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AN APPLICATION OF STAR POLYNOMIALS TO DISCRETE RANDOM ALLOCATION PROBLEMS

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Abstract

The star polynomial of a graph can be used to investigate certain models associated with random space filling problems -a type of allocation problem. In particular, various discrete forms of Renyi's Random Parking Problem (see Grimaldi [3], Renyi [4] and Rosen[5]) can be modeled by star covers of chains. There are also similar finite resource allocation problems in Computer Science which can be investigated, by means of star polynomials of graphs.

1. A Discrete Random Parking Model

The model used here, is a linear parking lot, with at least two parking spaces; for example, parking parallel to a curb. Each vehicle which uses the parking lot, can be accommodated in one parking space. As vehicles arrive, they park randomly in any available parking space. We impose the restriction that each parked vehicle must have at least one empty parking space incident with it (so that it is not "blocked in"). Such a vehicle is properly parked. We will assume that all parked vehicles are properly parked. The process terminates, when no more vehicles can properly park. The parking lot is then said to the **saturated**; and if m parking spaces are occupied when this occurs, then the parking lot is said to be **m-saturated**. In general, we are interested in, the mean density of the number of vehicles in a saturated parking lot.

Definitions

(i) The **size** of a parking lot P, is the number of parking spaces in P.
(ii) If P has size t (≥ 2), and there are m occupied spaces in P, then the **coverage** of P is $\frac{m}{t}$.

Let **a(m)** be the number of m-saturations of P; and let N be the total number of possible saturations of P. Then, by using the coverage as a random variable -denoted by V, the **expected coverage** of P is

$$\varepsilon = \sum_m \frac{m}{t} \Pr(V = \frac{m}{t}) = \sum_m \frac{m}{t} \frac{a(m)}{N},$$

where the summation is taken over all the m- saturations of P. This can be written as

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$$\varepsilon = \frac{\sum_m m a(m)}{tN} \quad 0 \, . \qquad \ldots (1)$$

In this investigation, we will be interested in finding the expected coverage of saturated parking lots.

We will associate with a parking lot P of size n-1, the chain (a tree with nodes of valency 1 and 2 only) $\mathbf{P_n}$ ($n \geq 3$)with n nodes (and therefore n-1 edges). Throughout this paper, we will assume that the chain, associated with a parking lot, has at least two edges. Each edge of P_n will represent a parking space. Therefore the size of P would be the number of edges in P_n. Since it is possible for two vehicles to park in adjacent spaces, a saturation of P, will be represented by a covering of the nodes of P_n with subgraphs which are either nodes, edges or paths of length 2. In such a covering, every component will be separated from the next, by at least one edge and at most two. We will give bounds for ε in terms of parameters associated with star polynomials of the chain.

2. The Star Polynomial of a Graph

The graphs considered here are finite and have neither loops nor multiple edges. Let G be a graph with n nodes.

Definitions

(i) An m- **star** $\mathbf{S_m}$($m \geq 2$) is a tree with m+1 nodes, consisting of a node (called its **centre**) joined to m other nodes (called its **tips**).

(ii) The **0-star** S_0 is a node and the **1-star** S_1 is a complete graph with two nodes.

A **star cover** of G is a spanning subgraph of G, whose components are all stars. Throughout this paper, the word "cover" will mean star cover, unless otherwise specified.

With each star in G with r nodes, let us associate an indeterminate (over the complex numbers) or **weight** w_r. We define the **weight of a cover** C in G, as

$$w(C) = \prod_{\alpha} w_{\alpha},$$

where the product is taken over all the components a in C. The **(general) star polynomial** of G is

$$E(G;\mathbf{w}) = \sum w(C),$$

where **w** is a vector of indeterminate, and the summation is taken over all the star covers C in G. It will be a polynomial in the indeterminates w_1, w_2, w_3, ... and w_n. If we put $w_i = w$ for all i, then the resulting polynomial in the one variable w, is called the **simple star polynomial** of G; and is denoted by E(G;w). The star polynomial of a graph was introduced in Farrell [1].

