [9] that the space $\overset{\circ}{C}{}_{ev}^{\infty}(\mathbb{R}_+^n)$ normalized by $\sqrt{d_{\alpha,\gamma}}$ cannot be a functional space. One of the simplest norms on $\overset{\circ}{C}{}_{ev}^{\infty}(\mathbb{R}_+^n)$ equivalent to $\sqrt{d_{\alpha,\gamma}}$ has the form

$$\|u\|_{\alpha,\gamma} = \left(\int_{\mathbb{R}_+^n} \left(1 + |\xi|^2\right)^{2\alpha} \left|\mathbf{F}_\gamma[u](\xi)\right|^2 \xi^\gamma d\xi \right)^{1/2}. \tag{10}$$

Further, we show that the norm (10) can be represented using convolution kernels (7) that generate the generalized Bessel potential.

Theorem 1. *The norm* $\|u\|_{\alpha,\gamma}$ *admits the following representation*:

$$\begin{aligned} \|u\|_{\alpha,\gamma}^2 = 2^{|\gamma|-n+1} \prod_{i=1}^n \Gamma^2\left(\frac{\gamma_i + 1}{2}\right) \Bigg(\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \frac{\left|{}^\gamma\mathbf{T}_x^y u(x) + u(y)\right|^2}{|x|^{n+|\gamma|+4\alpha}} \left(\omega_{-4\alpha,\gamma}(|x|) - \omega_{-4\alpha,\gamma}(0)\right) x^\gamma dx dy^\gamma dy \\ - \int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \frac{\left|{}^\gamma\mathbf{T}_x^y u(x) - u(y)\right|^2}{|x|^{n+|\gamma|+4\alpha}} \left(\omega_{-4\alpha,\gamma}(|x|) + \omega_{-4\alpha,\gamma}(0)\right) x^\gamma dx dy^\gamma dy \Bigg). \end{aligned} \tag{11}$$

Proof. Let $\gamma_{n+1} \geq 0$ be arbitrary. We consider the expression

$$\mathbf{J} = \int_0^\infty \int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \frac{\left| j_{\frac{\gamma_{n+1}-1}{2}}(z_0) u(x) - u(y) \right|^2}{\left[{}^\gamma\mathbf{T}_x^y |x|^2 + z_0^2\right]^{\frac{n+|\gamma'|+1+4\alpha}{2}}} x^\gamma dx y^\gamma dy z_0^{\gamma_{n+1}} dz_0.$$

We transform the integral $\mathbf{J}$ as

$$\mathbf{J} = \int_0^\infty z_0^{\gamma_{n+1}} dz_0 \int_{\mathbb{R}_+^n} \frac{x^\gamma dx}{\left[|x|^2 + z_0^2\right]^{\frac{n+|\gamma|+1+2\alpha}{2}}} \int_{\mathbb{R}_+^n} \left| j_{\frac{\gamma_{n+1}-1}{2}}(z_0)\, {}^\gamma\mathbf{T}_x^y u(x) - u(y) \right|^2 y^\gamma dy$$

$$= \frac{2^{n-|\gamma|}}{\prod_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int_0^\infty dz_0 \int_{\mathbb{R}_+^n}\frac{x^\gamma dx}{\left[|x|^2+z_0^2\right]^{\frac{n+|\gamma|+1+2\alpha}{2}}}\int_{\mathbb{R}_+^n}\left|j_{\frac{\gamma_{n+1}-1}{2}}(z_0)\mathbf{j}_\gamma(x;\xi)-1\right|^2 \left|\mathbf{F}_\gamma[u](\xi)\right|^2 \xi^\gamma d\xi.$$

Assuming $(x, z_0) = \tilde{z}$, $\tilde{\xi} = (\xi, 1)$, $\gamma' = (\gamma, \gamma_{n+1})$, we have

$$\mathbf{J} = \int_{\mathbb{R}_+^n}\mathcal{B}(\xi)\left|\mathbf{F}_\gamma[u](\xi)\right|^2\xi^\gamma d\xi,$$

where

$$\mathcal{B}(\xi) = \frac{2^{n-|\gamma|}}{\prod_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int_{\mathbb{R}_+^{n+1}}\frac{\left|\mathbf{j}_{\gamma'}(\tilde{z};\tilde{\xi})-1\right|^2}{|\tilde{z}|^{n+|\gamma'|+1+4\alpha}}\tilde{z}^{\gamma'}d\tilde{z}.$$

