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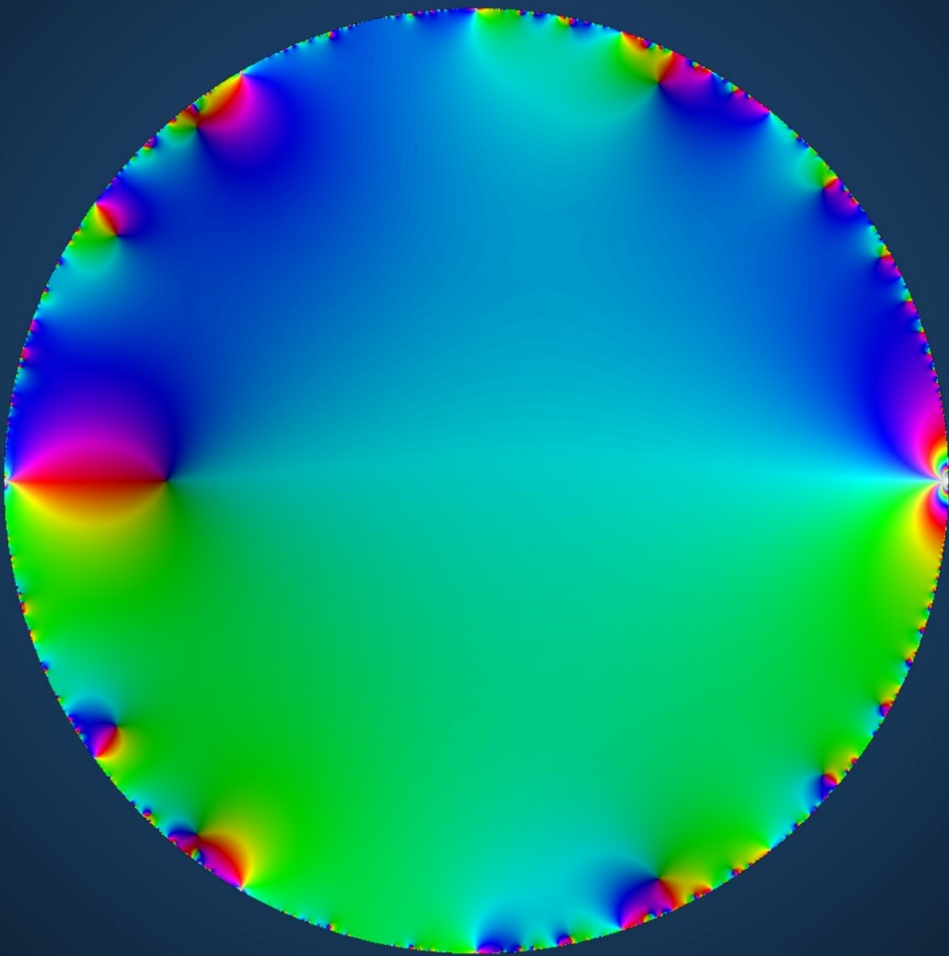

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On Some Aspects of Mock and Quantum Modularity in Theoretical Physics

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Publications

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1 *Miranda C.N. Cheng, Francesca Ferrari, Gabriele Sgroi.*
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Presented in Chapter 3.

2 *Miranda C.N. Cheng, Gabriele Sgroi.*
"Cone Vertex Algebras, Mock Theta Functions, and Umbral Moonshine Modules."
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3 *Miranda C. N. Cheng, Ioana Coman, Davide Passaro, Gabriele Sgroi.*
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Other publications by the author:

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The author participated in the theoretical and conceptual discussions for all the publications.
The order of the authors in the publications presented in this thesis is purely alphabetical. More details on the contribution for each publication are listed below.

In [1] the author contributed to the writing and editing various parts of the manuscript and to the computations of the regularisation procedure presented in chapter 3.

In [2] the author contributed to all the computations as well as writing and editing the entire manuscript.

In [3] the author contributed to various computations present in the manuscript and in particular to the proofs of section 4.2 and to writing and editing some parts of the manuscript.

Contents

1 Introduction

The relationship between Physics and Mathematics is one of the most durable, successful, and perhaps surprising synergies between different fields of human knowledge. On one side, Mathematics has been the preferred language to describe nature ever since Physics departed from Natural Philosophy to become an experimental science. Galileo Galilei, one of the founding fathers of modern science, wrote about the necessity of mathematical formalism to understand the universe:

[The universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word.

(Galileo Galilei, Il Saggiatore, 1623)

In the following centuries, representing Physical reality through Mathematics has allowed humankind to achieve unprecedented precision in the description and forecast of natural phenomena.
On the other side, Physics has always been one of the major sources of inspiration for the discovery and development of mathematical theories. In fact, despite the stunning level of abstractness that Mathematics has reached in recent times, the origin of almost all its branches can be traced back to some problems that humankind had to solve in the real world.
String Theory is a striking example of the symbiosis between Physics and Mathematics. It is a promising candidate for a theory of everything, allowing a description of the universe from Planck to cosmological scales. The core idea that led to the development of String Theory is changing the dimension of the elementary entity of Physics: moving from point particles to one-dimensional strings. The reasons for the special and fundamental role that Mathematics plays in String Theory are manifold, but one of the factors that contributed the most is the lack of guidance from experiments in the search for new Physics that we have witnessed in the last decades.

While the Standard Model has proved itself to be the most successful theory currently at our disposal in terms of predictive power, we believe it cannot be the ultimate theory of the universe. One of its major deficiencies is the failure to conciliate the theories of General Relativity and Quantum Mechanics in a unique and coherent framework. Unfortunately, at present time, there is little to no experimental evidence on which direction to look for in the search for a more fundamental theory of nature.
Left with the help of logical consistency alone, Mathematics has become a great tool to shape and limit the possible candidates to a (perhaps) ultimate theory of Physics. In fact, it turns out that the features of String Theory are quite constrained by the requirement of mathematical consistency alone. On the other side, the surprisingly rich and complex structure of String Theory has provided an invaluable source of inspiration for the discovery of new Mathematics.
While the relationship between the two fields is broad and wide, in this thesis we will restrict our focus mainly on some aspects of Number Theory that have recently shown up interesting connections with some topics originating from String Theory. In particular, we will study some appearances of mock modularity and quantum modularity (both exotic deviations from modular invariance) in some Physical quantities of interest. In this respect, the main research question motivating this thesis is to understand to what extent these complicated modular behaviors can give fruitful insights about those quantities. As we will see in this thesis, starting from this very particular aspect of Number Theory, in the context of String Theory, we will encounter further relations to other different areas of Mathematics, such as Group Theory and Topology, showing how deep and complex its interconnection with Physics is.
The origin of the concept of mock modularity can be traced back to the genius of the world-renowned mathematician Srinivasa Ramanujan at the beginning of the 20th century. In his last letter, he provided 17 examples of what he called "mock" theta functions, but he did not provide a precise definition nor instructions on how he constructed those distinguished examples. In essence, a mock theta function is a function f on the complex upper-half plane $\mathbb{H} := \{\tau \in \mathbb{C} | \ \mathrm{Im}(\tau) > 0\}$ that:

1. has singularities with exponential growth in an infinite number of roots of unity;
2. for each root of unity ζ there is a modular form f_ζ such that the difference $f - f_\zeta$ is bounded as $q := e^{2\pi i\tau} \to \zeta$ radially;
3. f is not the sum of a modular form and a function that is bounded radially at all the roots of unity.

The theory underlying mock theta functions had remained obscure for almost a

century, until some light was shed by Zwegers in his PhD thesis [4] through the theory of indefinite theta functions. We now know that mock theta functions belong to the family of mock modular forms which are, roughly speaking, holomorphic functions that can be "completed" to produce a modular form by adding to them a suitable non-holomorphic function. Recently, mock modular forms have been shown to be related to another class of objects with even more exotic modular properties: quantum modular forms [5]. In particular, fixed a weight and a multiplier, there is a linear map from the space of mock modular forms to the space of quantum modular forms with the same weight and multiplier whose kernel is given by weakly holomorphic modular forms [6].
Before delving further into the details coming from Number Theory, we will give a brief high-level overview of the two major topics that we will encounter in the rest of this thesis, trying to underline the deep interconnections between Mathematics and Physics that lay at their core.

1.1 Moonshine

The term Moonshine is used to describe a plethora of surprising and unexpected relations between representations of finite groups and q-series possessing particular transformation properties under the action of the modular group. The known examples of Moonshine differ greatly not only for the finite groups involved but also for the peculiar modular properties of the q-series. Whether the plethora of knows Moonshine instances are related to one another and what is the general theory (if any) encompassing all Moonshine phenomena is still one of the most important open questions in the field.
One of the aspects that makes Moonshine interesting to study from a Physics point of view, is that the relations connecting the finite group theory side with the modular objects for various Moonshine instances usually exhibit a very rich structure and in most cases involve objects of interest in Mathematical Physics. This phenomenon has occurred since the very first example of Moonshine: Monstrous Moonshine [7].
Monstrous Moonshine was born from the observation, due to McKay, that the generator of (meromorphic) modular functions J, called the Hauptmodul, has a q-series expansion

$$J(\tau) = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \cdots \tag{1.1}$$

in which the coefficient of $q := e^{2\pi i\tau}$ can be decomposed as $1 + 196883$, which are the dimensions of the two smallest representations of the largest sporadic simple group: the monster group $\mathbb{M}$. Later Thompson observed similar decompositions for some of the subsequent coefficients [8]. This striking observation was later enriched by extending it to graded characters, now commonly known as McKay-Thompson series, of some monster group representations V_n

$$T_g(\tau) := \sum_{n\geq -1} \mathrm{Tr}_{V_n}(g)q^n \tag{1.2}$$

conjecturing that they should correspond to the Hauptmodul J_{Γ_g} of some subgroup $\Gamma_g \leq SL_2(\mathbb{R})$. In particular, when g is the identity element, we recover the original case

$$J(\tau) := \sum_{n\geq -1} \dim(V_n)q^n. \tag{1.3}$$

For the sake of exposition, we will not go into the technical details of the statement of the Monstrous Moonshine conjecture, referring the interested reader to [9] for a detailed exposition.
The quest for the proof of Monstrous Moonshine has shown a very interesting interplay between Physics and Mathematics, providing new ideas to both fields. In particular, it has contributed to the development of the theory of Vertex Operator Algebras, a rigorous formulation of the chiral algebra of conformal field theories.
A milestone in the understanding of Monstrous Moonshine was, in fact, the construction of the monster module $V^\natural$ [10], [11] using ideas coming from Conformal Field Theory. The main ingredient of the construction is a CFT consisting of 24 chiral bosons compactified on the torus $\mathbb{R}^{24}/\Lambda$ specified by the Leech lattice Λ, the unique 24-dimensional even unimodular lattice with no roots. This provides a CFT with central charge $c = 24$ and thus its partition function will start with q^{-1}. Since the partition function is invariant under modular transformations, this fixes it to be equal to the Hauptmodul J up to an additive constant. To remove this constant, one has to get rid of all the primary fields of weight one ∂X^i: this is done by taking a $\mathbb{Z}_2$ orbifold with respect to the action $X^i \to -X^i$. $V^\natural$ is then constructed as a direct sum of the twisted and untwisted sectors of this orbifold theory. It was shown that the automorphism group of this theory is precisely the monster and the proof was completed by Borcherds [12] by proving that the graded characters

$$T_g^\natural := \mathrm{Tr}_{V^\natural} g q^{L_0 - \frac{c}{24}} \tag{1.4}$$

indeed coincide with the Hauptmoduls J_{Γ_g} predicted by the conjecture.
Around 20 years after Monstrous Moonshine was understood, a new type of moonshine was observed: Mathieu Moonshine. This time, the insight that led to the formulation of this new type of moonshine came directly from Physics in the context of 2-dimensional supersymmetric sigma models with target manifold a $K3$ surface. The elliptic genus of these theories is given by the trace over the Ramond

sector (for both left- and right-movers)

$$\mathcal{E}(\tau, z) = \mathrm{Tr}_{\mathcal{H}_{RR}}(-1)^{J_0+\bar{J}_0} y^{J_0} q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}} \tag{1.5}$$

where $y := e^{2\pi i z}$ and J_0 is the zero-mode of the $U(1)$ current of the $\mathcal{N} = 2$ superconformal algebra (and analogously $\bar{J}_0$ for the right-movers).
In [13] it was noted that expanding the elliptic genus for $K3$ surfaces as [1]

$$\mathcal{E}(\tau, z) = \frac{\theta(\tau, z)^2}{\eta(\tau)^3} \left[24\mu(\tau, z) - 2q^{-\frac{1}{8}} \left(1 - \sum_{n=1}^{\infty} A_n q^n \right) \right], \tag{1.6}$$

the first coefficients A_n are non-negative and can be decomposed in terms of dimensions of representations of the largest Mathieu group M_{24}. It was also noted that this decomposition is very similar to the observation of that led to Monstrous Moonshine. Equivalent versions of the McKay-Thompson series were proposed shortly after [14] to connect the twining genera

$$\phi_g(\tau, z) = \mathrm{Tr}_{\mathcal{H}_{RR}} g(-1)^{J_0+\bar{J}_0} y^{J_0} q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}} \tag{1.7}$$

to graded characters of M_{24} and of the $\mathcal{N} = 4$ superconformal algebra. Mathieu Moonshine would have a natural explanation in terms of decomposition of representation if M_{24} was a symmetry of non-linear sigma model on $K3$, however it was proven that M_{24} cannot be the symmetry group of any individual sigma model with target space $K3$ [15]. While the existence of a module with a M_{24} action has been proven [16], a direct construction as the one for $V^\natural$, let alone its physical interpretation, is still missing.
To conclude this subsection, we mention that Mathieu Moonshine revealed itself to be just a particular example of a family of 23 groups giving rise to Umbral Moonshine [17], [18]. These example also have relations to nonlinear sigma models on $K3$ [19] and, as for Mathieu Moonshine, direct constructions of modules producing the relevant McKay-Thompson series as graded characters are missing in many instances. We will delve into the matter in chapter 2, where we will provide a vertex operator algebra realization of modules for some instances of Umbral Moonshine.

1.2 Topological Invariants and Physics

The study of topological invariants in connection to physical theories has a long history and one of the most studied and famous examples born out of it is Chern-

[1] We will encounter the theta function $\theta(\tau, z)$ and the Appell-Lerch sum $\mu(\tau, z)$ in chapter 2 and provide definitions there.

Simons theory [20]. Given a closed, connected, oriented 3-manifold M_3 and a gauge group G, the building blocks of Chern-Simons theory are the connections A on a principal bundle of G. The Chern-Simons functional is given by

$$S_{CS}(A) = \frac{1}{4\pi} \int_{M_3} \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \tag{1.8}$$

and the partition function for level k Chern-Simons theory is

$$Z_{CS}[k; M_3] = \int \mathcal{D}A e^{i(k-2)S_{CS}(A)}. \tag{1.9}$$

Gauge invariance of the partition function requires k to be integer. As the partition function does not depend on the metric of the manifold M_3, it should be determined by its topological properties alone. In fact, the following normalization of the Chern-Simons partition function

$$\tau_k(M_3) := \frac{i\sqrt{2k}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} Z_{CS}[k; M_3] \tag{1.10}$$

with $q = e^{\frac{2\pi i}{k}}$, coincides to what is know in the literature as the Witten-Reshetikhin-Turaev (WRT) invariant τ_k [21]. More generally, in the context of Chern-Simons theory, a broader family of topological invariants can be associated to links in M_3. Given an irreducible representation R of the gauge group G and a knot K, we can define an observable, the Wilson loop, as the trace of the path-ordered exponential of holonomy of the connection A around K

$$W_R(K) = \mathrm{Tr}_R \left\{ \mathcal{P} e^{\oint_K A_i dx^i} \right\} \tag{1.11}$$

where $\mathcal{P}$ stands for the path ordering of the exponential.
Given a loop $L = \bigcup_{\alpha=1}^{n} K_\alpha$ as the union of the non-intersecting knots K_α, and chosen an irreducible representation R_α for each K_α we can define the following invariants

$$Z[k; M_3, L] = \frac{1}{Z_{CS}[k; M_3]} \int \mathcal{D}A e^{i(k-2)S_{CS}(A)} \prod_{\alpha=1}^{n} W_{R_\alpha}(K_\alpha). \tag{1.12}$$

For $G = SU(2)$, these coincide with the famous Jones polynomials of L [20].
It turns out that the invariants τ_k (1.10) admit a mathematical definition for q being an arbitrary root of unity [21]. It is thus natural to ask whether invariants of 3-manifolds can be defined, more generally, for any complex value of q. Such invariants have been proposed in [22] as a refinement of the WRT invariants.

The root source of inspiration for these new invariants came, once again, form Physics and it originated in the context of 3d-3d correspondence. We will give here only a brief overview of these invariants focusing on their physical origin and motivation, leaving most of the technicalities to chapters 3 and 4. In essence, the 3d-3d correspondence associates to a 3-manifold M_3, a 3d $\mathcal{N}=2$ supersymmetric gauge theory $T[M_3]$ by compactifying a 6d $\mathcal{N}=(2,0)$ superconformal field theory on M_3. For the sake of exposition, we will restrict here to gauge group $SU(2)$. We will discuss generalizations to higher rank gauge groups when we will encounter them in chapter 4. The quantity realizing the new invariants is the supersymmteric partition function of the theory $T[M_3]$ on $D^2 \times S^1$, also called the "half-index", associated to particular boundary conditions $\mathcal{B}_a$ along the boundary of the disk. Such boundary conditions correspond to the choice of an Abelian flat connection a and thus we will have a family of invariants $\widehat{Z}_a$ labelled by Abelian flat connections. One of the interesting properties of these new invariants is that the coefficients of their q-series are integers, i.e.

$$\widehat{Z}_a(q) = q^\delta \sum_n c_n q^n \quad c_n \in \mathbb{Z}$$

for some $\delta \in \mathbb{R}$. In the context of 3d-3d correspondence the integers c_n can be identified (up to signs) with some homological invariants given by the BPS sector of the Hilbert space of the 3d theory. Because of this, the $\widehat{Z}_a$ invariants are also known as homological blocks. These new invariants have been studied under various different aspects in the recent literature. A full recount of the status of the undergoing research is out of the scope of this thesis, we will refer the reader to [23, 24, 25, 26, 27, 28, 29, 30, 31, 32] for a list of references. This list is definitely incomplete considered the size and breadth of the recent literature on the topic, but its elements have been selected by relevance and similarity to the contents of subsequent chapters.
In this thesis we will focus on the peculiar behaviors that the $\widehat{Z}_a$ invariants can exhibit under modular transformation. In this respect, these new invariants can be seen as a 3d generalization of the elliptic genus of 2d theories. In fact, the $\widehat{Z}_a$ receives contribution from the 3d "bulk" theory as well as from the 2d edge modes of the boundary theory and, when the contribution from the former is trivial, it reduces to the elliptic genus of the boundary theory. However, in the general case the 3d contribution will spoil the modularity properties of the elliptic genus leading to more complicated behaviours. Conjecturally the $\widehat{Z}_a$ will still exhibit some properties under modular transformations, albeit in a more complicated form. More specifically, in many known classes of examples, it has been shown that the $\widehat{Z}_a$ is a (higher-depth) quantum modular form [33], [34] [35], [36]. We will delve into this topic and its implications in chapters 3 and 4.

1.3 Background: modularity

In this section we will summarize a series of facts regarding modularity that will be needed in the rest of this thesis. The purpose of this section is to establish a common ground of definitions and conventions needed throughout the thesis as well as to introduce some basic notions to a reader (and in particular a physicist) new to these concepts. When needed, we will repeat the relevant definitions and properties in later chapters, possibly adapting the conventions as suitable. Given the variety and broadness of the notions presented, it is not intended to give here an exhaustive and organic overview encompassing the different areas of the field. The interested reader is encouraged to read the references provided in each subsection to delve deeper into the corresponding concept.

1.3.1 A plethora of modular properties

In this section we give a brief account of a variety of objects exhibiting different kinds and degrees of modular properties. The aim is to introduce the definitions needed for later sections. There are many standard reference covering these topics, we will mostly follow [37], [9], [38].
In the following we will use the standard notation $q := e^{2\pi i\tau}$ with $\tau \in \mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$.
We start by recalling that the special linear group $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{H}$ through Möbius transformations

$$\tau \to \gamma\tau := \frac{a\tau + b}{c\tau + d} \tag{1.13}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$. In what follows we will consider a variety of objects with interesting transformations under the action of $\mathrm{SL}(2,\mathbb{Z})$ or its congruence subgroups, i.e. subgroups of finite index of $\mathrm{SL}(2,\mathbb{Z})$ that are determined by congruence conditions on the matrix entries. Notice that the matrices $\pm\gamma$ have the same action, so we can restrict the attention to $\mathrm{PSL}(2,\mathbb{Z}) := \mathrm{SL}(2,\mathbb{Z})/\{\pm 1\}$.

Modular forms

The simplest objects we can define are functions that are invariant under (1.13) with respect to the full modular group, i.e. functions $f : \mathbb{H} \to \mathbb{C}$ such that $f(\gamma\tau) = f(\tau)\ \forall \tau \in \mathbb{H}, \gamma \in \mathrm{SL}(2,\mathbb{Z})$. However, this definition turns out to be too restrictive if we want also to require the functions to be holomorphic. In fact, as all the cusps $i\infty \cup \mathbb{Q}$ are equivalent under the action of the modular group, the only functions satisfying these requirements are the constant functions. There are many ways to generalize the previous definition in order to get more interesting functions, we will focus here on generalizations that trade the simplicity of modular invariance with more complicated transformation properties. As we will see in the rest of this thesis, this is the silver lining that relates the various modular objects that we will encounter along the way: by considering more involved modular transformations we discover objects with richer properties. To this extent, for any Γ congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$, we define the slash operator $(\cdot)|_{k,\chi}\gamma$ for weight k, multiplier $\chi : \mathrm{SL}(2,\mathbb{Z}) \to \mathbb{C}$, and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, acting on functions $f : \mathbb{H} \to \mathbb{C}$ as

$$f(\tau)|_{k,\chi}\gamma := (c\tau + d)^{-k}\chi(\gamma) f\left(\frac{a\tau + b}{c\tau + d}\right). \tag{1.14}$$

The definition can be easily extended to the case of vector-valued functions $f : \mathbb{H} \to \mathbb{C}^n$ in which case $\chi(\gamma)$ is a $n \times n$ matrix. With this notation, we will call a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ ($f : \mathbb{H} \to \mathbb{C}^n$, respectively) satisfying

$$f(\tau)|_{k,\chi}\gamma - f(\tau) = 0 \tag{1.15}$$

$\forall \gamma \in \Gamma$ and $\forall \tau \in \mathbb{H}$, a (vector-valued) modular form for Γ of weight k and multiplier χ. For consistency, the multiplier must be a representation of Γ when k is integer, while otherwise it must be a projective representation of Γ. Furthermore, for the existence of non-trivial examples we also need $\chi(\mathbf{1}_2)$ and $e^{-\pi i k}\chi(-\mathbf{1}_2)$ to be the identity matrix $\mathbf{1}_n$. In what follows we will restrict to the case $k \in \frac{1}{2}\mathbb{Z}$.
As an example let's consider the Eisenstein series

$$E_{2k}(\tau) := 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1 - q^n} \tag{1.16}$$

where $\zeta(z)$ is the famous Riemann's zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.17}$$

For $k \geq 2$, it can be shown that E_{2k} is a modular form of weight $2k$. The significance of Eisenstein series in the context of modular forms is given by the fact that E_4 and E_6 are the generators of the ring of modular forms over the full modular group $\mathrm{SL}(2,\mathbb{Z})$. Thus, every modular form M_k of weight $k \in \mathbb{Z}$ for $\mathrm{SL}(2,\mathbb{Z})$ admits a unique expansion as a sum

$$M_k(\tau) = \sum_{\substack{\alpha,\beta \in \mathbb{N} \\ 4\alpha + 6\beta = k}} c_{\alpha,\beta} E_4^{\alpha} E_6^{\beta} \tag{1.18}$$

where the coefficient $c_{\alpha,\beta} \in \mathbb{C}$ are non-zero only for a finite number of α, β.

Jacobi forms

In this subsection we will introduce Jacobi forms. We will limit the account to the basic definitions and properties needed in later sections, following closely §3 of [39]. A complete account of the theory of Jacobi forms can be found in [40].
We begin by defining elliptic forms. We call a smooth function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ an elliptic form of index m if $z \to \phi(\tau, z)$ is holomorphic[2] $\forall \tau \in \mathbb{H}$ and $\forall (\lambda, \mu) \in \mathbb{Z}^2$

$$\phi(\tau, z) = e^{2\pi i(m\lambda^2\tau + 2m\lambda z)}\phi(\tau, z + \lambda\tau + \mu). \tag{1.19}$$

Every elliptic form admits a theta-decomposition

$$\phi(\tau, z) = \sum_{r \mod 2m} h_r(\tau)\theta_{m,r}(z, \tau) \tag{1.20}$$

in terms of smooth functions $h : \mathbb{H} \to \mathbb{C}$, called the theta coefficient of ϕ, and the unary theta functions

$$\theta_{m,r}(\tau, z) := \sum_{\substack{n\in\mathbb{Z} \\ n=r \mod 2m}} e^{2\pi i z n} q^{\frac{n^2}{4m}}. \tag{1.21}$$

We will also define the modular and skew-modular action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ on the elliptic functions of index m as

$$\phi|_{k,m}\gamma(\tau, z) := \frac{e^{-2\pi i \frac{cmz^2}{c\tau+d}}}{(c\tau + d)^k}\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) \tag{1.22}$$

$$\phi|^{sk}_{k,m}\gamma(\tau, z) := \frac{e^{-2\pi i \frac{cmz^2}{c\tau+d}}}{(c\overline{\tau} + d)^k}\frac{c\overline{\tau} + d}{|c\overline{\tau} + d|}\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) \tag{1.23}$$

Given an elliptic function ϕ with theta decomposition as (1.20), we will say it is a weak (skew-)holomorphic Jacobi form of weight k and index m if all its theta coefficients h_r are (anti-)holomorphic on $\mathbb{H}$, it is invariant under the (skew-)modular action for all $\gamma \in \mathrm{SL}(2, \mathbb{Z})$ and the function $\tau \to \phi(\tau, z)$ is bounded as $\mathrm{Im}(\tau) \to \infty$ $\forall z \in \mathbb{C}$. We notice that holomorphicity and translation invariance imply that a weak Jacobi form admits a Fourier expansion

$$\phi(\tau, z) = \sum_{\substack{D,l\in\mathbb{Z} \\ D=l^2 \mod 4m}} C_\phi(D, l)q^{-\frac{D}{4m}}q^{\frac{l^2}{4m}}y^l \tag{1.24}$$

for some functions $C_\phi(\cdot, l)$. We will say that a weak holomorphic Jacobi form is a holomorphic Jacobi form if $C_\phi(D, l) = 0$ for $D > 0$ and a cuspidal Jacobi form if

[2] As in [39], we only require ϕ to be real-analytic in τ and not holomorphic.

$C_\phi(D,l) = 0$ for $D \geq 0$. Analogous definitions can be given for skew-holomorphic weak Jacobi forms.
An important example of Jacobi forms, that will be relevant for the rest of the thesis, are theta functions. Let A be a $r \times r$ positive definite symmetric matrix with integral coefficients. Define the associated bilinear form $B(v,w) := v^T Aw$ and quadratic form $Q(v) := \frac{1}{2}v^T Av$. Given $\mathbf{x}_0 \in \mathbb{Z}^r$ we can define

$$\Theta_{Q,\mathbf{x}_0}(\tau; z) := \sum_{\mathbf{n}\in\mathbb{Z}^r} q^{Q(\mathbf{n})} y^{B(\mathbf{n},\mathbf{x}_0)}. \tag{1.25}$$

This is a Jacobi form of weight $\frac{r}{2}$ and index $Q(\mathbf{x}_0)$ [37].

Mock modular forms

In this section we will introduce the concept of mock modular forms. We will follow section §2.2 of [9] and refer to some notions from section §2 of [38].
Given a cusp form g, i.e. a modular form with zero constant coefficient in the Fourier expasion $g(\tau) = \sum_{n\geq 0} c_g(n)q^n$, of weight $2-w$ with $w \in \frac{1}{2}\mathbb{Z}$, we define the non-holomorphic weight w Eichler integral of g

$$g^*(\tau) = C\int_{-\overline{\tau}}^{\infty} (\tau' + \tau)^{-w}\overline{g(-\overline{\tau'})}d\tau'. \tag{1.26}$$

There is no canonical normalization C of the shadow, we choose here for simplicity $C = 2^{w-1}i^{w+1}$ following the conventions in [38]. In later sections, we will specify when a different choice of the normalization is made.
Let $h : \mathbb{H} \to \mathbb{C}$ be a holomorphic function on $\mathbb{H}$ with at most exponential growth at all cusps. We say that h is a mock modular form of weight $w \in \frac{1}{2}\mathbb{Z}$ if there exists modular form g of weight $2-w$ such that, defining its completion as $\hat{h} := h + g^*$, $\hat{h}$ transforms like a modular form for some subgroup $\Gamma \in \mathrm{SL}(2,\mathbb{Z})$, i.e. $\hat{h}(\tau)|_{w,\chi}\gamma = \hat{h}(\tau)\ \forall\tau \in \mathbb{H},\ \gamma \in \Gamma$. Notice that, by construction, $\hat{h}$ is not holomorphic. If we define the "shadow operator"

$$\xi_w := 2i\mathrm{Im}(\tau)^w \overline{\frac{\partial}{\partial\overline{\tau}}}, \tag{1.27}$$

its action on the Eichler integral of a cusp form g of weight $2-w$ will be given by

$$\xi_w(g^*(\tau)) = g(\tau) \tag{1.28}$$

and thus, when applied to a mock modular form of weigth w it will return its shadow. Furthermore, this implies that given a mock modular form h of weight

w, the completion $\hat{h}$ of h is annihilated by the the weight w Laplacian

$$\Delta_w := \mathrm{Im}(\tau)^{2-w}\partial_\tau \mathrm{Im}(\tau)^w \partial_{\overline{\tau}} \tag{1.29}$$

i.e. $\Delta_w \hat{h}(\tau) = 0 \ \forall \tau \in \mathbb{H}$.
Smooth functions transforming as modular forms annihilated by the Laplacian (1.29) (growing at most exponentially at the cusps) are called harmonic Maass forms. Given a harmonic Mass form f of weight w, it will have a Fourier expansion of the form [38]

$$f(\tau) = \sum_{n \gg -\infty} c_f^+(n)q^n + c_f^-(0)\mathrm{Im}(\tau)^{1-w} + \sum_{\substack{n\ll\infty \\ n\neq 0}} c_f^-(n)\Gamma(1-w, -4\pi n \mathrm{Im}(\tau))q^n. \tag{1.30}$$

We will call

$$f^+(\tau) := \sum_{n \gg -\infty} c_f^+(n)q^n \tag{1.31}$$

the holomorphic part of f, and

$$f^-(\tau) := c_f^-(0)\mathrm{Im}(\tau)^{1-w} + \sum_{\substack{n\ll\infty \\ n\neq 0}} c_f^-(n)\Gamma(1-w, -4\pi n \mathrm{Im}(\tau))q^n \tag{1.32}$$

the non-holomorphic part of f.
In this language, we can more generally define a mock modular form h as the holomorphic part f^+ of a harmonic Maass form f. Furthermore we will define its shadow g as the action of the shadow operator (1.27) $g := \xi_w(f) = \xi_x(f^-)$.

Quantum modular forms

We will now introduce quantum modular forms and their higher depth generalization. We will follow closely section §2.2 of [1] and supplement it with notions from the original references for quantum modular forms [5] and higher depth quantum modular forms [41].
The concept of quantum modular forms (QMF) was first introduced by D. Zagier [5]. Roughly speaking, a quantum modular form is a function defined on $\mathbb{Q}$ with a certain modular-like property: the deviation from modularity, measured by a modular difference function denoted by p_γ, has nice analytic properties that are not a priori manifest or expected. More specifically, a function $Q : \mathbb{Q} \to \mathbb{C}$ is called a quantum modular form of weight w and multiplier χ for Γ if for every $\gamma \in \Gamma$ the modular difference function $p_\gamma(x) : \mathbb{Q}\backslash\{\gamma^{-1}(\infty)\} \to \mathbb{C}$, defined by

$$p_\gamma(x) := Q(x) - Q|_{w,\chi}\gamma(x) \tag{1.33}$$

is a real-analytic function of $\mathbb{R} \setminus S_\gamma$ where S_γ is a finite set depending on γ.

Most of the cases that will be considered in this thesis belong to a family of quantum modular forms satisfying stronger conditions. These are called strong quantum modular form. A strong quantum modualar form associates to every rational number $x \in \mathbb{Q}$ not just a complex number but a complex formal power series $Q(x + i\epsilon)$. Since p_γ was required to be real-analytic on $\mathbb{R} \setminus S_\gamma$, it extends holomorphically to a neighbourhood of $\mathbb{R} \setminus S_\gamma$ and in particular it has a power series expansion around each point $x \in \mathbb{Q}$ that is convergent for some positive radius $r_x > 0$. We require strong quantum modular forms to satisfy the stronger requirement that

$$p_\gamma(x + i\epsilon) = Q(x + i\epsilon) - Q|_{w,\chi}\gamma(x + i\epsilon) \tag{1.34}$$

holds as an identity between countable collections of formal power series. Before giving examples of quantum modular forms, we define their higher depth generalizations. We will define depth n quantum modular forms recursively starting from quantum modular forms that will constitute the depth-1 quantum modular forms. In words, a depth-N quantum modular form will be a function on the rationals such that its modular difference is a linear combination (with coefficients in the space of analytic functions) of quantum modular forms of depth smaller than N. To give a precise definition, we will write $\mathcal{Q}_k^N(\Gamma, \chi)$ to indicate the space of depth-N quantum modular forms of weight k, multiplier χ for Γ. We will use the convention $\mathcal{Q}_k^0(\Gamma, \chi) = 1$. We will denote with $\mathcal{O}(\mathbb{R})$ the space of analytic functions on $\mathbb{R}$. We will say that a function $Q : \mathbb{Q} \to \mathbb{C}$ is a quantum modular form of depth-N if

$$Q(x) - Q|_{w,\chi}\gamma(x) \in \bigoplus_{j \in \mathcal{J}} \mathcal{Q}_{k_j}^{N_j}(\Gamma, \chi_j)\mathcal{O}(\mathbb{R}) \tag{1.35}$$

for some finite set $\mathcal{J}$, and $N_j < N \ \forall j \in \mathcal{J}$. As an example of higher depth quantum modular forms, we notice that the product of two depth one quantum modular forms is a depth two quantum modular forms. To be more precise, if $Q_1 \in \mathcal{Q}_{k_1}^1(\Gamma_1, \chi_1)$ and $Q_2 \in \mathcal{Q}_{k_2}^1(\Gamma_2, \chi_2)$ then $Q_1 Q_2 \in \mathcal{Q}_{k_1+k2}^1(\Gamma_1 \cap \Gamma_2, \chi_1\chi_2)$. We will encounter and discuss in more detail examples of quantum modular forms in following chapters of this thesis.

1.3.2 Indefinite theta functions

In this section we will review some basic properties of indefinite theta functions of signature $(r - 1, 1)$. We will mostly follow the exposition present in chapter 8 of [38].