The 0-star is called "trivial". If we restrict the components of the covers to be non-trivial stars, then the resulting polynomial is called the **proper star polynomial** of G; and is denoted by $E(G;\mathbf{w'})$, where $\mathbf{w'} = (0, w_2, w_3, \ldots w_n)$. The equivalent simple proper polynomial is denoted by $E_1(G;w)$.

In general, the star polynomial of a graph can be useful in any investigation involving the numbers of star covers of the graph. The sum of the coefficients of E(G;w) is the total number of covers of G. The coefficient of w in E(G;w) is the number of spanning stars in G. The existence of a star cover with k stars of specified lengths, is determined by the existence of the appropriate term in E(G;w).

3. The Fundamental Edge Theorem for Star Polynomials

Let a be an edge in G. By (**star**) **incorporating** α, we mean that α is required to belong to every star cover of G. The edge α is then called an **incorporated edge**.

The set of all star covers of G can be partitioned into two classes, (i) those containing a specified edge α, and (ii) those which do not. The covers which do not contain α will be covers of the graph G' obtained from G by deleting α. The covers which contain α, will be covers of the graph G* obtained from G by incorporating α. Thus we have the following theorem.

Theorem 1 (The Fundamental Edge Theorem for Star Polynomials)

Let G be a graph and α an (unincorporated) edge of G. Let G' be the graph obtained from G by deleting α, and G* the graph obtained from G by (star) incorporating α. Then

$$E(G;w) = E(G';w) + E(G^*;w).$$

The **Fundamental Edge Algorithm for Star Polynomials**, consists of recursive applications of the above theorem, until we obtain graphs whose star polynomials can be written down. We will refer to this algorithm as the **reduction process** for star polynomials.

We now give a result about the graph G*, which will be useful in applications of the reduction process. Since a cycle cannot be a subgraph, of any star cover, we have the following result.

Lemma 1

If G* contains an incorporated cycle, then $E(G^*;w) = 0$.

Lemma 1 provides a useful simplification to the reduction process. At any stage of the reduction process, we can immediately remove from an intermediate graph, all final edges which complete cycles with incorporated edges.

4. Some Basic Properties of Star Polynomials

The general star polynomial of G, with p nodes, will be a polynomial in the indeterminates w_1, w_2, w_3,...and w_n. Its terms will be of the form $A\,w_1^{i_1} w_2^{i_2} \ldots w_n^{i_n}$, where A is the number of covers of G consisting of i_1 0-stars (component nodes), i_2 1-stars (edges), i_3 2-stars ; etc. In the case of the simple star polynomial E(G;w), the terms are of the form $A_r w^r$, where A_r is the number of star covers of G with r components.

The following properties of E(G;w), for a graph with n nodes, are immediate consequences of the definitions

Property 1

Each term of E(G;w) is of the form $A\,w_1^{i_1} w_2^{i_2} \dots w_n^{i_n}$, where A is the coefficient and $\sum_{j=1}^{n} j i_j = n$.

Property 2

The coefficient of w_1^n is 1.

Property 3

The coefficient of w_n is the number of spanning stars, in G.

Property 4

The coefficient of $w_1^{n-2} w_2$ is q -the number of edges in G.

Property 5 (The Component Theorem)

If G has k components G_1, G_2, G_3 , ... and G_k, then

$$E(G;w) = \prod_{i=1}^{k} E(G_i; w).$$

Some immediate consequences of the above properties are the following results.

Lemma 2

G is connected if the coefficient of w_n in E(G;w); or the coefficient of w in E(G;w), is non zero.

The following theorem shows that the simple polynomial has no gaps.

Theorem 2

Let G be a graph with n nodes. Let i (≤ n)be the smallest exponent of w in E(G;w). Then all terms with exponents higher than i; up to w^n, must occur in E(G;w) with non zero coefficients.

Proof

The existence of a term in w^i implies the existence of a cover with i components. By deleting an appropriate number of edges, we can obtain a cover with r components, for all $i < r \leq n$. The results therefore follows.