Making a change of variables $\tilde{z} = z/|\tilde{\xi}|$ in $\mathcal{B}(\xi)$, we obtain

$$\mathcal{B}(\xi) = \frac{2^{n-|\gamma|}}{\prod_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int_{\mathbb{R}_+^{n+1}}\frac{\left|\mathbf{j}_{\gamma'}(\tilde{z};\tilde{\xi})-1\right|^2}{|\tilde{z}|^{n+|\gamma'|+1+4\alpha}}\tilde{z}^{\gamma'}d\tilde{z}$$

$$= \frac{2^{n-|\gamma|}}{\prod_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\left(1+|\xi|^2\right)^{2\alpha}\int_{\mathbb{R}_+^{n+1}}\frac{\left|\mathbf{j}_{\gamma'}\left(z;\frac{\tilde{\xi}}{|\tilde{\xi}|}\right)-1\right|^2}{|z|^{n+|\gamma'|+1+4\alpha}}z^{\gamma'}dz$$

and, by virtue of the Parseval equality for the Hankel transform,

$$\mathbf{J} = D(n,\gamma',\alpha)\int_{\mathbb{R}_+^n}\left(1+|\xi|^2\right)^{2\alpha}\left|\mathbf{F}_\gamma[u](\xi)\right|^2\xi^\gamma d\xi,$$

where

$$D(n,\gamma',\alpha) = \frac{\Gamma^2\left(\frac{\gamma_{n+1}+1}{2}\right)}{2^{1-\gamma_{n+1}}}C(n+1,\gamma',\alpha),$$

$$C(n+1,\gamma',\alpha) = \frac{2^{1-|\gamma'|-4\alpha}\pi}{\sin(2\alpha\pi)\Gamma(2\alpha+1)\Gamma\left(\frac{n+|\gamma'|+1}{2}+2\alpha\right)\prod_{i=1}^{n+1}\Gamma\left(\frac{\gamma_i+1}{2}\right)}.$$

Hence, for arbitrary $\gamma_{n+1} \geq 0$, we get

$$\|u\|^2_{\alpha,\gamma} = \frac{1}{D(n,\gamma',\alpha)}\int_0^\infty\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n}\frac{\left|j_{\frac{\gamma_{n+1}-1}{2}}(z_0)u(x)-u(y)\right|^2}{\left[{}^\gamma\mathbf{T}_x^y|x|^2+z_0^2\right]^{\frac{n+|\gamma|+1+2\alpha}{2}}}x^\gamma dx y^\gamma dy z_0^{\gamma_{n+1}}dz_0.$$

Passing to $\gamma_{n+1} = 0$ and assuming

$$E(n,\gamma,\alpha) = D(n,\gamma',\alpha)\Big|_{\gamma_{n+1}=0} = \frac{\pi\sqrt{\pi}2^{-|\gamma|-4\alpha}}{\sin(2\alpha\pi)\Gamma(2\alpha+1)\Gamma\left(\frac{n+|\gamma|+4\alpha+1}{2}\right)\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)},$$

we write

$$\|u\|^2_{\alpha,\gamma} = \frac{1}{E(n,\gamma,\alpha)} \int_{-\infty}^{\infty} \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \frac{\left| e^{\frac{iz_0}{2}} u(x) - e^{-\frac{iz_0}{2}} u(y) \right|^2}{\left[{}^\gamma \mathbf{T}^y_x |x|^2 + z_0^2 \right]^{\frac{n+|\gamma|+1+2\alpha}{2}}} x^\gamma dx y^\gamma dy dz_0$$