We will start by introducing some notation. Let us set, as it is usual, $q := e^{2\pi i\tau}$, $y := e^{2\pi iz}$. Given a symmetric matrix A with integer coefficients, we define the associated bilinear form $B(\mathbf{v},\mathbf{w}) := \mathbf{v}^T A\mathbf{w}$ and, correspondingly, the quadratic form $Q(\mathbf{v}) := \frac{1}{2}B(\mathbf{v},\mathbf{v})$. As mentioned in section 1.3.1, when Q is positive definite, fixed $\mathbf{x}_0 \in \mathbb{Z}^r$, we have that

$$\Theta_{Q,\mathbf{x}_0}(\tau; z) := \sum_{\mathbf{n}\in\mathbb{Z}^r} q^{Q(\mathbf{n})} y^{B(\mathbf{n},\mathbf{x}_0)} \tag{1.36}$$

is a Jacobi form of weight $\frac{r}{2}$ and index $Q(\mathbf{x}_0)$. This result, however, does not hold when Q is not positive definite. In fact, when the quadratic form is non-positive definite the series in (1.36) does not converge. For quadratic forms of type $(r-1,1)$, i.e. the largest linear subspace on which Q is negative definite has dimension 1, generalizations of (1.36) where studied by Zwegers [4]. In this case, the set $\{\mathbf{c}\in\mathbb{R}^r : Q(\mathbf{c}) < 0\}$ splits into two connected components, we fix one of these and denote it C_Q. Explicitly we choose a $\mathbf{c}_0$ such that $Q(\mathbf{c}_0) < 0$ and define

$$C_Q := \{\mathbf{c}\in\mathbb{R}^r : Q(\mathbf{c}) < 0, B(\mathbf{c},\mathbf{c}_0) < 0\}. \tag{1.37}$$

We also define

$$S_Q := \{\mathbf{c} = (c_1,\ldots,c_r) \in \mathbb{Z}^r : \gcd(c_1,\ldots,c_r) = 1, Q(\mathbf{c}) = 0, B(\mathbf{c},\mathbf{c}_0) < 0\} \tag{1.38}$$

and consider the compactification of C_Q, $\overline{C}_Q := C_Q \cup S_Q$. We furthermore define, $\forall \mathbf{c}\in\overline{C}_Q$,

$$R(\mathbf{c}) := \begin{cases} \mathbb{R}^r & \text{if } \mathbf{c}\in C_Q, \\ \{\mathbf{a}\in\mathbb{R}^r : B(\mathbf{a},\mathbf{c}) \notin \mathbb{Z}\} & \text{if } \mathbf{c}\in S_Q. \end{cases} \tag{1.39}$$

With the above notation, given a symmetric matrix A, $\mathbf{c}_1,\mathbf{c_2}\in\overline{C}_Q$, $\mathbf{a}\in R(\mathbf{c}_1)\cap R(\mathbf{c}_2)$, and $\mathbf{b}\in\mathbb{R}^r$, we can define the indefinite theta functions

$$\Theta_{\mathbf{a},\mathbf{b}}(\tau) := \sum_{\mathbf{n}\in\mathbf{a}\in\mathbb{Z}^r} \rho(\mathbf{n};\tau) e^{2\pi i B(\mathbf{b},\mathbf{n})} q^{Q(\mathbf{n})} \tag{1.40}$$

where we have written

$$\rho(\mathbf{n};\tau) = \rho^{\mathbf{c}_1}(\mathbf{n};\tau) - \rho^{\mathbf{c}_2}(\mathbf{n};\tau) \tag{1.41}$$

with

$$\rho^{\mathbf{c}}(\mathbf{n};\tau) := \begin{cases} E\left(\frac{B(\mathbf{c},\mathbf{n})v^{\frac{1}{2}}}{\sqrt{-Q(\mathbf{c})}}\right) & \text{if } \mathbf{c}\in C_Q \\ \operatorname{sgn}(B(\mathbf{c},\mathbf{n})) & \text{if } \mathbf{c}\in S_Q \end{cases} \tag{1.42}$$

in which $v = \mathrm{Im}(\tau)$ and E is the error function

$$E(z) := 2\int_0^z e^{-\pi t^2} dt. \tag{1.43}$$

Furthermore, defining

$$\mathcal{D}(\mathbf{c}) := \left\{ (\mathbf{z}, \tau) \in \mathbb{C}^r \times \mathbb{H} : \frac{\mathrm{Im}(\mathbf{z})}{\mathrm{Im}(\tau)} \in R(\mathbf{c}) \right\} \tag{1.44}$$

setting $\mathbf{z} := \mathbf{a}\tau + \mathbf{b}$, for $(\mathbf{z}, \tau) \in \mathcal{D}(\mathbf{c})$ we can also define

$$\Theta_{A,\mathbf{c}_1,\mathbf{c}_2}(\mathbf{z}; \tau) := \sum_{\mathbf{n}\in\mathbb{Z}^r} \rho(\mathbf{n} + \mathbf{a}; \tau) e^{2\pi i B(\mathbf{n},\mathbf{z})} q^{Q(\mathbf{n})}. \tag{1.45}$$

Notice that the following relation to the one-variable indefinite theta (1.40) holds

$$\Theta_{A,\mathbf{c}_1,\mathbf{c}_2}(\mathbf{z}; \tau) = e^{-2\pi i B(\mathbf{a},\mathbf{b})} q^{-Q(\mathbf{a})} \Theta_{\mathbf{a},\mathbf{b}}(\tau). \tag{1.46}$$

It has been shown [4] that, with the assumptions above, the series in (1.40) and (1.45) converge absolutely. For convenience, we will sometimes omit to write the matrix A and the vectors $\mathbf{c}_1$, $\mathbf{c}_2$ and write $\Theta(z; \tau)$. It has been shown in [4] that the function $\Theta(z; \tau)$ satisfies the following transformation properties

- For all $\lambda \in \mathbb{Z}^r$ and $\mu \in A^{-1}\mathbb{Z}^r$

$$\Theta(\mathbf{z} + \lambda\tau + \mu; \tau) = e^{-2\pi i B(\mathbf{z},\lambda)} q^{-Q(\lambda)} \Theta(\mathbf{z}; \tau). \tag{1.47}$$

- Writing $A^* = (A_{11}, \ldots, A_{rr})^T$,

$$\Theta(\mathbf{z}; \tau + 1) = \Theta\left(\mathbf{z} + \frac{1}{2}A^{-1}A^*; \tau\right). \tag{1.48}$$

- For $(\mathbf{z}, \tau) \in \mathcal{D}(\mathbf{c}_1) \cap \mathcal{D}(\mathbf{c}_2)$

$$\Theta\left(\frac{\mathbf{z}}{\tau}; -\frac{1}{\tau}\right) = \frac{i(-i\tau)^{\frac{r}{2}}}{\sqrt{-\det(A)}} \sum_{\mathbf{n}\in A^{-1}\mathbb{Z}^r/\mathbb{Z}^r} e^{\frac{2\pi i}{\tau} Q(\mathbf{z}+\mathbf{n}\tau)} \Theta(\mathbf{z} + \mathbf{n}\tau; \tau) \tag{1.49}$$

The analogous properties for the one-variable indefinite theta defined in (1.40) can be easily obtained using equation (1.46). Furthermore, it has been shown [38] that for $\mathbf{c}_1, \mathbf{c}_2 \in \mathbb{Z}^r \cap \overline{C}_Q$ with relatively prime coordinates, $\mathbf{a}, \mathbf{b} \in R(\mathbf{c}_1) \cap R(\mathbf{c}_2)$ the indefinite theta in (1.40) is the component of a vector valued mixed harmonic

Maass form of weight $\frac{r}{2}$ with holomorphic part given by

$$\Theta^+_{\mathbf{a},\mathbf{b}}(\tau) = \sum_{\mathbf{n}\in\mathbb{Z}^r+\mathbf{a}} \left[\mathrm{sgn}(B(\mathbf{c}_1,\mathbf{n})) - \mathrm{sgn}(B(\mathbf{c}_2,\mathbf{n}))\right] e^{2\pi i B(\mathbf{b},\mathbf{n})} q^{Q(\mathbf{n})}. \tag{1.50}$$

In particular, the action of the shadow operator (1.27) on the indefinite theta function (1.40) is given by

$$\begin{aligned}\xi_{\frac{r}{2}}(\Theta_{\mathbf{a},\mathbf{b},\mathbf{c}_1,\mathbf{c}_2}(\tau)) =& \sqrt{2\mathrm{Im}(\tau)^{r-1}} \sum_{j\in\mathcal{J}} (-1)^j \sum_{\ell_0\in P_{0,j}} g_{\frac{B(\mathbf{c}_j,\ell_0)}{2Q(\mathbf{c}_j)},-B(\mathbf{c}_j,\mathbf{b})}(-2Q(\mathbf{c}_j)\tau) \\ &\times \overline{\sum_{\nu\in\ell_0^\perp+\langle\mathbf{c}_j\rangle_{\mathbb{Z}}^\perp} e^{2\pi i B(\nu,\mathbf{b}^\perp)} q^{Q(\nu)}}\end{aligned} \tag{1.51}$$

where we have indicated with $\perp$ the component orhogonal to $\mathbf{c}_j$, so that e.g. $\ell_0^\perp := \ell_0 - \frac{B(\mathbf{c}_j,\ell_0)}{2Q(\mathbf{c}_j)}\mathbf{c}_j$, $\langle\mathbf{c}_j\rangle_{\mathbb{Z}}^\perp := \{\lambda \in \mathbb{Z}^r : B(\mathbf{c}_j,\lambda) = 0\}$, $\mathcal{J} := \{j \in \{1,2\} : \mathbf{c}_j \in C_Q\}$, $P_{0,j}$ is a finite set such that

$$\left\{\ell \in \mathbf{a} + \mathbb{Z}^r : \frac{B(\mathbf{c}_j,\ell)}{2Q(\mathbf{c}_j)} \in [0,1)\right\} = \dot{\bigcup}_{\ell_0\in P_{0,j}} (\ell_0 + \langle\mathbf{c}_j\rangle_{\mathbb{Z}}^\perp) \tag{1.52}$$

and $g_{a,b}$ is the unary theta function

$$g_{a,b}(\tau) := \sum_{n\in a+\mathbb{Z}} n e^{2\pi i n b} q^{\frac{n^2}{2}}. \tag{1.53}$$

It is also shown [4], [42], [43], [38] that Ramanujan's mock theta functions (and a further number of mock theta functions discovered later) can be written in terms of a linear combination of modular forms and indefinite theta functions (1.50) with $r = 2$. This can be viewed as a generalization of the following relation between the indefinite theta functions and the Appell-Lerch sum

$$\mu(z_1,z_2;\tau) := \frac{y_1^{\frac{1}{2}}}{\theta(z_2;\tau)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n y_2^n q^{\frac{n(n+1)}{2}}}{1-y_1q^n}, \tag{1.54}$$

where $z_1, z_2 \in \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, $y_j = e^{2\pi i z_j}$ for $j = 1, 2$, and $\theta(z;\tau)$ is the Jacobi theta function

$$\theta(z;\tau) := \sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{\pi i n^2\tau + 2\pi i n\left(z+\frac{1}{2}\right)}. \tag{1.55}$$

Namely, it can be shown [38] that, for $0 < \frac{\mathrm{Im}(z_1)}{\mathrm{Im}(\tau)}, \frac{\mathrm{Im}(z_2)-\mathrm{Im}(z_1)}{\mathrm{Im}(\tau)} + \frac{1}{2} < 1$, we have the following relation

$$\mu(z_1, z_2; \tau) = \frac{y_1^{\frac{1}{2}}}{2\theta(z_2; \tau)} \Theta^+_{A,\mathbf{c}_1,\mathbf{c}_2} \left(z_1, z_2 - z_1 + \frac{\tau+1}{2}; \tau \right) \tag{1.56}$$

for $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\mathbf{c}_1 = (0, 1)$, $\mathbf{c}_2 = (-1, 1)$.
Notice that, writing $\mathbf{z} = \mathbf{a}\tau + \mathbf{b}$ with $\mathbf{a} \in R(\mathbf{c}_1) \cap R(\mathbf{c}_2), \mathbf{b} \in \mathbb{R}^r$, equation (1.56) can be related to expression (1.50) through

$$\Theta^+_{A,\mathbf{c}_1,\mathbf{c}_2}(\mathbf{a}\tau + \mathbf{b}, \tau) = e^{-2\pi i B(\mathbf{a},\mathbf{b})} q^{-Q(\mathbf{a})} \Theta^+_{\mathbf{a},\mathbf{b}}(\tau). \tag{1.57}$$

The relation to Ramanujan's mock theta functions follows via their expression in terms of the universal mock theta functions g_2, g_3 [42] which in turn can be related to the Appell-Lerch sum (1.54) [44].

2 Cone Vertex Algebras, Mock Theta Functions, and Umbral Moonshine Modules

In this chapter we will present the work of [2]. We will explore how the relation between Ramanujan's mock theta functions and indefinite theta functions can be exploited to construct vertex operator algebra modules whose graded characters reproduce some instances of the McKay–Thompson series predicted by Umbral Moonshine. Since vertex operator algebras describe the chiral algebra of 2d Conformal Field Theories, we will catch a glimpse of some connections between Group Theory, Physics and Number Theory. The explicit construction of modules for Umbral Moonshine is, in fact, interesting not only from the mathematical perspective but also to gain better insight of the appearance of the Umbral Moonshine phenomenon in the context of non-linear sigma models on $K3$ surfaces. Furthermore, the construction of modules whose trace functions reproduce a certain family of indefinite theta functions can be interesting to get more insights on the role of mock modularity in Physics.
We will begin this chapter by giving some context and a brief introduction to Umbral Moonshine and recalling some basics definitions of lattice Vertex Algebras before delving into the construction of the vertex algebra modules.

Umbral Moonshine consists of a family of 23 moonshine instances associated to appropriate quotients of the automorphism groups of Niemeier lattices, the 23 even unimodular positive-definite lattices of rank 24 with non-trivial root systems. Given the root system X of a Niemeier lattice L^X, the umbral group G^X associated to it is given by the quotient of the automorphisms group of L^X by the Weyl group W^X associated to the root system

$$G^X := \mathrm{Aut}(L^X)/W^X. \tag{2.1}$$

Following [17, 18], we will often refer to twenty-three instances as the different *lambencies* of umbral moonshine. To each conjugacy class $[g]$ of G^X is associated a vector-valued mock modular form, the umbral McKay–Thompson series H_g^X. The umbral moonshine conjecture predicts, for each Niemeier lattice, the existence of a naturally defined bi-graded G^X-module

$$\check{K}^X := \bigoplus_{r\in I^X} \bigoplus_{\substack{D\in\mathbb{Z}, D\leq 0 \\ D=r^2 \mod 4m}} \check{K}^X_{r,-D/4m} \tag{2.2}$$

such that the corresponding McKay–Thompson series $H^X_g = (H^X_{g,r})$ is related to the graded trace of g over $\check{K}$ by

$$H^X_{g,r}(\tau) = -2q^{-\frac{1}{4m}}\delta_{r,1} + \sum_{\substack{D\in\mathbb{Z}, D\leq 0 \\ D=r^2 \mod 4m}} \mathrm{tr}_{\check{K}^X_{r,-D/4m}}(g)q^{-\frac{D}{4m}} \tag{2.3}$$

where m is the Coxeter number of any simple component of the Niemeier root system X, and $I^X \subset \mathbb{Z}/2m\mathbb{Z}$ is specified by

$$I^X := \begin{cases} \{1,2,3,\ldots,m-1\} & \text{if } X \text{ has an A-type component,} \\ \{1,3,5,\ldots,\frac{m}{2}\} & \text{if } X \text{ has a D-type and no A-type components,} \\ \{1,4,5\} & \text{if } X = E_6^4, \\ \{1,7\} & \text{if } X = E_8^3. \end{cases} \tag{2.4}$$

The existence of the modules (2.2) has been proven in [16] for the case of Mathieu Moonshine, and then in [45] for the remaining cases. These proofs, however, do not prescribe how such modules can be built nor offer much insight on possible further algebraic structure.

Since Monstrous Moonshine [7], the very first example of a moonshine phenomenon, vertex operator algebras have proven to be an invaluable tool to understand the underlying structure behind the moonshine properties [10], [11], [46], [12]. It is thus natural to ask if a similar approach could provide interesting insights in the case of umbral moonshine. For some instances of umbral moonshine, it has already been shown that suitable (super) vertex operator algebras can be used to explicitly construct the modules $\check{K}^X$ [47], [48] or to solve the so called "meromorphic module problem", i.e. building modules such that specific trace functions give the meromorphic Jacobi forms associated to the H_g of Umbral Moonshine [49], [50], [51]. In particular, in [47] the authors built the module $\check{K}^{E_8^3}$ through the means of particular vertex operator algebras obtained from lattice cones. Their construction makes use of the relations between the umbral McKay–Thompson series for E_8^3, the fifth order Ramanujan's mock theta functions ϕ_0, ϕ_1 and their expressions in terms of indefinite theta functions. It is natural to ask if the techniques of [47] can be extended to build modules for other instances of umbral moonshine. In this work we employ a particular class of cone vertex algebras and construct modules for instances of umbral moonshine corresponding to root systems $A_7^2D_5^2$, $A_{11}D_7E_6$,

$A_{15}D_9$. In order to achieve this, we will establish intermediate results relating cone vertex algebras to indefinite theta functions that are mock theta functions. In particular, we first describe a specific family of indefinite theta functions which can be expressed in terms of trace functions of cone vertex algebras. Then, expressing the umbral McKay–Thompson series H_g^X in terms of indefinite theta functions, we relate H_g^X to suitable linear combinations of the traces of cone vertex algebra and other known (super) vertex operator algebras. In the cases considered, we find that the respective umbral groups act trivially on the underlying cone vertex algebra modules. Thus the modules realizing the McKay–Thompson series appearing in these examples have the structure of a tensor product $R \otimes M$ of a finite group representation R and a (super) vertex algebra module M. In particular, the umbral finite group G acts on $R \otimes M$ as $G \otimes \mathbf{1}_{\mathcal{V}}$, while the vertex algebra $\mathcal{V}$ acts as $\mathbf{1}_G \otimes \mathcal{V}$. This makes the analysis particularly simple as the representation of the umbral group can be determined independently from the relevant cone vertex algebra structure.
As an intermediate result, we also show that the following Appell-Lerch sums

$$\mu(z_1, z_2; \tau) := \frac{y_1^{\frac{1}{2}}}{\theta(z_2; \tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n y_2^n q^{\frac{n(n+1)}{2}}}{1 - y_1 q^n}, \tag{2.5}$$

$$\mu_{m,0}(z, \tau) := \sum_{k \in \mathbb{Z}} y^{2km} q^{mk^2} \frac{yq^k + 1}{yq^k - 1}, \tag{2.6}$$

admit an expression in terms of indefinite theta functions and cone vertex algebra characters. These are distinguished examples connecting cone vertex algebras to mock theta functions and umbral moonshine. In fact, all Ramanujan mock thetas can be expressed in terms of (2.5) [4], [42], [43], [38], while (2.6) appears in the construction of the optimal meromorphic Jacobi forms associated to the umbral McKay–Thompson series [18], [45]. The latter fact allows us to draw a connection between the construction of modules for the McKay–Thompson series (as considered in this chapter) and the meromorphic module problem considered in [49], [51]. Furthermore, the specialized Appell-Lerch sum (2.6) is also interesting because it captures the non-modular part of the elliptic genus of non-compact supersymmetric coset models, as featured in [52], [53], [19]. The techniques used in this chapter can be easily used to build an alternative module for the elliptic genus of such theories in terms of cone vertex algebras trace functions.
This chapter is organized as follows: in subsection 2.1 we recall basic notions and notations of cone vertex algebras that will be used in the rest of the chapter; in section 2.2 we present a core result (Theorem 2.2.1) relating trace functions of cone vertex algebras to indefinite theta functions; in section 2.3 we give expression specifying the umbral McKay–Thompson series for lambencies $\ell = 8, 12, 16$ in

terms of indefinite theta functions and modular forms; finally, in section 2.4 we specify the umbral McKay–Thompson series considered in section 2.3 in terms of trace functions of vertex algebra modules (Theorems 2.4.2, 2.4.3, 2.4.4).

2.1 Background: Lattice Vertex Algebras

In this subsection we will briefly summarize the construction of vertex algebras associated to lattices, closely following the exposition in [47]. The main goal is to introduce the notation and conventions that will be used in the rest of the chapter. A full introduction to Vertex Operator Algebras is out of the scope of this thesis, we refer the interested reader to, e.g, [46], [54], and [55] for an overview.
Consider a lattice L. Let's define $\mathfrak{h} := L \otimes_{\mathbb{Z}} \mathbb{C}$ with the symmetric $\mathbb{C}$-bilinear form $\langle \cdot, \cdot \rangle$ naturally inherited from the bilinear form on L. Given a formal variable t, define $\hat{\mathfrak{h}} := \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C}\mathbf{c}$ with the Lie algebra structure given by $[u \otimes t^m, v \otimes t^n] = m \langle u, v \rangle \delta_{m+n,0} \mathbf{c}$ with $\mathbf{c}$ a central element. The algebra $\hat{\mathfrak{h}}$ has a natural decomposition as $\hat{\mathfrak{h}} = \hat{\mathfrak{h}}^- \oplus \hat{\mathfrak{h}}^0 \oplus \hat{\mathfrak{h}}^+$ with $\hat{\mathfrak{h}}^\pm := \mathfrak{h}[t^{\pm 1}]t^\pm$ and $\hat{\mathfrak{h}}^0 := \mathfrak{h} \oplus \mathbb{C}\mathbf{c}$. Given an ordered integral basis $\{\epsilon_j\}$ for the lattice L, define

$$b(\epsilon_i, \epsilon_j) = \begin{cases} 0 & \text{if } i \leq j \\ 1 & \text{if } i > j \end{cases} \tag{2.7}$$

extended linearly over L, and set $\beta(\lambda, \mu) := (-1)^{b(\lambda,\mu)}$. We then consider the ring $\mathbb{C}_\beta[L]$ generated by $\mathbf{v}_\lambda$, $\lambda \in L$, satisfying $\mathbf{v}_\lambda \mathbf{v}_\mu = \beta(\lambda, \mu)\mathbf{v}_{\lambda+\mu}$. Give $\mathbb{C}_\beta[L]$ a $\hat{\mathfrak{h}}^0 \oplus \hat{\mathfrak{h}}^+$-module structure by setting, for $h \in \mathfrak{h}$ and $\lambda \in L$, $\mathbf{c}\mathbf{v}_\lambda = \mathbf{v}_\lambda$ and $u(m)\mathbf{v}_\lambda = \delta_{m,0} \langle u, \lambda \rangle \mathbf{v}_\lambda$, where we have used the standard notation $u(m) = u \otimes t^m$. Finally, we consider the module

$$V_L := U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}^0 \otimes \hat{\mathfrak{h}}^+)} \mathbb{C}_\beta[L]. \tag{2.8}$$

We can equip this module with a (unique) vertex algebra structure with vacuum vector $1 \otimes \mathbf{v}_0$, vertex operator map $Y : V_L \to (\mathrm{End} V_L)[[z, z^{-1}]]$ given by, for $u \in \mathfrak{h}$ and $\lambda \in L$,

$$\begin{aligned} Y(u(-1) \otimes \mathbf{v}_0, z) &= \sum_{n \in \mathbb{Z}} u(n) z^{-n-1} \\ Y(\mathbf{1} \otimes \mathbf{v}_\lambda, z) &= \exp\left(-\sum_{n<0} \frac{\lambda(n)}{n} z^{-n} \right) \exp\left(-\sum_{n>0} \frac{\lambda(n)}{n} z^{-n} \right) \mathbf{v}_\lambda z^{\lambda(0)} \end{aligned} \tag{2.9}$$

where $\mathbf{v}_\lambda$ in the right hand side denotes the operator $p \otimes \mathbf{v}_\mu \to \beta(\lambda, \mu) p \otimes \mathbf{v}_{\lambda+\mu}$, and $z^{\lambda(0)}(p \otimes \mathbf{v}_\mu) := (p \otimes \mathbf{v}_\mu) z^{\langle \lambda, \mu \rangle}$. Furthermore, given the basis $\{\epsilon_j\}$ for L and the dual basis $\{\epsilon'_j\}$, $\epsilon'_j \in L \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfying $\langle \epsilon'_i, \epsilon_j \rangle = \delta_{ij}$, we can define the conformal element

$$\omega := \frac{1}{2} \sum_i \epsilon'_i(-1)\epsilon_i(-1) \otimes \mathbf{v}_0. \tag{2.10}$$

Writing $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$, we have $[L(0), v(n)] = -nv(n)$ and $L(0)1\otimes\mathbf{v}_\lambda = \frac{\langle\lambda,\lambda\rangle}{2} 1 \otimes \mathbf{v}_\lambda$. In particular, when the bilinear form on L is positive definite, this give V_L the structure of a vertex operator algebra. In the more general case, vector of zero length give infinite dimensional eigenspaces for $L(0)$. We can define a finite order automorphism of V_L by choosing $h \in L \otimes_{\mathbb{Z}} \mathcal{Q}$ acting as $h(0)p \otimes \mathbf{v}_\lambda = \langle h, \lambda \rangle\, p \otimes \mathbf{v}_\lambda$ with $p \in S(\hat{\mathfrak{h}}^-)$ and defining

$$g_h := e^{2\pi i h(0)}. \tag{2.11}$$

In order to build twisted modules for the lattice vertex algebra , let's consider $\mathbb{C}_\beta[L+h]$ generated by $\mathbf{v}_{\mu+h}$, with $\mu \in L$ and $h \in L\otimes_{\mathbb{Z}}\mathbb{Q}$, equipped with the $\mathbb{C}_\beta[L]$-module structure given by $\mathbf{v}_\lambda \mathbf{v}_{\mu+h} = \beta(\lambda,\mu)\mathbf{v}_{\lambda+\mu+h}$ and the $U(\hat{\mathfrak{h}}^0 \otimes \hat{\mathfrak{h}}^+)$-module structure $\mathbf{c}\mathbf{v}_{\mu+h} = \mathbf{v}_{\mu+h}$, $u(m)\mathbf{v}_{\mu+h} = \delta_{m,0} \langle u, \mu + h\rangle \mathbf{v}_{\mu+h}$ for $u \in \mathfrak{h}$, $\mu, \lambda \in L$. We can then define g_h-twisted modules for the lattice vertex algebra V_L by setting $V_{L+h} := U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}^0\otimes\hat{\mathfrak{h}}^+)} \mathbb{C}_\beta[L+h]$ and defining $Y_h := V_L \to (\mathrm{End} V_{L+h})[[z, z^{-1}]]$ similarly as before but with $\mathbf{v}_\lambda$ acting as $\mathbf{v}_\lambda(p \otimes \mathbf{v}_{\mu+h}) = \beta(\lambda,\mu)p \otimes \mathbf{v}_{\lambda+\mu+h}$. When h belongs to the dual lattice $L^* = \{\lambda \in L \otimes_{\mathbb{Z}} Q | \langle \lambda, L\rangle \in \mathbb{Z}\}$, the modules are untwisted. Furthermore, all the g_h-twisted modules of V_L are given by $V_{L+h'}$ for some $h' \in L \otimes_{\mathbb{Z}} \mathbb{Q}$ congruent to h modulo L^*. The action of $L \otimes_Z \mathbb{Q}$ on V_L specified by g_h extends to $g_{h'}$-twisted modules through

$$g_h(p \otimes \mathbf{v}_{\lambda+h'}) := e^{2\pi i \langle h,\lambda\rangle} p \otimes \mathbf{v}_{\lambda+h'}. \tag{2.12}$$

In order to include vertex algebras associated to cones, as opposed to the full lattice, we will describe a family of sub-vertex algebras of V_L. For a $K \subset L$ that is closed under addition that contains 0, the submodule V_K of V_L generated by $\mathbf{v}_\lambda$ for $\lambda \in K$ has the structure of a sub-vertex algebra of V_L with the same conformal element. Furthermore, given $\gamma \in L \otimes_{\mathbb{Z}} \mathcal{Q}$, for any $K' \subset L + \gamma$ such that $K + K' \subset K'$, the corresponding $V_{K'}$ with the restriction of the vertex operators $a \otimes b \mapsto Y(a, z)b$ to $V_K \otimes V_{K'}$ has the structure of a twisted module over V_K.

2.2 Indefinite Theta Functions and Cone Vertex Algebras

In this section we will describe a family of trace functions of vertex algebras modules that can be expressed in terms of indefinite theta functions.
Consider a symmetric 2×2 matrix A with integer coefficients, the associated bilinear and quadratic forms B and Q as in section 1.3.2, and the vectors $\mathbf{c}_1, \mathbf{c}_2 \in \bar{C}_Q$ satisfying

$$\mathbf{c}_1^{\mathrm{T}} A = k(1,0), \quad \mathbf{c}_2^{\mathrm{T}} A = k'(0,-1), \tag{2.13}$$

with $k, k' \in \mathbb{R}^*$ and $\mathrm{sgn}(k) = \mathrm{sgn}(k')$. With the above constraints, we will consider the family of rank 2 indefinite theta functions [1]

$$\Theta_{\mathbf{a},\mathbf{b}}(N\tau) = \sum_{\mathbf{n} \in \mathbb{Z}^2 + \mathbf{a}} \left[\mathrm{sgn}(B(\mathbf{c}_1, \mathbf{n})) - \mathrm{sgn}(B(\mathbf{c}_2, \mathbf{n}))\right] e^{2\pi i B(\mathbf{n},\mathbf{b})} q^{NQ(\mathbf{n})} \tag{2.14}$$

with N a positive integer, $\mathbf{b} \in \mathbb{R}^2$, and $\mathbf{a} = (a_1, a_2) \in \mathbb{Q}^2 \cap R(\mathbf{c}_1) \cap R(\mathbf{c}_2)$. As we will show in later sections, in the cases considered the components of umbral McKay–Thompson series can be rewritten in terms of indefinite theta functions with such quadratic form A and vectors $\mathbf{c}_1, \mathbf{c}_2$.
Let's now define the relevant cone vertex algebras. Following the construction in section 2.1, we start by defining the underlying lattice. We consider the rank 2 lattice $L^{(N)}$ generated by ϵ_1, ϵ_2 with the bilinear form $\langle \cdot, \cdot \rangle$ specified by the matrix A and a positive integer N as

$$\langle \epsilon_i, \epsilon_j \rangle = N A_{ij}. \tag{2.15}$$

Consider the cone inside of $L^{(N)}$

$$P^{(N)} = \left\{ \sum_{i=1}^{2} \alpha_i \epsilon_i \in L^{(N)} \otimes \mathbb{Q} : \alpha_i \geq 0, \forall i = 1, 2 \right\} \tag{2.16}$$

and its shifted version $P^{(N)} + \gamma := \{\mu + \gamma | \mu \in P^{(N)}\}$. As described in section 2.1, $V_{P^{(N)}}$, generated by $\mathbf{v}_\lambda$ for $\lambda \in P^{(N)}$, is a sub-vertex operator algebra of $V_{L^{(N)}}$. For $\mathbf{a} := (a_1, a_2) \in \mathbb{Q}^2$ let's define $\rho_{\mathbf{a}}^+ := a_1 \epsilon_1 + a_2 \epsilon_2$ and $\rho_{\mathbf{a}}^- := (1 - a_1)\epsilon_1 + (1 - a_2)\epsilon_2$. To any lattice $L^{(N)}$ we thus associate a module $V_{\mathbf{a}}^{(N)}$ given by the following direct sum

$$V_{\mathbf{a}}^{(N)} := V_{P^{(N)} + \rho_{\mathbf{a}}^+} \oplus V_{P^{(N)} + \rho_{\mathbf{a}}^-} \tag{2.17}$$

where $V_{P^{(N)} + \rho_{\mathbf{a}}^+}$ and $V_{P^{(N)} + \rho_{\mathbf{a}}^-}$ are the modules of the vertex algebra $V_{P^{(N)}}$ built from $P^{(N)} + \rho_{\mathbf{a}}^+$ and $P^{(N)} + \rho_{\mathbf{a}}^-$ respectively. Notice that, when $\rho_{\mathbf{a}}^{\pm} \in {L^{(N)}}^*$, as is

[1] From now on we will omit the + apex from the symbol $\Theta_{\mathbf{a},\mathbf{b}}^+$

the case when $a_1, a_2 \in \left\{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\right\}$, the modules $V_{P^{(N)}+\rho_{\mathbf{a}}^{\pm}}$ are untwisted. For $\lambda = n_1\epsilon_1 + n_2\epsilon_2 + \rho_{\mathbf{a}}^{\pm} \in P^{(N)} + \rho_{\mathbf{a}}^{\pm}$, $\mathbf{b} = (b_1, b_2) \in \mathbb{Q}^2$ and write $\mathbf{n} = (n_1, n_2)$, let's define the operator $g_{\mathbf{b}} : V_{\mathbf{a}}^{(N)} \to V_{\mathbf{a}}^{(N)}$ acting as

$$g_{\mathbf{b}}(p \otimes \mathbf{v}_\lambda) := \begin{cases} e^{2\pi i B(\mathbf{n},\mathbf{b})} p \otimes \mathbf{v}_\lambda & \text{if } \mathbf{v}_\lambda \in V_{P^{(N)}+\rho_{\mathbf{a}}^{+}} \\ -e^{-2\pi i B(\mathbf{n}+\mathbf{1},\mathbf{b})} p \otimes \mathbf{v}_\lambda & \text{if } \mathbf{v}_\lambda \in V_{P^{(N)}+\rho_{\mathbf{a}}^{-}} \end{cases} \tag{2.18}$$

The main object we will be interested in is the trace function

$$T_{\mathbf{a},\mathbf{b}}^{(N)}(\tau) := \mathrm{Tr}_{V_{\mathbf{a}}^{(N)}}\left(g_{\mathbf{b}} q^{L(0)-c/24}\right). \tag{2.19}$$

To ensure that $T_{\mathbf{a},\mathbf{b}}^{(N)}$ converges in the upper-half plane, we restrict to the case where $v^T A v > 0 \; \forall v \in P^{(N)} + \rho_{\mathbf{a}}^{\pm}$, namely that the quadratic form associated to A is positive definite in the two shifted cones. Under such assumptions, we have the following relation between the trace functions $T_{\mathbf{a},\mathbf{b}}^{(N)}(\tau)$ and $\Theta_{\mathbf{a},\mathbf{b}}(N\tau)$.

Lemma 2.2.0.1. *Let A and Q be as above, and consider $\mathbf{c}_1, \mathbf{c}_2 \in \overline{C}_Q$ satisfying*

$$\mathbf{c}_1^{\mathrm{T}} A = k(1,0), \quad \mathbf{c}_2^{\mathrm{T}} A = k'(0,-1), \tag{2.20}$$

for some $k, k' \in \mathbb{R}^*$ with $kk' > 0$. Given $\mathbf{a} = (a_1, a_2) \in \mathbb{Q}^2 \cap R(\mathbf{c}_1) \cap R(\mathbf{c}_2)$ with $0 < a_1, a_2 < 1$, we have

$$T_{\mathbf{a},\mathbf{b}}^{(N)}(\tau) = \mathrm{sgn}(k) \frac{e^{-2\pi i B(\mathbf{a},\mathbf{b})}}{2\eta(\tau)^2} \Theta_{\mathbf{a},\mathbf{b}}(N\tau). \tag{2.21}$$

Proof. Explicitly, (2.19) equals

$$
\begin{aligned}
T^{(N)}_{\mathbf{a},\mathbf{b}}(\tau) &= \frac{1}{\eta(\tau)^2}\left(\sum_{\mu\in P^{(N)}+\rho^+_{\mathbf{a}}} e^{2\pi i B(\mathbf{n},\mathbf{b})}q^{\frac{\langle\mu,\mu\rangle}{2}} - \sum_{\mu\in P^{(N)}+\rho^-_{\mathbf{a}}} e^{-2\pi i B(\mathbf{n}+\mathbf{1},\mathbf{b})}q^{\frac{\langle\mu,\mu\rangle}{2}}\right)\\
&= \frac{1}{\eta(\tau)^2}\left(\sum_{\substack{\mathbf{n}\in\mathbb{Z}^2\\ n_1,n_2\geq 0}} e^{2\pi i B(\mathbf{n},\mathbf{b})}q^{Q(\mathbf{n}+\mathbf{a})} - \sum_{\substack{\mathbf{n}\in\mathbb{Z}^2\\ n_1,n_2\geq 0}} e^{-2\pi i B(\mathbf{n}+\mathbf{1},\mathbf{b})}q^{Q(\mathbf{n}+\mathbf{1}-\mathbf{a})}\right)\\
&= \frac{1}{\eta(\tau)^2}\left(\sum_{\substack{\mathbf{n}\in\mathbb{Z}^2\\ n_1,n_2\geq 0}} e^{2\pi i B(\mathbf{n},\mathbf{b})}q^{Q(\mathbf{n}+\mathbf{a})} - \sum_{\substack{\mathbf{n}\in\mathbb{Z}^2\\ n_1,n_2\leq 0}} e^{2\pi i B(\mathbf{n}-\mathbf{1},\mathbf{b})}q^{Q(\mathbf{n}-\mathbf{1}+\mathbf{a})}\right)\\
&= \frac{1}{\eta(\tau)^2}\left(\sum_{\substack{\mathbf{n}\in\mathbb{Z}^2\\ n_1,n_2\geq 0}} - \sum_{\substack{\mathbf{n}\in\mathbb{Z}^2\\ n_1,n_2< 0}}\right) e^{2\pi i B(\mathbf{n},\mathbf{b})}q^{Q(\mathbf{n}+\mathbf{a})}
\end{aligned}
\tag{2.22}
$$

where we have written $\mu = (n_1+a_1)\epsilon_1 + (n_2+a_2)\epsilon_2$ and $\mathbf{n} = (n_1, n_2)$.
On the other hand, since $0 < a_1, a_2 < 1$, the factor $\rho^{\mathbf{c},\mathbf{c}'}(\mathbf{n})$ in (2.14) equals, using (2.13),

$$
\begin{aligned}
\mathrm{sgn}(B(\mathbf{c},\mathbf{n})) - \mathrm{sgn}(B(\mathbf{c}',\mathbf{n})) &= \mathrm{sgn}(k)\mathrm{sgn}(n_1+a_1) + \mathrm{sgn}(k')\mathrm{sgn}(n_2+a_2)\\
&= \begin{cases} 2\mathrm{sgn}(k) & \text{if } n_1,n_2\geq 0\\ -2\mathrm{sgn}(k) & \text{if } n_1,n_2<0\\ 0 & \text{otherwise}\end{cases}
\end{aligned}
\tag{2.23}
$$

where we have also used $\mathrm{sgn}(k) = \mathrm{sgn}(k')$. By comparison it is immediate to see that the difference between the two sums in (2.22) equals the indefinite theta function defined in (2.14) up to the overall $\frac{e^{-2\pi i B(\mathbf{a},\mathbf{b})}}{2}\mathrm{sgn}(k)$ factor. □

The result can be easily generalized to the cases where a_1 or a_2 is equal to 0. In that case we will have an extra one-dimensional theta series appearing in the right hand side of (2.21). In fact, let's consider for example the case $a_1 = 0, a_2 \neq 0$. We have

$$\begin{aligned}\operatorname{sgn}(B(\mathbf{c},\mathbf{n})) - \operatorname{sgn}(B(\mathbf{c}',\mathbf{n})) &= \operatorname{sgn}(k)\operatorname{sgn}(n_1) + \operatorname{sgn}(k')\operatorname{sgn}(n_2+a_2) \\ &= \begin{cases} 2\operatorname{sgn}(k) & \text{if } n_1 \geq 0, n_2 > 0 \\ -2\operatorname{sgn}(k) & \text{if } n_1, n_2 < 0 \\ \operatorname{sgn}(k) & \text{if } n_1 \geq 0, n_2 = 0 \\ -\operatorname{sgn}(k) & \text{if } n_1 < 0, n_2 = 0 \\ 0 & \text{otherwise} \end{cases}\end{aligned} \tag{2.24}$$

so we have to add some series to account the case $n_2 = 0$ correctly. An easy calculation for the general case shows that we have the following

Theorem 2.2.1. *Let A, Q, $\mathbf{c}_1$, $\mathbf{c}_2$, be as in Lemma 2.2.0.1 and $\mathbf{a} = (a_1, a_2) \in \mathbb{Q}^2 \cap R(\mathbf{c}_1) \cap R(\mathbf{c_2})$ with $0 \leq a_1, a_2 < 1$. We have*

$$\begin{aligned}T^{(N)}_{\mathbf{a},\mathbf{b}}(\tau) =& sgn(k)\frac{e^{-2\pi i B(\mathbf{a},\mathbf{b})}}{2\eta(\tau)^2}\Bigg[\Theta_{\mathbf{a},\mathbf{b}}(N\tau) + \delta_{a_1}\sum_{\substack{n_1=0\\ n_2\in\mathbb{Z}}} e^{2\pi i B(\mathbf{n}+\mathbf{a},\mathbf{b})}q^{NQ(\mathbf{n}+\mathbf{a})} \\ &+ \delta_{a_2}\sum_{\substack{n_1\in\mathbb{Z}\\ n_2=0}} e^{2\pi i B(\mathbf{n}+\mathbf{a},\mathbf{b})}q^{NQ(\mathbf{n}+\mathbf{a})} - \delta_{a_1}\delta_{a_2}e^{2\pi i B(\mathbf{a},\mathbf{b})}q^{Q(\mathbf{a})}\Bigg],\end{aligned} \tag{2.25}$$

where δ_i is the Kronecker delta $\delta_{i,0}$.