5. Application of Star Polynomials

A star cover of a chain only contains nodes (S_0), edges(S_2) or 2-stars (S_3). It follows that every saturation of the parking lot P can be represented by a star cover of the associated chain P_n($n \geq 3$). We will call these covers, **saturated covers**; and the corresponding star polynomial, the **saturated star polynomial**. The saturated star polynomial of P_n will be denoted by $E'(P_n;w)$. Saturated star covers are therefore star covers, to which no more edges can be added, without violating the "properly parked" condition for the corresponding parking problem. Since components of a star cover have at least one separating edge, the essential condition that no vehicle is "blocked in" is automatically satisfied, for saturated star covers.

The following diagram shows the saturated covers for chains with up to 8 nodes.

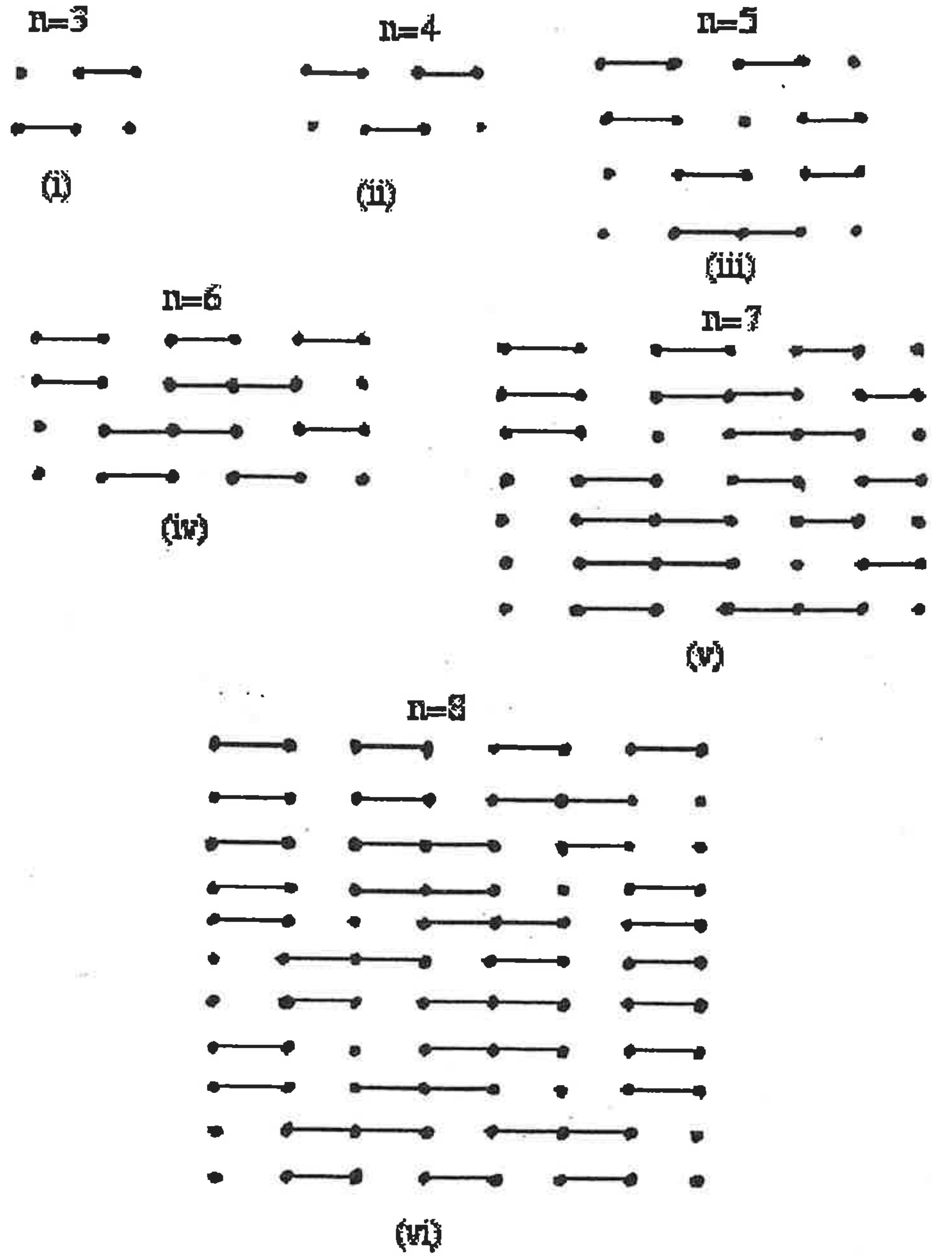

Figure 1

The following is a table of saturated star polynomials of chains with up to 8 nodes.