$$= \frac{1}{E(n,\gamma,\alpha)} \int_{-\infty}^{\infty} \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \frac{\left| {}^\gamma \mathbf{T}^y_x u(x) - u(y) \right|^2 \cos^2 \frac{z_0}{2}}{\left[|x|^2 + z_0^2 \right]^{\frac{n+|\gamma|+1+2\alpha}{2}}} x^\gamma dx y^\gamma dy dz_0 \tag{12}$$

$$+ \frac{1}{E(n,\gamma,\alpha)} \int_{-\infty}^{\infty} \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \frac{\left| {}^\gamma \mathbf{T}^y_x u(x) + u(y) \right|^2 \sin^2 \frac{z_0}{2}}{\left[|x|^2 + z_0^2 \right]^{\frac{n+|\gamma|+1+2\alpha}{2}}} x^\gamma dx y^\gamma dy dz_0.$$

Using formula (2.5.9.4) from [10] and (7), we obtain

$$\int_{-\infty}^{\infty} \frac{\cos^2 \frac{z_0}{2} dz_0}{\left[|x|^2 + z_0^2 \right]^{\frac{n+|\gamma|+1+2\alpha}{2}}} = \frac{\sqrt{\pi} 2^{1-n-4\alpha} \Gamma(-2\alpha) \prod_{i=1}^n \Gamma\left(\frac{\gamma_i + 1}{2}\right)}{|x|^{n+|\gamma|+4\alpha} \Gamma\left(\frac{n + |\gamma| + 4\alpha + 1}{2}\right)} \left(\omega_{-4\alpha,\gamma}(0) + \omega_{-4\alpha,\gamma}(|x|)\right),$$

$$\int_{-\infty}^{\infty} \frac{\sin^2 \frac{z_0}{2} dz_0}{\left[|x|^2 + z_0^2 \right]^{\frac{n+|\gamma|+1+2\alpha}{2}}} = \frac{\sqrt{\pi} 2^{1-n-4\alpha} \Gamma(-2\alpha) \prod_{i=1}^n \Gamma\left(\frac{\gamma_i + 1}{2}\right)}{|x|^{n+|\gamma|+4\alpha} \Gamma\left(\frac{n + |\gamma| + 4\alpha + 1}{2}\right)} \left(\omega_{-4\alpha,\gamma}(0) - \omega_{-4\alpha,\gamma}(|x|)\right).$$

Substituting the calculated integrals into (12) and simplifying the constant, we obtain representation (11). □

We note that there are two possible approaches to the definition of the class $\mathbf{B}^\alpha_\gamma(L^\gamma_p)$ of generalized Bessel potentials of order α in $\mathbb{R}^n_+$. The first is that $u \in \mathbf{B}^\alpha_\gamma(L^\gamma_p)$ if u is a generalized convolution $(G^\gamma_\alpha * \varphi)_\gamma$ for a certain $\varphi \in L^\gamma_p(\mathbb{R}^n_+)$. This approach was presented in [3], where B-hypersingular integrals were used. The second approach consists in introducing the following norm into $\mathbf{B}^\alpha_\gamma(L^\gamma_p)$:

$$\|u\|^2_{\alpha,\gamma} = \int_{\mathbb{R}^n_+} \left(1 + |\xi|^2\right)^{2\alpha} \left|\mathbf{F}_\gamma[u](\xi)\right|^2 \xi^\gamma d\xi, \tag{13}$$

which can be written using the convolution kernel that generates the generalized Bessel potential. The norm $\|u\|_{\alpha,\gamma}$ is most convenient for studying the class of generalized Bessel potentials in $\mathbb{R}^n_+$.

4. PERFECT FUNCTIONAL COMPLETION OF THE CLASS $\mathcal{F}^\gamma_\alpha$

In this section, we study the normalized functional class $\mathcal{F}^\gamma_\alpha$ obtained by introducing the norm $\|u\|_{\alpha,\gamma}$ of the form (13) in the class $\overset{\circ}{C}{}^\infty_{ev}$ and show that the normalized functional class $\mathcal{F}^\gamma_\alpha$ has a perfect functional completion. For $\alpha > 0$, the exceptional class of a perfect completion is the class of sets on which the potential $\mathbf{G}^\alpha_\gamma\varphi$ of the function $\varphi \in L^\gamma_2$ may be undefined. The functions in a saturated perfect completion are equal to this potential, except for the exceptional set.