We will now show that the Appell–Lerch sums (2.5) and (2.6) can be written in terms of the trace functions (2.19). These functions will also be important for later sections. Let's first consider the Appell–Lerch sum (2.5). We have the following

$$\mu(z_1, z_2; \tau) = \frac{y_1^{\frac{1}{2}}}{\theta(z_2;\tau)}\sum_{n\in\mathbb{Z}}\frac{(-1)^n y_2^n q^{\frac{n(n+1)}{2}}}{1 - y_1 q^n}. \tag{2.26}$$

We have the following

Corollary 2.2.1.1. *Let $\tilde{\mathbf{a}} = (\tilde{a}_1, \tilde{a}_2) \in \mathbb{Q}^2$ such that $0 < \tilde{a}_1 < 1$, $0 \leq \tilde{a}_2 - \tilde{a}_1 + \frac{1}{2} < 1$, $\tilde{\mathbf{b}} = (\tilde{b}_1, \tilde{b}_2) \in \mathbb{R}^2$, $N \in \mathbb{N}^*$. Let $T^{(N)}_{\mathbf{a},\mathbf{b}}$ be the trace function (2.19) associated to the lattice with quadratic form $N\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)$. We have*

$$\mu(\tilde{\mathbf{a}}N\tau + \tilde{\mathbf{b}}; N\tau) = \frac{2q^{\frac{Na_1}{2}}\eta(\tau)^2}{\theta((a_2+a_1-1/2)\tau + b_2 + b_1 - 1/2; N\tau)}q^{-NQ(\mathbf{a})}T^{(N)}_{\mathbf{a},\mathbf{b}}(\tau). \tag{2.27}$$

where $\mathbf{a} := (a_1, a_2) = \left(\tilde{a}_1, \tilde{a}_2 - \tilde{a}_1 + \frac{1}{2}\right)$ and $\mathbf{b} := (b_1, b_2) = \left(\tilde{b}_1, \tilde{b}_2 - \tilde{b}_1 + \frac{1}{2}\right)$.

Proof. The result follows by the rewriting of μ in terms of indefinite theta functions. In fact, using equations (1.56) and (1.57), we have

$$\mu(\tilde{\mathbf{a}}\tau + \tilde{\mathbf{b}}; \tau) = \frac{q^{\frac{a_1}{2}} e^{\pi i b_1}}{\theta((a_2 + a_1 - 1/2)\tau + b_2 + b_1 - 1/2; \tau)} e^{-2\pi i B(\mathbf{a},\mathbf{b})} q^{-Q(\mathbf{a})} \Theta_{\mathbf{a},\mathbf{b}}(\tau). \tag{2.28}$$

The choice $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\mathbf{c}_1 = (0,1)$, $\mathbf{c}_2 = (-1,1)$ satisfies (2.13). Furthermore, while $P^{(N)}$ has infinitely many vectors of the form $n_2\epsilon_2$ $\forall n_2 \in \mathbb{Z}$, that have null norm, the scalar product $\langle \lambda, \lambda \rangle = N(n_1 + a_1)^2 + 2N(n_1 + a_1)(n_2 + a_2)$ is strictly positive $\forall \lambda \in P^{(N)} + \rho^{\pm}_{\mathbf{a}}$ with for $0 \leq a_2 < 1$, $0 < a_1 < 1$. Thus, using Theorem 2.2.1, the conclusion follows. □

As already mentioned, all Ramanujan's mock theta functions can be written in terms of the Appell-Lerch sum (1.54) (up to modular functions) with the choice of $\mathbf{z} = \tilde{\mathbf{a}}\tau + \tilde{\mathbf{b}}$ discussed above [38], [4], [42], [43], thus they can be expressed in terms of cone vertex algebras trace functions using the previous Corollary.

Let's now consider the specialized Appell-Lerch sum (2.6). This function appears in the definition of the meromorphic Jacobi forms associated to the umbral McKay–Thompson series [18], [45], [56]. We will see that it also admits an expression in terms of the trace function (2.19). Specifically, we have the following

Corollary 2.2.1.2. *Let $a \in \mathbb{Q}^*$, $|a| < 1$, $b \in \mathbb{R}$, $N \in \mathbb{N}^*$. Consider the lattice with quadratic form $A = N\left(\begin{smallmatrix} 2m & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with $m \in \mathbb{N}$, and the trace function* (2.19) $T^{(N)}_{\mathbf{a},\mathbf{b}}$ *associated to it. We have*

$$\mu_{m,0}(aN\tau + b, N\tau) = -2f(b)q^{-2mNa^2}\eta(\tau)^2 T^{(N)}_{\mathbf{a},\mathbf{b}}(\tau) - \sum_{n \in \mathbb{Z}+a} e^{2\pi i n b} q^{2mNn^2} \tag{2.29}$$

where $\mathbf{b} = (b,0)$ and $\mathbf{a} = (a,0)$, $f(b) = 1$ when $a > 0$ while $\mathbf{a} = (1+a, 0)$, $f(b) = e^{-4\pi i b}$ when $a < 0$.

Proof. Let's show first that, for $\left|\frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\right| < 1$, $\mathrm{Im}(z) \neq 0$, we can write $\mu_{m,0}(z,\tau)$ in terms of indefinite theta functions satisfying the conditions of Theorem 2.2.1. We write $\mu_{m,0}(z,\tau) = f_1(z,\tau) + f_2(z,\tau)$, with

$$f_1(z,\tau) := -\sum_{k\in\mathbb{Z}} \frac{y^{2km} q^{mk^2}}{1 - yq^k}, \quad f_2(z,\tau) := -\sum_{k\in\mathbb{Z}} \frac{y^{2km+1} q^{mk^2+k}}{1 - yq^k} \tag{2.30}$$

Let us also set

$$A = \begin{pmatrix} 2m & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{c}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{c}_2 = \begin{pmatrix} -1 \\ 2m \end{pmatrix}. \tag{2.31}$$

Let us first focus on the domain $0 < \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)} < 1$. Using the geometric series expansion for the denominator in the range $0 < \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)} < 1$, we can rewrite f_1 as

$$\begin{aligned} f_1(z,\tau) &= -\left(\sum_{k,l\geq 0} - \sum_{k,l<0}\right) y^{2km+l} q^{mk^2+kl} \\ &= -\frac{1}{2}\sum_{(k,l)\in\mathbb{Z}^2}\left[\operatorname{sgn}\left(k+\frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\right)+\operatorname{sgn}(l)\right] e^{2\pi i B[(k,l),(z,0)]} q^{Q((k,l))} \\ &\quad -\frac{1}{2}\sum_{k\in\mathbb{Z}} y^{2mk} q^{mk^2} \end{aligned} \tag{2.32}$$

where the second sum has to be introduced to fix the contributions for $l = 0$. It is then immediate to see that we can write

$$f_1(z,\tau) = -\frac{1}{2}\Theta_{A,\mathbf{c}_1,\mathbf{c}_2}(z,0;\tau) - \frac{1}{2}\sum_{k\in\mathbb{Z}} y^{2mk} q^{mk^2}. \tag{2.33}$$

Analogously

$$\begin{aligned} f_2(z,\tau) &= -\left(\sum_{k,l\geq 0} - \sum_{k,l<0}\right) y^{2km+l+1} q^{mk^2+k(l+1)} \\ &= -\left(\sum_{k\geq 0, l\geq 1} - \sum_{k<0, l<1}\right) y^{2km+l} q^{mk^2+kl} \\ &= -\frac{1}{2}\sum_{(k,l)\in\mathbb{Z}^2}\left[\operatorname{sgn}\left(k+\frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\right)+\operatorname{sgn}(l)\right] e^{2\pi i B[(k,l),(z,0)]} q^{Q((k,l))} \\ &\quad +\frac{1}{2}\sum_{k\in\mathbb{Z}} y^{2mk} q^{mk^2} \\ &= -\frac{1}{2}\Theta_{A,\mathbf{c}_1,\mathbf{c}_2}(z,0;\tau) + \frac{1}{2}\sum_{k\in\mathbb{Z}} y^{2mk} q^{mk^2} \end{aligned} \tag{2.34}$$

where in the second line we have sent $l + 1 \to l$ and the second sum is again due to the $l = 0$ terms. Interestingly, when summing f_1 and f_2, only the contribution of the indefinite theta survives, and we have

$$\mu_{m,0}(z,\tau) = -\Theta_{A,\mathbf{c}_1,\mathbf{c}_2}(z,0;\tau). \tag{2.35}$$

In particular, notice that $\mathbf{c}_1$ and $\mathbf{c}_2$ satisfy (2.13). When $z = a\tau + b$ with $a \in \mathbb{Q}$,

$0 < a < 1$, $b \in \mathbb{R}$, for any $N \in \mathbb{N}$, using Theorem 2.2.1 we have

$$\begin{aligned}\mu_{m,0}(aN\tau + b, N\tau) &= -2e^{-4\pi imab}q^{-2mNa^2}\Theta_{\mathbf{a},\mathbf{b}}(N\tau) \\ &= -2q^{-2mNa^2}\eta(\tau)^2 T^{(N)}_{\mathbf{a},\mathbf{b}}(\tau) - \sum_{n\in\mathbb{Z}+a} e^{2\pi inb}q^{2mNn^2}\end{aligned} \tag{2.36}$$

with $\mathbf{a} = (a, 0)$ and $\mathbf{b} = (b, 0)$.
The same result still holds in the domain $0 < -\frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)} < 1$. In this case we have $|yq^k| < 1$ for $k > 0$ and $|yq^k| > 1$ for $k \leq 0$. Thus we get

$$f_1(z,\tau) = -\left(\sum_{\substack{k>0\\ l\geq 0}} - \sum_{\substack{k\leq 0\\ l<0}}\right) y^{2km+l}q^{mk^2+kl}. \tag{2.37}$$

On the other side, in this domain

$$\mathrm{sign}\left(k + \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\right) + \mathrm{sign}(l) = \begin{cases} 2 & \text{if } k > 0, l > 0, \\ 1 & \text{if } k > 0, l = 0, \\ -1 & \text{if } k \leq 0, l = 0, \\ -2 & \text{if } k \leq 0, l < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.38}$$

So we have again

$$\begin{aligned} f_1(z,\tau) = &-\frac{1}{2}\sum_{(k,l)\in\mathbb{Z}^2}\left[\mathrm{sgn}\left(k + \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\right) + \mathrm{sgn}\,(l)\right] e^{2\pi iB[(k,l),(z,0)]}q^{Q((k,l))} \\ &-\frac{1}{2}\sum_{k\in\mathbb{Z}} y^{2mk}q^{mk^2}. \end{aligned} \tag{2.39}$$

Proceeding in the same way for $f_2(z,\tau)$, it is possible to show that equation (2.35) still holds in the domain $0 < -\frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)} < 1$. In particular, we have

$$\mu_{m,0}(a\tau + b, \tau) = -2e^{-4\pi imab}q^{-2ma^2}\Theta_{\mathbf{a},\mathbf{b}}(\tau) \tag{2.40}$$

where $\mathbf{a} = (1 + a, 0)$ and $\mathbf{b} = (b, 0)$ and we have used the property $\Theta_{\mathbf{a},\mathbf{b}} = \Theta_{\mathbf{a}+\mathbf{s},\mathbf{b}}$ for all $\mathbf{s} \in \mathbb{Z}^2$. Notice that $1 + a > 0$ and thus we can use Theorem 2.2.1. We then get

$$\mu_{m,0}(aN\tau + b, N\tau) = -2e^{-4\pi imb}q^{-2mNa^2}\eta(\tau)^2 T^{(N)}_{\mathbf{a},\mathbf{b}}(\tau) - \sum_{n\in\mathbb{Z}+a} e^{2\pi inb}q^{2mNn^2}. \tag{2.41}$$

□

2.3 Umbral McKay–Thompson Series, Mock Theta Functions and Indefinite Thetas

In this section we will write the umbral McKay–Thompson series appearing for lambency $\ell = 8, 12, 16$ in terms of mock theta functions, eta quotients and Jacobi theta functions. In particular, all the mock theta functions encountered in these cases can be rewritten in terms of the indefinite theta functions [4], [38], [42], [43], with data satisfying the properties of Theorem 2.2.1. All the indefinite theta function have bilinear form $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and vectors $\mathbf{c}_1 = (0, 1)\,, \mathbf{c}_2 = (-1, 1)$. The relations between mock theta functions and indefinite theta functions relevant for the cases considered are collected in appendix A.
In some cases it is not possible to directly specify the individual Umbral McKay–Thompson series in terms of mock theta functions. When this happens, we will specify suitable linear combinations of the umbral McKay–Thompson series with disjoint sets of q-powers. In this way, the individual series can be retrieved by projecting onto the desired set of q-powers. In fact, given an instance of umbral moonshine with Coxeter number m, the r-th component of the corresponding mock modular form will have a series expansion in which the appearing q-powers will have the general form $q^{-\frac{r^2}{4m}+N}$ with $N \in \mathbb{N}$. Thus, the q-series of components with different values of $r^2 \mod 4m$ will have no common q-powers and therefore a linear combination of such components contains the same information as the set of the individual components.
The expressions provided are obtained by making use of the explicit specification of some umbral McKay–Thompson series in terms of mock theta functions combined with the multiplicative relations among different lambencies, as provided in [18].

2.3.1 Lambency Eight

Lambency $\ell = 8$ corresponds to the Niemeier root system $A_7^2 D_5^2$ with umbral group Dih_4. The McKay–Thompson series appearing for $\ell = 8$ can be expressed in terms of mock theta functions and eta quotients by making use of the multiplicative relations with $\ell = 4$ and the explicit specifications in [18]. In particular, we encounter the order 2 mock theta functions

$$A(q) := \sum_{n=0}^{\infty} \frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}, \tag{2.42}$$

$$B(q) := \sum_{n=0}^{\infty} \frac{q^{n}(-q;q^2)_n}{(q;q^2)_{n+1}}, \tag{2.43}$$

and the order 8 mock theta functions

$$S_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n}, \tag{2.44}$$

$$S_1(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)}, \tag{2.45}$$

$$T_0(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}}, \tag{2.46}$$

$$T_1(q) := \sum_{n=0}^{\infty} q^{n(n+1)} \frac{(-q^2;q^2)_n}{(-q;q^2)_{n+1}}. \tag{2.47}$$

The expressions specifying all the components for all conjugacy classes of Dih_4 in terms of the previous functions are

$$\begin{aligned}
&(H^{(8)}_{1A,1} - H^{(8)}_{1A,7})(2\tau) = H^{(4)}_{2C,1}(\tau) = q^{-\frac{1}{16}}(-2S_0(q) + 4T_0(q)),\\
&H^{(8)}_{1A,2}(\tau) = H^{(8)}_{1A,6}(\tau) = 4q^{-\frac{1}{4}}A(q),\\
&H^{(8)}_{1A,4}(\tau) = 4q^{\frac{1}{2}}B(q),\\
&(H^{(8)}_{1A,3} - H^{(8)}_{1A,5})(2\tau) = H^{(4)}_{2C,3}(\tau) = q^{\frac{7}{16}}(2S_1(q) - 4T_1(q)),\\
&(H^{(8)}_{2BC,1} - H^{(8)}_{2BC,7})(2\tau) = H^{(4)}_{4C,1}(\tau) = -2q^{-\frac{1}{16}}S_0(q)\\
&(H^{(8)}_{2BC,3} - H^{(8)}_{2BC,5})(2\tau) = H^{(4)}_{4C,3}(\tau) = 2q^{\frac{7}{16}}S_1(q)\\
&(H^{(8)}_{4A,1} - H^{(8)}_{4A,7} - H^{(8)}_{4A,3} + H^{(8)}_{4A,5})(2\tau) = H^{(4)}_{4B,1}(\tau) - H^{(4)}_{4B,3}(\tau),
\end{aligned} \tag{2.48}$$

together with the identities $H^{8}_{2BC,r} = H^{8}_{4A,r} = 0$ for r even, and the pairing relation

$$H^{(8)}_{2A,r} + (-1)^r H^{(8)}_{1A,r} = 0. \tag{2.49}$$

We can furthermore express the difference between components $r = 1$ and $r = 3$ for $\ell = 4$ appearing in the relation for class 4A in terms of an eta quotient [2] [18]

$$H^{(4)}_{4B,1}(\tau) - H^{(4)}_{4B,3}(\tau) = -2\frac{\eta(\frac{\tau}{2})\eta(2\tau)^4}{\eta(\tau)^2\eta(4\tau)^2}. \tag{2.50}$$

We can further simplify the previous expressions by making use of the following lemma

Lemma 2.3.0.1. *The order 8 mock theta functions S_0, S_1, T_0, T_1 satisfy*

$$\begin{aligned} S_0(q) + 2T_0(q) &= \frac{q^{\frac{1}{16}}}{2}\left(\frac{\eta(\frac{\tau}{2})^3}{\eta(\tau)\eta(2\tau)} + \frac{\eta(\tau)^8}{\eta(\frac{\tau}{2})^3\eta(2\tau)^4}\right), \\ S_1(q) + 2T_1(q) &= \frac{q^{-\frac{7}{16}}}{2}\left(-\frac{\eta(\frac{\tau}{2})^3}{\eta(\tau)\eta(2\tau)} + \frac{\eta(\tau)^8}{\eta(\frac{\tau}{2})^3\eta(2\tau)^4}\right). \end{aligned} \tag{2.51}$$

Proof. Using the expression in appendix A of [38]

$$\begin{aligned} S_0(q) &= -2iq^{\frac{1}{2}}g_2(iq^{\frac{1}{2}};q^4) + \frac{(-iq^{\frac{1}{2}};-q)^2_\infty(-q;-q)_\infty(-q^3;q^8)_\infty(-q^5;q^8)_\infty}{(-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty}, \\ S_1(q) &= -2iq^{\frac{1}{2}}g_2(-iq^{\frac{3}{2}};q^4) + \frac{(-iq^{\frac{1}{2}};-q)^2_\infty(-q;-q)_\infty(-q;q^8)_\infty(-q^7;q^8)_\infty}{(-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty}, \\ T_0(q) &= iq^{\frac{1}{2}}g_2(iq^{\frac{1}{2}};q^4) - \frac{(-iq^{\frac{1}{2}};-q)^2_\infty(-q;-q)_\infty(-q^3;q^8)_\infty(-q^5;q^8)_\infty}{2(-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty} \\ &\quad + \frac{1}{4}\frac{(q^{\frac{1}{2}};q^{\frac{1}{2}})^3_\infty}{(q;q)_\infty(q^2;q^2)_\infty} + \frac{1}{4}\frac{(q;q)^8_\infty}{(q^{\frac{1}{2}};q^{\frac{1}{2}})^3_\infty(q^2;q^2)^4_\infty}, \\ T_1(q) &= iq^{\frac{1}{2}}g_2(-iq^{\frac{3}{2}};q^4) - \frac{(-iq^{\frac{1}{2}};-q)^2_\infty(-q;-q)_\infty(-q;q^8)_\infty(-q^7;q^8)_\infty}{2(-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty} \\ &\quad - \frac{q^{-\frac{1}{2}}}{4}\frac{(q^{\frac{1}{2}};q^{\frac{1}{2}})^3_\infty}{(q;q)_\infty(q^2;q^2)_\infty} + \frac{q^{-\frac{1}{2}}}{4}\frac{(q;q)^8_\infty}{(q^{\frac{1}{2}};q^{\frac{1}{2}})^3_\infty(q^2;q^2)^4_\infty}, \end{aligned} \tag{2.52}$$

where g_2 is the universal mock theta function

$$g_2(\zeta;q) = \sum_{n=0}^{\infty}\frac{(-q)_n q^{\frac{n(n+1)}{2}}}{(\zeta)_{n+1}(\zeta^{-1}q)_{n+1}}, \tag{2.53}$$

we can express S_0 (S_1 respectively) in terms of T_0 (T_1) and eta quotients. In fact,

[2] This formula has a typo in the original paper.

we can express the linear combinations $S_0 + 2T_0$, $S_1 + 2T_1$ as

$$\begin{aligned} S_0(q) + 2T_0(q) &= \frac{1}{2}\frac{(q^{\frac{1}{2}};q^{\frac{1}{2}})^3_\infty}{(q;q)_\infty(q^2;q^2)_\infty} + \frac{1}{2}\frac{(q;q)^8_\infty}{(q^{\frac{1}{2}};q^{\frac{1}{2}})^3_\infty(q^2;q^2)^4_\infty}, \\ S_1(q) + 2T_1(q) &= -\frac{q^{-\frac{1}{2}}}{2}\frac{(q^{\frac{1}{2}};q^{\frac{1}{2}})^3_\infty}{(q;q)_\infty(q^2;q^2)_\infty} + \frac{q^{-\frac{1}{2}}}{2}\frac{(q;q)^8_\infty}{(q^{\frac{1}{2}};q^{\frac{1}{2}})^3_\infty(q^2;q^2)^4_\infty}, \end{aligned} \tag{2.54}$$

from which the conclusion since $\eta(\tau) = q^{\frac{1}{24}}(q;q)_\infty$. □

Using the previous relations we can rewrite the expressions for the components specifying the Umbral McKay–Thompson series for all conjugacy class of the Umbral group Dih_4 as

$$\begin{aligned} &(H^{(8)}_{1A,1} - H^{(8)}_{1A,7} - H^{(8)}_{1A,3} + H^{(8)}_{1A,5})(2\tau) = q^{-\frac{1}{16}}8T_0(q) + q^{\frac{7}{16}}8T_1(q) - 2\frac{\eta(\tau)^8}{\eta(\frac{\tau}{2})^3\eta(2\tau)^4}, \\ &(H^{(8)}_{2A,1} - H^{(8)}_{2A,7} - H^{(8)}_{2A,3} + H^{(8)}_{2A,5})(2\tau) = q^{-\frac{1}{16}}8T_0(q) + q^{\frac{7}{16}}8T_1(q) - 2\frac{\eta(\tau)^8}{\eta(\frac{\tau}{2})^3\eta(2\tau)^4}, \\ &(H^{(8)}_{2BC,1} - H^{(8)}_{2BC,7} - H^{(8)}_{2BC,3} + H^{(8)}_{2BC,5})(2\tau) = 4q^{-\frac{1}{16}}T_0(q) + 4q^{\frac{7}{16}}T_1(q) \\ &\qquad\qquad -2\frac{\eta(\tau)^8}{\eta(\frac{\tau}{2})^3\eta(2\tau)^4}, \\ &(H^{(8)}_{4A,1} - H^{(8)}_{4A,7} - H^{(8)}_{4A,3} + H^{(8)}_{4A,5})(2\tau) = -2\frac{\eta(\frac{\tau}{2})\eta(\tau)^4}{\eta(\tau)^2\eta(4\tau)^2}, \\ &H^{(8)}_{1A,2}(\tau) = H^{(8)}_{1A,6}(\tau) = 4q^{-\frac{1}{8}}A(q), \\ &H^{(8)}_{1A,4}(\tau) = 4q^{\frac{1}{2}}B(q). \end{aligned} \tag{2.55}$$

We can finally use the relations collected in appendix A to write all the appearing mock theta functions in terms of indefinite theta functions.

Proposition 2.3.0.1. *The expression specifying all the Mc-Kay Thompson series* *for $\ell = 8$ at all conjugacy classes of the umbral group Dih_4 are*

$$
\begin{aligned}
&(H^{(8)}_{1A,1} - H^{(8)}_{1A,7} - H^{(8)}_{1A,3} + H^{(8)}_{1A,5})(2\tau) = \\
&8e^{-\frac{3\pi i}{4}} \frac{\eta(4\tau)}{2\eta(2\tau)\eta(8\tau)} \left[\Theta_{(\frac{5}{8},\frac{1}{8}),(\frac{1}{2},0)}(8\tau) - i\Theta_{(\frac{7}{8},\frac{3}{8}),(\frac{1}{2},0)}(8\tau)\right] - 2\frac{\eta(\tau)^8}{\eta(\frac{\tau}{2})^3\eta(2\tau)^4}, \\
&(H^{(8)}_{2A,1} - H^{(8)}_{2A,7} - H^{(8)}_{2A,3} + H^{(8)}_{2A,5})(2\tau) = \\
&8e^{-\frac{3\pi i}{4}} \frac{\eta(4\tau)}{2\eta(2\tau)\eta(8\tau)} \left[\Theta_{(\frac{5}{8},\frac{1}{8}),(\frac{1}{2},0)}(8\tau) - i\Theta_{(\frac{7}{8},\frac{3}{8}),(\frac{1}{2},0)}(8\tau)\right] - 2\frac{\eta(\tau)^8}{\eta(\frac{\tau}{2})^3\eta(2\tau)^4}, \\
&(H^{(8)}_{2BC,1} - H^{(8)}_{2BC,7} - H^{(8)}_{2BC,3} + H^{(8)}_{2BC,5})(2\tau) = \\
&4e^{-\frac{3\pi i}{4}} \frac{\eta(4\tau)}{2\eta(2\tau)\eta(8\tau)} \left[\Theta_{(\frac{5}{8},\frac{1}{8}),(\frac{1}{2},0)}(8\tau) - i\Theta_{(\frac{7}{8},\frac{3}{8}),(\frac{1}{2},0)}(8\tau)\right] - 2\frac{\eta(\tau)^8}{\eta(\frac{\tau}{2})^3\eta(2\tau)^4}, \\
&(H^{(8)}_{4A,1} - H^{(8)}_{4A,7} - H^{(8)}_{4A,3} + H^{(8)}_{4A,5})(2\tau) = -2\frac{\eta(\frac{\tau}{2})\eta(2\tau)^4}{\eta(\tau)^2\eta(4\tau)^2}, \\
&H^{(8)}_{1A,2}(\tau) = H^{(8)}_{1A,6}(\tau) = 2e^{-\frac{3\pi i}{4}} \frac{\eta(4\tau)}{\eta(2\tau)^2}\Theta_{(\frac{3}{4},\frac{1}{4}),(0,\frac{1}{2})}(4\tau), \\
&H^{(8)}_{1A,4}(\tau) = 2e^{-\frac{3\pi i}{4}} \frac{\eta(2\tau)}{\eta(\tau)\eta(4\tau)}\Theta_{(\frac{3}{4},\frac{1}{2}),(0,\frac{1}{2})}(4\tau).
\end{aligned}
\tag{2.56}
$$

Note that for the same components $H^{(8)}_g, r$ with different conjugacy class g, the same indefinite theta function appears multiplied by a (possibly vanishing) prefactor. Thus the vertex algebra structure is invariant under the action of the umbral group.

2.3.2 Lambency Twelve

At lambency $\ell = 12$, we have Niemeier root system $A_{11}D_7E_6$ and umbral group $\mathbb{Z}_2$. The mock theta functions relevant in this case are the order 3

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}, \tag{2.57}$$

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}, \tag{2.58}$$

and the order 6

$$\sigma(q) := \sum_{n=0}^{\infty} \frac{q^{\frac{(n+1)(n+2)}{2}}(-q;q)_n}{(q;q^2)_{n+1}}, \tag{2.59}$$

$$\psi_6(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q)_{2n+1}}. \tag{2.60}$$

All the McKay–Thompson series for conjugacy class $2A$ are specified in terms of the ones for conjugacy class $1A$ by the pairing relation $H^{(12)}_{2A,r} + (-1)^r H^{(12)}_{1A,r} = 0$. In [18] we find the following identities in terms of mock theta functions

$$\begin{aligned} H^{(12)}_{1A,2}(\tau) = H^{(12)}_{1A,10}(\tau) &= -2q^{-\frac{4}{48}}\sigma(q), \\ H^{(12)}_{1A,4}(\tau) = H^{(12)}_{1A,8}(\tau) &= 2q^{\frac{2}{3}}\omega(q). \end{aligned} \tag{2.61}$$

The multiplicative relations between $\ell = 12$ and $\ell = 6$

$$\begin{aligned} (H^{(12)}_{1A,1} - H^{(12)}_{1A,11})(2\tau) &= H^{(6)}_{2B,1}(\tau), \\ (H^{(12)}_{1A,5} - H^{(12)}_{1A,7})(2\tau) &= H^{(6)}_{2B,5}(\tau), \\ (H^{(12)}_{1A,3} - H^{(12)}_{1A,9})(2\tau) &= H^{(6)}_{2B,3}(\tau), \end{aligned} \tag{2.62}$$

together with the multiplicative relations for $\ell = 6$

$$\begin{aligned} H^{(6)}_{2B,1}(3\tau) - H^{(6)}_{2B,3}(3\tau) + H^{(6)}_{2B,5}(3\tau) &= H^{(2)}_{6A,1}(\tau), \\ H^{(6)}_{2B,1}(2\tau) - H^{(6)}_{2B,5}(2\tau) &= H^{(3)}_{4C}(\tau), \end{aligned} \tag{2.63}$$

and the following further explicit expressions in terms of mock theta functions

$$\begin{aligned} H^{(3)}_{4C,1}(\tau) &= -2q^{-\frac{1}{12}}f(q^2), \\ H^{(6)}_{2B,3}(\tau) &= -2q^{-\frac{3}{8}}\psi_6(q), \end{aligned} \tag{2.64}$$

allow to specify all the components with r odd in terms of mock theta functions and the function $H^{(2)}_{6A,1}$ as

$$\begin{aligned} \left(H^{(12)}_{1A,1} - H^{(12)}_{1A,11}\right)(2\tau) &= \frac{1}{2}H^{(2)}_{6A,1}\left(\frac{\tau}{3}\right) - q^{-\frac{3}{8}}\psi_6(q) - q^{-\frac{1}{24}}f(q), \\ \left(H^{(12)}_{1A,5} - H^{(12)}_{1A,7}\right)(2\tau) &= \frac{1}{2}H^{(2)}_{6A,1}\left(\frac{\tau}{3}\right) - q^{-\frac{3}{8}}\psi_6(q) + q^{-\frac{1}{24}}f(q), \\ \left(H^{(12)}_{1A,3} - H^{(12)}_{1A,9}\right)(2\tau) &= H^{(6)}_{2B,3}(\tau) = -2q^{-\frac{3}{8}}\psi_6(q). \end{aligned} \tag{2.65}$$

Finally, the multiplicative relations with $\ell = 4$ give the component $r = 6$

$$H^{(12)}_{1A,6}(3\tau) = H^{(12)}_{1A,2}(3\tau) + H^{(12)}_{1A,10}(3\tau) - H^{(4)}_{3A,2}(\tau) = -4q^{\frac{1}{4}}\sigma(q^3) - H^{(4)}_{3A,2}(\tau). \quad (2.66)$$

For $H^{(4)}_{3A,2}$ and $H^{(2)}_{6A,1}$ a simple expression in terms of mock theta functions and/or eta quotients is not known, so we need to deal with them separately. It is convenient to write everything in terms of $\ell = 4$ functions by using the multiplicative relation

$$\left(H^{(4)}_{3A,1} - H^{(4)}_{3A,3}\right)(2\tau) = H^{(2)}_{6A,1}(\tau). \quad (2.67)$$

Components of the McKay–Thompson series at $\ell = 4$ for conjugacy class $3A$ are specified by different powers of $y = e^{2\pi iz}$ in [45]

$$\begin{aligned} 2i\theta_1(3\tau, 6z)\theta_1(z,\tau)^{-1}\theta_1(3\tau,3z)^{-1}\eta(\tau)^3 = &- 2\mu^0_{4,0}(z,\tau) - 2\mu^1_{4,0}(z,\tau) \\ &+ \sum_{r \mod 8} H^{(4)}_{3A,r}\theta_{4,r}(z,\tau) \end{aligned} \quad (2.68)$$

Where we have made use of the following functions

$$\begin{aligned} \theta_1(z,\tau) &:= -iq^{\frac{1}{8}}y^{\frac{1}{2}}\prod_{n>0}(1-y^{-1}q^{n-1})(1-yq^n)(1-q^n), \\ \theta_2(z,\tau) &:= q^{\frac{1}{8}}y^{\frac{1}{2}}\prod_{n>0}(1+y^{-1}q^{n-1})(1+yq^n)(1-q^n), \\ \theta_{m,r}(z,\tau) &:= \sum_{k\in\mathbb{Z}} y^{2mk+r}q^{\frac{(2mk+r)^2}{4m}}, \\ \mu^k_{m,0}(z,\tau) &:= \frac{1}{2}\left(\mu_{m,0}(z,\tau) + (-1)^k\mu_{m,0}\left(z,\tau+\frac{1}{2}\right)\right). \end{aligned} \quad (2.69)$$

We recall that the function $\mu_{m,0}(z,\tau)$, defined in (2.6), has an expression in terms of indefinite theta functions. In fact, for $\left|\frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\right| < 1$, $\mathrm{Im}(z) \neq 0$, setting $z = a\tau + b$ with $a \in \mathbb{Q}^*$, $|a| < 1$, $b \in \mathbb{R}$ we can use the result in equation (2.35) to write

$$\begin{aligned} \sum_{r \mod 8} H^{(4)}_{3A,r}(\tau)\theta_{4,r}(a\tau+b,\tau) &= -2\Theta^+_{A^{(4)},\mathbf{c}^{(4)}_1,\mathbf{c}^{(4)}_2}(a\tau+b,0;\tau) \\ &+ 2i\theta_1(6a\tau+6b,3\tau)\theta_1(a\tau+b,\tau)^{-1}\theta_1(3a\tau+3b,3\tau)^{-1}\eta(\tau)^3. \end{aligned} \quad (2.70)$$

with $A^{(m)} = \left(\begin{smallmatrix} 2m & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\mathbf{c}^{(m)}_1 = (0,1)$, $\mathbf{c}^{(m)}_2 = (-1, 2m)$. Notice also that equation (2.68) implies that $H^{(4)}_{3A,r}$ have even coefficients. We can thus rewrite the umbral McKay Thompson series in terms of indefinite theta functions using the relations in appendix A as follows.

Proposition 2.3.0.2. *The expression specifying all the McKay–Thompson series* *for $\ell = 12$ at all conjugacy classes of the umbral group $\mathbb{Z}_2$ are given by*

$$
\begin{aligned}
(H^{(12)}_{1A,1} - H^{(12)}_{1A,11})(2\tau) =& - e^{-\frac{7\pi i}{6}} \frac{\eta(\tau)\eta(6\tau)}{2\eta(2\tau)\eta(3\tau)^2}\Theta_{(\frac{1}{3},\frac{1}{2}),(\frac{1}{2},\frac{1}{2})}(3\tau) \\
&+ \frac{2e^{-\frac{5\pi i}{6}}}{\eta(\tau)}\Theta_{(\frac{2}{3},\frac{1}{6}),(\frac{1}{2},0)}(3\tau) \\
&- \frac{\eta(3\tau)^4}{\eta(\tau)\eta(6\tau)^2} + \frac{\left(H^{(4)}_{3A,1} - H^{(4)}_{3A,3}\right)}{2}\left(\frac{2}{3}\tau\right) \\
(H^{(12)}_{1A,5} - H^{(12)}_{1A,7})(2\tau) =& - e^{-\frac{7\pi i}{6}} \frac{\eta(\tau)\eta(6\tau)}{2\eta(2\tau)\eta(3\tau)^2}\Theta_{(\frac{1}{3},\frac{1}{2}),(\frac{1}{2},\frac{1}{2})}(3\tau) \\
&- \frac{2e^{-\frac{5\pi i}{6}}}{\eta(\tau)}\Theta_{(\frac{2}{3},\frac{1}{6}),(\frac{1}{2},0)}(3\tau) \\
&+ \frac{\eta(3\tau)^4}{\eta(\tau)\eta(6\tau)^2} + \frac{\left(H^{(4)}_{3A,1} - H^{(4)}_{3A,3}\right)}{2}\left(\frac{2}{3}\tau\right), \\
(H^{(12)}_{1A,3} - H^{(12)}_{1A,9})(2\tau) =& -e^{-\frac{7\pi i}{6}} \frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)^2}\Theta_{(\frac{1}{3},\frac{1}{2}),(\frac{1}{2},\frac{1}{2})}(3\tau), \\
H^{(12)}_{1A,2}(\tau) = H^{(12)}_{1A,10}(\tau) =& -2e^{-i\frac{\pi}{2}} \frac{\eta(2\tau)\eta(3\tau)}{2\eta(\tau)\eta(6\tau)^2}\Theta_{(\frac{1}{2},\frac{1}{6}),(0,\frac{1}{2})}(6\tau), \\
H^{(12)}_{1A,4}(\tau) = H^{(12)}_{1A,8}(\tau) =& \frac{2e^{-\frac{\pi i}{2}}}{\eta(\tau)}\Theta_{(\frac{1}{2},\frac{1}{3}),(0,\frac{1}{2})}(6\tau) + 2\frac{\eta(6\tau)^4}{\eta(2\tau)\eta(3\tau)^2}, \\
H^{(12)}_{1A,6}(3\tau) =& -2e^{\frac{-\pi i}{2}} \frac{\eta(6\tau)\eta(9\tau)}{\eta(3\tau)\eta(18\tau)^2}\Theta_{(\frac{1}{2},\frac{1}{6}),(0,\frac{1}{2})}(18\tau) - H^{(4)}_{3A,2}(\tau).
\end{aligned}
\tag{2.71}
$$

together with the pairing relation $H^{(12)}_{2A,r} + (-1)^r H^{(12)}_{1A,r} = 0$.