Table 1
Saturated Star Polynomials of Chains

n	$E'(P_n;w)$
3	$2w_1w_2$
4	$w_1^2w_2 + w_2^2$
5	$w_1^2w_3 + 3w_1w_2^2$
6	$w_1^2w_2^2 + 2w_1w_2w_3 + w_2^3$
7	$4w_1^2w_2w_3 + 2w_1w_2^3 + w_2^2w_3$
8	$w_1^2w_2^3 + w_1^2w_3^2 + 8w_1w_2^2w_3 + w_2^4$

The number of vehicles in any saturation of P, is equal to the number of edges in the corresponding saturated cover. Since we are interested only in the number of edges in these covers, it will be sufficient to use the corresponding simple saturated star polynomial of the chain. We can write the simple saturated star polynomial of P_n, as

$$E'(P_n;w) = \sum_{r=1}^{n} a_r w^r,$$

where a_r is the number of saturated covers with r components.

Now, a saturated cover C of P_n with r components, contains r-1 edges less than P_n does, since there is one unused edge between "consecutive" pair of components. Therefore C contains (n-1) - (r-1) = n-r edges.

$$\Rightarrow \sum_m m\,a(m) = \sum_{r=1}^{n}(n-r)a_r = \sum_{r=1}^{n} na_r - \sum_{r=1}^{n} ra_r = n\sum_{r=1}^{n} a_r - \sum_{r=1}^{n} ra_r$$

$$= n\,E'(P_n;1) - \frac{d}{dw}E'(P_n;w)\Big|_{w=1}, \qquad \ldots (2)$$

where $E'(P_n;1)$ is the polynomial obtained from $E'(P_n;w)$ by putting w =1. The number of different saturations of P with r vehicles corresponds to the number of saturated covers of P_n with r components; that is, a_r. Therefore, the total number of different saturations of P is the sum of the coefficients in $E'(P_n;w)$. i.e.

$$N = E'(P_n;1).$$

We can therefore write Equation (2) as

$$\sum_m m\,a(m) = nN - D, \qquad \ldots (3)$$

where $D = \frac{d}{dw}E'(P_n;w)\Big|_{w=1}$

From Equation (1), we get (using Equation (3))

$$e = \frac{\sum_m m\,a(m)}{tN} = \frac{nN - D}{(n-1)N} = \frac{n}{n-1} - \frac{D}{(n-1)N}.$$

Hence we have the following result.

Theorem 3

Let P be a parking lot of size n-1. Then

$$\varepsilon = \tau\left(1 - \frac{D}{nN}\right),$$

where $\tau = \frac{n}{n-1}$.

We now give a table of values of the expected coverages for parking lots with up to 8 parking spaces.

Table 2

Expected Coverage of Parking Lots

n	$E'(P_n;w)$	$\frac{d}{dw}E'(P_n;w)$	N	D	τ	ε
3	$2w^2$	$4w$	2	4	3/2	0.50
4	$w^3 + w^2$	$3w^2 + 2w$	2	5	4/3	0.50
5	$4w^3$	$12w^2$	4	12	5/4	0.50
6	$w^4 + 3w^3$	$4w^3 + 9w^2$	4	13	6/5	0.55
7	$6w^4 + w^3$	$24w^3 + 3w^2$	7	27	7/6	0.52
8	$w^5 + 10w^4$	$5w^4 + 40w^3$	11	45	8/7	0.56

5. Star Polynomials of Chains

The following result is also given in [1](Lemma 1).