The abstract set $\mathcal{E}$ in which the functions of a linear functional class $\mathcal{F}$ are defined, is called the *basis set* of $\mathcal{F}$.

The *exceptional class* in the basic set $\mathcal{E}$ is the class $\mathcal{A}$ of subsets of the set $\mathcal{E}$ that satisfies two properties:

(i) hereditarity: if $A \subset \mathcal{A}$ and $B \in A$, then $B \in \mathcal{A}$,

(ii) σ additivity: if $A_n \in \mathcal{A}$, $n = 1, 2, \ldots$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

We say that any sentence is true, exc. $\mathcal{A}$, if the set of points at which it is false belongs to the exceptional class $\mathcal{A}$.

A linear functional class $\mathcal{F}$ is a *linear functional class* in relation to $\mathcal{A}$ if $\mathcal{A}$ is an exceptional class that contains the exceptional set of each f from $\mathcal{F}$. If $\mathcal{F}$ is a functional class relative to $\mathcal{A}$ (we write rel. $\mathcal{A}$ in this case), then $\mathcal{A}$ is called the exceptional class for $\mathcal{F}$, while the sets of $\mathcal{A}$ are called exceptional sets.

If there exists the smallest exceptional class $\mathcal{A}$ in relation to which $\mathcal{F}$ has a functional completion, then the saturated completion rel. $\mathcal{A}$ is called the *perfect completion* of $\mathcal{F}$.

If $\mathcal{F}$ is the functional class rel. $\mathcal{A}$, then so is the class $\mathcal{F}'$ of all functions defined exc. $\mathcal{A}$ and equal exc. $\mathcal{A}$ to a certain function from $\mathcal{F}$. In this case, the class $\mathcal{F}'$ is called a *saturated extension* of $\mathcal{F}$ rel. $\mathcal{S}$.

For $\alpha > 0$, we denote by $\mathcal{A}_{2\alpha}^{\gamma}$ the class of all sets of A such that, for some function $\varphi \in L_2^{\gamma}$ such that $\varphi \geq 0$, we have

$$A \subset \bigcup_{x} \left\{ x \in \mathbb{R}_{+}^{n} : (\mathbf{G}_{\gamma}^{2\alpha}\varphi)(x) = +\infty \right\}.$$

Let $\mathbf{P}_{\gamma}^{\alpha}$ denote the class of all functions u defined by exc. $\mathcal{A}_{2\alpha}^{\gamma}$ such that, for some function $\varphi \in L_{\gamma}^{2}$,

$$u(x) = (\mathbf{G}_{\gamma}^{2\alpha}\varphi)(x) \,\mathrm{exc}.\mathcal{A}_{2\alpha}^{\gamma}.$$

Since the kernel $G_{\alpha}^{\gamma}(x) \in L_1^{\gamma}(\mathbb{R}_{+}^{n})$, it follows that, for any $\varphi \in L_2^{\gamma}$, the function $(\mathbf{G}_{\gamma}^{\alpha}\varphi)(x)$ is defined and finite almost everywhere and $(\mathbf{G}_{\gamma}^{\alpha}\varphi)(x) \in L_2^{\gamma}$. In particular, each set of $\mathcal{A}_{2\alpha}^{\gamma}$ has Lebesgue weight measure zero. In addition, the Hankel transform of $(\mathbf{G}_{\gamma}^{\alpha}\varphi)(x)$ has the form $\mathbf{F}_{\gamma}(\mathbf{G}_{\gamma}^{\alpha}\varphi)(x) = (1 + |\xi|^2)^{-\alpha/2}\mathbf{F}_{\gamma}[\varphi](x)$. By virtue of the last equality and the Parseval equality for the Hankel transform, we have

$$\left\| \mathbf{G}_{\gamma}^{2\alpha}\varphi \right\|_{\alpha,\gamma} = \left(\int_{\mathbb{R}_{+}^{n}} \left| \mathbf{F}_{\gamma}[\varphi](x) \right|^2 \xi^{\gamma} d\xi \right)^{1/2} = c \left\| \varphi \right\|_{L_2^{\gamma}}, \quad \varphi \in L_2^{\gamma},$$

which proves the following lemma.