Again, we observe that the vertex algebra structure is invariant under the action of the umbral group in this case. The only difference between conjugacy class $1A$ and conjugacy class $2A$ is an overall minus sign thanks to the pairing relation.

2.3.3 Lambency Sixteen

At $\ell = 16$ we have Niemeier root system $A_{15}D_9$ and umbral group $\mathbb{Z}_2$. Again, all the McKay–Thompson series for conjugacy class $2A$ are related to the one for class $1A$ by the pairing relation $H^{(16)}_{2A,r} + (-1)^r H^{(16)}_{1A,r} = 0$. As a result, we only need to specify $H^{(16)}_{1A,r}$ explicitly. Using the expressions in [18] we can specify all the components of the Umbral McKay–Thompson series for class $1A$ in terms of order 8 mock thetas: $T_0(q)$ and $T_1(q)$ already defined in the previous section and

$$U_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^4,q^4)_n}, \tag{2.72}$$

$$V_0(q) := -1 + 2\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_{n+1}}, \tag{2.73}$$

$$V_1(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}}, \tag{2.74}$$

as

$$\begin{aligned} &H^{(16)}_{1A,2}(\tau) = H^{(16)}_{1A,14}(\tau) = 2q^{-\frac{1}{16}}T_0(-q),\\ &H^{(16)}_{1A,4}(\tau) = H^{(16)}_{1A,12}(\tau) = 2q^{-\frac{1}{4}}V_1(q),\\ &H^{(16)}_{1A,6}(\tau) = H^{(16)}_{1A,10}(\tau) = 2q^{\frac{7}{16}}T_1(-q),\\ &H^{(16)}_{1A,8}(\tau) = V_0(q),\\ &\sum_{n=0,7}(-1)^n H^{(16)}_{1A,2n+1}(8\tau) = H^{(2)}_{8A,1}(\tau) = -2q^{-\frac{1}{8}}U_0(q). \end{aligned} \tag{2.75}$$

Using the relations in appendix A we easily obtain

Proposition 2.3.0.3. *The expression specifying all the Mc-Kay Thompson series for $\ell = 16$ at all conjugacy classes of the umbral group $\mathbb{Z}_2$ are*

$$\begin{aligned} &H^{(16)}_{1A,2}\left(\tau-\frac{1}{2}\right) = H^{(16)}_{1A,14}\left(\tau-\frac{1}{2}\right) = 2e^{-\frac{3\pi i}{4}}\frac{\eta(4\tau)}{2\eta(2\tau)\eta(8\tau)}\Theta_{(\frac{5}{8},\frac{1}{8}),(\frac{1}{2},0)}(8\tau),\\ &H^{(16)}_{1A,4}(\tau) = H^{(16)}_{1A,12}(\tau) = 2ie^{-\frac{3\pi i}{8}}\frac{q^{-\frac{1}{16}}}{2\theta_1(-\tau,8\tau)}\Theta_{(\frac{3}{8},\frac{1}{4}),(0,\frac{1}{2})}(8\tau),\\ &H^{(16)}_{1A,6}\left(\tau-\frac{1}{2}\right) = H^{(16)}_{1A,10}\left(\tau-\frac{1}{2}\right) = 2e^{-\frac{5\pi i}{4}}\frac{\eta(4\tau)}{2\eta(2\tau)\eta(8\tau)}\Theta_{(\frac{7}{8},\frac{3}{8}),(\frac{1}{2},0)}(8\tau),\\ &H^{(16)}_{1A,8}(\tau) = -ie^{-\frac{\pi i}{8}}\frac{q^{-\frac{1}{16}}}{\theta_1(-\tau,8\tau)}\Theta_{(\frac{1}{8},\frac{1}{2}),(0,\frac{1}{2})}(8\tau) - \frac{\eta(2\tau)^3\eta(4\tau)}{\eta(\tau)^2\eta(8\tau)},\\ &\sum_{n=0,7}(-1)^n H^{(16)}_{1A,2n+1}(8\tau) = H^{(2)}_{8A,1}(\tau) = -2\frac{\eta(4\tau)}{2\eta(8\tau)^2}\Theta_{(\frac{1}{4},\frac{1}{4}),(0,0)}(4\tau) \end{aligned} \tag{2.76}$$

together with the pairing relation $H^{(16)}_{2A,r} + (-1)^r H^{(16)}_{1A,r} = 0$.

We observe that also in this case the indefinite thetas appearing in all components are invariant under the action of the umbral group.

Remark 2.3.1. The quantity $q^{\frac{1}{16}}\theta_1(-\tau, 8\tau)$ is modular under the congruence subgroup

$$\Gamma_1(8) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z}) : a, d = 1 \bmod 8, c = 0 \bmod 8 \right\} \tag{2.77}$$

generated by the transformations $T : \tau \to \tau + 1$, $\tilde{S} : \tau \to \frac{\tau}{8\tau+1}$, as is easy to see that from the transformation properties

$$\theta_1(z, \tau+1) = e^{\frac{\pi i}{4}}\theta_1(z,\tau), \quad \theta_1\left(\tfrac{z}{\tau}, -\tfrac{1}{\tau}\right) = -i\sqrt{-i\tau}e^{\frac{i\pi z^2}{\tau}}\theta_1(z,\tau). \tag{2.78}$$

2.4 Moonshine Modules

In this section we will build modules whose trace functions reproduce the specifying expressions for the McKay–Thompson series provided in the previous section for lambency $\ell = 8, 12, 16$. As mentioned in the previous section, in these cases we found that the the umbral groups act trivially on all the indefinite theta functions appearing in the McKay–Thompson series. Thus, we can construct modules that have the structure of a tensor product between an appropriate linear representation of the umbral group and a direct sum of vertex algebras modules on which the umbral group acts trivially. In the following, all the trace functions defined as in (2.19) are trace functions of modules of subalgebras of the vertex algebra associated to the two-dimensional lattice with the indefinite quadratic form $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.
We first start by introducing some vertex algebra modules that will appear in our construction, and then provide explicit expressions for the relevant umbral moonshine modules.

2.4.1 Heisenberg, Clifford and Weyl Characters

In this section we collect formulas for characters of (super) vertex algebras that will recover some of the functions appearing in the McKay–Thompson series specified in the previous section. Here we will follow the notation and definitions in [47], [49], [50] for the super vertex operator algebras and their modules.
The simplest character we will need is the character of the Heisenberg vertex operator algebra $\mathcal{H}$

$$\chi^H(\tau) := \mathrm{tr}_{\mathcal{H}}\left(q^{L(0)-\frac{c}{24}}\right) = \frac{1}{q^{\frac{1}{24}}\prod_{n>0}(1-q^n)} = \frac{1}{\eta(\tau)}. \tag{2.79}$$

Next, we will consider the graded characters of the irreducible canonically-twisted modules of the Clifford vertex operator algebra $A^{\pm}_{\text{tw}}$ [47]

$$\chi^{A^{\pm}}(\tau) := \mathrm{tr}_{A^{\pm}_{\text{tw}}}\left(p(0)q^{L(0)-\frac{c}{24}}\right) = \pm q^{\frac{1}{24}}\prod_{n>0}(1-q^n) = \pm\eta(\tau) \tag{2.80}$$

as well as the character of the (d-dimensional) Clifford super vertex operator algebra canonically-twisted module A_{tw} [49]

$$\chi^{A_{\text{tw}}}(z,\tau) := \mathrm{tr}_{A_{\text{tw}}}\left(y^{J(0)}q^{L(0)-\frac{d}{24}}\right) = y^{\frac{d}{4}}q^{\frac{d}{24}}\prod_{n>0}(1+y^{-1}q^{n-1})^{\frac{d}{2}}(1+yq^n)^{\frac{d}{2}}. \tag{2.81}$$

Finally, we will also make use of the canonically twisted d-dimensional Weyl modules V_{tw} [49]

$$\chi^{V_{\text{tw}}}(z,\tau) := \mathrm{tr}_{V_{\text{tw}}}(y^{J(0)}q^{L(0)-\frac{d}{24}}) = y^{-\frac{d}{4}}q^{-\frac{d}{24}}\prod_{n>0}(1-y^{-1}q^{n-1})^{-\frac{d}{2}}(1-yq^n)^{-\frac{d}{2}}. \tag{2.82}$$

Remark 2.4.1. The previous formula holds when each factor $(1-X)^{-1}$ is interpreted as $\sum_{n\geq 0} X^n$, which is possible in the domain $0 < -\mathrm{Im}(z) < \mathrm{Im}(\tau)$.

From now on, let's fix $d = 2$ since this is the case that will be needed the following subsections. In particular, for $d = 2$, we get the following relations with the Jacobi theta functions defined in (2.69)

$$\begin{aligned} \chi^{V_{\text{tw}}}(z,\tau) &= -i\frac{\eta(\tau)}{\theta_1(z,\tau)}, \\ \chi^{A_{\text{tw}}}\left(z+\frac{1}{2},\tau\right) &= -\frac{\theta_1(z,\tau)}{\eta(\tau)}. \end{aligned} \tag{2.83}$$

We will also need characters of 1-dimensional lattice vertex algebras. Let's consider the general 1-dimensional (even) lattice $L^1 := \{\alpha\epsilon : \alpha \in \mathbb{Z}\}$ with scalar product $\langle\epsilon,\epsilon\rangle = 2m$. Let's recall the operator g_h for $h := \epsilon \otimes h \in L^1 \otimes_{\mathbb{Z}} \mathbb{Q}$ defined in (2.12). We have

$$\chi_h^{L^1}(\tau) := \mathrm{Tr}_{V_{L^1}}(g_h q^{L_0-\frac{c}{24}}) = \frac{1}{\eta(\tau)}\sum_{n\in\mathbb{Z}} e^{4\pi i m h n}q^{mn^2}. \tag{2.84}$$

Furthermore, the characters of the modules $V_{L^1+\frac{r}{2m}}$, for $0 < r < 2m$

$$\chi_h^{L^1+\frac{r}{2m}}(\tau) := \mathrm{Tr}_{V_{L^1+\frac{r}{2m}}}(g_h q^{L_0-\frac{c}{24}}) = \frac{1}{\eta(\tau)}\sum_{n\in\mathbb{Z}} e^{2\pi i h(2mn+r)}q^{\frac{(2mn+r)^2}{4m}}. \tag{2.85}$$

give the theta functions $\theta_{m,r}$ defined in (2.69). Since they will appear frequently

Table 2.1: Character table of Dih_4

	1A	2A	2B	2C	4A
A_1	1	1	1	1	1
A_2	1	1	−1	−1	1
B_1	1	1	−1	1	−1
B_2	1	1	1	−1	−1
E	2	−2	0	0	0

later, let's give special names to the following characters of the vertex algebra V_{L^1} associated to the 1 dimensional lattice $L^1 := \{\alpha\epsilon : \alpha \in \mathbb{Z}\}$ with scalar product $\langle\epsilon,\epsilon\rangle = 2$, and the vertex algebra V_K associated to $K \subset L^1 = \{\alpha\epsilon : \alpha \in \mathbb{Z}_{\geq 0}\}$. Introducing the operator

$$g_{\frac{1}{4}}(p \otimes n\epsilon) = (-1)^n (p \otimes n\epsilon) \tag{2.86}$$

which corresponds to (2.12) with the choice $h = \frac{1}{4}\epsilon$, we define

$$\begin{aligned}
\chi^{L^1}(\tau) &:= \mathrm{Tr}_{V_{L^1}}\left(q^{L_0-\frac{c}{24}}\right) = \frac{1}{\eta(\tau)}\sum_{n\in\mathbb{Z}} q^{n^2},\\
\chi^{K}(\tau) &:= \mathrm{Tr}_{V_K}\left(q^{L_0-\frac{c}{24}}\right) = \frac{1}{\eta(\tau)}\sum_{n\geq 0} q^{n^2},\\
\tilde{\chi}^{K}(\tau) &:= \mathrm{Tr}_{V_K}\left(g_{\frac{1}{4}}q^{L_0-\frac{c}{24}}\right) = \frac{1}{\eta(\tau)}\sum_{n\geq 0}(-1)^n q^{n^2}.
\end{aligned} \tag{2.87}$$

2.4.2 Lambency Eight

The umbral group for lambency $\ell = 8$ is $G = Dih_4$. We will use the conventions for the names of conjugacy classes and irreducible representations that are specified in the character table 2.1. Using the results of the previous sections, we can specify the McKay–Thompson series for $\ell = 8$ in terms of characters of the VOAs introduced before. The even components can be directly rewritten as

$$\begin{aligned}
H^{(8)}_{g,2}(\tau) = H^{(8)}_{g,6}(\tau) &= 2\mathrm{tr}_{E_2}(g)\chi^{A^+}(4\tau)\chi^{A^+}(\tau)^2\chi^{H}(2\tau)^2 T^{(4)}_{\left(\frac{3}{4},\frac{1}{4}\right),\left(0,\frac{1}{2}\right)}(\tau),\\
H^{(8)}_{g,4}(\tau) &= 2\mathrm{tr}_{E_2}(g)\chi^{A^+}(2\tau)\chi^{H}(\tau)\chi^{H}(4\tau)\chi^{A^+}(\tau)^2 T^{(4)}_{\left(\frac{3}{4},\frac{1}{2}\right),\left(0,\frac{1}{2}\right)}(\tau),
\end{aligned} \tag{2.88}$$

while the odd components are specified by

$$
\begin{aligned}
&(H^{(8)}_{g,1} - H^{(8)}_{g,7} - H^{(8)}_{g,3} + H^{(8)}_{g,5})(2\tau) = \\
&2\mathrm{tr}_{2A_1\oplus B_1\oplus B_2}(g)\chi^{A^+}(4\tau)\chi^{H}(2\tau)\chi^{H}(8\tau)\chi^{A^+}(\tau) \\
&\times\left[\chi^{A^+}(\tau)T^{(8)}_{(\frac{5}{8},\frac{1}{8}),(\frac{1}{2},0)}(\tau) + \chi^{A^-}(\tau)T^{(8)}_{(\frac{7}{8},\frac{3}{8}),(\frac{1}{2},0)}(\tau)\right] \\
&+\left[\chi^{A^+}\left(\frac{\tau}{2}\right)tr_{A_1\oplus A_2}(g) + \chi^{A^-}\left(\frac{\tau}{2}\right)tr_{B_1\oplus B_2}(g)\right]\Bigg[\tilde{\chi}^{K}(\tau)\chi^{L^1}(\tau) + \\
&\chi^{A^-}\left(\frac{\tau}{2}\right)\chi^{A^+}\left(\frac{\tau}{2}\right)\chi^{\mathcal{H}}(\tau)^2\chi^{K}\left(\frac{\tau}{2}\right)\chi^{L^1}\left(\frac{\tau}{2}\right) + \chi^{\mathcal{H}}(\tau)\chi^{K}(\tau) + \\
&\chi^{A^+}\left(\frac{\tau}{2}\right)\chi^{\mathcal{H}}(\tau)^2\chi^{K}\left(\frac{\tau}{2}\right)\Bigg] + 2\mathrm{tr}_{A_1}(g)\chi^{A^-}(\tau)\chi^{A^+}(\tau)^7\chi^{\mathcal{H}}\left(\frac{\tau}{2}\right)^3\chi^{H}(2\tau)^4.
\end{aligned}
\tag{2.89}
$$

In rewriting the second addend we have used the following lemma so that the prefactor multiplying the characters is integer

Lemma 2.4.1.1.

$$
\begin{aligned}
&\frac{\eta(\frac{\tau}{2})\eta(2\tau)^4}{\eta(\tau)^2\eta(4\tau)^2} - \frac{\eta(\tau)^8}{\eta(\frac{\tau}{2})^3\eta(2\tau)^4} = \\
&2\chi^{A^+}\left(\frac{\tau}{2}\right)\Bigg[\tilde{\chi}^{K}(\tau)\chi^{L^1}(\tau) - \chi^{A^+}\left(\frac{\tau}{2}\right)^2\chi^{\mathcal{H}}(\tau)^2\chi^{K}\left(\frac{\tau}{2}\right)\chi^{L^1}\left(\frac{\tau}{2}\right) \\
&\qquad + \chi^{\mathcal{H}}(\tau)\chi^{K}(\tau) + \chi^{A^+}\left(\frac{\tau}{2}\right)\chi^{\mathcal{H}}(\tau)^2\chi^{K}\left(\frac{\tau}{2}\right)\Bigg]
\end{aligned}
\tag{2.90}
$$

Proof. Using the identities [57]

$$
\frac{\eta(2\tau)^5}{\eta(\tau)^2\eta(4\tau)^2} = \sum_{n\in\mathbb{Z}} q^{n^2} =: \theta^1(\tau), \quad \frac{\eta(\tau)^2}{\eta(2\tau)} = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}, \tag{2.91}
$$

we get

$$
\begin{aligned}
&\frac{\eta(\frac{\tau}{2})\eta(2\tau)^4}{\eta(\tau)^2\eta(4\tau)^2}-\frac{\eta(\tau)^8}{\eta(\frac{\tau}{2})^3\eta(2\tau)^4}=\frac{\eta\left(\frac{\tau}{2}\right)}{\eta(2\tau)}\frac{\eta(2\tau)^5}{\eta(\tau)^2\eta(4\tau)^2}-\frac{\eta\left(\frac{\tau}{2}\right)}{\eta(\tau)^2}\left(\frac{\eta(\tau)^5}{\eta\left(\frac{\tau}{2}\right)^2\eta(2\tau)^2}\right)^2\\
&=\frac{\eta\left(\frac{\tau}{2}\right)}{\eta(\tau)^2}\left[\frac{\eta(\tau)^2}{\eta(2\tau)}\theta_1(\tau)-\theta_1\left(\frac{\tau}{2}\right)^2\right]=\frac{\eta\left(\frac{\tau}{2}\right)}{\eta(\tau)^2}\left[\sum_{m,n\in\mathbb{Z}}(-1)^n q^{m^2+n^2}-q^{\frac{m^2+n^2}{2}}\right]\\
&=2\frac{\eta(\frac{\tau}{2})}{\eta(\tau)^2}\left[\sum_{\substack{n,m\in\mathbb{Z}\\ n\geq 0}}(-1)^n q^{m^2+n^2}-\sum_{\substack{n,m\in\mathbb{Z}\\ n\geq 0}}q^{\frac{m^2+n^2}{2}}-\sum_{n\geq 0}q^{n^2}+\sum_{n\geq 0}q^{\frac{n^2}{2}}\right]
\end{aligned}
\tag{2.92}
$$

and the conclusion follows easily using the expressions for the characters provided in (2.87). □

In order to specify the trace functions that will give us the relevant umbral McKay–Thompson series at $\ell = 8$, let's define the modules

$$
\begin{aligned}
\mathfrak{M}^{(8)}_{1,1} &:= {A^{+}_{tw}}^{\otimes 3}\otimes\mathcal{H}^{\otimes 2}\otimes V^{(8)}_{\left(\frac{5}{8},\frac{1}{8}\right)},\\
\mathfrak{M}^{(8)}_{1,2} &:= {A^{+}_{tw}}^{\otimes 2}\otimes A^{-}_{tw}\otimes\mathcal{H}^{\otimes 2}\otimes V^{(8)}_{\left(\frac{7}{8},\frac{3}{8}\right)},\\
\mathfrak{M}^{(8)}_{1,3} &:= A^{+}_{tw}\otimes K\otimes L^1,\\
\mathfrak{M}^{(8)}_{1,4} &:= {A^{+}_{tw}}^{\otimes 2}\otimes A^{-}_{tw}\otimes\mathcal{H}^{\otimes 2}\otimes K\otimes L^1,\\
\mathfrak{M}^{(8)}_{1,5} &:= A^{+}_{tw}\otimes\mathcal{H}\otimes K,\\
\mathfrak{M}^{(8)}_{1,6} &:= {A^{+}_{tw}}^{\otimes 2}\otimes\mathcal{H}^{\otimes 2}\otimes K,\\
\mathfrak{M}^{(8)}_{1,7} &:= A^{-}_{tw}\otimes K\otimes L^1,\\
\mathfrak{M}^{(8)}_{1,8} &:= {A^{-}_{tw}}^{\otimes 2}A^{+}_{tw}\otimes\mathcal{H}^{\otimes 2}\otimes K\otimes L^1\\
\mathfrak{M}^{(8)}_{1,9} &:= A^{-}_{tw}\otimes\mathcal{H}\otimes K,\\
\mathfrak{M}^{(8)}_{1,10} &:= A^{+}_{tw}\otimes A^{-}_{tw}\otimes\mathcal{H}^{\otimes 2}\otimes K,\\
\mathfrak{M}^{(8)}_{1,11} &:= A^{-}_{tw}\otimes {A^{+}_{tw}}^{\otimes 7}\otimes\mathcal{H}^{\otimes 7},\\
\mathfrak{M}^{(8)}_{2} &:= {A^{+}_{tw}}^{\otimes 3}\otimes\mathcal{H}^{\otimes 2}\otimes V^{(4)}_{\left(\frac{3}{4},\frac{1}{4}\right)},\\
\mathfrak{M}^{(8)}_{4} &:= {A^{+}_{tw}}^{\otimes 3}\otimes\mathcal{H}^{\otimes 2}\otimes V^{(4)}_{\left(\frac{3}{4},\frac{1}{2}\right)},
\end{aligned}
$$

and for each of them let's define the vectors

$$
\begin{aligned}
\omega_{1,1}^{(8)} &:= 2\hat{\omega}^{(1)} + \frac{1}{2}\hat{\omega}^{(2)} + \frac{1}{2}\hat{\omega}^{(3)} + \hat{\omega}^{(4)} + 4\hat{\omega}^{(5)} + \frac{1}{2}\hat{\omega}^{(6)},\\
\omega_{1,2}^{(8)} &:= 2\hat{\omega}^{(1)} + \frac{1}{2}\hat{\omega}^{(2)} + \frac{1}{2}\hat{\omega}^{(3)} + \hat{\omega}^{(4)} + 4\hat{\omega}^{(5)} + \frac{1}{2}\hat{\omega}^{(6)},\\
\omega_{1,3}^{(8)} &:= \frac{1}{4}\hat{\omega}^{(1)} + \frac{1}{2}\hat{\omega}^{(2)} + \frac{1}{2}\hat{\omega}^{(3)},\\
\omega_{1,4}^{(8)} &:= \frac{1}{4}\hat{\omega}^{(1)} + \frac{1}{4}\hat{\omega}^{(2)} + \frac{1}{4}\hat{\omega}^{(3)} + \frac{1}{2}\hat{\omega}^{(4)} + \frac{1}{2}\hat{\omega}^{(5)} + \frac{1}{4}\hat{\omega}^{(6)} + \frac{1}{4}\hat{\omega}^{(7)},\\
\omega_{1,5}^{(8)} &:= \frac{1}{4}\hat{\omega}^{(1)} + \frac{1}{4}\hat{\omega}^{(2)} + \frac{1}{4}\hat{\omega}^{(3)},\\
\omega_{1,6}^{(8)} &:= \frac{1}{4}\hat{\omega}^{(1)} + \frac{1}{4}\hat{\omega}^{(2)} + \frac{1}{2}\hat{\omega}^{(3)} + \frac{1}{2}\hat{\omega}^{(4)} + \frac{1}{4}\hat{\omega}^{(5)},\\
\omega_{1,7}^{(8)} &:= \frac{1}{4}\hat{\omega}^{(1)} + \frac{1}{2}\hat{\omega}^{(2)} + \frac{1}{2}\hat{\omega}^{(3)},\\
\omega_{1,8}^{(8)} &:= \frac{1}{4}\hat{\omega}^{(1)} + \frac{1}{4}\hat{\omega}^{(2)} + \frac{1}{4}\hat{\omega}^{(3)} + \frac{1}{2}\hat{\omega}^{(4)} + \frac{1}{2}\hat{\omega}^{(5)} + \frac{1}{4}\hat{\omega}^{(6)} + \frac{1}{4}\hat{\omega}^{(7)},\\
\omega_{1,9}^{(8)} &:= \frac{1}{4}\hat{\omega}^{(1)} + \frac{1}{2}\hat{\omega}^{(2)} + \frac{1}{2}\hat{\omega}^{(3)},\\
\omega_{1,10}^{(8)} &:= \frac{1}{4}\hat{\omega}^{(1)} + \frac{1}{4}\hat{\omega}^{(2)} + \frac{1}{2}\hat{\omega}^{(3)} + \frac{1}{2}\hat{\omega}^{(4)} + \frac{1}{4}\hat{\omega}^{(5)},\\
\omega_{1,11}^{(8)} &:= \frac{1}{2}\hat{\omega}^{(1)} + \frac{1}{2}\hat{\omega}^{(2)} + \frac{1}{2}\hat{\omega}^{(3)} + \frac{1}{2}\hat{\omega}^{(4)} + \frac{1}{2}\hat{\omega}^{(5)} + \frac{1}{2}\hat{\omega}^{(6)} + \frac{1}{2}\hat{\omega}^{(7)} + \frac{1}{2}\hat{\omega}^{(8)} +\\
&\quad \frac{1}{4}\hat{\omega}^{(9)} + \frac{1}{4}\hat{\omega}^{(10)} + \frac{1}{4}\hat{\omega}^{(11)} + \hat{\omega}^{(12)} + \hat{\omega}^{(13)} + \hat{\omega}^{(14)} + \hat{\omega}^{(15)},\\
\omega_{2}^{(8)} &:= 4\hat{\omega}^{(1)} + \hat{\omega}^{(2)} + \hat{\omega}^{(3)} + 2\hat{\omega}^{(4)} + 2\hat{\omega}^{(5)} + \hat{\omega}^{(6)},\\
\omega_{4}^{(8)} &:= 2\hat{\omega}^{(1)} + \hat{\omega}^{(2)} + \hat{\omega}^{(3)} + \hat{\omega}^{(4)} + 4\hat{\omega}^{(5)} + \hat{\omega}^{(6)},
\end{aligned}
$$

where, for brevity, we have written $\hat{\omega}^{(i)} = \mathbf{v} \otimes \cdots \otimes \left(\omega^{(i)} - \frac{c^{(i)}}{24}\mathbf{v}\right) \otimes \cdots \otimes \mathbf{v}$ to indicate the tensor product of vectors that at position i has the factor $\omega^{(i)} - \frac{c^{(i)}}{24}\mathbf{v}$, where ω and c are respectively the conformal vector and central charge of the module at the i-th position, and the remaining factors are the vacuum vectors $\mathbf{v}$ of the other modules. Let's consider the operators[3] $\hat{L}(0)$ corresponding to the 0-modes of the vertex operators associated to the previous vectors. With this notation we get

Theorem 2.4.2. *The umbral McKay–Thompson series at lambency $\ell = 8$ are*

[3]To make the notation lighter we will not write the indices in $\hat{L}$. It is understood that, for each module, $\hat{L}$ corresponds to the vector associated to the module.

specified by

$$
\begin{aligned}
&H^{(8)}_{g,2}(\tau) = H^{(8)}_{g,6}(\tau) = 2\mathrm{tr}_{E_2}(g)\mathrm{tr}_{\mathfrak{M}^{(8)}_2}\left(g_{(0,\frac{1}{2})}q^{\hat{L}(0)}\right),\\
&H^{(8)}_{g,4}(\tau) = 2\mathrm{tr}_{E_2}(g)\mathrm{tr}_{\mathfrak{M}^{(8)}_4}\left(g_{(0,\frac{1}{2})}q^{\hat{L}(0)}\right),\\
&(H^{(8)}_{g,1} - H^{(8)}_{g,7} - H^{(8)}_{g,3} + H^{(8)}_{g,5})(\tau) = 2\mathrm{tr}_{2A_1\oplus B_1\oplus B_2}(g)\mathrm{tr}_{\mathfrak{M}^{(8)}_{1,1}\oplus\mathfrak{M}^{(8)}_{1,2}}\left(g_{(\frac{1}{2},0)}q^{\hat{L}(0)}\right)\\
&+\mathrm{tr}_{A_1\oplus A_2}(g)\mathrm{tr}_{\mathfrak{M}^{(8)}_{1,3}}\left(g_{\frac{1}{4}}q^{\hat{L}(0)}\right) + \mathrm{tr}_{B_1\oplus B_2}(g)\mathrm{tr}_{\mathfrak{M}^{(8)}_{1,7}}\left(g_{\frac{1}{4}}q^{\hat{L}(0)}\right)\\
&+\mathrm{tr}_{A_1\oplus A_2}(g)\mathrm{tr}_{\mathfrak{M}^{(8)}_{1,4}\oplus\mathfrak{M}^{(8)}_{1,5}\oplus\mathfrak{M}^{(8)}_{1,6}}\left(q^{\hat{L}(0)}\right) + \mathrm{tr}_{B_1\oplus B_2}(g)\mathrm{tr}_{\mathfrak{M}^{(8)}_{1,8}\oplus\mathfrak{M}^{(8)}_{1,9}\oplus\mathfrak{M}^{(8)}_{1,10}}\left(q^{\hat{L}(0)}\right)\\
&+2\mathrm{tr}_{A_1}(g)\mathrm{tr}_{\mathfrak{M}^{(8)}_{1,11}}\left(q^{\hat{L}(0)}\right),
\end{aligned}
\tag{2.93}
$$

where $g_{\mathbf{b}}$ *acts as specified in* (2.18) *on the cone vertex algebra module in the tensor product and trivially on all the others. Analogously* $g_{\frac{1}{4}}$ *is specified by* (2.86) *and only acts non-trivially on the module* K*.*

2.4.3 Lambency Twelve

The umbral group corresponding to $\ell = 12$ is $\mathbb{Z}/2\mathbb{Z}$. There are only 2 irreducible representations, we will call A the trivial representation and B the sign representation.

We can specify the McKay–Thompson series in terms of characters of vertex algebras and $H^{(4)}$ functions. Let's write

$$e_4(\tau) = -H^{(4)}_{3A,2}(\tau) \tag{2.94}$$

$$o_4(\tau) = \left(\frac{H^{(4)}_{3A,1} - H^{(4)}_{3A,3}}{2}\right)\left(\frac{2}{3}\tau\right) \tag{2.95}$$

The odd components are specified by

$$\begin{aligned}
&\left(H^{(12)}_{g,1} - H^{(12)}_{g,11}\right)(2\tau) = \\
&\mathrm{tr}_A(g)\Bigg[\chi^{A^-}(\tau)\chi^{A^+}\left(\frac{\tau}{2}\right)^2\chi^{A^+}(6\tau)\chi^{\mathcal{H}}(2\tau)\chi^{\mathcal{H}}(3\tau)T^{(6)}_{(\frac{1}{3},\frac{1}{2}),(\frac{1}{2},\frac{1}{2})}\left(\frac{\tau}{2}\right) \\
&\qquad + 4\chi^{A^+}\left(\frac{\tau}{2}\right)^2\chi^{\mathcal{H}}(\tau)T^{(6)}_{(\frac{2}{3},\frac{1}{6}),(\frac{1}{2},0)}\left(\frac{\tau}{2}\right) \\
&\qquad + \chi^{A^+}(3\tau)^2\chi^{A^-}(3\tau)\chi^{\mathcal{H}}(\tau)\chi^{\mathcal{H}}(6\tau)^2 + o_4(\tau)\Bigg], \\
&\left(H^{(12)}_{g,5} - H^{(12)}_{g,7}\right)(2\tau) = \\
&\mathrm{tr}_A(g)\Bigg[\chi^{A^-}(\tau)\chi^{A^+}\left(\frac{\tau}{2}\right)^2\chi^{A^+}(6\tau)\chi^{\mathcal{H}}(2\tau)\chi^{\mathcal{H}}(3\tau)T^{(6)}_{(\frac{1}{3},\frac{1}{2}),(\frac{1}{2},\frac{1}{2})}\left(\frac{\tau}{2}\right) \\
&\qquad + 4\chi^{A^-}\left(\frac{\tau}{2}\right)\chi^{A^+}\left(\frac{\tau}{2}\right)\chi^{\mathcal{H}}(\tau)T^{(6)}_{(\frac{2}{3},\frac{1}{6}),(\frac{1}{2},0)}\left(\frac{\tau}{2}\right) \\
&\qquad + \chi^{A^+}(3\tau)^2\chi^{A^+}(3\tau)\chi^{\mathcal{H}}(\tau)\chi^{\mathcal{H}}(6\tau)^2 + o_4(\tau)\Bigg], \\
&\left(H^{(12)}_{g,3} - H^{(12)}_{g,9}\right)(2\tau) = \\
&2\mathrm{tr}_A(g)\chi^{A^-}(\tau)\chi^{A^+}\left(\frac{\tau}{2}\right)\chi^{A^+}(6\tau)\chi^{\mathcal{H}}(2\tau)\chi^{\mathcal{H}}(3\tau)^2T^{(6)}_{(\frac{1}{3},\frac{1}{2}),(\frac{1}{2},\frac{1}{2})}\left(\frac{\tau}{2}\right).
\end{aligned} \tag{2.96}$$

The even components are instead given by

$$\begin{aligned}
&H^{(12)}_{g,2}(\tau) = H^{(12)}_{g,10}(\tau) = 2tr_B(g)\chi^{\mathcal{H}}(6\tau)^2\chi^{A^-}(\tau)\chi^{A^+}(2\tau)\chi^{A^+}(3\tau)T^{(6)}_{(\frac{1}{2},\frac{1}{6}),(0,\frac{1}{2})}(\tau), \\
&H^{(12)}_{g,4}(\tau) = H^{(12)}_{g,8}(\tau) = \\
&4\mathrm{tr}_B(g)\left[\chi^{A^+}(\tau)T^{(6)}_{(\frac{1}{2},\frac{1}{3}),(0,\frac{1}{2})}(\tau) + 2\chi^{A^+}(6\tau)^4\chi^{\mathcal{H}}(2\tau)\chi^{\mathcal{H}}(3\tau)^2\right], \\
&H^{(12)}_{g,6}(3\tau) = \\
&\mathrm{tr}_B(g)\left[4\chi^{A^-}(6\tau)\chi^{A^+}(9\tau)\chi^{A^+}(\tau)^2\chi^{\mathcal{H}}(3\tau)\chi^{\mathcal{H}}(18\tau)^2T^{(18)}_{(\frac{1}{2},\frac{1}{6}),(0,\frac{1}{2})}(\tau) + e_4(\tau)\right].
\end{aligned} \tag{2.97}$$