Lemma 3

The star polynomial of the chain P_n -denoted by P(n) , satisfies the recurrence relation

$$P(n) = w_1P(n-1) + w_2P(n-2) + w_3P(n-3) \quad (n \geq 3),$$

with $P(0) = 1$, $P(1) = w_1$ and $P(2) = w_1^2 + w_2$.

Proof

Let us apply the reduction process to P_n by deleting a terminal edge α. Then the graph G' will consist of a component node and the chain P_{n-1}. The incorporated edge α can belong to either a 1-star or a 2-star. If α belongs to a 1-star, then the 1-star has weight w_2 and the remaining graph is P_{n-2}. If α belongs to a 2-star, then the 2-star has weight w_3, and the remaining graph is P_{n-3}. The boundary conditions can be easily verified. Hence the result follows.

This recurrence has been used to produce the following table of values for P(n), for n = 1 up to n = 8. This table is also given in [2].

Table 3
Star Polynomials of Chains

n	P(n)
1	w_1
2	$w_1^2 + w_2$
3	$w_1^3 + 2w_1w_2 + w_3$
4	$w_1^4 + 3w_1^2 w_2 + 2w_1w_3 + w_2^2$
5	$w_1^5 + 4w_1^3w_2 + 3w_1^2 w_3 + 3w_1w_2^2 + 2w_2w_3$
6	$w_1^6 + 5w_1^4w_2 + 4w_1^3w_3 + 6w_1^2 w_2^2 + 6w_1w_2 w_3 + w_2^3 + w_3^2$
7	$w_1^7 + 6w_1^5w_2 + 5w_1^4w_3 + 10w_1^3w_2^2 + 12w_1^2w_2w_3 + 4w_1w_2^3$ $+ 3w_1w_3^2 + 3w_2^2w_3$
8	$w_1^8 + 7w_1^6w_2 + 6w_1^5w_3 + 15w_1^4 w_2^2 + 20w_1^3w_2 w_3 + 10w_1^2w_2^3$ $+ 6w_1^2w_3^2 + 12w_1w_2^2w_3 + w_2^4 + 3w_2w_3^2$

The corresponding proper star polynomials ($w_1 = 0$) of chains with up to 8 nodes can be immediately obtained from the above table. They are given in the following table.

Table 4
Proper Star Polynomials of Chains

n	$E(P_n;w')$
1	0
2	w_2
3	w_3
4	w_2^2
5	$2w_2 w_3$
6	$w_2^3 + w_3^2$
7	$3w_2^2w_3$
8	$w_2^4 + 3w_2w_3^2$

We can obtain a generating function and hence an explicit formula for P(n), from the recurrence given in Lemma 3. The results are given in the following theorem, which was proven in [1].

Theorem 4

(i) $$E(P_n;w) = \sum \frac{(i+j+k)!}{i!j!k!} w_1^i w_2^j w_3^k,$$

where the summation is taken over all non-negative integral solutions of $i + 2j + 3k = n$.

(ii) Its generating function is

$$E(P_n;w,t) = (1 - w_1 t - w_2 t^2 - w_3 t^3)^{-1}.$$

By putting $w_i = w$, for all i, we obtain the following result.

Lemma 4

The simple star polynomial of the chain satisfies the recurrence

$$P(n) = w\,[\,P(n-1) + P(n-2) + P(n-3)\,]\ (n \geq 3),$$

with $P(0) = 1$, $P(1) = w$ and $P(2) = w^2 + w$.

From Theorem 3, we get, by putting $w_i = w$, for all i,

$$E(P_n;w) = \sum \frac{(i+j+k)!}{i!j!k!} w_1^i w_2^j w_3^k \Big|_{w_i = w} = \sum \frac{r!}{i!j!k!} w^r$$

$$= \sum_{r=1}^{n} b_r w^r,$$

where b_r is the number of covers with r components. It can be easily deduced that $b_r = \sum \frac{r!}{i!j!k!}$, where the summation is taken over all non-negative integral solutions of the pair of simultaneous equations $i + 2j + 3k = n$ and $i + j + k = r$. Hence we have the following result.

