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CONSTRUCTIONS OF CONNECTED GRAPHS WITH A GIVEN MATCHABLE RATIO

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Abstract

Given a positive rational number $\frac{a}{b}(0 \le a \le b)$, we identify families of connected graphs G, such that the ratio of the number of matchable edges to the total number of edges in G -denoted by $\pi(G)$, is $\frac{a}{b}$. We call $\pi(G)$, the matchable ratio of G. For certain kinds of rational numbers, we identify the smallest connected graphs with the property. This problem was initially discussed in [1].

1. Introduction

The graphs considered here are finite, and contain neither loops nor multiple edges. Let G be a graph. An edge of G is called **matchable**, if and only if it belongs to a perfect matching in G. This paper continues the investigation into matchable edges started in [1], where we focused on **totally matchable graphs**, that is, graphs in which every edge is matchable. We now turn our attention to graphs which are not totally matchable. These are graphs which contain non-matchable edges.

Definition

Let G be a graph with b edges; i.e. of size b, and with a matchable edges. Then the ratio $\pi(G) = \frac{a}{b}$ is called the **matchable ratio** of G.

The following are additional definitions, which will apply to the material that follows.

Definitions

(i) A **1-cycle** and a **2-cycle** is a vertex and an edge respectively. A cycle with more than two vertices is called a **proper cycle**.

In the material that follows, "cycle" will mean "proper cycle", unless otherwise specified.

(ii) A graph G is **non-matchable (matchable) saturated** if and only if no more non-matchable (matchable)edges can be added to it.

(iii) An edge joining two non-adjacent vertices of a cycle is a **chord** (The 3-cycle has no chords).

(iv) An even n-cycle is **canonically labeled,** if an only if its vertices are labeled in some agreed order, with the consecutive integers from 1 to n. For purposes of this paper, we will take the order to be a clockwise.

(v) In a canonically labeled cycle, a chord joining two odd (even) labeled vertices is called **odd (even)**; otherwise it is **mixed**.

(vi) A graph, consisting of a canonically labeled r-cycle C_r, with s odd chords and no even chords added, will be denoted by $G_{r,s}$. The subgraph C_r, is called its **boundary**.

(vii) A **chain** is a tree with nodes of valency 1 and 2 only. A **boundary chain** of $G_{r,s}$ is any connected subgraph of its boundary.

From the definition, the graph $G_{r,s}$ contains $r \geq 4$ vertices and $r+s$ edges. The graph $G_{r,0}$ is the cycle C_r.

In [1] we identified graphs with certain matchable ratios. We also established the existence of a graph for any given matchable ratio (Theorem 3.3). The proof of this result also provides an algorithm for the construction of such graphs. The graphs obtained from this theorem, may be disconnected. It is therefore interesting to be able to construct connected graphs with a given matching ratio $\frac{a}{b}$. Even more interesting, is the construction of a connected graph having precisely size b and with a matchable edges. In this paper, we give constructions for such graphs. Moreover, we identify the smallest order graphs with this property.

2. Graphs with matchable ratio a/b, where $a = 2n$ $(n > 1)$, $0 \leq a \leq b$ and $b - a \leq \binom{n}{2}$

Lemma 1

In the graph $G_{r,s}$, every chord is non-matchable.

Proof

Figure 1 shows a canonical drawing of $G_{r,s}$.