We define the modules

$$\begin{aligned}
\mathfrak{M}_{1,1}^{(12)} &:= A_{tw}^{-} \otimes {A_{tw}^{+}}^{\otimes 3} \otimes \mathcal{H}^{\otimes 2} \otimes V^{(6)}_{\left(\frac{1}{3},\frac{1}{2}\right)},\\
\mathfrak{M}_{1,2}^{(12)} &:= {A_{tw}^{+}}^{\otimes 2} \otimes \mathcal{H} \otimes V^{(6)}_{\left(\frac{2}{3},\frac{1}{6}\right)},\\
\mathfrak{M}_{1,3}^{(12)} &:= {A_{tw}^{+}}^{\otimes 2} \otimes A_{tw}^{-} \otimes \mathcal{H}^{\otimes 3},\\
\mathfrak{M}_{2}^{(12)} &:= A_{tw}^{-} \otimes {A_{tw}^{+}}^{\otimes 2} \otimes \mathcal{H}^{\otimes 2} \otimes V^{(6)}_{\left(\frac{1}{2},\frac{1}{6}\right)},\\
\mathfrak{M}_{3}^{(12)} &:= A_{tw}^{-} \otimes {A_{tw}^{+}}^{\otimes 2} \otimes \mathcal{H}^{\otimes 3} \otimes V^{(6)}_{\left(\frac{1}{3},\frac{1}{2}\right)},\\
\mathfrak{M}_{4,1}^{(12)} &:= A_{tw}^{+} \otimes V^{(6)}_{\left(\frac{1}{2},\frac{1}{3}\right)},\\
\mathfrak{M}_{4,2}^{(12)} &:= {A_{tw}^{+}}^{\otimes 4} \otimes \mathcal{H}^{\otimes 3}\\
\mathfrak{M}_{5,1}^{(12)} &:= A_{tw}^{-} \otimes {A_{tw}^{+}}^{\otimes 3} \otimes \mathcal{H}^{\otimes 2} \otimes V^{(6)}_{\left(\frac{1}{3},\frac{1}{2}\right)},\\
\mathfrak{M}_{5,2}^{(12)} &:= A_{tw}^{-} \otimes A_{tw}^{+} \otimes \mathcal{H} \otimes V^{(6)}_{\left(\frac{2}{3},\frac{1}{6}\right)}\\
\mathfrak{M}_{5,3}^{(12)} &:= {A_{tw}^{+}}^{\otimes 3} \otimes \mathcal{H}^{\otimes 3}\\
\mathfrak{M}_{6}^{(12)} &:= A_{tw}^{-} \otimes {A_{tw}^{+}}^{\otimes 3} \otimes \mathcal{H}^{\otimes 3} \otimes V^{(18)}_{\left(\frac{1}{2},\frac{1}{6}\right)},
\end{aligned}$$

and, to account for the different coefficients in front of τ, the vectors

$$\begin{aligned}
\omega_{1,1}^{(12)} &:= \frac{1}{2}\hat{\omega}^{(1)} + \frac{1}{4}\hat{\omega}^{(2)} + \frac{1}{2}\hat{\omega}^{(3)} + 3\hat{\omega}^{(4)} + \hat{\omega}^{(5)} + \frac{3}{2}\hat{\omega}^{(6)} + \frac{1}{4}\hat{\omega}^{(7)},\\
\omega_{1,2}^{(12)} &:= \frac{1}{4}\hat{\omega}^{(1)} + \frac{1}{4}\hat{\omega}^{(2)} + \frac{1}{2}\hat{\omega}^{(3)} + \frac{1}{4}\hat{\omega}^{(4)},\\
\omega_{1,3}^{(12)} &:= \frac{3}{2}\hat{\omega}^{(1)} + \frac{3}{2}\hat{\omega}^{(2)} + \frac{3}{2}\hat{\omega}^{(3)} + \frac{1}{2}\hat{\omega}^{(4)} + 3\hat{\omega}^{(5)} + 3\hat{\omega}^{(6)},\\
\omega_{2}^{(12)} &:= \hat{\omega}^{(1)} + 2\hat{\omega}^{(2)} + 3\hat{\omega}^{(3)} + 6\hat{\omega}^{(4)} + 6\hat{\omega}^{(5)} + \hat{\omega}^{(6)},\\
\omega_{3}^{(12)} &:= \frac{1}{2}\hat{\omega}^{(1)} + \frac{1}{4}\hat{\omega}^{(2)} + 3\hat{\omega}^{(3)} + \hat{\omega}^{(4)} + \frac{3}{2}\hat{\omega}^{(5)} + \frac{3}{2}\hat{\omega}^{(6)} + \frac{1}{4}\hat{\omega}^{(7)},\\
\omega_{4,1}^{(12)} &:= \hat{\omega}^{(1)} + \hat{\omega}^{(2)},\\
\omega_{4,2}^{(12)} &:= 6\hat{\omega}^{(1)} + 6\hat{\omega}^{(2)} + 6\hat{\omega}^{(3)} + 6\hat{\omega}^{(4)} + 2\hat{\omega}^{(5)} + 3\hat{\omega}^{(6)} + 3\hat{\omega}^{(7)},\\
\omega_{5,1}^{(12)} &:= \frac{1}{2}\hat{\omega}^{(1)} + \frac{1}{4}\hat{\omega}^{(2)} + \frac{1}{4}\hat{\omega}^{(3)} + 3\hat{\omega}^{(4)} + \hat{\omega}^{(5)} + \frac{3}{2}\hat{\omega}^{(6)} + \frac{1}{4}\hat{\omega}^{(7)},\\
\omega_{5,2}^{(12)} &:= \frac{1}{4}\hat{\omega}^{(1)} + \frac{1}{4}\hat{\omega}^{(2)} + \frac{1}{2}\hat{\omega}^{(3)} + \frac{1}{4}\hat{\omega}^{(4)}\\
\omega_{5,3}^{(12)} &:= \frac{3}{2}\hat{\omega}^{(1)} + \frac{3}{2}\hat{\omega}^{(2)} + \frac{3}{2}\hat{\omega}^{(3)} + \frac{1}{2}\hat{\omega}^{(4)} + 3\hat{\omega}^{(5)} + 3\hat{\omega}^{(6)},\\
\omega_{6}^{(12)} &:= 2\hat{\omega}^{(1)} + 3\hat{\omega}^{(2)} + \frac{1}{3}\hat{\omega}^{(3)} + \frac{1}{3}\hat{\omega}^{(4)} + 1\hat{\omega}^{(5)} + 6\hat{\omega}^{(6)} + 6\hat{\omega}^{(7)} + \frac{1}{3}\hat{\omega}^{(8)},
\end{aligned}$$

where again we have written $\hat{\omega}^{(i)} = \mathbf{v}\otimes\cdots\otimes\left(\omega^{(i)} - \frac{c^{(i)}}{24}\mathbf{v}\right)\otimes\cdots\otimes\mathbf{v}$. As before we write $\hat{L}(0)$ to indicate the 0-mode of the vertex operators associated to the previous vectors. We also need modules for $e_4(\tau)$ and $o_4(\tau)$. It is possible to specify these modules implicitly by making use of equation (2.70). In fact, using Corollary 2.2.1.2 we can rewrite $\mu_{m,0}(z,\tau)$ in terms of characters of cone vertex algebras and 1-dimensional lattice vertex algebras. Furthermore, the theta functions $\theta_{m,r}$ also admits expressions in terms of trace functions of 1d lattice vertex algebras as described in section 2.4.1. It remains to find a module for the meromorphic Jacobi form

$$\psi_{3A}^{(4)}(z,\tau) := 2i\theta_1(6z,3\tau)\theta_1(z,\tau)^{-1}\theta_1(3z,3\tau)^{-1}\eta(\tau)^3 \tag{2.98}$$

featuring in equation (2.68). Notice that constructing modules for these meromorphic functions is what is referred to as the "meromorphic module problem" in [49]. It is easy to see that (2.98) also admits an expression in terms of characters of the modules discussed in 2.4.1. In fact we have, for $0 < -\mathrm{Im}(z) < \mathrm{Im}(\tau)$,

$$\psi_{3A}^{(4)}(z,\tau) = 2i\chi^{A^+}(\tau)\chi^{A^-}(\tau)\chi^{A_{tw}}\left(6z+\frac{1}{2},3\tau\right)\chi^{V_{tw}}(z,\tau)\chi^{V_{tw}}(3z,3\tau). \tag{2.99}$$

Using the relations $H^{(4)}_{3A,r}(\tau) = -H^{(4)}_{3A,-r}(\tau)$, and $\theta_{m,r}(z,\tau) = \theta_{m,-r}(z,\tau)$, we can give a prescription for the construction of modules[4] for $H^{(4)}_{3A,r}$ starting from equation (2.70). In fact, we can write Θ^+ as

$$\Theta^+_{A^{(4)},\mathbf{c}_1^{(4)},\mathbf{c}_2^{(4)}}(a\tau+b,0;\tau) = 2\sum_{(n_1,n_2)\in\mathcal{C}} (-1)^{s(n_1,n_2)} y^{8n_1+n_2} q^{4n_1^2+n_1n_2} - \sum_{n\in\mathbb{Z}} y^{8n} q^{4n^2} \tag{2.101}$$

where $\mathcal{C}$ is the cone

$$\mathcal{C} := \left\{(n_1,n_2)\in\mathbb{Z}^2 : n_1\geq 0, n_2\geq 0\right\} \cup \left\{(n_1,n_2)\in\mathbb{Z}^2 : n_1<0, n_2<0\right\} \tag{2.102}$$

and s corresponds to the sign automorphism

$$s(n_1,n_2) := \begin{cases} 1 & \text{if } n_1\geq 0, n_2\geq 0, \\ -1 & \text{if } n_1<0, n_2<0. \end{cases} \tag{2.103}$$

The vector space interpretation of the indefinite theta function (2.101), the vertex algebra interpretation of $\psi^{(4)}_{3A}$ (2.99), together with (2.70) give a definition of a bi-graded vector space $\mathcal{H} = \bigoplus_{n,l} \mathcal{H}_{n,l}$ with an additional $\mathbb{Z}_2$-grading, that satisfies

$$\sum_{r=1}^{3} \frac{H^{(4)}_{3A,r}(\tau)}{2} \left[\theta_{4,r}(z,\tau) - \theta_{4,r}(-z,\tau)\right] = \sum_{n,l} \mathrm{sdim}(\mathcal{H}_{n,l}) q^n y^l \tag{2.104}$$

where sdim stands for the super dimension that takes the $\mathbb{Z}_2$ grading into account by including additional sign factors. We now define the operators $\tilde{L}_0$ and $\tilde{J}_0$ acting as $\tilde{L}_0\mathbf{v} = n\mathbf{v}$, $\tilde{J}_0\mathbf{v} = l\mathbf{v}$ $\forall \mathbf{v}\in\mathcal{H}_{n,l}$. We can thus define a supertrace on $\mathcal{H}$ through

$$\mathrm{sTr}_{\mathcal{H}} q^{\tilde{L}_0} y^{\tilde{J}_0} := \sum_{n,l} \mathrm{sdim}(\mathcal{H}_{n,l}) q^n y^l. \tag{2.105}$$

---

[4]We can also express modules for $H^{(4)}$ implicitly in terms of vertex algebra modules by writing, for $z = a\tau + b$ with $a\in\mathbb{Q}^*$, $|a|<1$, $b\in\mathbb{R}$

$$\begin{aligned}\sum_{r=1}^{3} H^{(4)}_{3A,r}(\tau)\left[\theta_{4,r}(z,\tau) - \theta_{4,r}(-z,\tau)\right] &= -4e^{-16\pi i b} q^{-2ma^2} \chi^{\mathcal{H}}(\tau)^2 \tilde{T}^{(1)}_{\mathbf{a},\mathbf{b}}(\tau) \\ &+ 2i\chi^{A^+}(\tau)\chi^{A^-}(\tau)\chi^{A_{tw}}\left(6z+\frac{1}{2},3\tau\right)\chi^{V_{tw}}(z,\tau)\chi^{V_{tw}}(3z,3\tau) - 2\chi^{L^1+a}_{\frac{b}{16}}(\tau)\chi^{H}(\tau)\end{aligned} \tag{2.100}$$

where $\mathbf{a} = (1+a,0)$, $\mathbf{b} = (b,0)$ and we have written $\tilde{T}_{\mathbf{a},\mathbf{b}}$ to indicate the cone vertex algebra trace function with quadratic form $\tilde{A} = \left(\begin{smallmatrix} 8 & 1 \\ 1 & 0\end{smallmatrix}\right)$ in order to distinguish it from the trace functions with respect to $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0\end{smallmatrix}\right)$. Thus the McKay–Thompson series $H^{(4)}_{3A,r}$ are specified by the different y-powers in the right hand side of (2.100). Notice that the z dependence influences, through a, which cone vertex algebra and one dimensional lattice modules will appear in the right hand side of (2.100)

Noticing that

$$[\theta_{4,r}(z,\tau)-\theta_{4,r}(-z,\tau)]=\sum_{k\in\mathbb{Z}}\left(y^{4k+r}-y^{-(4k+r)}\right)q^{\frac{(4k+r)^2}{16}} \tag{2.106}$$

we can specify $H^{(4)}_{3A,r}$ for $r=1,2,3$ with the previous notation through

$$\frac{H^{(4)}_{3A,r}(\tau)}{2}=\mathrm{sTr}_{\tilde{\mathcal{H}}_r}q^{\tilde{L}_0-\left(\frac{\tilde{J}_0}{4}\right)^2} \tag{2.107}$$

where

$$\mathcal{H}_r=\bigoplus_n \mathcal{H}_{n,l=r}. \tag{2.108}$$

With this notation, we can rewrite the functions (2.94), and (2.95) as

$$e_4(\tau)=-2\mathrm{sTr}_{\tilde{\mathcal{H}}_2}q^{\tilde{L}_0-\left(\frac{\tilde{J}_0}{4}\right)^2}, \tag{2.109}$$

$$o_4\left(\frac{3}{2}\tau\right)=\mathrm{sTr}_{\tilde{\mathcal{H}}_1}q^{\tilde{L}_0-\left(\frac{\tilde{J}_0}{4}\right)^2}-\mathrm{sTr}_{\tilde{\mathcal{H}}_3}q^{\tilde{L}_0-\left(\frac{\tilde{J}_0}{4}\right)^2}. \tag{2.110}$$

We thus have

Theorem 2.4.3. *The umbral McKay–Thompson series at lambency $\ell=12$ are specified by*

$$\begin{aligned}
\left(H^{(12)}_{g,1}-H^{(12)}_{g,11}\right)(\tau)&=\mathrm{tr}_A(g)\Bigg[\mathrm{tr}_{\mathfrak{M}^{(12)}_{1,1}}\left(g_{\left(\frac{1}{2},\frac{1}{2}\right)}q^{\hat{L}(0)}\right)+4\mathrm{tr}_{\mathfrak{M}^{(12)}_{1,2}}\left(g_{\left(\frac{1}{2},0\right)}q^{\hat{L}(0)}\right)\\
&\qquad+\mathrm{tr}_{\mathfrak{M}^{(12)}_{1,3}}\left(q^{\hat{L}(0)}\right)\Bigg]+\mathrm{tr}_A(g)o_4\left(\frac{\tau}{2}\right),\\
\left(H^{(12)}_{g,3}-H^{(12)}_{g,9}\right)(\tau)&=2\mathrm{tr}_A(g)tr_{\mathfrak{M}^{(12)}_{3}}\left(g_{\left(\frac{1}{2},\frac{1}{2}\right)}q^{\hat{L}(0)}\right),\\
\left(H^{(12)}_{g,5}-H^{(12)}_{g,7}\right)(\tau)&=\mathrm{tr}_A(g)\Bigg[\mathrm{tr}_{\mathfrak{M}^{(12)}_{5,1}}\left(g_{\left(\frac{1}{2},\frac{1}{2}\right)}q^{\hat{L}(0)}\right)+4\mathrm{tr}_{\mathfrak{M}^{(12)}_{5,2}}\left(g_{\left(\frac{1}{2},0\right)}q^{\hat{L}(0)}\right)\\
&\qquad+\mathrm{tr}_{\mathfrak{M}^{(12)}_{5,3}}\left(q^{\hat{L}(0)}\right)\Bigg]+\mathrm{tr}_A(g)o_4\left(\frac{\tau}{2}\right),\\
H^{(12)}_{g,2}(\tau)=H^{(12)}_{g,10}(\tau)&=2\mathrm{tr}_B(g)\mathrm{tr}_{\mathfrak{M}^{(12)}_{2}}\left(g_{\left(0,\frac{1}{2}\right)}q^{\hat{L}(0)}\right),\\
H^{(12)}_{g,4}(\tau)=H^{(12)}_{g,8}(\tau)&=4\mathrm{tr}_B(g)\left[\mathrm{tr}_{\mathfrak{M}^{(12)}_{4,1}}\left(g_{\left(0,\frac{1}{2}\right)}q^{\hat{L}(0)}\right)+2\mathrm{tr}_{\mathfrak{M}^{(12)}_{4,2}}\left(q^{\hat{L}(0)}\right)\right],\\
H^{(12)}_{g,6}(\tau)&=4\mathrm{tr}_B(g)\mathrm{tr}_{\mathfrak{M}^{(12)}_{6}}\left(g_{\left(0,\frac{1}{2}\right)}q^{\hat{L}(0)}\right)+\mathrm{tr}_B(g)e_4\left(\frac{\tau}{3}\right),
\end{aligned} \tag{2.111}$$

where $g_{\mathfrak{b}}$ acts as specified in (2.18) *on the cone vertex algebra module in the tensor product and trivially on all the others.*

2.4.4 Lambency Sixteen

The umbral group is $G = \mathbb{Z}/2\mathbb{Z}$. Using the same notation as before for the irreducible representations, we can write all the McKay–Thomposon series in terms of characters as

$$
\begin{aligned}
&H^{(16)}_{g,2}\left(\tau-\frac{1}{2}\right) = H^{(16)}_{g,14}\left(\tau-\frac{1}{2}\right) = \\
&2\mathrm{tr}_B(g)\chi^{A^+}(4\tau)\chi^{A^+}(\tau)^2\chi^{H}(2\tau)\chi^{H}(8\tau)T^{(8)}_{(\frac{5}{8},\frac{1}{8}),(\frac{1}{2},0)}(\tau),\\
&H^{(16)}_{g,4}(\tau) = H^{(16)}_{g,12}(\tau) = 2\mathrm{tr}_B(g)q^{-\frac{1}{16}}\chi^{A^+}(\tau)\chi^{V}(8\tau,-\tau)T^{(8)}_{(\frac{3}{8},\frac{1}{4}),(0,\frac{1}{2})}(\tau),\\
&H^{(16)}_{g,6}\left(\tau-\frac{1}{2}\right) = H^{(16)}_{g,10}\left(\tau-\frac{1}{2}\right) = \\
&2\mathrm{tr}_B(g)\chi^{A^+}(4\tau)\chi^{A^+}(\tau)^2\chi^{H}(2\tau)\chi^{H}(8\tau)T^{(8)}_{(\frac{7}{8},\frac{3}{8}),(\frac{1}{2},0)}(\tau),\\
&H^{(16)}_{g,8}(\tau) = \mathrm{tr}_B(g)\Bigg(2q^{-\frac{1}{16}}\chi^{A^+}(\tau)\chi^{V}(8\tau,-\tau)T^{(8)}_{(\frac{1}{8},\frac{1}{2}),(0,\frac{1}{2})}(\tau)\\
&\qquad + \chi^{A^+}(2\tau)^3\chi^{A^-}(4\tau)\chi^{H}(\tau)^2\chi^{H}(8\tau)\Bigg),\\
&\sum_{n=0,7}(-1)^n H^{(16)}_{g,2n+1}(8\tau) = 2\mathrm{tr}_A(g)\chi^{A^+}(4\tau)\chi^{A^+}(\tau)^2\chi^{H}(8\tau)T^{(8)}_{(\frac{1}{4},\frac{1}{4}),(0,0)}(\tau).
\end{aligned}
\tag{2.112}
$$

Let's now consider the following tensor products of modules

$$
\begin{aligned}
\mathfrak{M}^{(16)}_1 &:= {A^+_{tw}}^{\otimes 3}\otimes\mathcal{H}\otimes V^{(8)}_{(\frac{1}{4},\frac{1}{4})},\\
\mathfrak{M}^{(16)}_2 &:= {A^+_{tw}}^{\otimes 3}\otimes\mathcal{H}^{\otimes 2}\otimes V^{(8)}_{(\frac{5}{8},\frac{1}{8})},\\
\mathfrak{M}^{(16)}_4 &:= A^+_{tw}\otimes V_{tw}\otimes V^{(8)}_{(\frac{3}{8},\frac{1}{4})}\\
\mathfrak{M}^{(16)}_6 &:= {A^+_{tw}}^{\otimes 3}\otimes\mathcal{H}^{\otimes 2}\otimes V^{(8)}_{(\frac{7}{8},\frac{3}{8})},\\
\mathfrak{M}^{(16)}_{8,1} &:= A^+_{tw}\otimes V_{tw}\otimes V^{(8)}_{(\frac{1}{8},\frac{1}{2})},\\
\mathfrak{M}^{(16)}_{8,2} &:= {A^+_{tw}}^{\otimes 3}\otimes A^-_{tw}\otimes\mathcal{H}^{\otimes 3},
\end{aligned}
$$

and the respective vectors

$$\begin{aligned}
\omega_1^{(16)} &:= \frac{1}{2}\hat{\omega}^{(1)} + \frac{1}{8}\hat{\omega}^{(2)} + \frac{1}{8}\hat{\omega}^{(3)} + \hat{\omega}^{(4)} + \frac{1}{8}\hat{\omega}^{(5)},\\
\omega_2^{(16)} &:= 4\hat{\omega}^{(1)} + \hat{\omega}^{(2)} + \hat{\omega}^{(3)} + 2\hat{\omega}^{(4)} + 8\hat{\omega}^{(5)} + \hat{\omega}^{(6)},\\
\omega_4^{(16)} &:= \hat{\omega}^{(1)} + 8\hat{\omega}^{(2)} + \hat{\omega}^{(3)},\\
\omega_6^{(16)} &:= 4\hat{\omega}^{(1)} + \hat{\omega}^{(2)} + \hat{\omega}^{(3)} + 8\hat{\omega}^{(4)} + \hat{\omega}^{(5)},\\
\omega_{8,1}^{(16)} &:= \hat{\omega}^{(1)} + 8\hat{\omega}^{(2)} + \hat{\omega}^{(3)},\\
\omega_{8,2}^{(16)} &:= 2\hat{\omega}^{(1)} + 2\hat{\omega}^{(2)} + 2\hat{\omega}^{(3)} + 4\hat{\omega}^{(4)} + \hat{\omega}^{(5)} + \hat{\omega}^{(6)} + 8\hat{\omega}^{(7)},
\end{aligned}$$

using the same notation as before. Defining $\hat{L}(0)$ as usual we get

Theorem 2.4.4. *The umbral McKay–Thompson series at lambency $\ell = 16$ are specified by*

$$\begin{aligned}
&H_{g,2}(\tau) = H_{g,14}(\tau) = 2\mathrm{tr}_B(g)\mathrm{tr}_{\mathfrak{M}_2^{(16)}}\left(g_{\left(\frac{1}{2},0\right)}e^{\pi i\hat{L}(0)}q^{\hat{L}(0)}\right),\\
&H_{g,4}(\tau) = H_{g,12}(\tau) = 2q^{-\frac{1}{16}}\mathrm{tr}_B(g)\mathrm{tr}_{\mathfrak{M}_4^{(16)}}\left(g_{\left(0,\frac{1}{2}\right)}q^{-J(0)}q^{\hat{L}(0)}\right),\\
&H_{g,6}(\tau) = H_{g,10}(\tau) = 2\mathrm{tr}_B(g)\mathrm{tr}_{\mathfrak{M}_6^{(16)}}\left(g_{\left(\frac{1}{2},0\right)}e^{\pi i\hat{L}(0)}q^{\hat{L}(0)}\right),\\
&H_{g,8}(\tau) = \mathrm{tr}_B(g)\left[2q^{-\frac{1}{16}}\mathrm{tr}_{\mathfrak{M}_{8,1}^{(16)}}\left(g_{\left(0,\frac{1}{2}\right)}q^{-J(0)}q^{\hat{L}(0)}\right) + \mathrm{tr}_{\mathfrak{M}_{8,2}^{(16)}}\left(q^{\hat{L}(0)}\right)\right],\\
&\sum_{n=0,7}(-1)^n H_{g,2n+1}(\tau) = 2\mathrm{tr}_A(g)\mathrm{tr}_{\mathfrak{M}_1^{(16)}}\left(g_{(0,0)}q^{\hat{L}(0)}\right),
\end{aligned} \tag{2.113}$$

where $g_{\mathbf{b}}$ acts as specified in (2.18) *on the cone vertex algebra module in the tensor product and trivially on all the others. Analogously, $J(0)$ acts non-trivially only on the Weyl module V_{tw}.*

2.5 Conclusion

In this chapter we showed how certain trace functions of cone vertex algebras are related to a certain family of indefinite theta functions of signature $(1, 1)$. This family possesses interesting number theoretic properties and it is related to Appell-Lerch sums and Ramanujan's mock theta functions. For three instances of umbral moonshine, those with lambency $\ell = 8, 12, 16$, this allowed us to construct modules for the relevant finite groups in terms of cone vertex algebras and other known super vertex operator algebras modules. We end this chapter with a collection of open questions and possible future directions.

- We expect that the family of indefinite theta functions expressible as trace functions of cone vertex algebras can be extended by studying vertex algebras associated to cones with a more general shape than what used in (2.16). The condition in (2.13) on the choice of $\mathbf{c}$ is chosen to restrain the sum over the lattice vectors on the first and third quadrant of the plane. More general choices for the vectors $\mathbf{c}$ will lead to a sum on different cones.

- Another natural generalization is to investigate more general cone vertex algebras that can reproduce, through trace functions, indefinite theta functions of general signature $(r-n,n)$. In particular, it is worth investigating whether cone vertex algebras could be useful to gain a better understanding of the umbral moonshine phenomenon more generally, including the potential moonshine phenomenon involving all the optimal Jacobi theta functions classified in [39]. As remarked in previous sections, all mock theta functions appearing in the McKay–Thompson series of umbral moonshine can be written in terms of the traces of cone vertex algebras discussed in this chapter. The remaining challenge is thus to find expressions of the McKay–Thompson series that are compatible with the umbral group actions. While here we have limited our analysis to three instances of umbral moonshine with small umbral groups that turn out to act trivially on the cone vertex algerba structure, more involved group actions can certainly appear in other examples, akin to what happens in [47]. Furthermore, we note that the trace functions of the cone vertex algebras seem to connect the McKay–Thompson series to the meromorphic Jacobi forms associated to various instances of umbral moonshine, as a consequence of Corollary 2.2.1.2.

- Finally, it would be interesting to investigate the physical significance of the cone vertex algebras. Vertex operator algebras provide a mathematical axiomatization of the chiral algebra of conformal field theories in two dimension and it would be interesting to understand what kind of conformal fields theories cone vertex algebras are related to. For instance, it is known that the specialized Appell-Lerch sum (2.6) captures the non-modular part of the elliptic genus of non-compact supersymmetric coset models [53], [19]. This could shed light on the still mysterious relation between umbral moonshine and string theories compactified on $K3$ surfaces [19] (see also [9] for more complete references).

3 Three-Manifold Quantum Invariants and Mock Theta Functions

In this chapter we will explore the appearance of mock modularity in the context of three-manifold quantum invariants and their relation to indefinite theta functions. In particular, we will discuss the modular behavior of the $\widehat{Z}$ invariants introduced in [22] and discuss a conjecture relating the invariants of manifolds with opposite orientation. Inspired by this conjecture, we will propose a regularisation procedure, exploiting the theory of indefinite theta functions, to construct the invariants for the manifold with the opposite orientation.
We will present here the contents of [1] and discuss a further regularisation for the quantum invariants of Brieskorn spheres that can be tuned to yield examples explicitly satisfying conjecture 3.1.4.1.

To see how mock modular forms appear in the study of three-manifolds, we first introduce a set of topological invariants, noted by $\widehat{Z}_a$, defined in [22] for *weakly negative plumbed manifolds*. Roughly speaking, these are three-manifolds obtained through surgeries along links that are in turn determined by weighted graphs (cf. Figure 3.1), which moreover satisfy a certain negativity condition [33].

More precisely, the data we need is a plumbing graph, which is a weighted simple graph (V, E, λ) specified by the set V of vertices, the set E of edges, and an integral weight function $\lambda : V \to \mathbb{Z}$. Equivalently, the data can be captured by an adjacency matrix M, which is a square matrix of size $|V|$ with entries given by $M_{vv'} = \lambda(v)$ if $v = v'$, 1 if $(v, v') \in E$ and 0 otherwise. The data determines a three-manifold[1]. We say that M_3 is a weakly negative plumbed manifold if M^{-1} is negative-definite when restricted to the subspace generated by all vertices with degree larger than 2.

Definition 3.0.1. For M_3 a weakly negative plumbed three-manifold and using the above notation, we define the quantum invariants $\widehat{Z}_a(M_3; \tau)$ via the following

[1] Different weighted graph related by the so-called Kirby moves can lead to the same topological three-manifold M_3. See [58] for a proof the Kirby-invariance of the quantum invariants $\widehat{Z}_a(M_3)$ defined in Definition 3.0.1.

principal value $|V|$-dimensional integral:

$$\begin{aligned}\widehat{Z}_a(M_3;\tau) &:= (-1)^\pi q^{\frac{3\sigma-\sum_{v\in V}\lambda(v)}{4}} \\ &\times \sum_{\mathbf{n}\in 2M\mathbb{Z}^{|V|}+\mathbf{a}} \mathrm{vp}\prod_{v\in V}\oint_{|w_v|=1}\frac{dw_v}{2\pi i w_v}\left(w_v-\frac{1}{w_v}\right)^{2-\deg(v)} q^{-\frac{\mathbf{n}^T M^{-1}\mathbf{n}}{4}}\, e^{2\pi i \mathbf{z}^T\mathbf{n}}\end{aligned} \tag{3.0.1}$$

where we write $q := e^{2\pi i\tau}$ and $w_v := e^{2\pi i z_v}$ as usual, and use the bold-faced letters to denote elements in $\mathbb{Z}^{|V|}$. When M^{-1} is moreover negative definite, the above can be rewritten as

$$\begin{aligned}\widehat{Z}_a(M_3;\tau) :=&(-1)^\pi q^{\frac{3\sigma-\sum_{v\in V}\lambda(v)}{4}} \\ &\times \mathrm{vp}\prod_{v\in V}\oint_{|w_v|=1}\frac{dw_v}{2\pi i w_v}\left(w_v-\frac{1}{w_v}\right)^{2-\deg(v)} \Theta_a^{-M}(\tau,\mathbf{z})\end{aligned} \tag{3.0.2}$$

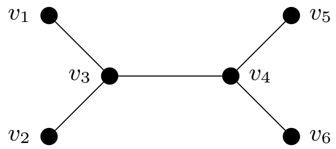
In the above, π denotes the number of positive eigenvalues, and σ is the signature of M^{-1}. The label a of the quantum invariants $\widehat{Z}_a(M_3)$ can be identified with elements of the set $\mathrm{Spin}^c(Y) \cong \pi_0\mathcal{M}_{\mathrm{ab}}(M_3) \cong (2\mathbb{Z}^{|V|}+\delta)/(2M\mathbb{Z}^{|V|})$, where $\delta \in \mathbb{Z}^{|V|}/2\mathbb{Z}^{|V|}$ is defined by $\delta_v = \deg(v) \bmod 2$, and $\mathcal{M}_{\mathrm{ab}}(M_3)$ denotes the moduli space of Abelian flat connections. Denote by $\mathbf{a}$ the corresponding element of $(2\mathbb{Z}^{|V|}+\delta)/(2M\mathbb{Z}^{|V|})$, the theta function reads

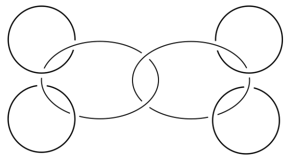
$$\Theta_a^{-M}(\tau,\mathbf{z}) = \sum_{\mathbf{n}\in 2M\mathbb{Z}^{|V|}+\mathbf{a}} q^{-\frac{\mathbf{n}^T M^{-1}\mathbf{n}}{4}}\, e^{2\pi i\mathbf{z}^T\mathbf{n}}. \tag{3.0.3}$$

A well-known topological invariant for three-manifolds is the Witten-Reshetikhin-Turaev (WRT) invariant, defined for all three-manifolds. Physically speaking, it is (up to certain well-understood prefactors) the partition function of Chern-Simons theory on the three-manifold M_3 which we denote by $Z_{\mathrm{CS}}(M_3)$. For a given three-manifold M_3 (and a simple Lie group G which we will take to be $G = SU(2)$ for the sake of concreteness), we obtain a function $Z_{\mathrm{CS}}(M_3) : \mathbb{Z} \to \mathbb{C}$ defined on all integers, namely the (shifted) Chern-Simons levels. Analogous to knot theory, it would be desirable to have a q-series version of the invariants defined on a continuous domain, such as the upper-half plane. This would then be the first step towards a categorification of three-manifold invariants, analogous to the categorification programme of knot invariants. It was shown for weakly negative plumbed manifolds that the quantum invariants $\widehat{Z}_a$ provide exactly such a q-series generalisation of the WRT invariants. To be more precise, for weakly negative

Plumbing graph Γ,
with weights $\lambda : V \to \mathbb{Z}$

Framed link L,
with framing coefficients $\lambda(v_i)$





Adjacency matrix M

$$\begin{pmatrix} \lambda(v_1) & 0 & 1 & 0 & 0 & 0 \\ 0 & \lambda(v_2) & 1 & 0 & 0 & 0 \\ 1 & 1 & \lambda(v_3) & 1 & 0 & 0 \\ 0 & 0 & 1 & \lambda(v_4) & 1 & 1 \\ 0 & 0 & 0 & 1 & \lambda(v_5) & 0 \\ 0 & 0 & 0 & 1 & 0 & \lambda(v_6) \end{pmatrix}$$

Plumbed M_3
obtained from surgery along L

Figure 3.1: Weighted graphs, adjacency matrices, links and plumbed three-manifolds.

plumbed manifolds one has [22]

$$(i\sqrt{2k})Z_{\text{CS}}(M_3;k) = \sum_{a,b} X_{ab} e^{2\pi i \text{CS}(a)k} \lim_{\tau\to\frac{1}{k}} \widehat{Z}_b(M_3;\tau). \tag{3.0.4}$$

In the above equation, the sum over a is over the set $((2\mathbb{Z}^{|V|}+\delta)/2M\mathbb{Z}^{|V|}))/\mathbb{Z}_2$, which can be identified with the space of gauge-inequivalent $SU(2)$ Abelian flat connections on M_3, and $\text{CS}(a)$ denotes the corresponding Chern-Simons invariant when we regard a as a label for Abelian $SU(2)$ flat connections. The sum over b is over the set $(\mathbb{Z}^{|V|}/M\mathbb{Z}^{|V|})/\mathbb{Z}_2$, and the matrix X has as elements

$$X_{ab} = \frac{\sum_{(a',b')\in\{\mathbb{Z}_2\times\mathbb{Z}_2 \text{ orbit of } (a,b)\}} e^{2\pi i (a', M^{-1}b')}}{2\sqrt{|\text{Det}M|}}. \tag{3.0.5}$$

To summarise, two steps need to be taken in order to retrieve Z_{CS} from $\widehat{Z}_a$. First, $\widehat{Z}_a$ has an extra label a indexing the $SU(2)$ Abelian flat connections while Z_{CS} does not, and this label needs therefore to be summed over. Second, a so-called radial limit $\tau \to \frac{1}{k}$ taking $\tau \in \mathbb{H}$ to the boundary $\mathbb{Q}\cup\{i\infty\}$ of the upper-half plane needs to be taken in order to relate the continuous variable τ and the (shifted) Chern-Simons level.

The modular-like properties of the quantum invariants $\widehat{Z}_a$ is a rich subject that has been in development since [33]. So far it develops in parallel to the study of modular-like properties of knot invariants (see for instance [5], [59], [60], [61] for a sample of work in this direction), although it is expected that the two topics are related both in their physical and mathematical contexts.

For concreteness and in order to make direct contact with Ramanujan's mock theta functions, here we restrict our attention to the simplest non-trivial plumbing graphs: the so-called three-star weighted graphs. These are, as the name suggests, weighted simple graphs with one vertex of degree three, three vertices of degree one, while the rest of the vertices (if any) have degree two. See Figure 3.2. We will denote the unique vertex with degree three by v_0. Such graphs are either weakly negative or not, depending on the sign of $(M^{-1})_{00}$. When $(M^{-1})_{00} < 0$, Definition 3.0.1 is readily applicable and it is not hard to show that the quantum invariants $\widehat{Z}_a$ are always holomorphic functions on the upper-half plane with well-defined q-expansions and moreover have integral coefficients. In fact, a lot more is true: up to a possible addition of a polynomial, the quantum invariants $\widehat{Z}_a$ are linear combinations of false theta functions multiplied by a rational q-power (cf. §3.1.1 and [34]).

A puzzle immediately arises given the simple result for weakly negative three-star graphs: what happens when one flips the orientation of the three-manifold? While this might sound like an innocuous operation, it can in fact have rather dramatic consequences due to the *pseudo-chiral symmetry* (or CP symmetry in physical terms)

$$Z_{\mathrm{CS}}(M_3;k) = Z_{\mathrm{CS}}(-M_3;-k) \tag{3.0.6}$$

of Chern-Simons theory. From the relation (3.0.4) between the quantum invariants $\widehat{Z}_a(M_3;\tau)$ and $Z_{\mathrm{CS}}(M_3;k)$, and in particular the relation "$\tau \to \frac{1}{k}$" between the two variables, one is led to the guess

$$\widehat{Z}_a(-M_3;\tau) \text{ “=” } \widehat{Z}_a(M_3;-\tau). \tag{3.0.7}$$

There are a few immediate problems with this guess. Recall that for a weakly negative plumbed manifold M_3, Definition 3.0.1 defines a function $\widehat{Z}_a(M_3;\tau)$ on the upper-half plane $\mathbb{H}$, which is not preserved by the action $\tau \mapsto -\tau$. As a result it is not clear what the right-hand side of the equation (3.0.7) even means. More concretely, it is clear from (3.0.1) that for plumbed manifolds one has $\tau \mapsto -\tau \Leftrightarrow M \mapsto -M$, which flips the sign of the adjacency matrix and hence flips the signature of the lattice for which the theta function Θ_a^{-M} should be defined, and as a result does not render a function on $\mathbb{H}$ when one tries to literally apply Definition 3.0.1.

This is when the question starts to become interesting from the perspective of mock modular forms. To be concrete, we let M_3 be a weakly negative three-star plumbed three-manifold. As mentioned before, for such cases $\widehat{Z}_a(M_3;\tau)$ are basically false theta functions, which are known to furnish (rather simple) examples of the so-called *quantum modular forms*, as will be explained in §3.1. The quantum modular properties of the quantum invariants $\widehat{Z}_a$ are essentially what makes their relation (3.0.4) to Z_{CS} possible. At the same time, it can be shown that a mock theta function and the corresponding false theta function lead to a pair of quantum modular forms that are in fact basically equivalent (cf. Lemma 3.1.4.1), in a way that precisely leads to the radial limit relation (3.0.6). This leads to the natural guess that the quantum invariants $\widehat{Z}_a(-M_3;\tau)$ for the orientation-reversed three-star plumbed manifold are given by mock theta functions. This conjecture, proposed in [33], will be discussed in §3.1.4.1.