Theorem 5

The simple star polynomial of the chain P_n can be written as

$$E(P_n;w) = \sum_{r=1}^{n} b_r w^r,$$

where $b_r = \sum \frac{r!}{i!j!k!}$; and the summation is taken over all non-negative integral solutions of the pair of simultaneous equations $i + 2j + 3k = n$ and $i + j + k = r$.

The following table gives the simple star polynomials, and the simple proper star polynomials of chains, with up to eight nodes.

Table 5
Simple Star Polynomials and Simple Proper Star Polynomials of Chains

n	$E(P_n;w)$	$E_1(P_n;w)$
2	$w^2 + w$	w
3	$w^3 + 2w^2 + w$	w
4	$w^4 + 3w^3 + 3w^2$	w^2
5	$w^5 + 4w^4 + 6w^3 + 2w^2$	$2w^2$
6	$w^6 + 5w^5 + 10w^4 + 7w^3 + w^2$	$w^3 + w^2$
7	$w^7 + 6w^6 + 15w^5 + 16w^4 + 6w^3$	$3w^3$
8	$w^8 + 7w^7 + 21w^6 + 30w^5 + 19w^4 + 3w^3$	$w^4 + 3w^3$

6. The Expected Coverage of Chains

We define the terms "size" and "coverage" for chains in an analogous manner as for parking lots. We also define the **expected coverage of a chain** P_n in an analogous manner as Equation (1). We denote by ε_1 and ε_2, the expected coverage of P_n with star covers and proper star covers respectively.

Let us now define the following numerical quantities:

$$D_1 = \frac{d}{dw}E(P_n;w)\Big|_{w=1},\quad D_2 = \frac{d}{dw}E_1(P_n;w)\Big|_{w=1},\quad N_1 = E(P_n;1) \text{ and}$$

$N_2 = E_1(P_n;1)$.

Then we have the following result, which is analogous to Theorem 3.

Lemma 5

$$\text{(i)}\qquad \varepsilon_1 = \tau\left(1-\frac{D_1}{nN_1}\right)$$

and

$$\text{(ii)}\qquad \varepsilon_2 = \tau\left(1-\frac{D_2}{nN_2}\right).$$

The following is a table of some of the numerical quantities defined above. The derivatives of the polynomials are obtained from Table 5.

Table 6

Expected Coverage (With Star Covers) of Chains with up to 8 Nodes

n	$\frac{d}{dw}E(P_n;w)$	N_1	D_1	τ	ε_1
2	$2w+1$	2	3	2	0.50
3	$3w^2+4w+1$	4	8	3/2	0.50
4	$4w^3+9w^2+6w$	7	19	4/3	0.43
5	$5w^4+16w^3+18w^2+4w$	13	43	5/4	0.42
6	$6w^5+25w^4+40w^3+21w^2+2w$	24	94	6/5	0.42
7	$7w^6+36w^5+75w^4+64w^3+18w^2$	44	200	7/6	0.41
8	$8w^7+49w^6+126w^5+150w^4+76w^3+9w^2$	81	418	8/7	0.41

The analogous table for proper star covers is the following.

Table 7

Expected Coverage (With Proper Star Covers) of Chains with up to 8 Nodes

n	$\frac{d}{dw}E_1(P_n;w)$	N_2	D_2	τ	e_2
2	1	1	1	2	1.00
3	1	1	1	3/2	1.00
4	$2w$	1	2	4/3	0.67
5	$4w$	2	4	5/4	0.75
6	$3w^2+2w$	2	5	6/5	0.70
7	$9w^2$	3	9	7/6	0.67
8	$4w^3+9w^2$	4	13	8/7	0.68

7. Relationships Between the Coefficients of the Simple Star Polynomials

In this section, we will assume the following forms of the three simple star polynomials introduced above.

(i) $E'(P_n;w) = \sum_{r=1}^{n} a_r w^r$ - the simple saturated star polynomial of P_n

(ii) $E(P_n;w) = \sum_{r=1}^{n} b_r w^r$ - the simple star polynomial of P_n

and (iii) $\overline{E}_1(P_n;w) = \sum_{r=1}^{n} c_r w^r$ - the simple proper star polynomial of P_n.