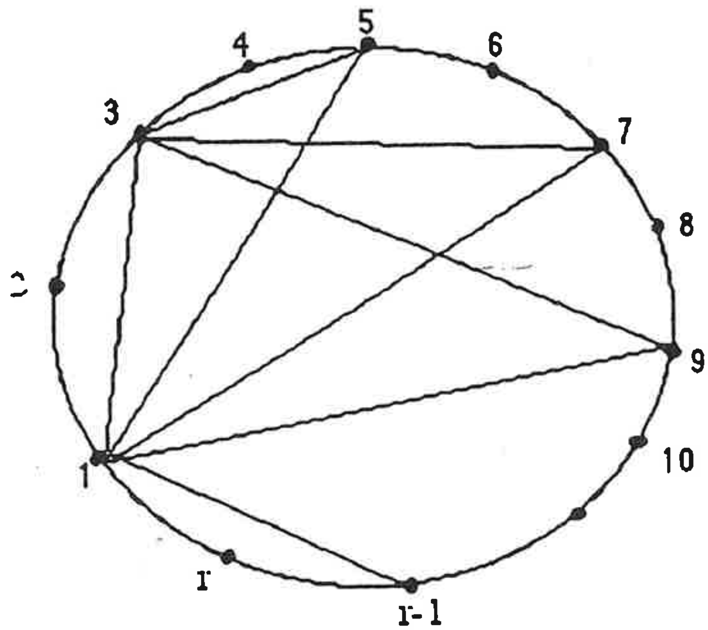

Figure 1

All chains referred to, will be boundary chains. It can be easily seen that a chain has odd order, if and only if the labels of its end-vertices have the same parity. Let us assume that there exists a perfect matching M containing the (odd) chord xy Now, remove vertices x and y from $G_{r,s}$. Since x and y are odd and there are no even or mixed chords in $G_{r,s}$, we have in $G_{r,s}$-x-y vertices x-1, x +1, y -1 and y+1 each having valency 1 (see Figure 2) and all other even vertices have valency 2. In particular, the edges (x-1, x-2), (x+1, x+2),(y -1, y -2) and (y+1, y+2) must all be in M.

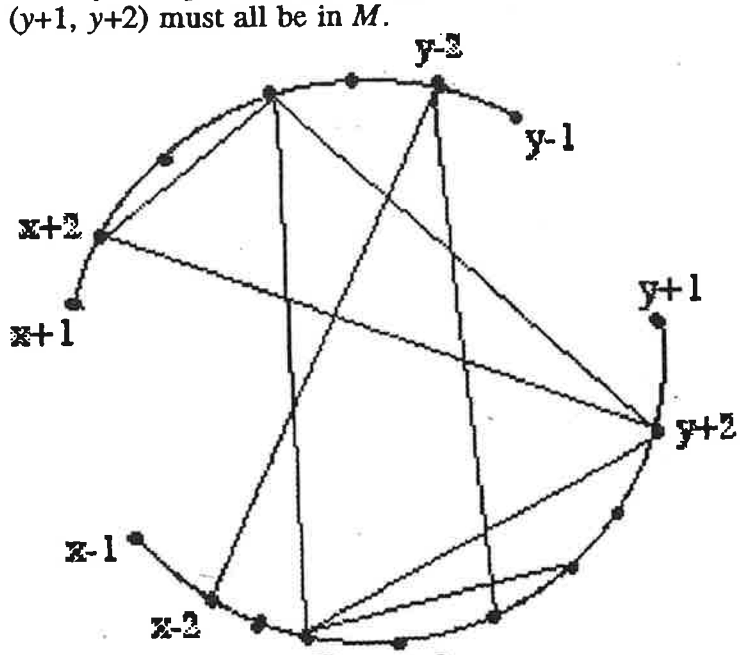

Figure 2

The edge joining x+1 to x+2 must belong to M. This forces the edge joining x+3 to x+4 to be in M. By continuing in this manner, we get that the edge joining y-3 and y-2 must belong to M. But the edge (y-1, y-2) is in M. This is a contradiction; since(by assumption) M is a perfect matching. Thus our assumption is false. The chord xy is non-matchable. Hence the result follows. □

Theorem 1

Let $r = 2n$ ($n \geq 1$). If $s = \binom{n}{2}$, then the graph $G_{r,s}$ is non-matchable saturated.

Proof

From the lemma, all the chords of $G_{r,s}$ are non-matchable. There are n odd labelled vertices in $G_{r,s}$. Any pair of these can be joined to form an odd chord. The number of such pairs is $\binom{n}{2}$. Therefore, when s takes this value, every pair of odd labelled vertices are joined by an edge, so that all odd chords are included.

We must now show that no more non-matchable chords can be added to $G_{r,s}$, that is, every new chord is matchable. Let us add a new chord xy. Then xy must either be (i) even or (ii) mixed. Call the resulting graph G.