In §3.2, we will review some recent results supporting the conjecture. The first involves building the relevant orientation-reversed three-manifold via Dehn surgeries on knot complements [58], and the second involves employing the indefinite theta series to extend the definition of $\widehat{Z}_a(M_3)$ to general plumbed manifolds [62]. To illustrate the various ideas discussed in this chapter, we will discuss in details the specific example of the Brieskorn sphere $M_3 = \Sigma(2,3,7)$.

3.1 False, Mock, and Three-Manifolds

In this section we argue that mock modular forms play a role in three-manifold quantum invariants. In §3.1.1 we introduce the relevant class of quantum invariants and review their relation to false theta functions. In §3.1.2 we review the quantum modular properties of false and mock theta functions and explain their relevance for three-manifold topology. In §3.1.3 we discuss a mock conjecture for $\widehat{Z}_a$ and its motivation and consequences.

3.1.1 False Theta Functions and Negative Three-Star Graphs

For concreteness, we focus on the simplest type of non-trivial graph: the three-star graphs (see Figure 3.2). These type of graphs correspond via plumbing (cf. Figure 3.1) to Seifert manifolds with three singular fibers. The relation between false theta functions and the WRT invariants for this family of three-manifolds was first pointed out in [63] and later extensively studied in [64], [65], [66]. Here we are interested in their quantum invariants $\widehat{Z}_a(M_3)$. It is easy to see [33] that Definition 3.0.1 leads to a function well-defined on $\mathbb{H}$ if and only if $(M^{-1})_{00} < 0$, namely when the resulting plumbed three-manifold M_3 is weakly negative.

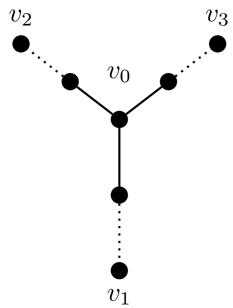


Figure 3.2: A three-star graph.

In order to describe the quantum modular properties of $\widehat{Z}_a(M_3)$, we will need the following definitions.

Definition 3.1.1. Let $m \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}/2m$. Define false theta functions

$$\widetilde{\theta^1_{m,r}}(\tau) := \sum_{\substack{k\in\mathbb{Z} \\ k\equiv r \pmod{2m}}} \operatorname{sgn}(k)\, q^{\frac{k^2}{4m}}. \tag{3.1.1}$$

Note that this is nothing but the usual theta function for one-dimensional lattice $\sqrt{2m}\mathbb{Z}$ when the sign factor in the summand is removed. This leads to the nomenclature [67, 68].

It will also be convenient to define, after [34], the following functions for $m \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}$

$$F_{j,m}(\tau) := \sum_{k\in\mathbb{Z}} \operatorname{sgn}\left(k+\frac{1}{2}\right) q^{\left(k+\frac{j}{2m}\right)^2}. \tag{3.1.2}$$

Furthermore, we have

$$F_{j,m}(m\tau) = \widetilde{\theta^1_{m,j}}(\tau) + p_{m,j}(\tau), \tag{3.1.3}$$

where $p_{m,j}(\tau)$ is the polynomial in q given by

$$p_{m,j}(\tau) = \begin{cases} -2\sum_{k=1}^{\lfloor \frac{j}{2m}\rfloor} q^{\frac{(j-2mk)^2}{4m}} & , \text{ if } \ j \geq 2m \\ 0 & , \text{ if } \ 0 \leq j < 2m \\ 2\sum_{k=0}^{-\lfloor \frac{j}{2m}\rfloor -1} q^{\frac{(j+2mk)^2}{4m}} & , \text{ if } \ j < 0. \end{cases} \tag{3.1.4}$$

Note that the definition (3.1.2) can be extended to $m, j \in \mathcal{Q}^*$ since the right hand side only depends on their ratio $\frac{j}{m}$.

In terms of the above building blocks, it can be shown that given a 3-star weighted graph, the corresponding $\widehat{Z}_a(M_3)$ can be written in terms of $F_{j,p}$ for some p and j. See Theorem 4.2 in [34] for the result on a closely related quantity, denoted $Z(q)$ in [34], and [33] for numerous examples.

In particular, in what follows we will further restrict our attention to weakly negative plumbed manifolds with three-star plumbing graphs with four nodes. Denote by M the corresponding adjacency matrix, let $A := -\frac{1}{2}M^{-1}$ and let v_0 be the unique vertex with degree three. Moreover, assume that the corresponding adjacency matrix M is unimodular. As a result there is only one quantum invariant $\widehat{Z}_0(M_3;\tau) := \widehat{Z}_a(M_3;\tau)$ with $a = \delta \bmod 2M\mathbb{Z}^{|V|}$, as defined in 3.0.1. Write also

$$\begin{aligned} &m = 2A_{00} \\ &b_0 = 2\sum_{j=1}^{3} A_{j0}, \quad b_i = 4A_{i0} - 2\sum_{j=1}^{3} A_{j0} \\ &c_0 = A_{12} + A_{23} + A_{31} + \frac{1}{2}\sum_{j=1}^{3} A_{jj}, \quad c_i = c_0 - 2\sum_{\substack{j\in\{1,2,3\}\\ j\neq i}} A_{ij} \end{aligned} \tag{3.1.5}$$

for $i = 1, 2, 3$. Note that $d_i := -\frac{b_i^2}{4m} + c_j$ satisfy $d_i = d_j =: d$ for all $i, j \in \{0, 1, 2, 3\}$. In the above notation we have the following Proposition.

Proposition 3.1.1.1. *[34] Consider a weakly negative three-star plumbing graph with four nodes and unimodular adjacency matrix, denote by M_3 the corresponding plumbed three-manifold. Its unique quantum invariant satisfies*

$$(-1)^\pi q^{-c} \widehat{Z}_0(M_3; \tau) = \sum_{j=0}^{3} F_{m-b_j,m}(m\tau) \tag{3.1.6}$$

with $c = d + \frac{3\sigma - \sum_v m(v)}{4}$, where m, b_j and d are defined as above and where σ and π as defined as in Definition 3.0.1.

Note that, using (3.1.3) this immediately shows

$$q^{-c} \widehat{Z}_0(M_3; \tau) = \sum_{j=1}^{4} \widetilde{\theta^1_{m,m-b_j}}(\tau) + p(\tau) \tag{3.1.7}$$

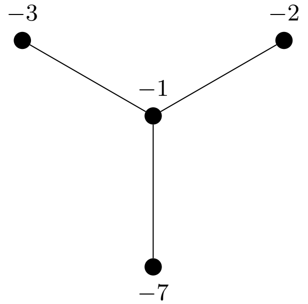
where $p(\tau)$ is a polynomial which one can work out explicitly using (3.1.3). Often times, one has $-m < b_j \leq m$ for all $j \in \{0, 1, 2, 3\}$ and $p(\tau) = 0$. In other words, up to an overall rational power of q and possibly the addition of a polynomial, the quantum invariants $\widehat{Z}_0$ is given by a false theta function.

We mention that the same result discussed above can be proven more generally for manifolds M_3 that are Brieskorn spheres using the methods of [58]. We will use this more general result in later sections.

Example. In this section, we will illustrate the computation of the quantum invariant and in particular Proposition 3.1.1.1, with the example of the Brieskorn sphere $M_3 = \Sigma(2, 3, 7)$, which can be described as the intersection between the algebraic surface $\{x^2 + y^3 + z^7 = 0\}$ and the five sphere $\{|x|^2 + |y|^2 + |z|^2 = 1\}$. It can be obtained as a plumbed manifold with the plumbing graph shown in Figure 3.3. Note that M is indeed unimodular, consistent with the fact that Brieskorn sphere is a integral homology sphere with trivial $H_1(M_3; \mathbb{Z}) \cong \mathbb{Z}^4/M\mathbb{Z}^4$ and there is hence just one quantum invariant $\widehat{Z}_0(M_3; \tau)$.

Plugging the adjacency matrix in (3.1.5) one obtains

$$m = 42 \ , \quad (b_j, 4c_j) = (1, 1), (-13, 5), (-29, 21), (41, 41) \text{ for } j = 0, 1, 2, 3. \tag{3.1.8}$$



$$M = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 0 & 0 & -7 \end{pmatrix}$$

Figure 3.3: Plumbing graph and adjacency matrix for $\Sigma(2,3,7)$

Using Definition 3.0.1 (or (3.1.6)), one obtains that

$$\begin{aligned} \widehat{Z}_0(\Sigma(2,3,7);\tau) &= q^{\frac{83}{168}} \sum_{\substack{k\geq 0 \\ k^2\equiv 1 \pmod{42}}} \left(\frac{k}{21}\right) q^{\frac{k^2}{168}} \\ &= q^{\frac{83}{168}} \left(\widetilde{\theta^1_{42,1}} - \widetilde{\theta^1_{42,13}} - \widetilde{\theta^1_{42,29}} + \widetilde{\theta^1_{42,41}}\right)(\tau) \\ &= q^{\frac{1}{2}} \left(1 - q - q^5 + q^{10} - q^{11} + \dots\right) \end{aligned} \tag{3.1.9}$$

which is indeed a false theta function of weight 1/2. The fact that $Z_{\mathrm{CS}}(\Sigma(2,3,7))$ is given by the above function by taking the limit

$$Z_{\mathrm{CS}}(\Sigma(2,3,7);k) = \frac{1}{i\sqrt{2k}} \lim_{t\to 0^+} (\widehat{Z}_0(\Sigma(2,3,7); \frac{1}{k} + t)) \tag{3.1.10}$$

was first established by [63].

3.1.2 False, Mock, and Quantum

As we have seen, a pre-requisite for a q-series to play the role of the quantum invariants $\widehat{Z}_a(M_3)$ is to have a specific behaviour when taking the radial limit, so that it gives the WRT invariants via (3.0.4). This is demonstrated in the $\Sigma(2,3,7)$ example in (3.1.10). This leads us to the concept of *quantum modular forms* (QMF), first introduced by D. Zagier [5] and discussed in the introduction of this thesis.

We recall here that, roughly speaking, a quantum modular form is a function defined on $\mathcal{Q}$ with a certain modular-like property: the deviation from modularity, measured by a modular difference function denoted by p_γ, has nice analytic properties that are not a priori manifest or expected. Here we work with a specific version of the definition that is often referred to as *strong quantum modular forms.*

We refer to §7.3 of [33] for details about modular forms in the current context.

Let us also recall the usual definition of the slash operator acting on the space of holomorphic functions on $\mathbb{H}$ for weight w and multiplier χ on Γ, which we take to be a subgroup of the modular group $SL_2(\mathbb{Z})$:

$$f(\tau)|_{w,\chi}\gamma := f\left(\frac{a\tau+b}{c\tau+d}\right)\chi(\gamma)(c\tau+d)^{-w}\,, \qquad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma. \tag{3.1.11}$$

Definition 3.1.2. [5] Consider a function $Q : \mathcal{Q} \to \mathbb{C}$. It is called a *strong quantum modular form* of weight w and multiplier χ for Γ if for every $\gamma \in \Gamma$ the modular difference function $p_\gamma(x) : \mathcal{Q}\backslash\{\gamma^{-1}(\infty)\} \to \mathbb{C}$, defined by

$$p_\gamma(x) := Q(x) - Q|_{w,\chi}\gamma(x) \tag{3.1.12}$$

is a real-analytic function of $\mathbb{R}$ minus finitely many points.

The false theta functions we encountered in §3.1.1 are examples of quantum modular forms. The simplest way to see this is to note that false theta functions defined in (3.1.1) are examples of Eichler integrals. Given a cusp form $g = \sum_{n>0} a_g(n)q^n$ of weight $w \in \frac{1}{2}\mathbb{Z}$, its Eichler integral is defined as

$$\widetilde{g}(\tau) := \sum_{n>0} n^{1-w} a_g(n) q^n. \tag{3.1.13}$$

It is easy to see that the false theta function $\widetilde{\theta^1_{m,r}}$ is the Eichler integral, up to a constant, of the weight 3/2 unary theta function

$$\theta^1_{m,r}(\tau) := \frac{1}{\sqrt{2}m} \sum_{\substack{\ell\in\mathbb{Z} \\ \ell = r \bmod 2m}} \ell\, q^{\ell^2/4m}, \tag{3.1.14}$$

as the notation suggests.

Note that this is equal to the following integral [2]

$$\widetilde{g}(\tau) = C\int_\tau^{i\infty} g(z')(z'-\tau)^{w-2}dz', \tag{3.1.15}$$

where $C = \frac{(2\pi i)^{w-1}}{\Gamma(w-1)}$. Letting $Q(x) := \lim_{t\to 0^+} \widetilde{g}(x+it)$ in Definition 3.1.2, one immdiately sees that the modular difference function $p_\gamma(x)$ admits an expression

[2]We choose the branch to be the principal branch $-\pi < \arg x \leq \pi$.

as a period integral

$$p_\gamma(x) = C \int_{\gamma^{-1}(i\infty)}^{i\infty} g(z')(z'-x)^{w-2} dz' \tag{3.1.16}$$

and is hence equipped with the desired analytic properties.

An analogous argument demonstrates that mock modular forms also lead to quantum modular forms. To see that, we first recall the definition of mock modular forms, adapted to the classes of functions that are relevant for our context. In particular, we assume that the shadow g is a cusp form with real Fourier coefficients, namely $\overline{g(-\bar{\tau})} = g(\tau)$.

Definition 3.1.3. We say that a holomorphic function f on $\mathbb{H}$ is a *mock modular form* of weight k and multiplier χ on Γ if there exists a weight $2-k$ cusp form g on Γ such that the non-holomorphic *completion* of f, defined as

$$\hat{f}(\tau) = f(\tau) - g^*(\tau)$$

satisfies $\hat{f} = \hat{f}|_{k,\chi}\gamma$ for every $\gamma \in \Gamma$. In the above, g^* denotes the non-holomorphic Eichler integral

$$g^*(\tau) := C \int_{-\bar{\tau}}^{i\infty} (\tau' + \tau)^{-k} g(\tau')\, d\tau', \tag{3.1.17}$$

defined for $\tau \in \mathbb{H}$.

Note that there is no canonical normalization for the shadow and we choose ours to simplify the comparison between mock modular forms and Eichler integrals (3.1.13). Assuming that the limit $\lim_{t\to 0^+} f(x+it)$ exists for a given $x \in \mathcal{Q}$, let $Q(x) := \lim_{t\to 0^+} f(x+it)$, and consider $\gamma \in \Gamma$ in the notation of Definition 3.1.2. The γ-invariance of the completion $\hat{f}$ leads to an expression for the corresponding modular difference function $p_\gamma(x)$ given by the modular difference function associated to $g^*(\tau)$. Through the related function (given by $\tau \mapsto -\tau$)

$$\tilde{g}^*(\tau) = C \int_{\bar{\tau}}^{i\infty} (\tau' - \tau)^{-k} g(\tau')\, d\tau' \tag{3.1.18}$$

and the fact that the modular difference function associated with $\tilde{g}^*$ is again a period integral completely analogous to (3.1.16), it follows that mock modular forms indeed lead to quantum modular forms. Moreover, the quantum modular forms arising from a mock modular form and the Eichler integral of its shadow are clearly closely related.

However, just as Ramanujan already pointed out in his original work [69], mock modular forms inevitably encounter infinities when approaching certain rational numbers from within the upper-half plane. Nonetheless, there exists a finite collection of weakly holomorphic modular forms that can be used to “cut out” these infinities and render a well-defined radial limit. More explicitly, we have the following theorem.

Theorem 3.1.4. *[70, 6] Let f be a mock modular form of weight k and multiplier system χ for $\Gamma_0(N)$ with non-vanishing shadow g, and let $\{x_1, \ldots, x_t\} \subset \mathcal{Q} \cup \{i\infty\}$ be a set of representatives of $\Gamma_0(N)$-inequivalent cusps, then*

1. *the function $f(\tau)$ has exponential singularities at infinitely many rational numbers,*
2. *for every weakly holomorphic modular form G of weight k and multiplier system χ for $\Gamma_0(N)$, $f - G$ has exponential singularities at infinitely many rational numbers,*
3. *there exits a collection $\{G_j\}_{j=1}^t$ of weakly holomorphic modular forms with the following property. Given any cusp x, let x_j be the cusp representative that is $\Gamma_0(N)$-equivalent to x and write $G_x = G_j$. Then $f - G_x$ is bounded towards x.*

Moreover, following the arguments sketched above, the mock modular form and the Eichler integral of its shadow leads to a pair of closely related strong quantum modular forms.

Lemma 3.1.4.1. *[33] With the notation of Theorem 3.1.4, let g be the shadow of f, the asymptotic expansions of the Eichler integral $\tilde{g}$ and the mock modular fomr $f - G_x$ near x take the form*

$$(f - G_x)(-x + it) \sim \sum_{n\geq 0} \alpha_x(n)(-t)^n \text{ and } \tilde{g}(x + it) \sim \sum_{n\geq 0} \alpha_x(n)t^n. \tag{3.1.19}$$

In particular, when the shadow g is a weight 3/2 unary theta function, the mock modular forms are (up to an overall rational power of q) called mock theta functions in the terminology of [5], and the Eichler integral are the false theta functions encountered in §3.1.1. The false-mock pair satisfies

$$\lim_{t\to 0^+} (f - G_x)(x + it) = \lim_{t\to 0^+} \tilde{g}(-x + it), \tag{3.1.20}$$

reminiscent of the relation (3.0.6) between $Z_{\rm CS}(M_3)$ and $Z_{\rm CS}(-M_3)$ when taking $x = \frac{1}{k}$. Focusing on the cusp $x = 0$, we can see that the false and mock forms have the “same” asymptotic series, approaching from the upper- and lower-half plane, in the sense that the asymptotic expansions in the limit $t \to 0^+$ satisfy

$$(f - G_0)(it) \sim \sum_{n\geq 0} \alpha_0(n)(-t)^n \text{ and } \tilde{g}(it) \sim \sum_{n\geq 0} \alpha_0(n)t^n. \tag{3.1.21}$$

On the three-manifold side, the cusp $x = 0$ is relevant for the perturbative invariants (the so-called Ohtsuki series), capturing the expansion around the semi-classical $k \to \infty$ limit.

To end this subsection, we provide an explicit example of such a false-mock pair.

Example. Consider the order seven mock theta function $F_0(q)$ by Ramanujan [69]. It is, up to an overall power of $q^{-\frac{1}{168}}$, a mock modular form of weight 1/2

$$f(\tau) = q^{-\frac{1}{168}} F_0(q) = q^{-\frac{1}{168}} \left(1 + q + q^3 + q^4 + q^5 + 2q^7 + O(q^8)\right), \tag{3.1.22}$$

whose shadow is given by the unary theta function

$$g(\tau) = \left(\theta^1_{42,1} - \theta^1_{42,13} - \theta^1_{42,29} + \theta^1_{42,41}\right)(\tau). \tag{3.1.23}$$

Compared to (3.1.9), we see that the Eichler integral is (up to a factor $q^{\frac{83}{168}}$) precisely the quantum invariant of the Brieskorn sphere $\Sigma(2, 3, 7)$:

$$\widehat{Z}_0(\Sigma(2, 3, 7); \tau) = q^{\frac{83}{168}} \tilde{g}(\tau). \tag{3.1.24}$$

3.1.3 A Mock Conjecture

In [33], the following relation between mock modular forms and three-manifold quantum invariants is proposed[3].

Conjecture 3.1.4.1. *Let M_3 be a three-manifold whose quantum invariants take the form*

$$\widehat{Z}_a(M_3; \tau) = q^c \left(\tilde{g}(\tau) + p(\tau)\right) \tag{3.1.25}$$

where $c \in \mathcal{Q}$, $\tilde{g}(\tau)$ is the Eichler integral of a theta function $g(\tau)$ of weight $w = \frac{3}{2}$ and $p(\tau)$ is a polynomial in q, then

$$\widehat{Z}_a(-M_3; \tau) = q^{-c} \left(f(\tau) + p(-\tau)\right), \tag{3.1.26}$$

where $f(\tau)$ is a weight 1/2 mock modular form whose shadow is given by $g(\tau)$.

[3]Note that when $-M_3$ is not a weakly negative plumbed manifold, the mathematical definition 3.0.1 does not apply and this conjecture can be seen as rather a definition. However, recall that a physical definition of $\widehat{Z}_a(M_3)$ does exist for all closed three-manifolds [22]. As a result, independent computations can in principle be carried out for $-M_3$, as we will demonstrate in §3.2.1 for certain classes of $-M_3$. With this in mind we regard (3.1.26) as a conjecture.

The relevance of the above conjecture can be seen in Proposition 3.1.1.1, which guarantees the existence of M_3 satisfying the condition of the conjecture. More generally, we also expect mixed weight and higher-depth mock modular forms to play a role in three-manifolds quantum invariants. See [33] and [31]. In what follows we briefly describe the three general motivations for the above conjecture, first discussed in [33]. In §3.2 we will present explicit calculations which render results predicted by Conjecture 3.1.4.1, and hence constitute further evidence for it.

- As mentioned in the previous subsection, the asymptotic values (3.1.20) and expansions (3.1.21) of a false-mock pair are analogous to the relation (3.0.6) among the WRT invariants of a pair of three-manifolds related by a flip in orientation.

- Some false theta functions have known expressions as q-hypergeometric series, which converge not only inside but also outside the unit circle (when considered as a function of q). In some cases the expression on the other side is given by a mock theta function. See §7.4 of [33] for details.

- When a weight 1/2 mock modular form can be expressed as a so-called Rademacher sum, one can prove in general that the same Rademacher sum, now performed in the lower rather than the upper half-plane, yields precisely the corresponding Eichler integral. In other words, the Rademacher sum yields a function defined on both $\mathbb{H}$ and $\mathbb{H}^-$, where they coincide with the mock resp. false theta function in question.

We refer to §7.4 of [33] for a detailed discussion of the third point above. To illustrate the second point, let us consider an example that is again relevant for the Brieskorn sphere $M_3 = \Sigma(2,3,7)$.

Example. Let us define a function $\psi : \mathbb{H} \cup \mathbb{H}^-$ in terms of the q-hypergeometric series:

$$\psi(\tau) := \sum_{n\geq 0} \frac{q^{n^2}}{(q^{n+1};q)_n}. \tag{3.1.27}$$

Note that the q-hypergeometric series converges both for $|q| < 1$ and $|q| > 1$. It can be shown that [65]

$$\psi(\tau) = \begin{cases} q^{-\frac{1}{168}}\tilde{g}(\tau) & , \quad \tau \in \mathbb{H} \\ F_0(q^{-1}) & , \quad \tau \in \mathbb{H}^- \end{cases}. \tag{3.1.28}$$

See also [71] for a more general discussion. As a result, since $\widehat{Z}_0(\Sigma(2,3,7),\tau) = q^{\frac{1}{2}}\psi(\tau)$ for $\tau \in \mathbb{H}$, we can try to extend the definition of LHS to $\mathbb{H}^-$ using the

RHS. It is hence natural to guess that (cf. (3.0.7))

$$\widehat{Z}_0(-\Sigma(2,3,7),\tau) \text{ “ = ” } \widehat{Z}_0(\Sigma(2,3,7),-\tau)\text{“ = ”}q^{-\frac{1}{2}}F_0(q). \tag{3.1.29}$$

We now end this section with a discussion on certain important open questions. First, note that Conjecture 3.1.4.1 does not specify, given a shadow, *which* mock modular form f should be the correct quantum invariant for the orientation-reversed manifold M_3. Recall that two mock modular forms differing by a (weakly holomorphic) modular form have the same shadow. This question is of crucial importance since, as proposed in [22], the Fourier coefficients of the quantum invariants $\widehat{Z}_a$ are (up to a possible factor of 2) integers which have the physical interpretation of counting supersymmetric quantum states in the underlying quantum physical theory. This said, we do expect the leading term of $\widehat{Z}_a$ in the $\tau \to i\infty$ expansion to obey the naive $q \leftrightarrow q^{-1}$ relation and this puts meaningful constraints on the mock modular forms. Second, as we have seen in §3.1.2, mock and false theta functions relate to the WRT invariants in a slightly different way. While the radial limit of false theta functions are well-defined, for many cusps x one has to subtract the singular terms (by subtracting a modular form G_x which cuts out the singularity for instance) of the mock form in order to have a well-defined limit when approaching x from within the upper-half plane (cf. Lemma 3.1.4.1). The asymmetry might not be so surprising from the physical point; the $M5$-brane theory is known to be a chiral theory. It would be extremely interesting to understand the physical or topological interpretation of the singular terms when taking radial limit of mock theta functions.

3.2 Explicit Calculations

In this section we summarise recent developments which make it possible to define and to compute the quantum invariants $\widehat{Z}_a(-M_3)$ for certain three-manifolds $-M_3$ that are relevant for the mock conjecture discussed in §3.1.3. We illustrate these methods with explicit computations for the Brieskorn sphere $M_3 = \Sigma(2,3,7)$.

3.2.1 Quantum Invariants via Knots

In this subsection, we review a (conjectural) way, introduced in [58], to compute the quantum invariants $\widehat{Z}_a$ for some of the three-manifolds that are relevant for the mock conjecture 3.1.4.1, by constructing them via Dehn surgeries of knot complements.

Consider a knot K. Let $Y(K)$ be the knot complement of K in an integral homology sphere $\hat{Y}$. A closed manifold $Y_{p/r}(K)$ can be obtained by $Y(K)$ via Dehn surgery with coefficient $p/r \in \mathcal{Q}^*$. Roughly speaking, p/r specifies the diffeomorphism of $\partial Y(K)$, dictating the way a solid torus is glued along $\partial Y(K)$ to obtain $Y_{p/r}(K)$.

Now consider the special case when $\hat{Y} = S^3$. Given this choice, one associates to a knot K a two variable series

$$F_K(x,q) \in 2^{-c}q^{\Delta}\mathbb{Z}[x^{1/2,}, x^{-1/2}][q^{-1}, q]] \tag{3.2.1}$$

where $c \in \mathbb{Z}_+$ and $\Delta \in \mathbb{Q}$. For instance, for K a positive torus knot, an explicit expression for $F_K(x,q)$ has been given in [58]. Define a "Laplace transform" $\mathcal{L}^{(a)}_{p/r}$, given by (see also [72])

$$\mathcal{L}^{(a)}_{p/r} : x^u q^v \to \begin{cases} q^{-u^2 r/p} \cdot q^v & \text{if } ru - a \in p\mathbb{Z}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{3.2.2}$$

It has been shown for positive torus knots K (Theorem 1.2 of [58]) and conjectured for general knots (Conjecture 1.7 of [58]) that, for values of p/r such that the right hand side is well defined and for some $d \in \mathbb{Q}$ and $\varepsilon \in \{\pm 1\}$, one has

$$\widehat{Z}_a(\tau, S^3_{p/r}(K)) = \varepsilon q^d \cdot \mathcal{L}^{(a)}_{p/r}[(x^{\frac{1}{2r}} - x^{-\frac{1}{2r}})F_K(x,q)] \tag{3.2.3}$$

where we canonically identify the Spin^c-structure of $S^3_{p/r}(K)$ with

$$a \in \mathbb{Z} + \frac{r+1}{2} \bmod p\mathbb{Z}. \tag{3.2.4}$$

Now it remains to compute F_K for general knots. It is convenient to define a rescaled version of $F_K(q,x)$:

$$f_K(x,q) := \frac{F_K(x,q)}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}. \tag{3.2.5}$$

Based on physical expectations, a relation between the Borel resummation of the colored Jones polynomial $J_n(e^{\hbar})$ and $f_K(x,q)$, where $q = e^{\hbar}$ and $x = e^{n\hbar}$, is conjectured in (Conjecture 1.5 of) [58]. Drawing inspiration from the analogous conjectures for the Chern-Simons partiton function on the knot complement [73] or for the colored Jones polynomials [74], the following was proposed in [58]

Conjecture 3.2.0.1. *For any knot $K \subset S^3$ the quantum polynomial $\widehat{A}$ of K annihilates the series $f_K(x,q)$*

$$\widehat{A} f_K(x,q) = 0 \tag{3.2.6}$$

and

$$\lim_{q\to 1} f_K(x,q) = s.e. \frac{1}{\Delta_K(x)} \tag{3.2.7}$$

where the symmetric expansion s.e. denotes the average of the expansions of the given rational function as $x \to 0$ (as a Laurent power series in x) and as $x \to \infty$ (as a Laurent power series in x^{-1}).

Note that (3.2.6) sets up a recursion relation for the coefficients $f_m(q)$ in $f_K(q,x) = \sum_m f_m(q)x^m$, while the relation (3.2.7) to the Alexander polynomial $\Delta_K(x)$ provides a boundary condition for the recursion equation. This is often sufficient to determine F_K to any desired order.

Example. For the figure-eight knot $K = \mathbf{4_1}$, the above-mentioned procedure leads to the leading order expansion[58]

$$F_{\mathbf{4_1}}(x,q) = \frac{1}{2}\left(\Xi(x,q) - \Xi(x^{-1},q)\right) \tag{3.2.8}$$

where

$$\Xi(x,q) = x^{1/2}+2\,x^{3/2}+(q^{-1}+3+q)x^{5/2}+(2q^{-2}+2q^{-1}+5+2q^{1}+2q^{2})+\dots. \tag{3.2.9}$$

The orientation-flipped Brieskorn sphere $-\Sigma(2,3,7)$ can be constructed through surgery on the complement in S^3 of the figure-eight knot $\mathbf{4_1}$, namely $-\Sigma(2,3,7) = S^3_{-1}(\mathbf{4_1})$. Exploiting the conjecture (3.2.3) and plugging in (3.2.8)-(3.2.9), we obtain the result:

$$\widehat{Z}_0(-\Sigma(2,3,7)) = -q^{-\frac{1}{2}}(1+q+q^3+q^4+q^5+2q^7+q^8+2q^9+q^{10}+2q^{11}+\dots). \tag{3.2.10}$$

Note that the above leading terms in the q-expansion coincide (up to a sign) with the guess (3.1.29) based on quantum modular properties and on q-hypergeometric identities. However, the procedure outlined in this subsection does not immediately lead to a way to prove the modularity of (3.2.10). We will see yet another way to compute $\widehat{Z}_0(-\Sigma(2,3,7))$ in the following subsection.

3.2.2 Relation to indefinite theta functions

As mentioned earlier, one immediate problem with the proposal

$$\widehat{Z}_a(-M_3;\tau) \text{ “=” } \widehat{Z}_a(M_3;-\tau)$$

from (3.0.7) is the fact that in Definition 3.0.1 one has $\tau \leftrightarrow -\tau \Leftrightarrow M \leftrightarrow -M$, and after this flipping of signature one no longer obtains a theta function Θ_a^{-M} (3.0.3) (and an integral (3.0.1)) that makes sense on the upper-half plane.

While it seems to be the end of the road as far as Definition 3.0.1 is concerned, a natural possibility is to replace the naive theta series with a regularised theta function. Indeed, building on previous work by Vignéras [75], Zwegers [4] has devised a way to define a regularisation for theta functions of signature $(1, n)$ which retains its holomorphicity, and moreover established the relation to mock theta functions. The regularisation of general indefinite theta functions and the relation to higher-depth mixed mock modular forms has recently been developed in [76], [77], [78], [79]. In [62], we apply these results to define and to compute quantum invariants for plumbed three-manifolds that are not weakly negative.

For the sake of concreteness and in order to establish a direct relation to Ramanujan's mock theta function, we focus on the class of three-manifolds discussed in Proposition 3.1.1.1. In the notation of Proposition 3.1.1.1 and of Figure 3.2, after performing the integration over w_{v_i} for $i \in \{1, 2, 3\}$ and write the $w_{v_0} = w$, we obtain

$$\widehat{Z}_0(M_3;\tau) = (-1)^\pi q^{\frac{3\sigma - \sum_v m(v)}{4}} \text{ vp} \oint_{|w|=1} \frac{dw}{2\pi i w(w - w^{-1})} h(\tau, z) \tag{3.2.11}$$

where

$$h(\tau, z) = \sum_{j=0}^{3} \sum_{\varepsilon \in \{\pm 1\}} \varepsilon \sum_{k \in 1+2\mathbb{Z}} q^{\frac{m}{4}k^2 - \frac{\varepsilon b_j}{2}k + c_j} w^k. \tag{3.2.12}$$

Note that naively taking $\tau \mapsto -\tau$ in $h(\tau, z)$ gives

$$\begin{aligned} &\frac{q^{\frac{1}{24}}}{\eta(\tau)} \sum_{j=0}^{3} \sum_{\varepsilon \in \{\pm 1\}} \varepsilon \sum_{k \in 1+2\mathbb{Z}} \sum_{n \in \mathbb{Z}} (-1)^n q^{-\frac{m}{4}k^2 + \frac{\varepsilon b_j}{2}k - c_j + \frac{3n^2 - n}{2}} w^k \\ &= \frac{q^{-d} e^{\frac{\pi i}{6}}}{\eta(\tau)} \sum_{j=0}^{3} \sum_{\varepsilon \in \{\pm 1\}} \varepsilon w^{\frac{\varepsilon b_j}{m}} \sum_{\mathbf{v} \in \Lambda_{j,\varepsilon}} q^{\frac{(\mathbf{v},\mathbf{v})}{2}} e^{2\pi i (z \ \frac{1}{2}) . \mathbf{v}} \end{aligned} \tag{3.2.13}$$

where we have inserted $1 = \frac{\eta(\tau)}{\eta(\tau)} = \frac{q^{\frac{1}{24}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{3n^2 - n}{2}}}{\eta(\tau)}$, and the bilinear form in the second line is given by

$$(\mathbf{v}', \mathbf{v}) := \mathbf{v}^T K \mathbf{v}, \quad K := \begin{pmatrix} -\frac{m}{2} & 0 \\ 0 & 3 \end{pmatrix}, \tag{3.2.14}$$

and the set of summation is given by

$$\Lambda_{j,\varepsilon} = \left\{ \mathbf{v} = \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix} \right) | v_1 \in 2\mathbb{Z} + 1 - \frac{\varepsilon b_j}{m}, v_2 \in \mathbb{Z} - \frac{1}{6} \right\}. \tag{3.2.15}$$

In other words, the key ingredient $h(\tau, z)$ of the integrand becomes, after taking $\tau \mapsto -\tau$ and multiplying by $\eta(\tau)$, a sum of theta functions of signature $(1,1)$ that we would like to make sense of. As a result, we propose the following definition for Brieskorn spheres:

$$\widehat{Z}_0(-M_3;\tau) := \frac{(-1)^\pi q^{-\frac{3\sigma - \sum_v m(v)}{4}}}{\eta(\tau)} \mathrm{vp} \oint_{|w|=1} \frac{dw}{2\pi i w(w-w^{-1})} \vartheta_a^M(\tau, z) \quad (3.2.16)$$

where

$$\vartheta_a^M(\tau, z) := q^{-d} e^{\frac{\pi i}{6}} \sum_{j=0}^{3} \sum_{\varepsilon \in \{\pm 1\}} \varepsilon w^{\frac{\varepsilon b_j}{m}} \sum_{\mathbf{v} \in \Lambda_{j,\varepsilon}} \rho(\mathbf{v})\, q^{\frac{(\mathbf{v},\mathbf{v})}{2}} e^{2\pi i (z \ \frac{1}{2}).\mathbf{v}} \quad (3.2.17)$$

for an appropriately chosen "regularisation factor" $\rho(\mathbf{v})$ which will be described explicitly in the example below.

Example. We will again take the example of $M_3 = \Sigma(2,3,7)$, with the plumbing graph and the adjacency matrix given in Figure 3.3. The relevant parameters m, b_j, and c_j are given in (3.1.8). Adapting [4] to preserve the symmetry[4] $\vartheta_a^M(\tau, z) = -\vartheta_a^M(\tau, -z)$, we choose the regularising factor

$$\rho(\mathbf{v}) = \rho^{c,c'}(\mathbf{v}) := \frac{1}{2}\left(\mathrm{sgn}(\bar{\mathbf{v}}, c) - \mathrm{sgn}(\bar{\mathbf{v}}, c')\right) \quad (3.2.18)$$

where $\bar{\mathbf{v}} = \left(\begin{smallmatrix} |v_1| \\ v_2 \end{smallmatrix}\right)$ for $\mathbf{v} = \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right)$ and the two timelike vectors are chosen to be

$$c = (1,0), \quad c' = (8,21). \quad (3.2.19)$$

Putting things together, we have the following result

Proposition 3.2.0.1. *When the regularisation factor $\rho(\mathbf{v})$ is given as in (3.2.18), (3.2.19), the definition (3.2.16) leads to*

$$q^{\frac{1}{2}} \widehat{Z}_0(-\Sigma(2,3,7);\tau) = 1 + q + q^3 + q^4 + q^5 + 2q^7 + O(q^8) \quad (3.2.20)$$

and the q-series agrees with the order 7 mock theta function of Ramanujan.

Note that this result, given by the order 7 mock theta function, was precisely what was expected in [33] (3.1.29). Moreover, at least the leading orders of q-expansion also, up to a sign, coincides with the result (3.2.10) which was obtained via a totally independent computation. These results constitute supporting evidence

[4] In this case, as noted in [34], the principal value contour integral renders the same result as integrating over a contour lying inside (or outside) the unit disk.

for Conjecture 3.1.4.1.