Since every saturated star cover is a star cover, the coefficients of $E'(P_n;w)$ are bounded above by the corresponding coefficients of $E(P_n;w)$. Clearly, there are star covers of P_n, which are not saturations. For example, the trivial cover , for all n; and the covers with one edge, for all values of $n \geq 5$. Hence we have the following result.

Lemma 6

For all chains P_n $(n \geq 3)$, and for all r ($1 \leq r \leq n$), $a_r \leq b_r$.

Consider a proper star cover of the chain P_n. This is a saturated cover, since no more edges can be added to it, without creating a component, that is not a star. Thus, it is a "proper" saturation. But every saturation is not necessarily a proper cover (for example, see Figure 2). Hence we have the following result.

Lemma 7

For all chains P_n $(n \geq 3)$, and for all r ($1 \leq r \leq n$), $c_r \leq a_r$.

The following theorem, results from Lemmas 6 and 7. It gives both upper an lower bounds on the number of different saturated covers with r stars.

Theorem 6

For all chains P_n $(n \geq 3)$, and for all r ($1 \leq r \leq n$)

$$c_r \leq a_r \leq b_r .$$

In any saturation of a parking lot, a terminal parking space and its adjacent space, cannot be simultaneously occupied. The corresponding condition for the chain, is that its two penultimate nodes cannot be centres of 2-stars. Let us call these 2-stars β and γ. Then covers containing β or γ are **forbidden covers.** Thus, forbidden covers do not correspond to saturations of the parking lot. We can put these covers into three (non-disjoint) classes

(i) those containing β,

(ii) those containing γ

and (iii) those containing both β and γ.

If β belongs to a cover, then the rest of the cover will be a cover of P_{n-3}. Similarly, if γ belongs to a cover, then the rest of the cover will be a cover of P_{n-3}. If both β and γ belong to a cover, then the rest of the cover will be a cover of P_{n-6}. Let d_r be the number of forbidden covers with r components ; and let us write $b_r(P_n)$ for b_r (the coefficient of w^r in $E(P_n;w)$). Also let δ_r' be the corresponding number of forbidden proper covers with r components. Then our discussion yields the following result.

Lemma 8

For all n $(n \geq 6)$,

(i) $d_r = 2b_{r-1}(P_{n-3}) - b_{r-2}(P_{n-6})$

and (ii) $\delta_r' = 2c_{r-1}(P_{n-3}) - c_{r-2}(P_{n-6})$.

The following theorem is an improvement on the result given in Theorem 6.

Theorem 7

For all chains P_n ($n \geq 1$), and for all r ($1 \leq r \leq n$)

$$c_r \leq a_r \leq (b_r - \delta_r).$$

Proof

Every saturation is a cover that is not forbidden. But, not every cover that is not forbidden is a saturation. Therefore $a_r \leq (b_r - \delta_r)$. Hence the result follows.

8. Bounds on the Expected Coverage of Parking Lots

The following result can be easily established from Theorem 6.

Lemma 9

(i) $D_2 \leq D \leq D_1$

and (ii) $N_2 \leq N \leq N_1$.

Since $D_2 \leq D$, it follows that $\frac{D_2}{N} \leq \frac{D}{N}$.

But $N \leq N_1$. Therefore $\frac{D_2}{N_1} \leq \frac{D_2}{N} \leq \frac{D}{N}$.

$$\Rightarrow \quad \frac{D_2}{nN_1} \leq \frac{D}{nN} . \Rightarrow (1- \frac{D_2}{nN_1}) \geq (1- \frac{D}{nN}).$$

$$\Rightarrow \quad \tau(1- \frac{D_2}{nN_1}) \geq \tau(1- \frac{D}{nN}) = \varepsilon.$$

Similarly, we can easily show that

$$\tau(1- \frac{D_1}{nN_2}) \leq \tau(1- \frac{D}{nN}) = \varepsilon.$$

Let us put $\lambda_1 = \tau(1- \frac{D_1}{nN_2})$, $\lambda_2 = t(1- \frac{D_2}{nN_1})$, $\gamma_1 = \tau(1- \frac{D_1}{nN})$ and $\gamma_2 = \tau(1- \frac{D_2}{nN})$. Then we can easily establish the following result, using a similar analysis. It gives upper and lower bounds for ε, ε_1 and ε_2.