Case (i) (xy is even)

Let us remove vertices x and y from G. In the resulting graph G', vertices x-1 and x+1, being odd vertices, will be joined by an edge (being odd labelled vertices); and so too will be vertices y-1 and y+1. Thus, the resulting graph will contain a new boundary cycle C_{r-2}. Since r is even, r-2 is even. Thus, G' has a perfect matching; and the chord xy is matchable in $G_{r,s}$.

Case (ii) (*ij* is mixed)

Without loss in generality, we will assume that x is odd and that y is even. Again, let us remove vertices x and y from G to obtain a graph G'. Then, G' will contain two boundary chains-one chain connecting vertex x+1 (even) to vertex y-1 (odd); the other, connecting vertex x-1 to vertex y+1. Since the chains have endnodes with different parities. They will be even chains; and consequently, have perfect matchings. Hence G' has a perfect matching. Adding the chord xy to this matching, yields a perfect matching in $G_{r,s}$.

We conclude therefore, that no more non-matchable edges can be added to $G_{r,s}$. Hence $G_{r,s}$ is saturated. □

Corollary 1.1

Let $a = 2n$ ($n \geq 2$) and b be positive integers, with $a \leq b$ and $0 \leq b\text{-}a \leq \binom{n}{2}$.

Then the graph $G_{a,b-a}$ is a connected graph such that $\pi(G_{a,b-a}) = \frac{a}{b}$.

Proof

Let the number of added (odd) chords be ε. Then (from Theorem 1) we get

$$0 \leq \varepsilon \leq \binom{n}{2}.$$

Since C_a has a edges, total number of edges in G is $a+\varepsilon$.

$$\Rightarrow \pi(G) = \frac{a}{a+\varepsilon}.$$

But $\varepsilon = b\text{-}a$. Therefore, the result follows. □

3. Graphs with matchable ratio a/b, with $a = 2n+1$ ($n > 1$), $0 < a < b$ and $b\text{-}a \leq \frac{4}{400}$.

Definition

The graph H_r is a canonically labeled r-cycle, with a chain of length 2, attached to one of its vertices; and with the vertex of valency 2 and 1 , labeled r+2 and r+1 respectively. The graph $H_{r,s}$ is the labeled graph H_r, with s odd chords added (See Figure 3).

$H_{r,s}$:

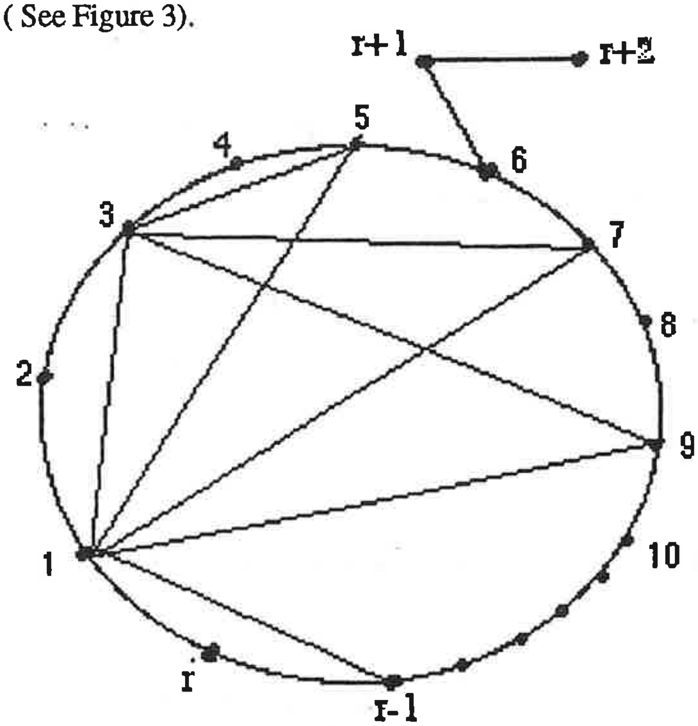

Figure 3

This graph is essentially the graph $G_{r,s}$, with a chain of length 2 attached to one of its boundary vertices., By definition, the graph $H_{r,s}$ contains r+2 vertices and r+2+s edges.