Lemma 1. *For* $\varphi \in L_2^{\gamma}$, *the following conditions are equivalent:*

1. $\varphi = 0$ *everywhere*, *except for sets of weighted Lebesgue measure zero*,

2. $\mathbf{G}_{\gamma}^{2\alpha}\varphi \equiv 0$,

3. $\mathbf{G}_{\gamma}^{2\alpha}\varphi = 0$ exc. $\mathcal{A}_{2\alpha}$,

4. $\mathbf{G}_{\gamma}^{2\alpha}\varphi = 0$ *everywhere*, *except for sets of weighted Lebesgue measure zero*,

5. $\left\| \mathbf{G}_{\gamma}^{2\alpha}\varphi \right\|_{\alpha,\gamma} = 0$.

Lemma 2. *The class* $\mathcal{A}_{2\alpha}^{\gamma}$ *is an exceptional class. The class* $\mathbf{P}_{\gamma}^{\alpha}$ *is the complete function space relative to* $\mathcal{A}_{2\alpha}^{\gamma}$.

Proof. To prove that $\mathcal{A}_{2\alpha}^{\gamma}$ is an exceptional class, one must prove that $\mathcal{A}_{2\alpha}^{\gamma}$ is hereditary (that is, if $A \in \mathcal{A}_{2\alpha}^{\gamma}$ and $B \subset A$, then $B \in \mathcal{A}_{2\alpha}^{\gamma}$) and σ additive. If $A \in \mathcal{A}_{2\alpha}^{\gamma}$ and $B \subset A$, then, for some function $\varphi \in L_2^{\gamma}$ such that $\varphi \geq 0$, it is true that

$$B \subset A \subset \bigcup_{x} \left\{ x \in \mathbb{R}_{+}^{n} : (\mathbf{G}_{\gamma}^{2\alpha}\varphi)(x) = +\infty \right\},$$

which means that $B \in \mathcal{A}_{2\alpha}^{\gamma}$. Next, let $A_n \in \mathcal{A}_{2\alpha}^{\gamma}$, and let $\varphi_n \geq 0$ be a function such that

$$A_n \subset \bigcup_{x} \left\{ x \in \mathbb{R}_{+}^{n} : (\mathbf{G}_{\gamma}^{2\alpha}\varphi_n)(x) = +\infty \right\}, \quad \left\| \varphi_n \right\|_{L_2^{\gamma}} \leq \frac{1}{2^{n+|\gamma|}}.$$

Then, if $A = \bigcup_{n=1}^{\infty} A_n$ and $\varphi = \sum_{n=1}^{\infty} \varphi_n$, it is clear that $\varphi_n \geq 0$ is a function in L_2^γ such that

$$A \subset \bigcup_x \left\{ x \in \mathbb{R}_+^n : (\mathbf{G}_\gamma^{2\alpha}\varphi)(x) = +\infty \right\};$$

therefore, $A \in \mathcal{A}_{2\alpha}^\gamma$. The lemma is proved.

□

Theorem 2. *The class $\mathbf{P}_\gamma^\alpha$ is the complete functional space relative to $\mathcal{A}_{2\alpha}^\gamma$.*

Proof. It follows from the above lemma that $\mathbf{P}_\gamma^\alpha$ is the normalized functional class rel. $\mathcal{A}_{2\alpha}^\gamma$; i.e., the conditions $u = 0$ exc. $\mathcal{A}_{2\alpha}^\gamma$ and $\|u\|_{\alpha,\gamma} = 0$ are equivalent. It follows from $\left\|\mathbf{G}_\gamma^{2\alpha}\varphi\right\|_{\alpha,\gamma} = c\,\|\varphi\|_{L_2^\gamma}$, $\varphi \in L_2^\gamma$ that $\mathbf{P}_\gamma^\alpha$ is complete; moreover, it is saturated. It remains only to prove the functional space property.