Lemma 10

For every parking lot P of size n-1 ($n \geq 3$),

(i) $\lambda_1 \leq \varepsilon \leq \lambda_2$

(ii) $\gamma_1 \leq \varepsilon_1 \leq \lambda_2$

(iii) $\lambda_1 \leq \varepsilon_2 \leq \gamma_2$

and (iv) $\gamma_1 \leq \varepsilon \leq \gamma_2$.

Theorem 8

$$\lambda_1 \leq \gamma_1 \leq \varepsilon \leq \gamma_2 \leq \lambda_2.$$

Proof

Since $N \geq N_2$,

$$\frac{D_1}{N} \leq \frac{D_1}{N_2}. \Rightarrow \frac{D_1}{nN} \leq \frac{D_1}{nN_2}. \Rightarrow (1- \frac{D_1}{nN}) \geq (1-\frac{D_1}{nN_2}).$$

$$\Rightarrow \tau(1- \frac{D_1}{nN}) \geq \tau(1-\frac{D_1}{nN_2}).$$

That is, $\gamma_1 \geq \lambda_1$. Similarly, we can show that $\gamma_2 \leq \lambda_2$. Hence the result follows.

Notice that the polynomials $E(P_n;w)$ and $E'(P_n;w)$ can be immediately obtained from $E(P_n;w)$ -the general star polynomial of P_n. Therefore, all the quantities D_1, D_2, N_1 and N_2 can be easily deduced from the general star polynomial of P_n. Hence, in practice, the bounds λ_1 and λ_2 can be easily calculated. However, note that D_1 is likely to be greater than nN_2; and so, the quantity $\frac{D_1}{nN_2}$ could be greater than 1. This could yield a negative value for λ_1; thereby making this lower bound (practically) useless.

The following table gives some values of ε, ε_1 and ε_2.

Table 8
Expected Coverage of Chains with up to 8 Nodes

n	ε_1	ε	ε_2
3	0.50	0.50	1.00
4	0.43	0.50	0.67
5	0.42	0.50	0.75
6	0.42	0.55	0.70
7	0.41	0.52	0.67
8	0.41	0.56	0.68

9. Discussion

In general, there are many covers, which contain components that are nodes, which are not saturations. But not all of them are unsaturated. In the following diagrams, we illustrate three saturations of P_{12} which contain component nodes.

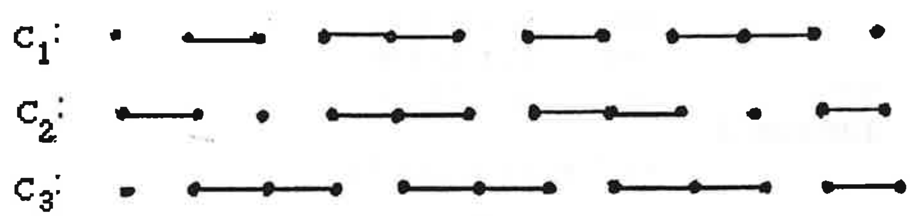

Figure 2

The characterization of these covers is crucial to an accurate count of the saturations of P_n. If there are Δ covers with component nodes, that are not saturations, then

$$a_r = b_r - \delta_r - \Delta.$$

If there are Δ' covers with component nodes, that are saturations, then

$$a_r = c_r - \delta_r' + \Delta'.$$

It is desirable to be able to find a_r, without having to enumerate all the saturated covers of P_n; as illustrated in Figure 1. Each of the above expressions for a_r, contain exactly one term;
(Δ and Δ' respectively), which cannot be obtained from $E(P_n;w)$. It is clear that a_r can be accurately counted if either Δ' or Δ could be counted. In lieu of an accurate count of a_r, it will be useful to obtain practical bounds for ε -perhaps in terms of ε_1 and ε_2. Table 8 suggests the inequality $\varepsilon_1 \leq \varepsilon \leq \varepsilon_2$; but we have been unable to prove this.

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