The following lemma is analogous to Lemma 1.

Lemma 2

In the graph $H_{r,s}$, every chord is non-matchable.

Proof

In Figure 3, we show a canonical drawing of a labeled graph $H_{r,s}$. The attached chain does not affect the arguments given in the proof of Lemma 1, since the edge joining r+1 to r+2 must be used in every perfect matching. Hence the result follows. □

This lemma implies that we can still add non-matchable odd (or even) chords to the subgraph $G_{r,s}$ of $H_{r,s}$. Also, it is clear that the edge joining the vertex $r+1$ to the boundary is non -matchable. Therefore, $H_{r,s}$ will have $r+1$ matchable edges and $s+1$ non-matchable edges. In $H_{r,s}$ new kinds of edges can join the" external" vertices $r+1$ and $r+2$ to vertices on the boundary of $G_{r,s}$. It is difficult to tell which ones are non-matchable, since the presence of these edges could even spoil the non-matchability of chords. We can however saturate the $G_{r,s}$ subgraph of $H_{r,s}$, so that the resulting graph is saturated with non-matchable chords. This yields the following analogy to Theorem 1.

Theorem 2

Let n be an integer greater than 2, and let $a = 2n+1$. Then the graph $H_{a-1,s}$ is saturated with non-matchable chords , when $s = \binom{n}{2}$.

The following result is immediate from Theorem 2; and is analogous to Corollary 1.1.

Corollary 2.1

Let $a = 2n+1$ $(n \geq 2)$ and b be positive integers, with $a \leq b$ and $0 \leq b\text{-}a \leq \binom{n}{2}$. Then the graph $H_{a-1,b-a+1}$ is a connected graph such that

$\pi(H_{a-1,b-a+1}) = \frac{a}{b}$. □

4. Graphs with matchable ratio a/b in which b-a is not bounded above by $\binom{n}{2}$.

We now consider the case in which $0 \leq a < b$ and $b\text{-}a$ is not bounded above by $\binom{n}{2}$. In the results above, $b\text{-}a$ is bounded above by $\binom{n}{2}$-the number of chords that can be added to the boundary cycle. This excludes many classes of rational numbers. For example, the rational number $\frac{1}{100}$ is not covered by the results above, since they all are based on the condition that C_a is a proper cycle. In fact, the smallest even value of a is 4 (Corollary 1.1) and the smallest odd value is 5 (Corollary 2.1). Therefore the numerator of the fraction must be at least 4. In this case the fraction will be $\frac{4}{400}$.

When b-a is bounded, as defined above, we have identified a connected graph G of size b with a matchable edges, such that $\pi(G) = \frac{a}{b}$. However, for some rational numbers, it will be impossible to find a graph; connected or not, with this property. For example, for the rational number $\frac{1}{100}$, one would have to find a graph with 100 edges in which exactly one edge belongs to a perfect matching. No such graph exists. We will therefore consider the related problem of finding a connected graph G, such that $\pi(G) = \frac{a}{b}$. This means that there are no restrictions on the size of the graph. Our technique is based on the simple fact that the rational numbers $\frac{a}{b}$ and $\frac{ka}{kb}$ are equal, for all non-zero values of k. This will allow us to use the construction given in Sections 2 and 3, since we can always arrange for ka to be even. We will do better than this. We will identify a smallest order graph obtained by our construction, that is, a smallest order $G_{r,s}$.

The following result is crucial.

Lemma 3

For all positive integers a and b, with $b \geq a$, there exists a positive integer k such that

$kb - ka \leq \binom{ka/2}{2}$, when ka is even. Furthermore, the smallest value of k for which the condition holds is $\left\lceil \frac{2}{a}\left(\frac{4}{a}(b-a)+1\right)\right\rceil$.

Proof

Since $b \geq a$, then $b-a \geq 0$ and $\left\lceil \frac{2}{a}\left(\frac{4}{a}(b-a)+1\right)\right\rceil$ is a positive integer.