From any sequence converging to 0, one can choose a subsequence $\{u_n\}$ such that $\sum_{n=1}^{\infty} \|u_n\|_{\alpha,\gamma} < \infty$. If $u_n = \mathbf{G}_\gamma^{2\alpha}\varphi_n$, except for the set $A_n \in \mathcal{A}_{2\alpha}^\gamma$, then we put $\varphi(x) = \sum_{n=1}^{\infty} |\varphi_n(x)|$. Then $\varphi \in L_2^\gamma$ and $\mathbf{G}_\gamma^{2\alpha}\varphi(x) \to 0$ for all $x \notin A_0 = \bigcup_x \left\{ x \in \mathbb{R}_+^n : (\mathbf{G}_\gamma^{2\alpha}\varphi)(x) = +\infty \right\}$. Since $u_n \to 0$ for all $x \notin \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{A}_{2\alpha}^\gamma$. This proves that $\mathbf{P}_\gamma^\alpha$ is a complete functional space relative to $\mathcal{A}_{2\alpha}^\gamma$.

□

Theorem 3. *The class $\mathbf{P}_\gamma^\alpha$ is the complete functional completion of the class $\mathcal{F}_\alpha^\gamma$.*

Proof. Let us show that $\mathcal{F}_\alpha^\gamma \subset \mathbf{P}_\gamma^\alpha$. Let $u \in \mathcal{F}_\alpha^\gamma$ and $\mathbf{F}_\gamma\varphi = (1 + |\xi|^2)^\alpha \mathbf{F}_\gamma u$, $\varphi = \mathbf{F}_\gamma^{-1}\varphi$. Since $u \in \overset{\circ}{C}{}_{ev}^{\infty}(\mathbb{R}_+^n)$, it follows that $\mathbf{F}_\gamma\varphi \in L_1^\gamma$ and $\mathbf{F}_\gamma\varphi \in L_2^\gamma$, then $\varphi \in L_1^\gamma$, $\varphi \in L_2^\gamma$, and is continuous and bounded. Therefore, $\mathbf{G}_\gamma^{2\alpha}\varphi$ is continuous and belongs to $\mathbf{P}_\gamma^\alpha$. Since $\mathbf{F}_\gamma u = \mathbf{F}_\gamma \mathbf{G}_\gamma^{2\alpha}\varphi$, we have $u = \mathbf{G}_\gamma^{2\alpha}\varphi$ except for the weighted Lebesgue zero measure, but since both functions are continuous, it follows that $u = \mathbf{G}_\gamma^{2\alpha}\varphi$ everywhere; thus, $u \in \mathbf{P}_\gamma^\alpha$ and, consequently, $\mathcal{F}_\alpha^\gamma \subset \mathbf{P}_\gamma^\alpha$. We denote by $\overline{\mathcal{F}_\alpha^\gamma}$ the closure of $\mathcal{F}_\alpha^\gamma$ in $\mathbf{P}_\gamma^\alpha$. Then $\overline{\mathcal{F}_\alpha^\gamma}$ is a functional completion of $\mathcal{F}_\alpha^\gamma$. We must show that $\overline{\mathcal{F}_\alpha^\gamma} = \mathbf{P}_\gamma^\alpha$ and that this completion is perfect. Since the norm $\|u\|_{\alpha,\gamma}$ is finite for every $u \in \mathbf{P}_\gamma^\alpha$, every $u \in \mathbf{P}_\gamma^\alpha$ is equal to some $v \in \overline{\mathcal{F}_\alpha^\gamma}$ everywhere except for the set of Lebesgue measure zero. However, both u and v lie in $\mathbf{P}_\gamma^\alpha$; therefore, the fact that u is equal to v everywhere, except for the set of Lebesgue measure zero, implies $\|u - v\|_{\alpha,\gamma} = 0$, and hence $u = v$ exc. $\mathcal{A}_{2\alpha}^\gamma$. Thus, $\overline{\mathcal{F}_\alpha^\gamma} = \mathbf{P}_\gamma^\alpha$.

□

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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