A Regularisation with a Controllable Shadow

In this paragraph we will discuss an alternative regularisation factor to (3.2.18) that has the advantage to make the computation of the shadow of the regularised $\widehat{Z}(-M_3,\tau)$ simpler. Using this regularisation it is possible to construct examples that explicitly satisfy conjecture 3.1.4.1.
In order to do so, we go back to equation (3.2.11) but this time we introduce the factor $1 = \frac{\eta(x\tau)}{\eta(x\tau)}$ in which we have scaled the argument of the Dedekind eta by a positive integer $x \in \mathbb{N}_{>0}$. This introduce an extra parameter that can be tuned in order to recover the desired shadow and/or q-series of the regularised $\widehat{Z}(-M_3,\tau)$, giving the regularisation more capacity to reproduce a wider range of mock modular forms.
We thus get

$$\widehat{Z}_0(-M_3;\tau) := \frac{(-1)^\pi q^{-\frac{3\sigma-\sum_v m(v)}{4}}}{\eta(x\tau)} \mathrm{vp} \oint_{|w|=1} \frac{dw}{2\pi i w(w-w^{-1})} \Theta_a^x(\tau,z) \tag{3.2.21}$$

with

$$\Theta_a^x(\tau,z) := q^{-d}\, e^{\frac{\pi i}{6}} \sum_{j=0}^{3} \sum_{\varepsilon\in\{\pm1\}} \varepsilon w^{\frac{\varepsilon b_j}{m}} \sum_{\mathbf{v}\in\Lambda_{j,\varepsilon}}, q^{\frac{(\mathbf{v},\mathbf{v})}{2}} e^{2\pi i (z\ \frac{1}{2}).\mathbf{v}}, \tag{3.2.22}$$

where the scalar product $(\cdot,\cdot)$ is the one associated to the matrix

$$\tilde{A} := \begin{pmatrix} -\frac{m}{2} & 0 \\ 0 & 3x \end{pmatrix}. \tag{3.2.23}$$

Forgetting the overall factor for simplicity, and expanding as in [58]

$$\left(w - \frac{1}{w}\right)^{-1} = -\frac{1}{2} \sum_{l \text{ odd}} \mathrm{sign}(l) w^l, \tag{3.2.24}$$

the integral in (3.2.21) equals

$$\mathrm{vp} \oint_{|w|=1} \frac{dw}{2\pi i w} \sum_{j=0}^{3} \sum_{\epsilon\in\{\pm1\}} \epsilon \sum_{k,l,n\in\mathbb{Z}} (-1)^n \mathrm{sign}(2l+1) q^{\tilde{Q}\left(2k+1-\epsilon\frac{b_j}{m}, n-\frac{1}{6}\right)} w^{2(k+l+1)} \tag{3.2.25}$$

where $\tilde{Q}(\mathbf{v}) = \frac{1}{2}\mathbf{v}^T \tilde{A}\mathbf{v}$ is the quadratic form[5] associated to $\tilde{A}$. For convenience, we consider the following scaled matrix

$$A := \begin{pmatrix} -2m & 0 \\ 0 & 3x \end{pmatrix} \tag{3.2.26}$$

and denote with Q its associated quadratic form. With this notation, after some manipulations, we can rewrite equation (3.2.25) as

$$\begin{aligned} \text{vp} \oint_{|w|=1} \frac{dw}{2\pi i w} \sum_{j=0}^{3} \sum_{k,l,n\in\mathbb{Z}} (-1)^n \Big[& \text{sign}(2l-1) q^{Q\left(k+\frac{m-b_j}{2m}, n-\frac{1}{6}\right)} \\ & - \text{sign}(2l+1) q^{Q\left(k-\frac{m-b_j}{2m}, n-\frac{1}{6}\right)} \Big] w^{2(k+l)} \end{aligned} \tag{3.2.27}$$

At this point we introduce two different regularisation factors for the two addends inside the square brackets in equation (3.2.27). In particular we multiply the first addend by $\text{sign}(2l-1)\rho_{\mathbf{c}_j}\left(k + \frac{m-b_j}{2m}, n - \frac{\epsilon_j}{6}\right)$ and the second by $-\text{sign}(2l+1)\rho_{\hat{\mathbf{c}}_j}\left(k - \frac{m-b_j}{2m}, n - \frac{\epsilon_j}{6}\right)$ with $\epsilon_j \in \{\pm 1\}$ that will be specified later to match the exact q-series. We have defined $\rho_{\mathbf{c}}(\mathbf{v}) = \frac{1}{2}\left(\text{B}(\mathbf{v}, \mathbf{c}) - \text{B}(\mathbf{v}, \mathbf{c}')\right)$ (notice that we now don't take the absolute value of the first entry of $\mathbf{v}$), and $\mathbf{c}_j$ and $\hat{\mathbf{c}}_j$ differ only for the sign of their first entry at different indices j as follows

$$c_j = \begin{cases} (c^{(1)}, c^{(2)}) & \text{for} \quad j = 1,2 \\ (-c^{(1)}, c^{(2)}) & \text{for} \quad j = 0,3 \end{cases}, \tag{3.2.28}$$

$$\hat{c}_j = \begin{cases} (-c^{(1)}, c^{(2)}) & \text{for} \quad j = 1,2 \\ (c^{(1)}, c^{(2)}) & \text{for} \quad j = 0,3 \end{cases}. \tag{3.2.29}$$

and analogously

$$c'_j = \begin{cases} (c'^{(1)}, c'^{(2)}) & \text{for} \quad j = 1,2 \\ (-c'^{(1)}, c'^{(2)}) & \text{for} \quad j = 0,3 \end{cases}, \tag{3.2.30}$$

$$\hat{c}'_j = \begin{cases} (-c'^{(1)}, c'^{(2)}) & \text{for} \quad j = 1,2 \\ (c'^{(1)}, c'^{(2)}) & \text{for} \quad j = 0,3 \end{cases}. \tag{3.2.31}$$

Up to now we just require $\mathbf{c}_j, \mathbf{c}'_j, \hat{\mathbf{c}}_j, \hat{\mathbf{c}}'_j$ to satisfy the conditions for the conver-

[5] With a slight abuse of notation we denote, for a bidimensional vector $\mathbf{v} = (x, y)$, Q((x,y)) as Q(x,y).

gence of the indefinite theta functions as specified in 1.3.2. We will impose further constraints later that will make the shadow easier to compute explicitly.
Notice that, in particular, the integrand is invariant when sending $w \to w^{-1}$. The integral thus reduce to the computation of the constant term in w [34]. Furthermore, since the sum in (3.2.27) is invariant sending $n \to -n$, we can exchange $n - \frac{1}{6}$ with $n - \frac{\epsilon_j}{6}$ inside the quadratic form. After some computations, we are left with

$$\begin{aligned} \widehat{Z_0}(-M_3;\tau) = \frac{(-1)^\pi q^\delta}{\eta(x\tau)} \sum_{j=0}^{3} \sum_{k,n\in\mathbb{Z}} \\ (-1)^n \rho_{\mathbf{c}_j}\left(k + \frac{m-b_j}{2m}, n - \frac{\epsilon_j}{6}\right) q^{Q\left(k+\frac{m-b_j}{2m}, n-\frac{\epsilon_j}{6}\right)} \\ = \frac{(-1)^\pi q^\delta}{\eta(x\tau)} \sum_{j=0}^{3} e^{\frac{i\pi\epsilon_j}{6}} \Theta^{j+}_{\mathbf{a_j},\mathbf{b}}(\tau), \end{aligned} \tag{3.2.32}$$

where we have written $\Theta^{j+}_{\mathbf{a_j},\mathbf{b}_j}$ for the indefinite theta function defined as in (1.50) with respect to the vectors $\mathbf{c}_j$ and $\mathbf{c'}_j$, $\delta := -\frac{3\sigma - \sum_v m(v)}{4} - d$ and

$$\mathbf{a}_j := \left(\tfrac{m-b_j}{2m}, -\tfrac{\epsilon_j}{6}\right), \quad \mathbf{b} := \left(0, \tfrac{1}{6x}\right) \tag{3.2.33}$$

We are now reduced, up to an overall factor, to a sum of 4 indefinite theta functions defined on a double-sided cone (rather than the single-sided cone given by (3.2.18)). The shadow can be extracted using the expression (1.51) for the action of the shadow operator ξ (1.27) on indefinite theta functions.
In order to do so, let's start with proving a useful lemma.

Lemma 3.2.0.1. *Let* $\mathbf{c}_i = (c_1^{(i)}, 2mc_2^{(i)})$ *with* $Q(\mathbf{c}_i) = -m$ *and* $c_l^{(i)} \in \mathbb{Z}$ *for* $i, l \in \{1, 2\}$. *With the notation of* (1.51), *the action of the shadow operator* (1.27) *on* $\Theta_{\mathbf{a}_j,\mathbf{b},\mathbf{c}_1,\mathbf{c}_2}$, *with* $\mathbf{a}_j$ *and* $\mathbf{b}$ *as in* (3.2.33) , *has the form*

$$\xi_1(\Theta^j_{\mathbf{a},\mathbf{b},\mathbf{c}_1,\mathbf{c}_2}(\tau)) = \frac{\overline{\eta(x\tau)}}{2}\sqrt{2Im(\tau)}e^{-\frac{i\pi}{6}\epsilon_j}\sum_{i\in\mathcal{I}}(-1)^{i+\tilde{c}} g_{-\frac{B(\mathbf{c}_i,\mathbf{a}_j)}{2m}, -B(\mathbf{c}_i,\mathbf{b})}(2m\tau) \tag{3.2.34}$$

where $\tilde{c} \in \mathbb{Z}$ *is determined by writing* $c_1 = \tilde{\epsilon} + 6\tilde{c}$ *with* $\tilde{\epsilon} \in \{\pm 1\}$.

Proof. Let us for simplicity focus on only one of the two summands in the sum over j in (1.51). To make the notation lighter, let us also drop the j index on the c-vector and write $\mathbf{c} = (c_1, 2mc_2)$ for its components. The norm condition

$Q(\mathbf{c}) = -m$ implies

$$c_1^2 - 6xmc_2^2 = 1. \tag{3.2.35}$$

In particular this implies that $\gcd(c_1, 3xc_2) = 1$ and $\gcd(c_1, 2mc_2) = 1$. Since, furthermore, $\mathbf{c}^\perp := (3xc_2, c_1)$ is orthogonal to $\mathbf{c}$, they constitute primitive vectors and we can write $\mathbb{Z}^2 = \{k\mathbf{c} + k'\mathbf{c}^\perp : k, k' \in \mathbb{Z}\}$. Thus

$$\left\{\ell \in \mathbf{a}_j + \mathbb{Z}^2 : \frac{B(\mathbf{c}, \ell)}{2Q(\mathbf{c})} \in [0,1)\right\}, \tag{3.2.36}$$

using $B(\mathbf{c}, \ell) = B(\mathbf{c}, \mathbf{a}_j) + kB(\mathbf{c}, \mathbf{c}) + k'B(\mathbf{c}, \mathbf{c}^\perp) = B(\mathbf{c}, \mathbf{a}_j) + 2kQ(\mathbf{c})$, has only one element modulo $\langle \mathbf{c} \rangle_{\mathbb{Z}}^\perp$. Thus ℓ_0 in (1.51) can be chosen to be $\ell_0 := \mathbf{a}_j + k\mathbf{c}$ with k fixed such that $B(\mathbf{c}, \mathbf{a}_j) + 2kQ(\mathbf{c}) \in [0,1)$.
We can now focus on the second sum

$$\sum_{\nu \in \ell_0^\perp + \langle \mathbf{c} \rangle_{\mathbb{Z}}^\perp} e^{2\pi i B(\nu, \mathbf{b}^\perp)} q^{Q(\nu)} = \sum_{n \in \mathbb{Z}} e^{2\pi i B(\ell_0^\perp + n\mathbf{c}^\perp, \mathbf{b}^\perp)} q^{Q(\ell_0^\perp + n\mathbf{c}^\perp)} \tag{3.2.37}$$

where $\ell_0^\perp = \mathbf{a}_j - \frac{B(\mathbf{c}, \mathbf{a_j})}{2Q(\mathbf{c})}\mathbf{c} = \mathbf{a}_j^\perp$. We start noticing that, since $\mathbf{c}$ and $\mathbf{c}^\perp$ constitute an orthonormal basis for $\mathbb{R}^2$ for the scalar product induced by the matrix A, we have, for the perpendicular component to $\mathbf{c}$, $\ell_0^\perp := \frac{B(\ell_0, c^\perp)}{2Q(\mathbf{c}^\perp)}\mathbf{c}^\perp$and thus $\ell_0^\perp + n\mathbf{c}^\perp = \left(n + \frac{B(\ell_0, \mathbf{c}^\perp)}{2Q(\mathbf{c}^\perp)}\right)\mathbf{c}^\perp$. With the same procedure, it is also easy to compute $B(\mathbf{c}^\perp, \mathbf{b}^\perp) = \frac{c_1}{2}$ and, using (3.2.35), also $Q(\mathbf{c}^\perp) = \frac{3x}{2}$. Thus equation (3.2.37) becomes

$$\sum_{\nu \in \ell_0^\perp + \langle \mathbf{c} \rangle_{\mathbb{Z}}^\perp} e^{2\pi i B(\nu, \mathbf{b}^\perp)} q^{Q(\nu)} = e^{2\pi i B(\ell_0^\perp, \mathbf{b}^\perp)} \sum_{n \in \mathbb{Z}} e^{\pi i n c_1} q^{\frac{3x}{2}\left(n + \frac{B(\ell_0, \mathbf{c}^\perp)}{3x}\right)}. \tag{3.2.38}$$

Since

$$\frac{1}{3x} B(\mathbf{c}^\perp, \mathbf{a}_j) = -(m - b_j)c_2 - \frac{\epsilon_j \tilde{\epsilon}}{6} c_1 \tag{3.2.39}$$

and, from (3.2.35), $c_1 = \tilde{\epsilon} + 6\tilde{c}$ for $\tilde{\epsilon} \in \{\pm 1\}$, $\tilde{c} \in \mathbb{Z}$ using the fact that

$$\sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{3x}{2}(n+\alpha)^2} = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{3x}{2}(n-\alpha)^2} \tag{3.2.40}$$

for all $\alpha \in \mathbb{R}$, we get

$$\sum_{\nu \in \ell_0^\perp + \langle \mathbf{c} \rangle_{\mathbb{Z}}^\perp} e^{2\pi i B(\nu, \mathbf{b}^\perp)} q^{Q(\nu)} = (-1)^\gamma e^{2\pi i B(\mathbf{a}_j^\perp, \mathbf{b}^\perp)} \eta(x\tau) \tag{3.2.41}$$

where we have used $\ell_0^\perp = \mathbf{a}_j^\perp$ and we have defined $\gamma := -(m - b_j)c_2 - \tilde{c}\epsilon_j$. Furthermore,

$$B(\mathbf{a}_j^{\perp}, \mathbf{b}^{\perp}) = \frac{B(\mathbf{b}, \mathbf{c}^{\perp})B(\mathbf{a}_j, \mathbf{c}^{\perp})}{2Q(\mathbf{c}^{\perp})} = \frac{c_1}{2}\left(\gamma - \frac{\epsilon_j \tilde{\epsilon}}{6}\right) \tag{3.2.42}$$

so

$$e^{2\pi i B(\mathbf{a}_j^{\perp}, \mathbf{b}^{\perp})} = (-1)^{\gamma + \tilde{c}} e^{-\frac{i\pi}{6}\epsilon_j}. \tag{3.2.43}$$

The claim follows using the property $g_{a,b} = g_{a+1,b}$. □

For convenience, we will write the function g in terms of the unary theta function $\theta^1_{m,r}$ defined in (3.1.1). It is easy in fact to see that, for any $N \in \mathbb{Z}$,

$$g_{\frac{r}{2m},N}(2m\tau) = \frac{e^{\frac{\pi i r N}{m}}}{\sqrt{2}} \theta^1_{m,r_i}(\tau). \tag{3.2.44}$$

With the conditions of the previous lemma, we have $B(\mathbf{c}_i, \mathbf{b}) = mc_2$, thus we can write

$$\xi_1(\Theta^j_{\mathbf{a},\mathbf{b},\mathbf{c_1},\mathbf{c_2}}(\tau)) = \frac{\overline{\eta(x\tau)}}{2\sqrt{2}} \sqrt{2\mathrm{Im}(\tau)} \sum_{i \in \mathcal{I}} (-1)^{i+\tilde{c}_i} e^{\frac{i\pi}{6}\epsilon_j} e^{-i\pi r_j^{(i)} c_2^{(i)}} \theta^1_{m, r_j^{(i)}(\tau)} \tag{3.2.45}$$

with $r_j^{(i)} = -B(\mathbf{c}_i, \mathbf{a}_j) \mod 2m = (m - b_j)c_1^{(i)} + xmc_2^{(i)} \mod 2m$ since it holds $\theta^1_{m,r}(\tau) = \theta^1_{m,r+2m}(\tau)$.
We now define $H(-M_3;\tau)$, the completed $q^{-\delta}\widehat{Z_0}(-M_3;\tau)$, as the result of substituting each indefinite theta function $\Theta^{j+}_{\mathbf{a_j},\mathbf{b}}$ in $\widehat{Z_0}(-M_3;\tau)$ with the corresponding (mixed) harmonic Maass form $\Theta^j_{\mathbf{a_j},\mathbf{b}}$ as specified in section 1.3.2. Combining equation (3.2.45) with equation (3.2.32), and using $\xi_{w'} = Im(\tau)^{(w'-w)}\xi_w$, we get that

$$\begin{aligned} \xi_{\frac{1}{2}}(H(-M_3;\tau)) &= -\frac{(-1)^{\pi}}{\overline{\eta(x\tau)}} \sum_{j=0}^{3} e^{-\frac{i\pi\epsilon_j}{6}} \xi_1(\Theta^j_{\mathbf{a_j},\mathbf{b}}(\tau)) \\ &= -\frac{(-1)^{\pi}}{2} \sum_{j=0}^{3} \sum_{i=1}^{2} (-1)^{i+\tilde{c}_i} e^{-i\pi r_j^{(i)} c_2^{(i)}} \theta^1_{m, r_j^{(i)}}(\tau) \end{aligned} \tag{3.2.46}$$

This shows that, with the above regularisation, $\widehat{Z_0}(-M_3;\tau)$ is a mock modular form of weight $\frac{1}{2}$.
As the next step, we will show that, with the proposed regularisation, we can recover the q-series and shadow of the order 7 Ramanujan mock theta function.
We choose $x = 3$, $\mathbf{c}_1 = (1, 0)$, $\mathbf{c}_2 = (55, 168)$ and $\epsilon_1 = -1$, $\epsilon_2 = 1$, $\epsilon_3 = -1$, $\epsilon_4 = 1$.
We get $m = 42$, $\tilde{\epsilon}_1 = \tilde{\epsilon}_2 = 1$ and $\mathbf{c}_1$, $\tilde{c}_1 = 0$, $\tilde{c}_2 = 9$ and

$$\begin{aligned} &r_0^{(1)} = 1, \quad r_1^{(1)} = 41, \quad r_2^{(1)} = 55, \quad r_3^{(1)} = 71, \\ &r_0^{(2)} = 55, \quad r_1^{(2)} = 71, \quad r_2^{(2)} = 1, \quad r_3^{(2)} = 41. \end{aligned} \tag{3.2.47}$$

Using $\theta^1_{m,r}(\tau) = -\theta^1_{m,-r}(\tau)$, we get

$$\xi_{\frac{1}{2}}(H(-\Sigma(2,3,7);\tau)) = \left(\theta^1_{42,1} - \theta^1_{42,13} - \theta^1_{42,29} + \theta^1_{42,41}\right)(\tau), \tag{3.2.48}$$

matching the shadow of the order seven Ramanujan mock theta function $F_0(q)$. Furthermore, expanding the regularised invariant (3.2.32) we get the expected q-series

$$q^{\frac{1}{2}}\widehat{Z_0}(-\Sigma(2,3,7);\tau) = 1 + q + q^3 + q^4 + q^5 + 2q^7 + O(q^8). \tag{3.2.49}$$

With the proposed regularisation, we are thus able to construct an expression for the invariant $\widehat{Z_0}(-\Sigma(2,3,7);\tau)$ and its shadow that explicitly satisfies conjecture 3.1.4.1.

3.3 Conclusion

In this chapter we have discussed a conjecture relating the quantum invariants $\widehat{Z}_a$ of a plumbed three manifold to the quantum invariants of the reversed oriented manifold. In doing so we have observed some aspects of quantum modularity, manifested in this case in the relation between the eichler integral of false theta functions and the shadow of the “companion” mock theta functions.
The relation between false/mock pairs inspired our proposals for regularisation prescriptions of the formula (3.0.1) for non weakly negative Brieskorn spheres. Using the theory of indefinite theta functions, we are able to show that the $\widehat{Z}$ invariants regularised using this prescription are mock modular forms. Furthermore, we have seen that. for the manifold $\Sigma(2,3,7)$. the proposed regularisation can reproduce the shadow predicted by conjecture 3.1.4.1 and that the corresponding q-series agrees with the one computed with independent methods.
The regularisation procedure described in this chapter needs as input the parameters ϵ_j, $j \in \{1,2,3,4\}$ and the vectors c_1, c_2 constrained only to satisfy Pell's equation $Q(\mathbf{c}_i) = -m$. These have to be fixed from the indices r of the false theta functions $\tilde{\theta}_{m,r}$ appearing in the invariant with the opposite orientation in order to reproduce the correct shadow. It is important to recall, however, that mock modular forms are not uniquely determined by their shadow. In fact, mock modular forms with the same shadow could differ by a modular form. So, even when it is possible to find $\mathbf{c_i}$, ϵ_j that reproduce the correct shadow, the obtained function could differ from the quantum invariant by a modular form and, furthermore, parameters leading to the same shadow could possibly yield different q-series.
Insights on how to choose the parameters to reproduce the correct quantum invariants can be derived from other characterizations and computation techniques used to compute $\widehat{Z}_a$. One interesting direction to explore is to understand the

construction presented here in terms of the seemingly closely related method of inverted Habiro series discussed in [29]. It would be also interesting to understand the expression for the regularised $\widehat{Z}$ invariants in terms of VOA characters similarly to [31]. With respect to this, it would be appealing to try to combine and interpret the construction of VOA modules for indefinite theta functions presented in chapter 2 in this setting. Finally, further insights could be gained from the Physical interpretation of these parameters in terms of the underlying 3d theory $T[M_3]$ that gives rise to the $\widehat{Z}$ invariants.

4 Quantum Modular $\widehat{Z}^G$-Invariants

In this chapter we continue the study of modular properties of certain q-series topological invariants of three-manifolds, presenting the work of [3]. More specifically, we focus on the $\widehat{Z}$-invariants (sometimes also referred to as the q-series invariant or the *homological blocks*), which we have already encountered in previous chapters. In particular, in this chapter we will consider $\widehat{Z}$-invariants for higher rank groups G and study their quantum modular properties as well as a recursive relation with the $\widehat{Z}$ invariants for lower rank groups.
In order to make this chapter self-contained, and to adapt the notation and convention to the setting, we will first briefly recall some basic facts and definitions about $\widehat{Z}$ invariants focusing on the aspects that will be more relevant to this chapter. We will then give an overview of the main results before discussing the detailed steps to obtain them. For conciseness of the exposition, we defer some of the proofs to section 4.6. Data for the examples presented in section 4.4 are collected in appendix B, definitions and properties of some of the functions encountered in this section are collected in appendix C.
We start by briefly recalling that $\widehat{Z}_b(M_3)$ is physically defined as the half-index (also called vortex partition function) of the three-dimensional $\mathcal{N}=2$ supersymmetric quantum field theory $T[M_3]$ obtained by compactifying a six-dimensional $\mathcal{N}=(2,0)$ superconformal field theory on the closed three-manifold M_3. b labels the specific choices of boundary conditions that leave some of the supersymmetries unbroken. That said, as the exact content of $T[M_3]$ is in general still not known, this physical definition does not always lead to a method to explicitly compute the $\widehat{Z}_b(M_3)$ invariants in practice. On the other hand, we recall that a relation is conjectured between $\widehat{Z}_b(M_3)$ and the Chern-Simons invariant $\mathrm{CS}(M_3)$ [22]. Specializing to the case $b_1(M_3)=0$, this relation reads

$$\mathrm{CS}(M_3;k)=\frac{1}{i\sqrt{2k}}\sum_{a,b} e^{2\pi i k\,\mathrm{CS}(a)}\, S^{(A)}_{ab}\,\widehat{Z}_b(M_3;\tau)|_{\tau\to\frac{1}{k}}\ , \tag{4.0.1}$$

where the sum can be thought of as over the connected components of the moduli spaces of Abelian flat connections up to Weyl group actions, or the inequivalent Spin^c structure and $S^{(A)}_{ab}$ is a concrete matrix whose form can be found in [22, 58, 23]. This relation suggests that $\widehat{Z}$-invariants can be viewed as a function that extends, and catergorifies via its BPS states counting interpretation, the WRT invariants. Using the above, the known expression for $\mathrm{CS}(M_3; k)$, and inspired by the localization expressions for the half-indices of certain known theories, a mathematical definition for $\widehat{Z}^G_{\underline{\vec{b}}}(M_3)$ invariants has been proposed for classes of three-manifolds M_3 [22], as well as knot complements in [58]. As the six-dimensional $\mathcal{N} = (2, 0)$ superconformal field theories are labelled by an ADE gauge group G, we expect $\widehat{Z}^G_{\underline{\vec{b}}}(M_3)$ to be similarly defined for all ADE gauge groups G. Indeed, the mathematical definition for an arbitrary simply-laced gauge group G is given in [25], generalizing the definition of [22] which corresponds to $G = SU(2)$.

For $G = SU(2)$, a relation between $\widehat{Z}^{SU(2)}_{\underline{\vec{b}}}(M_3)$ and quantum modular forms, in particular false [23, 80, 81] and mock theta functions [23, 82, 29] have been proposed. Generally, we have the following conjecture [23], which can be traced all the way back to the relation between false and mock theta functions and WRT invariants of three-manifolds [63, 83, 84, 85].

General Conjecture:
$\widehat{Z}^G_{\underline{\vec{b}}}(M_3)$ is closely related to a quantum modular form of some kind for any closed three-manifold M_3, any ADE gauge group G, and any boundary condition label $\underline{\vec{b}}$.

Consider for instance $G = SU(2)$ and M_3 a Seifert manifold, while $\widehat{Z}^G_{\underline{\vec{b}}}(M_3)$ is a linear combination of quantum modular forms when M_3 is a Seifert manifold with three or four exceptional fibers, it is given more generally by linear combinations of derivatives of quantum modular forms when M_3 has at least five exceptional fibers [80]. To expand our understanding of the above conjectural phenomenon, in this chapter we study quantum modularity of $\widehat{Z}^G_{\underline{\vec{b}}}(M_3)$ for gauge groups G with rank larger than one, and $G = SU(3)$ in particular. In short, in these cases the type of quantum modular forms will be *higher depth* quantum modular forms. Analogous to higher depth mock modular forms ([86, 87, 88, 89, 90, 91, 92, 93, 94, 95]), higher depth quantum modular forms can be defined recursively: the cocycles of a depth two quantum modular form are sums of depth one or zero quantum modular forms multiplied by analytic functions, and so on (cf. Definition 4.2.3). Before we go to the concrete results, let us mention a manifestation of quantum modularity in this context.

As discussed in [23] in the context of $\widehat{Z}$ invariants for $G = SU(2)$, the transseries expression of the WRT invariant at the semi-classical regime can be understood as a consequence of the following two facts: (1) the relation between WRT and $\widehat{Z}$ invariant (4.0.1), and (2) the quantum modularity of the $\widehat{Z}$ invariant. Schematically, when the rank-one $\widehat{Z}$ is a component of a vector-valued quantum modular form (see Definition 4.2.2) $z = (z_{b'})$ with weight w and S-matrix $S^{(q)}$, the above leads to

$$\begin{aligned}\mathrm{CS}(M_3;k) &= \frac{1}{i\sqrt{2k}}\sum_a e^{2\pi i k \mathrm{CS}(a)} \sum_b S^{(A)}_{ab} \lim_{\tau\to\frac{1}{k}} \widehat{Z}^{SU(2)}_{\underline{\vec{b}}}(M_3;\tau) \\ &= \frac{1}{i\sqrt{2k}}\sum_a e^{2\pi i k \mathrm{CS}(a)} \sum_b S^{(A)}_{ab} \left(k^w \sum_{b'} S^{(q)}_{bb'} \lim_{\tau\to -k} z_{b'}(\tau) + r_b(\tfrac{1}{k}) \right) .\end{aligned} \tag{4.0.2}$$

In the second line of the above equation, the first term inside the bracket arises from the S-transformation of $\widehat{Z}$, while $r_b(\frac{1}{k})$ is an asymptotic perturbative series in $\frac{1}{k}$ capturing the non-vanishing cocycle. The above equality turns out to capture many intricate structures related to flat $SL_2(\mathbb{C})$ connections on M_3. Note first that the terms involving $z_{b'}(-k)$ are responsible for the contributions from the saddle points corresponding to non-Abelian flat connections. Moreover, since the summation over a can be interpreted as a summation over the Abelian flat connections, it is clear from above that the transseries for semi-classical WRT invariants of this class of three-manifolds has the feature that only the saddle contributions from Abelian flat connections carry a factor given by a perturbative series, having the form $e^{-k\mathrm{CS}(a)}\frac{1}{\sqrt{k}}R_a(\frac{1}{k})$ where R_a is again a perturbative series. When $\widehat{Z}$ invariant is a depth N quantum modular forms, one sees that the above structure gets generalized. Now there are up to N "classes" of saddle points, with different complexity of the accompanying perturbative series. For instance, as before there will be no asymptotic series of $\frac{1}{k}$ multiplying the terms arising from the S-transformation of $\widehat{Z}_b$, and more generally there are saddle point contributions multiplied by products of ℓ perturbative series, for $0 \le \ell \le N$. Again, we expect that the quantum modularity structure controls the intricate topological structure of the flat connections on the 3-manifolds, and we will return to this point in §4.5. Finally, in light of the proposed relation between vertex algebras and $\widehat{Z}$-invariants [23, 31], we expect it to be also fruitful to understand quantum modularity of $\widehat{Z}$-invariants in the context of vertex algebras.

We will now briefly discuss the main contents and result contained in this chapter.

Quantum Modularity

First, we make the following conjecture about the quantum modular properties of $\widehat{Z}^G_{\underline{\vec{b}}}(M_3)$ invariants for weakly-negative or weakly-positive plumbed three-manifolds M_3, where we let G to be an arbitrary ADE gauge group and $\underline{\vec{b}}$ to be any allowed boundary condition as detailed in §4.3 (cf. (4.3.8)). As will be explained in §4.3, a plumbed three-manifold can be defined in terms of its plumbing matrix M encoding its plumbing graph. Following [23], we say a plumbed manifold is weakly-negative/positive when M^{-1}, when restricted to the subspace generated by the "junction vertices" (those with degree at least 3), is negative/positive-definite. A Seifert manifold can always be realized as a plumbed manifold with one junction vertex, and in this case we will simply refer to a Seifert manifold as a *negative/positive Seifert* manifold depending on the signature of the inverse plumbing matrix in the direction of the unique junction vertex.

We first make the following conjecture, specializing and refining the *General Conjecture* outlined earlier:

Conjecture 4.0.0.1. *Let G be an ADE group with rank r. For M_3 a weakly-negative or weakly-positive plumbed manifold with n junction vertices, the invariant $\widehat{Z}^G_{\underline{\vec{b}}}(M_3;\tau)$ is related to quantum modular forms of depth up to $r \times n$.*

We also prove the following special case, where $r = 2$ and $n = 1$.

Theorem 4.0.1. *Let $G = SU(3)$. For a negative Seifert manifold M_3 with three exceptional fibers and for all the allowed $\underline{\vec{b}}$, the invariant $\widehat{Z}^G_{\underline{\vec{b}}}(M_3;\tau)$ is a sum of depth-one and depth-two quantum modular forms.*

We will prove the above statement by studying the so-called companion function of $\widehat{Z}^G_{\underline{\vec{b}}}$, denoted by $\check{Z}^G_{\underline{\vec{b}}}$ which is defined as a function that has the same asymptotic expansion near $\tau \to \mathcal{Q}$ up to a naive $\tau \mapsto -\tau$ transformation. See Definition 4.2.1. We construct this in terms of iterated non-holomorphic Eichler integrals (cf. (4.2.50) and (4.2.52)), using a method similar to that of [96], and we will refer to the companion function constructed in this specific way simply as the companion function. One can also translate our analysis into the language of two-variable completion [97] instead of the companion in a relatively straightforward fashion. Specifically, in the language of [97], $\widehat{Z}^G_{\underline{\vec{b}}}$ is a sum of depth-one and depth-two false modular forms.

In §4.5 we briefly discuss the possible forms of quantum modularity for the cases beyond Conjecture 4.0.0.1.

A Recursive Structure

Next, consider changing the gauge group G while fixing the three-manifold M_3 in $\widehat{Z}^G(M_3)$, we ask the following question:

Question: Given a three-manifold M_3, are the quantum modular properties of $\widehat{Z}^{SU(N)}_{\underline{\vec{b}}}(M_3)$ for different N related?

To motivate this question, we find it illuminating to recall the following. Higher-depth quantum modular forms have been playing a prominent role in the study of the Vafa-Witten partition functions $Z^G_{\rm VW}(\tau; M_4)$ for twisted four-dimensional $\mathcal{N} = 4$ super Yang-Mills on four-dimensional manifolds M_4. In more details, when $b^+_1(M_4) = 1$, the invariant $Z^G_{\rm VW}(\tau; M_4)$ displays mock modular properties and the "depth" of the corresponding (mixed) modular forms is given in terms of the rank of the gauge group G. The mock modular properties in particular imply that there is a modular *completion* of $Z^G_{\rm VW}(\tau; M_4)$, denoted $\tilde{Z}^G_{\rm VW}(\tau; M_4)$, which is non-holomorphic with a canonically defined holomorphic part equaling $Z^G_{\rm VW}(\tau; M_4)$. While $\tilde{Z}^G_{\rm VW}(\tau; M_4)$ transforms as a modular object, it has a non-trivial $\bar{\tau}$-dependence referred to as its holomorphic anomaly. In other words, the $\bar{\tau}$-dependence of the completion function $\tilde{Z}^G_{\rm VW}(M_4)$ captures the mock modularity of the Vafa-Witten invariant $Z^G_{\rm VW}(M_4)$. Notably, the holomorphic anomaly of $\tilde{Z}^G_{\rm VW}(\tau; M_4)$ for $G = U(N)$ is given by $\tilde{Z}^G_{\rm VW}(\tau; M_4)$ for $G = U(n)$ with $0 < n < N$. Schematically, the conjecture states [98, 99]

$$\partial_{\bar{\tau}} \tilde{Z}^{U(N)}_{\rm VW} \sim \sum_{n_1+n_2=N} n_1 n_2 \tilde{Z}^{U(n_1)}_{\rm VW} \tilde{Z}^{U(n_2)}_{\rm VW}. \tag{4.0.3}$$

The above recursive relation, and more generally the mock modularity in this context, has been given a physical explanation from various perspectives including four-dimensional gauge theories, two-dimensional sigma models [100], curve counting [99], and DT invariants [101, 102, 103, 89]. Roughly speaking, the presence of a holomorphic anomaly is related to the presence of reducible connections from the gauge theory point of view, and to the possibility of separating multiple M5 branes from the M-theory point of view. The recursive structure (4.0.3) then naturally follows from these interpretations. The similar M5 brane origin of the three-manifold invariants $\widehat{Z}^G(\tau; M_3)$, as detailed in [104], in particular motivates the question on the recursive structure of the quantum modularity of $\widehat{Z}$ invariants that we mentioned earlier.

To explore this question, we now focus on negative Seifert manifolds with three exceptional fibers, corresponding to plumbing graphs with one junction vertex of degree three. Based on the relation between $\widehat{Z}$ invariants and VOA characters shown in [23, 31], we expect $\widehat{Z}^G_{\underline{\vec{b}}}(M_3)$ for $G = SU(r+1)$ to be a linear combination of rank-r' false theta functions, with $r' \leq r$, up to an overall rational power of q and possibly the addition of a finite polynomial in q and q^{-1}. Since many statements in the remaining part of the section are true up to an overall rational power of q and the addition of a finite polynomial in q and q^{-1}, for the sake of simplicity we will introduce the special notation $\overset{...}{=}$, $\overset{...}{\in}$, etc., where the ... is added on top of the symbols to denote that the relation holds when replacing $\widehat{Z}$ with $Cq^{\Delta}\widehat{Z} + f(q)$, for some $C \in \mathbb{C}$, $\Delta \in \mathcal{Q}$, $f(q) \in \mathbb{C}[q, q^{-1}]$, and similarly for $\check{Z}$.