Choose $k \geq \left\lceil \frac{2}{a}\left(\frac{4}{a}(b-a)+1\right)\right\rceil$. Then $\frac{ka}{2} \geq \left(\frac{4}{a}\right)(b-a)+1$. From this, we obtain $\frac{ka}{2} - 1 \geq \left(\frac{4}{a}\right)(b-a)$, which in turn, implies that $b - a \leq \frac{a}{4}(\frac{ka}{2} - 1)$.

Thus $k(b-a) \le \frac{ka}{4}(\frac{ka}{2}-1) = \binom{ka/2}{2}$, when ka is even. Hence there exists such a positive integer k. It can be easily shown that if $k < \left\lceil \frac{2}{a}\left(\frac{4}{a}(b-a)+1\right) \right\rceil$, the inequality no longer holds. Hence the result follows. □

This lemma identifies the range of values of *k* which would make the *ka*-cycle (when *ka* is even) large enough so that there would be enough "room" to add the necessary number (*kb-ka*) of chords. We can now construct a graph *G*, for which $\pi(G) = \frac{ka}{kb}$. This is the gist of the following theorem.

Theorem 3

Let $\frac{a}{b}$ be a positive rational number. Then, for $n = ka$ and $s = k(b-a)$, where $k \ge \left\lceil \frac{2}{a}\left(\frac{4}{a}(b-a)+1\right) \right\rceil$, the graph $G_{n,s}$ has the property that $\pi(G_{n,s}) = \frac{a}{b}$, when *ka* is even. When *ka* is odd, the graph $H_{n-1,s}$ has the property that $\pi(H_{n-1,s}) = \frac{a}{b}$.

Proof

The result follows immediately from Corollary 1.1, Corollary 2.1 and Lemma 3. □

This theorem gives the range of values of *k* for which graphs of the types $G_{r,s}$ and $H_{r,s}$ can be constructed with a given matchable ratio.

Example 1

Let $\frac{a}{b}$ =1. Then $a = b$. In this case, b-a = 0, so that $k = \frac{2}{a}$=1, $n = ka$ =2 and $s = k(b-a)$ =0. The resulting graph is $G_{n,s} = G_{2,0}$; which is a 2-cycle with no chords added. Therefore, the graph is an edge.

Example 2

Let $\frac{a}{b}=\frac{1}{100}$. Then a =1 and b =100. In this case, the smallest value of k is

$$\left\lceil \frac{2}{1}\left(\frac{4}{1}(100-1)+1\right)\right\rceil = 2(397)=794. \text{ Therefore } \frac{ka}{kb}=\frac{794}{79400}.$$

Therefore resulting graph is $G_{794,79400}$. This graph has 794 vertices, 79400 edges; and contains

79400 -794 =78606 chords.

It will be interesting to find out how good is this lowest value of k. We will therefore find the maximum number of (odd) chords that the 794-gon can contain. It is

$$\binom{794/2}{2}=\binom{397}{2}=78606.$$

This means that the graph $G_{794,79400}$ is saturated. Thus, we have indeed found the smallest order graph of the form $G_{r,s}$. In the above Example (ii), the smallest order graph belonging to the family of graphs of the form $G_{r,s}$ was found. However, it is a large graph. The natural question at this stage is the following. Can we find a smaller order graph; maybe from an entirely different family with $\frac{a}{b}=\frac{1}{100}$? This question motivates the material in the next section.

5. The smallest connected graphs with matchable ratio

If the matchable ratio $\frac{a}{b}$ is 0 or 1, then the smallest connected graphs are obvious, For $\frac{a}{b}$ = 0, the smallest connected graph is P_3 . For $\frac{a}{b}$ = 1, every edge is matchable. In this case, the smallest graph is an edge. We will therefore consider only matchable ratios which are neither 0 nor 1.

We will denote vertex and edge sets of a graph G, by V(G) and E(G), respectively.

Definition

Let A and B be graphs. We sill say that A is **smaller than** B if and only if $|V(A)| \leq |V(B)|$ and $|E(A)| \leq |E(B)|$ and at least one of the inequalities is strict.

Lemma 4

Let r be a positive integer.such that $r = \binom{n}{2}$, for some even positive integer n. Then the smallest order connected graph with r matchable edges is K_n. Otherwise, the smallest order connected graph with r matchable edges is $G_{r,0}$, if r is even and $r \geq 4$; and is H_{r-1}, if r is odd and $r \geq 5$.

Proof

If a graph has r matchable edges, then it has at least r edges. Hence, any connected graph G with $r > 0$ matchable edges must have the following properties.

(i) G has at least r edges.

(ii) G has an even number of vertices (since it must have a perfect matching).

Case (1) **r-even**

For $r= 2$, the smallest graph is P_4 - the chain with 4 vertices. When r is even, and greater than 2, two connected graphs satisfy the minimum value of Condition(i), i.e. every edge is matchable; and Condition (ii). They are the r-cycle and the complete graph K_n, where

$r = \binom{n}{2}$, for some even positive integer n. It follows that when r is even, and greater than 2, the smallest connected unsaturated graph with r matchable edges is the r-cycle. Case (2) **r-odd**

When $r = 1$, the smallest connected graph, must have at least two vertices, by Condition (ii). Hence it must be an edge, When $r=3$, two graphs satisfy the above conditions; the chain P_6 and the chain P_5, with a pendant edge attached to its centre vertex. Both graphs are trees with six vertices. For $r \geq 5$, we can start off the unique smallest connected matching unsaturated graph with r-1 ($\geq$4) matchable edges-which is C_{r-1}, and then add one more matchable edge, in the "cheapest" way. The desired graph G must have at least r+1 vertices. So ideally, we would like to add one matchable edge and exactly two vertices (if possible). This can be achieved in only one way; that is, by attaching a P_3 to a C_{r-1}. It follows that when r is odd, and greater than 3, the smallest connected graph with r matchable edges is an r-cycle, with a P_3 attached. Hence the result folllows. □

In order to construct a minimum graph with a prescribed number of matchable and non-matchable edges, we would start off with the smallest graph with the desired number of matchable edges and then add non-matchable edges, so as to minimize the number of additional vertices. Clearly, if we can add the non-matchable edges to the smallest graphs, without adding any new vertices, then the resulting graphs must be the smallest possible. This means that our initial smallest graph should be unsaturated. Thus, a complete graph cannot be used. Therefore, the smallest graphs with r matchable edges and s non-matchable edges and with appropriate restriction on s, are the graphs $G_{r,s}$ and $H_{r,s}$ defined in Section 2. Our discussion, together with Corollaries 1.1 and 2.1, lead to the following result.

Theorem 4

Let $a = 2n$ ($n > 1$) and b be positive integers, such that $a < b$ and $0 < b\text{-}a \leq \binom{n}{2}$. Then the smallest connected graph of size b with a matchable edges and with matching ratio $\frac{a}{b}$, is the graph $G_{a,b-a}$. If $a = 2n + 1$ ($n > 1$), then the smallest connected graph of size b with a matchable edges and with matching ratio $\frac{a}{b}$ is the graph $H_{a-1,b-a-1}$.

6. Discussion

At this stage, there is still one unanswered question.

<u>Problem</u> Given $\frac{a}{b}$ ($0 \leq a \leq b$), find a smallest graph G, relative to either order of size, such that $\pi(G) = \frac{a}{b}$.

As discussed in Section 4, for some matchable ratios $\frac{a}{b}$, it might be impossible for <u>any</u> graph with a matchable edges to have size b. Therefore, a smallest graph does not exist. So the next best thing, is to look for the smallest graph with <u>equal</u> matchable ratio. This case is still unsolved. Is the graph with 794 vertices and 79400 edges, in Example (ii), the smallest connected graph with matchable ratio $\frac{1}{100}$?

Lemma 3 gives the smallest multiple of the ratio, which will yield a cycle large enough to accommodate the necessary unmatchable edges. We are assured that the smallest graph of the $G_{r,s}$ and $H_{r,s}$ forms are found. But for the cases where $b\text{-}a > \binom{n}{2}$, we are not sure that these types of graphs are the smallest connected graphs.

7. References

[1] E. J. Farrell, M. I. Gargano and L. V. Quintas, An Edge Partition Problem Concerning Perfect Matchings, Congressus Numerantium 157(2002), 33-40.