More specifically, we expect $\widehat{Z}^G_b(M_3)$ to be a linear combination of functions of the following form

$$\begin{aligned} t^{(0),A_r} &= \sum_{\vec{n}\in\vec{\mu}+\Lambda_r} \left(\prod_i \mathrm{sgn}(\langle \vec{n}, \vec{v}_i\rangle) q^{|\vec{n}|^2/2}\right), \\ t^{(1),A_r} &= \sum_{\vec{n}\in\vec{\mu}+\Lambda_r} \langle \vec{n}, \vec{\sigma}\rangle \left(\prod_i \mathrm{sgn}(\langle \vec{n}, \vec{v}_i\rangle) q^{|\vec{n}|^2/2}\right) \end{aligned} \tag{4.0.4}$$

for some chosen $\vec{\mu}$, $\vec{v}_i$ and $\vec{\sigma}$ and rank r lattice Λ_r. We denote their companion functions, which we expect to be given by linear combinations of iterated Eichler integrals, by $\check{t}^{(\nu),A_r}$ [97]. Then the general structure of higher rank false theta functions suggests the following. Schematically,

$$\begin{aligned} &\frac{\partial}{\partial\bar{\tau}}\check{Z}^{SU(r+1)} \\ &\overset{...}{\in} \mathrm{span}\left(\{(\Im\tau)^{\nu_0-3/2}\,\overline{\theta^{\nu_0}}\,\check{t}^{(\nu_1),A_{r_1}}\check{t}^{(\nu_2),A_{r_2}}\dots|\nu_i\in\{0,1\}, r\geq 1+r_1+r_2+\dots\}\right) \end{aligned} \tag{4.0.5}$$

with θ^{ν} denotes the function of the type $\theta^{\nu}_{m,r}$, defined as the following. For m a positive integer, let Θ_m be the $2m$-dimensional Weil representation of the metaplectic group $\widetilde{SL_2(\mathbb{Z})}$ spanned by the column vector $\theta_m = (\theta_{m,r})_{r \bmod 2m}$ with theta function components

$$\theta_{m,r}(\tau,z) := \sum_{\ell\equiv r \bmod 2m} q^{\frac{\ell^2}{4m}} y^{\ell}\ , \qquad y := e^{2\pi i z}\ . \tag{4.0.6}$$

Derivatives of (4.0.6) define unary theta functions $\theta^{\nu}_{m,r} : \mathbb{H} \to \mathbb{C}$ for $\nu = 0, 1$, as

$$\theta^{\nu}_{m,r}(\tau) := \left(\left(\frac{1}{2\pi i}\frac{\partial}{\partial z}\right)^{\nu}\theta_{m,r}(\tau,z)\right)\Bigg|_{z=0}. \tag{4.0.7}$$

An important role in the study of $\widehat{Z}^{SU(2)}$ is played by Eichler integrals unary theta functions. The Eichler integral of a weight $w \in \frac{1}{2}\mathbb{Z}$ cusp form $g(\tau) = \sum_{n>0} a_g(n)q^n$

is a function given by[1]

$$\tilde{g}(\tau) := C(w) \sum_{n>0} a_g(n)\, n^{1-w} q^n, \qquad (4.0.8)$$

where $C(w) = i\frac{\Gamma(w-1)}{(-2\pi)^{w-1}}$, or

$$\tilde{g}(\tau) = \int_{\tau}^{i\infty} g(z')(-i(z'-\tau))^{-2+w} dz', \qquad (4.0.9)$$

with a carefully chosen contour. In particular, the Eichler integral of $\theta^1_{m,r}$ is proportional the false theta function

$$\tilde{\theta}^1_{m,r} := \sum_{k \equiv r\ (2m)} \mathrm{sgn}(k) q^{k^2/4m}. \qquad (4.0.10)$$

In this chapter we focus on the next simplest non-trivial case where $G = SU(3)$. What we find is an interesting recursion structure, which we will describe in terms of the Weil representations of the metaplectic group $\widetilde{SL_2(\mathbb{Z})}$ that are subrepresentations of Θ_m. We denote by Ex_m the group of exact divisors of m, where a divisor n of m is exact if $\left(n, \frac{m}{n}\right) = 1$ and the group multiplication for Ex_m is $n * n' := \frac{nn'}{(n,n')^2}$. In what follows we consider a subgroup K of Ex_m. The group K labels a subrepresentation of Θ_m, denoted Θ^{m+K} which we will describe in more details in §4.3. It has the property that $\Theta^{m+K'} \subset \Theta^{m+K}$ when $K' \supset K$, and in particular $\Theta^{m+K} = \Theta_m$ when $K = \{1\}$. In general, we have, in terms of the projectors P^{m+K} defined in, (4.3.18) and (4.3.19)

$$\theta^{m+K}_r(\tau, z) = \sum_{r' \in \mathbb{Z}/2m} P^{m+K}_{r,r'} \theta_{m,r'}(\tau, z)\ . \qquad (4.0.11)$$

We will write $\theta^{\nu,m+K}_r$, $\nu = 0, 1$, to be the corresponding linear combination of $\theta^{\nu}_{m,r}$ (see (4.3.21) for $\nu = 1$), and write $\sigma^{m+K} \subset \mathbb{Z}/2m$ as the set labelling (through r) the linearly independent $\theta^{m+K}_r(\tau, z)$. Similar notations $\tilde{\theta}^{\nu,m+K}_r$ and $(\theta^{\nu,m+K}_r)^*$ are used for the same linear combination (4.0.11) of the corresponding Eichler integrals and the non-holomorphic Eichler integral

$$(\theta^{\nu}_{m,r})^*(\tau) := \int_{-\bar{\tau}}^{i\infty} dw\ \frac{\overline{\theta^{\nu}_{m,r}(-\bar{w})}}{(-i(w+\tau))^{3/2-\nu}} \qquad (4.0.12)$$

(cf. (4.2.50)) which is up to an overall factor a companion for $\tilde{\theta}^{\nu}_{m,r}$. See §7.3 of

[1] To avoid an unnecessary proliferation of constants we adopt a different normalization of the Eichler integral than in previous chapters.

[23] for more details in the present context.

Now we explain the role of the representations Θ^{m+K} in the study of $\widehat{Z}^G$-invariants. It is shown [23, 80] that for any negative Seifert M_3 with three exceptional fibers, there exists a unique m and some $K \subset \mathrm{Ex}_m$ such that for all allowed choices of $\underline{\vec{b}}$

$$\widehat{Z}^{SU(2)}_{\underline{\vec{b}}}(M_3;\tau) \in \mathrm{span}\left(\{\tilde{\theta}^{1,m+K}_r | r \in \sigma^{m+K}\}\right), \tag{4.0.13}$$

which implies

$$\check{Z}^{SU(2)}_{\underline{\vec{b}}}(M_3;\tau,\bar{\tau}) \in \mathrm{span}\left(\{(\theta^{1,m+K}_r)^* | r \in \sigma^{m+K}\}\right). \tag{4.0.14}$$

From now on we will take the largest K such that the above is true. The following conjecture, based on observations and proven for homological spheres, indicates that the recursion of $\widehat{Z}^G$ have in fact finer structure than indicated in (4.0.5):

Conjecture 4.0.1.1. *Let M_3 be a negative Seifert manifold three exceptional fibers and let $\underline{\vec{b}}$ a choice of the boundary condition. Let m be the unique positive integer and K be the largest subgroup of Ex_m such that*

$$\widehat{Z}^{SU(2)}_{\underline{\vec{b}}}(M_3;\tau) \mathrel{\ddot{\in}} \mathrm{span}\left(\{\tilde{\theta}^{1,m+K}_r | r \in \sigma^{m+K}\}\right). \tag{4.0.15}$$

Let $\check{Z}^{SU(3)}_{\underline{\vec{b}}}(M_3;\tau,\bar{\tau})$ be the companion of $\widehat{Z}^{SU(3)}_{\underline{\vec{b}}}(\tau;M_3)$. Then it satisfies

$$\begin{aligned} &\frac{\partial}{\partial\bar{\tau}}\left(\check{Z}^{SU(3)}_{\underline{\vec{b}}}(M_3;\tau,\bar{\tau}) + z_{1\mathrm{d}}\right) \\ &\mathrel{\dddot{\in}} \frac{1}{\sqrt{\Im\tau}}\mathrm{span}\left(\{\overline{\theta^{1,m+K}_{r'}}(\theta^{\nu}_{m,r''})^* | \nu = 0,1,\ r'' \in \mathbb{Z}/2m,\ r' \in \sigma^{m+K}\}\right) \end{aligned} \tag{4.0.16}$$

where the 1d *piece is of the form*

$$z_{1\mathrm{d}} \mathrel{\ddot{\in}} \mathrm{span}\left(\{(\theta^{\nu}_{m,r})^* | r \in \mathbb{Z}/2m,\ \nu = 0,1\}\right). \tag{4.0.17}$$

We see that the same Weil representation Θ^{m+K} that governs the structure of $\widehat{Z}^{SU(2)}(M_3)$ also governs the structure of $\widehat{Z}^{SU(3)}(M_3)$. We will comment on its potential interpretation in §4.5.

When M_3 is moreover a homological sphere, namely when $H_1(M_3,\mathbb{Z})$ is trivial, it is topologically equivalent to a Brieskorn sphere $\Sigma(p_1,p_2,p_3)$ with coprime p_i's (4.3.4). In this case there is only one homological block $\underline{\vec{b}} = \underline{\vec{b}}_0$ and it is known that [23]

$$\frac{\partial}{\partial \bar{\tau}}(\check{Z}^{SU(2)}_{\underline{\vec{b}}}(M_3;\tau,\bar{\tau})) \overset{\cdots}{=} \overline{\theta^{1,m+K}_r} \tag{4.0.18}$$

for

$$m = p_1p_2p_3,\ K = \{1,\bar{p}_1,\bar{p}_3,\bar{p}_2\},\ r = m - \bar{p}_1 - \bar{p}_2 - \bar{p}_3 \tag{4.0.19}$$

where $\bar{p}_i := m/p_i$.

For this (infinite) family of M_3, we explicitly show that the conjecture is true, and we have

Theorem 4.0.2. *Conjecture 4.0.1.1 is true when M_3 is a homological sphere.*

In other words, in this case we have

$$\begin{aligned} \frac{\partial}{\partial \bar{\tau}}\check{Z}^{SU(2)}_{\underline{\vec{b}}}(M_3;\tau,\bar{\tau}) &\overset{\cdots}{=} \frac{1}{\sqrt{\Im\tau}}\overline{\theta^{1,m+K}_r} \\ \frac{\partial}{\partial \bar{\tau}}\left(\check{Z}^{SU(3)}_{\underline{\vec{b}}}(M_3;\tau,\bar{\tau}) + z_{1\mathrm{d}}\right) &\overset{\cdots}{=} \frac{1}{\sqrt{\Im\tau}} \sum_{r'\in\sigma^{m+K}} \overline{\theta^{1,m+K}_{r'}} B_{r'} \end{aligned} \tag{4.0.20}$$

where $B_{r'}$ is a linear combination of $(\theta^{\nu}_{m,r''})^*$ with $\nu = 0,1,\ r'' \in \mathbb{Z}/2m$. In particular, note that while only one component of Θ^{m+K} appears to play a role in the quantum modularity of $\widehat{Z}^{SU(2)}$, its modular images also play a role in $\widehat{Z}^{SU(3)}$. Here we have stated the recursive conjecture in terms of the companion function. As before, the above analysis on the recursive relation can be translated in the language of modular completions of higher-depth false theta functions [97]. Roughly speaking, the role of $\frac{\partial}{\partial\bar{\tau}}$ will be played by $\frac{\partial}{\partial w}$, acting on the two-variable completion that depends on $(\tau, w) \in \mathbb{H}\times\mathbb{H}$ and transforms as a bi-modular form. An analogous statement should then hold also for (the ∂_w derivatives of) the completion function defined in [97] in a natural fashion.

4.1 Notation Guide

For convenience, we collect here the notation that will be used in the rest of the chapter.

Table 4.1:

$e(x)$	Shorthand notation $e(x) = e^{2\pi i x}$.
$B_m(x)$	Bernoulli polynomials with generating function $\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$.

Continued on next page

Table 4.1: (Continued)

$\Lambda = \Lambda_{\mathfrak{g}}$	The root lattice associated to the simply-laced Lie algebra $\mathfrak{g}$.
$\Lambda^\vee$	The dual root lattice.
Φ_{s}	The set of simple roots $\{\vec{\alpha}_i\}_{i=1}^{\mathrm{rankG}}$.
$\Phi_\pm$	The sets of positive and negative roots.
$\vec{\rho}$	The Weyl vector of the root system $\vec{\rho} := \frac{1}{2}\sum_{\vec{\alpha}\in\Phi_+}\vec{\alpha}$.
$\langle\cdot,\cdot\rangle$	The scalar product in the dual space of the Cartan subalgebra of $\mathfrak{g}$.
$\lvert\vec{x}\rvert^2$	For $\vec{x}\in\mathbb{C}\otimes_{\mathbb{Z}}\Lambda$, the norm is defined by $\lvert\vec{x}\rvert^2 = \langle\vec{x},\vec{x}\rangle$.
$\{\vec{\omega}_i\}_{i=1}^{\mathrm{rankG}}$	The set of fundamental weights, satisfying $\langle\vec{\omega}_i,\vec{\alpha}_j\rangle = \delta_{i,j}$.
P^+	The set of dominant integral weights. See (4.2.1). For $\bar{P}^+$, see (4.2.2).
$\Delta\vec{\omega}$	The difference $\Delta\vec{\omega} := \vec{\omega}_1 - \vec{\omega}_2$ of the two fundamental weights in A_2 Lie algebra.
W	The Weyl group of the root system.
$w(\cdot)$	The action of the element $w\in W$.
$\ell(w)$	The length of w.
$Q(\mathbf{m})$	The norm $Q(\mathbf{m}) := \frac{1}{2}\lvert\vec{\mathbf{m}}\rvert^2 = (3m_1^2 + 3m_1m_2 + m_2^2)$ for $(m_1,m_2)\in\mathbb{R}^2$ (4.2.3).
ϱ	Shorthand notation $\varrho = (\vec{s},\vec{k},m,D)$ introduced in §4.2.
$\vec{\sigma}$	Shorthand notation $\vec{\sigma} = \vec{s} - \frac{m}{D}\vec{k}$.
$F^{(\varrho)}$	Generalized A_2 false theta function defined in equation (4.2.5).
$F_\nu^{(\varrho)}$	Partial theta functions, defined for $\nu = 0,1$ in (4.2.7) and (4.2.11).
$F_{\nu,\boldsymbol{\alpha}}$	Components of false theta functions, defined in equation (4.2.8).
$\mathcal{F}_\nu$	$\mathcal{F}_\nu(\boldsymbol{x}) = x_2^\nu e^{-Q(\boldsymbol{x})}$ (4.2.4)
$\mathcal{S}$	The set (4.2.12) of parameters $\boldsymbol{\alpha}$ (4.2.9) of the partial theta functions $F_{\nu,\boldsymbol{\alpha}}(\tau)$.
$\tilde{\mathcal{S}}$	Subset of the set $\mathcal{S}$ defined in equation (4.2.14).
$\mathbb{E}_\nu^{(\varrho)}(\tau)$	The companion functions of the functions $F_\nu^{(\varrho)}(\tau)$. See (4.2.40).
M	The adjacency matrix (4.3.1) of the weighted graph (V,E,a).

Continued on next page

Table 4.1: (Continued)

D	Smallest positive integer such that $DM^{-1}_{v_0,v} \in \mathbb{Z}$ for $\forall\ v \in V$; $m = D^2 \lvert M^{-1}_{v_0,v_0}\rvert$.
$\underline{\vec{b}}$	Generalised Spin^c structure (4.3.8) on a plumbed three-manifold M_3, labelling the boundary conditions of $T[M_3]$.
$\widehat{Z}^G_{\underline{\vec{b}}}(M_3)$	Topological invariant of a plumbed three-manifold M_3 (4.3.9).
$\check{Z}^G_{\underline{\vec{b}}}(M_3)$	Companion function of $\widehat{Z}^G_{\underline{\vec{b}}}(M_3)$.
$\dddot{=}$, $\dddot{\in}$, etc.	relations hold when replacing $\widehat{Z}$ with $Cq^{\Delta}\widehat{Z} + f(q)$, for some $C \in \mathbb{C}$, $\Delta \in \mathcal{Q}$, $f(q) \in \mathbb{C}[q, q^{-1}]$, and similarly for $\check{Z}$.

4.2 Generalized A_2 False Theta Functions

Let $\Lambda = \Lambda_{A_2}$ be the A_2 root lattice, W the corresponding Weyl group with $\ell : W \to \mathbb{Z}$ its length function. We denote by $W_+ \cong \mathbb{Z}/3$ the rotation subgroup of W given by the kernel of the map $w \mapsto (-1)^{\ell(w)}$. We also denote by $\Phi_{\rm s} = \{\vec{\alpha}_1, \vec{\alpha}_2\}$ a set of simple roots and $\{\vec{\omega}_1, \vec{\omega}_2\}$ the corresponding fundamental weights, $\Phi_\pm$ the set of positive resp. negative roots, and by

$$P^+ := \{\vec{\lambda} \in \Lambda^\vee | \langle \vec{\lambda}, \vec{\alpha}\rangle > 0\ \forall\ \vec{\alpha} \in \Phi_+\} \tag{4.2.1}$$

the set of dominant integral weights, where $\langle \cdot, \cdot \rangle$ is a quadratic form given by the A_2 Cartan matrix. For $\vec{x} \in \mathbb{C} \otimes_{\mathbb{Z}} \Lambda$, we define the norm $|\vec{x}|^2 := \langle \vec{x}, \vec{x}\rangle$ as usual. We will also define

$$\bar{P}^+ := \{\vec{\lambda} \in \Lambda^\vee | \langle \vec{\lambda}, \vec{\alpha}\rangle \geq 0\ \forall\ \vec{\alpha} \in \Phi_+\}\ . \tag{4.2.2}$$

It will be convenient to introduce the map

$$\mathbb{Z}^2 \to \Lambda, \quad \mathbf{m} = (m_1, m_2) \mapsto \vec{m} := m_2\,\vec{\omega}_1 + (3m_1 + m_2)\vec{\omega}_2\ ,$$

the corresponding norm

$$Q(\mathbf{m}) := \frac{1}{2}|\vec{m}|^2 = (3m_1^2 + 3m_1 m_2 + m_2^2)\ , \tag{4.2.3}$$

and the following functions on $\mathbb{R}^2$

$$\mathcal{F}_\nu\left(\mathbf{x}\right) := x_2^\nu e^{-Q(\mathbf{x})},\ \nu = 0, 1. \tag{4.2.4}$$

Then given a vector $\vec{s}$ in the root lattice, a positive integer m, a divisor D of m, and $\vec{k} \in \Lambda/D\Lambda$, we define the *generalized A_2 false theta function*

$$F^{(\varrho)}(\tau) = \sum_{w\in W} (-1)^{\ell(w)} \sum_{\substack{\vec{n}\in\Lambda\cap P^+ \\ \vec{n}\in w(\vec{k})+D\Lambda}} \min(n_1,n_2)\ q^{\frac{1}{2m}|-w(\vec{s})+\frac{m}{D}\vec{n}|^2} \tag{4.2.5}$$

where ϱ encodes the data $(\vec{s},\vec{k},m,D)$.The A_2 false theta functions, whose quantum modularity has been studied in [96] and which appear in the character of higher rank logarithmic vertex algebra log-$\mathcal{V}_{\bar{\Lambda}}(m)$ [105], always have $D=1$.

These generalized A_2 false theta functions are the building blocks of the $\widehat{Z}^{SU(3)}_{\underline{\vec{b}}}(M_3)$ invariants, when M_3 is a negative Seifert manifold with three exceptional fibers (4.3.13). The study of their explicit quantum modular properties will be the subject of this section.

In the above and elsewhere in this chapter, unless stated otherwise, we use the weight basis notation. For instance, we use (n_1,n_2) to denote $\vec{n} := n_1\vec{\omega}_1 + n_1\vec{\omega}_2 \in \Lambda^\vee$. We also write $\vec{k}|_i := \langle \vec{k}, \vec{\alpha}_i\rangle$ for $\vec{k} \in \mathbb{C}\otimes_{\mathbb{Z}} \Lambda$, so $\vec{n}|_i = n_i$ for $\vec{n} = (n_1,n_2)$.

4.2.1 Identities

We now rewrite the generalised A_2 false theta function in a form which allows us to determine its asymptotic behaviour in the limit where the modular parameter τ approaches a rational number. Similar to [96] we will first rewrite (4.2.5) as a sum over partial theta functions. Concretely, we have the following Lemma.

Lemma 4.2.0.1. *With the notation of* (4.2.5)*, we choose a representative of $\vec{k} \in \Lambda/D\Lambda$ such that $0 \le \langle \vec{k}, \vec{\omega}_i\rangle < D$ for $i = 1,2$, and write*

$$\vec{s} = \vec{\sigma} + \frac{m}{D}\vec{k}.$$

Then we have

$$F^{(\varrho)}(\tau) = F_0^{(\varrho)}(m\tau) + DF_1^{(\varrho)}(m\tau)\ , \tag{4.2.6}$$

where

$$F_0^{(\varrho)}(\tau) := \frac{D}{m} \sum_{w\in W_+} \sum_{i\in\{1,2\}} w(\vec{s})|_i\, F_{0,\boldsymbol{\alpha}_w^{(i)}}(\tau)\ ,\ F_1^{(\varrho)}(\tau) := \sum_{w\in W_+} \sum_{i\in\{1,2\}} F_{1,\boldsymbol{\alpha}_w^{(i)}}(\tau) \tag{4.2.7}$$

with

$$F_{\nu,\boldsymbol{\alpha}}(\tau) = \left(\sum_{\mathbf{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} + (-1)^\nu \sum_{\mathbf{n}\in\mathbf{1}-\boldsymbol{\alpha}+\mathbb{N}_0^2} \right) n_2^\nu q^{Q(\mathbf{n})}. \tag{4.2.8}$$

The $\boldsymbol{\alpha}_w^{(i)}$ vectors are defined in terms of $\vec{s}$, $\vec{k}$, and Weyl group element $w \in W$

$$\boldsymbol{\alpha}_w^{(1)} = \left(x + \frac{\Delta w(\vec{\sigma})}{m}, \xi_{w,1} - \frac{w(\vec{\sigma})|_1}{m} \right), \ \boldsymbol{\alpha}_w^{(2)} = \left(1 - x - \frac{\Delta w(\vec{\sigma})}{m}, \xi_{w,2} - \frac{w(\vec{\sigma})|_2}{m} \right) \tag{4.2.9}$$

in which

$$\xi_{w,i} := \left\lceil -\frac{w(\vec{k})|_i}{D} \right\rceil, \ \ \Delta w(\vec{\sigma}) := \frac{w(\vec{\sigma})|_1 - w(\vec{\sigma})|_2}{3}, \ \ x = \begin{cases} 0 & \text{when } w(\vec{k})|_2 \geq w(\vec{k})|_1 \\ 1 & \text{otherwise.} \end{cases} \tag{4.2.10}$$

The proof can be found in 4.6.

For later convenience, we will also use the following rewriting of (4.2.7)

$$F_0^{(\varrho)}(\tau) = \sum_{\boldsymbol{\alpha}\in\mathcal{S}} \eta_0(\boldsymbol{\alpha}) \sum_{\mathbf{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} q^{Q(\boldsymbol{n})}, \quad F_1^{(\varrho)}(\tau) = \sum_{\boldsymbol{\alpha}\in\mathcal{S}} \eta_1(\boldsymbol{\alpha}) \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} n_2 \, q^{Q(\mathbf{n})}. \tag{4.2.11}$$

where we write

$$\mathcal{S} = \bigcup_{w\in W_+} \left\{ \boldsymbol{\alpha}_w^{(1)}, \boldsymbol{\alpha}_w^{(2)}, \bar{\boldsymbol{\alpha}}_w^{(1)}, \bar{\boldsymbol{\alpha}}_w^{(2)} \right\}, \tag{4.2.12}$$

and

$$\bar{\boldsymbol{\alpha}}_w^{(i)} := \mathbf{1} - \boldsymbol{\alpha}_w^{(i)}, \ \ \eta_0(\boldsymbol{\alpha}_w^{(i)}) = \eta_0(\bar{\boldsymbol{\alpha}}_w^{(i)}) = \frac{D}{m} w(\vec{s})|_i \ , \ \ \eta_1(\boldsymbol{\alpha}_w^{(i)}) = -\eta_1(\bar{\boldsymbol{\alpha}}_w^{(i)}) = 1, \tag{4.2.13}$$

where $\mathbf{1} = (1,1)$. We will also write

$$\tilde{\mathcal{S}} = \bigcup_{w\in W_+} \left\{ \boldsymbol{\alpha}_w^{(1)}, \boldsymbol{\alpha}_w^{(2)} \right\}, \tag{4.2.14}$$

so that

$$F_\nu^{(\varrho)}(\tau) = \sum_{\boldsymbol{\alpha}\in\tilde{\mathcal{S}}} \eta_\nu(\boldsymbol{\alpha}) \, F_{\nu,\boldsymbol{\alpha}}. \tag{4.2.15}$$

Note that $m\boldsymbol{\alpha} \in \mathbb{Z}^2$, since $\vec{\sigma} \in \Lambda$ and hence $\Delta w(\vec{\sigma}) = \langle \Delta\vec{\omega}, w(\vec{\sigma}) \rangle \in \mathbb{Z}$, where we write $\Delta\vec{\omega} = \vec{\omega}_1 - \vec{\omega}_2 = \frac{1}{3}(\vec{\alpha}_1 - \vec{\alpha}_2)$.

To study the radial limit of $F^{(\varrho)}$, later we will be working with functions of the form $\sum_{\mathbf{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2}\mathcal{F}_\nu(\mathbf{n})$ for $\nu=0,1$ with $0\le\alpha_i\le 1$ for $i=1,2$. See (4.2.4). It will therefore be useful to note the following result on the effect of integral shifts of $\boldsymbol{\alpha}$.

Lemma 4.2.0.2. *Let $\boldsymbol{\beta}=\boldsymbol{\alpha}+(\delta\alpha_1,\delta\alpha_2)$ for $\delta\alpha_1,\delta\alpha_2\in\mathbb{Z}$. Consider $F_{\nu,\boldsymbol{\alpha}}(\tau)$ for $\nu=0,1$ as defined in (4.2.7). Then*

$$F_{\nu,\boldsymbol{\beta}}(\tau)-F_{\nu,\boldsymbol{\alpha}}(\tau)$$

is in the integral linear span of one-dimensional lattice sums $\{\tilde\theta^1[\kappa,a],\tilde\theta^0[\kappa,a]|\kappa,a\in\mathcal{Q}\}$, up to the addition of a finite polynomial $p(\tau)\in q^\Delta\mathbb{Z}[q]$, where

$$\begin{aligned}\tilde\theta^0[\kappa,a](\tau) &:= \sum_{n\in\mathbb{Z}}|n+a|\,q^{\kappa(n+a)^2}\\ \tilde\theta^1[\kappa,a](\tau) &:= \sum_{n\in\mathbb{Z}}\operatorname{sgn}(n+a)q^{\kappa(n+a)^2}\end{aligned}\;. \tag{4.2.16}$$

Note that $\tilde\theta^0$ and $\tilde\theta^1$ are themselves Eichler integrals of weight 1/2 resp. 3/2 theta functions, up to finite polynomials. See (4.0.8). The proof can be found in 4.6. Using the above lemma to shift vectors by integers, in the following section we will consider vectors $\boldsymbol{\mu}=(\mu_1,\mu_2)$ satisfying $\quad 0\le\mu_1,\mu_2\le 1$.

4.2.2 Radial Limits

In this subsection we aim to study the radial limit $\tau\to\frac{h}{k}\in\mathbb{Q}$, approached from the upper-half plane $\mathbb{H}$, of the generalized A_2 false theta functions $F^{(\varrho)}(\tau)$. To do so, we will use the Euler-Maclaurin summation formula, a strategy also employed in [96]. First we will recall the following asymptotic expansion formula, which goes back to [106][2].

For $\boldsymbol{\mu}=(\mu_1,\mu_2)$ with $\mu_1,\mu_2\ge 0$, and $F:\mathbb{R}^2_{\ge 0}\to\mathbb{R}$ a smooth rapidly decaying C^∞ function, the asymptotic expansion in the limit $t\to 0^+$ of F is given by [106]

$$\begin{aligned}&\sum_{\mathbf{n}\in\mathbb{N}_0^2}F((\mathbf{n}+\boldsymbol{\mu})t)\sim\frac{\mathcal{I}_F}{t^2}\\ &-\sum_{n\in\mathbb{N}}\frac{t^{n-2}}{n!}\int_0^\infty dx\left(B_n(\mu_1)F^{(n-1,0)}(0,x)+B_n(\mu_2)F^{(0,n-1)}(x,0)\right)\\ &+\sum_{\mathbf{n}\in\mathbb{N}^2}\frac{t^{n_1+n_2-2}}{n_1!n_2!}B_{n_1}(\mu_1)B_{n_2}(\mu_2)F^{(n_1-1,n_2-1)}(0,0)\;,\end{aligned} \tag{4.2.17}$$

[2] To apply the formulas in [106] correctly, it is important that the shift vector, denoted ϱ here, must satisfy $\mu_1,\mu_2\ge 0$.

where $\sim$ means that the two sides agree up to $O(t^N)$ terms for any $N \in \mathbb{N}$ and $\mathcal{I}_F$ is given by

$$\mathcal{I}_F := \int_0^\infty \int_0^\infty F(x_1, x_2) dx_1 dx_2 \ .$$

In the above expression, $B_m(x)$ are the Bernoulli polynomials whose generating function is given by $\frac{te^{xt}}{e^t-1} = \sum_{n=0}^\infty B_n(x)\frac{t^n}{n!}$. A key feature of these polynomials that follows directly from the generating function is their reflection property

$$B_m(x) = (-1)^m B_m(1-x) \ . \tag{4.2.18}$$

In order to apply (4.2.17) to derive the radial limit, we will further rewrite our generalized false theta function (4.2.7) for when $\mathrm{Re}\tau \in \mathcal{Q}$: for coprime integers h, k and $t \in \mathbb{R}_{>0}$, we have for $\nu = 0, 1$

$$F_\nu^{(\varrho)}(\tfrac{h}{k} + \tfrac{it}{2\pi}) = \\ (\sqrt{t})^{-\nu} \sum_{\boldsymbol{\mu} \in \mathcal{S}} \eta_\nu(\boldsymbol{\mu}) \sum_{\boldsymbol{\ell} \in (\mathbb{Z}/k\bar{m})^2} \mathrm{e}\left(\frac{h}{k} Q(\boldsymbol{\ell} + \boldsymbol{\mu})\right) \sum_{\substack{\boldsymbol{n} \in \frac{1}{k\bar{m}}(\boldsymbol{\ell}+\boldsymbol{\mu}) + \mathbb{Z}^2 \\ k\bar{m}\boldsymbol{n} \in \boldsymbol{\mu} + \mathbb{N}_0^2}} \mathcal{F}_\nu\left(k\bar{m}\sqrt{t}\,\boldsymbol{n}\right) \tag{4.2.19}$$

where we have defined

$$\delta := (h, m) \ , \quad \bar{m} := \frac{m}{\delta} \ , \tag{4.2.20}$$

and $\mathcal{F}_\nu(\boldsymbol{x})$ is given as in (4.2.4). To see that the sum over $\boldsymbol{\ell}$ is well-defined, note that $m\boldsymbol{\mu} \in \mathbb{Z}^2$ for all $\boldsymbol{\mu} \in \mathcal{S}$. To derive the asymptotic expansion (Proposition 4.2.0.1) of $F_\nu^{(\varrho)}$, we first establish the following lemma.

Lemma 4.2.0.3. *Let $\mathcal{S}$, η_ν, and $\bar{m}$ be as given in* (4.2.12)*,* (4.2.13) *and* (4.2.20)*. Then for $\nu = 0, 1$*

$$\sum_{\boldsymbol{\mu} \in \mathcal{S}} \eta_\nu(\boldsymbol{\mu}) \sum_{\boldsymbol{\ell} \in (\mathbb{Z}/k\bar{m})^2} \mathrm{e}\left(\frac{h}{k} Q(\boldsymbol{\ell} + \boldsymbol{\mu})\right) = 0 \ . \tag{4.2.21}$$

See Appendix 4.6 for the proof.

Lemma 4.2.0.4. *Given $w \in W_+$*

$$\sum_{\boldsymbol{\mu} \in \left\{\boldsymbol{\mu}_w^{(1)}, \boldsymbol{\mu}_w^{(2)}\right\}} \sum_{0 \le \ell_1, \ell_2 < k\bar{m}} B_n\left(\frac{\ell_1 + \mu_1}{k\bar{m}}\right) \mathrm{e}\left(\frac{h}{k} Q\left(\boldsymbol{\ell} + \boldsymbol{\mu}\right)\right) = 0 \tag{4.2.22}$$

for any odd positive integer n.

See Appendix 4.6 for the proof.

After establishing the above lemmas, upon using equations (4.2.17) and (4.2.19) we are now ready to prove the following asymptotic formula.

Proposition 4.2.0.1. *For $\nu = 0, 1$, the asymptotic limit near $\frac{h}{k}$ is given by*

$$F_\nu^{(\varrho)}(\tfrac{h}{k} + \tfrac{it}{2\pi}) \sim -2 \sum_{\boldsymbol{\alpha}\in\tilde{\mathcal{S}}} \eta_\nu(\boldsymbol{\mu}) \sum_{0\le \ell_1,\ell_2<k\bar{m}} \mathrm{e}\left(\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\mu})\right) \times \\ \Bigg[\sum_{\substack{n>1 \\ n\equiv\nu(2)}} \frac{(k\bar{m})^{n-2}\, t^{\frac{n-2-\nu}{2}}}{n!} \int_0^\infty dx \left(B_n\left(\frac{\ell_2+\mu_2}{k\bar{m}}\right) \mathcal{F}_\nu^{(0,n-1)}(x,0) \right. \\ \left. + B_n\left(\frac{\ell_1+\mu_1}{k\bar{m}}\right) \mathcal{F}_\nu^{(n-1,0)}(0,x)\right) - \sum_{\substack{\mathbf{n}\in\mathbb{N}^2 \\ n_1\equiv n_2+\nu \,(\mathrm{mod}\,2)}} \frac{(k\bar{m})^{n_1+n_2-2}\, t^{\frac{n_1+n_2-2-\nu}{2}}}{n_1!\, n_2!} \times \\ B_{n_1}\left(\frac{\ell_1+\mu_1}{k\bar{m}}\right) B_{n_2}\left(\frac{\ell_2+\mu_2}{k\bar{m}}\right) \mathcal{F}_\nu^{(n_1-1,n_2-1)}(\mathbf{0}) \Bigg] . \tag{4.2.23}$$

Proof. In (4.2.19), choose the sum over $\boldsymbol{\ell}$ to be over the range $0 \le \ell_1, \ell_2 < k\bar{m}$ and apply the Euler-Maclaurin summation formula (4.2.17) to $\sum_{\mathbf{n}\in\mathbb{N}_0^2} F((\mathbf{n}+\boldsymbol{\mu}')t')$, with

$$\boldsymbol{\mu}' = \frac{\boldsymbol{\ell}+\boldsymbol{\mu}}{k\bar{m}} \ , \quad t' = k\bar{m}\sqrt{t} \ , \quad F(\boldsymbol{x}) = \mathcal{F}_\nu(\boldsymbol{x}).$$

First note that the potential divergent, $\boldsymbol{\ell}$- and $\boldsymbol{\mu}$-independent term $\mathcal{I}_F/t^2$ actually vanishes contribution due to Lemma 4.2.0.3. Second, note that the reflection property (4.2.18) of the Bernoulli polynomials leads to the identity

$$\sum_{0\le \ell_1,\ell_2<k\bar{m}} \mathrm{e}\left(\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\mu})\right) B_n\left(\frac{\ell_i+\mu_i}{k\bar{m}}\right) \\ = (-1)^n \sum_{0\le \ell_1,\ell_2<k\bar{m}} \mathrm{e}\left(\frac{h}{k}Q(\boldsymbol{\ell}+\mathbf{1}-\boldsymbol{\mu})\right) B_n\left(\frac{\ell_i+1-\mu_i}{k\bar{m}}\right) \tag{4.2.24}$$

for $i = 1, 2$, and

$$\sum_{0 \le \ell_1, \ell_2 < k\bar{m}} \mathrm{e}\left(\frac{h}{k} Q(\boldsymbol{\ell} + \boldsymbol{\mu})\right) B_{n_1}\left(\frac{\ell_1 + \mu_1}{k\bar{m}}\right) B_{n_2}\left(\frac{\ell_2 + \mu_2}{k\bar{m}}\right) =$$
$$(-1)^{n_1+n_2} \sum_{0 \le \ell_1, \ell_2 < k\bar{m}} \mathrm{e}\left(\frac{h}{k} Q(\boldsymbol{\ell} + \mathbf{1} - \boldsymbol{\mu})\right) B_{n_1}\left(\frac{\ell_1 + 1 - \mu_1}{k\bar{m}}\right) B_{n_2}\left(\frac{\ell_2 + 1 - \mu_2}{k\bar{m}}\right) \tag{4.2.25}$$

where we have shifted the sum over $\boldsymbol{\ell}$ to $-\boldsymbol{\ell} + \mathbf{1}\,(k\bar{m} - 1)$. From (4.2.13) and (4.2.14), since $\boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}} := \mathbf{1} - \boldsymbol{\mu}$ appear in $\mathcal{S}$ in pairs, we can fold the sum into a sum over $\tilde{\mathcal{S}}$. Moreover, from $\eta_\nu(\bar{\boldsymbol{\mu}}) = (-1)^\nu \eta_\nu(\boldsymbol{\mu})$, the above identity implies that terms in the sum in the third line of (4.2.23) vanish unless $n_1 + n_2 \equiv \nu(2)$. Similarly, the terms in the second line of (4.2.23) vanish unless $n \equiv \nu(2)$. To show that the potentially divergent term with $n = 1$ when $\nu = 1$ vanishes, we first note that

$$\mathcal{F}_1^{(0,0)}(x, 0) = \mathcal{F}_1(x, 0) = 0 \tag{4.2.26}$$

and we are hence left to show that

$$\begin{aligned}
&\int_0^\infty dx \mathcal{F}_1(0, x) \sum_{\boldsymbol{\mu} \in \tilde{\mathcal{S}}} \eta_\nu(\boldsymbol{\mu}) \sum_{0 \le \ell_1, \ell_2 < k\bar{m}} \mathrm{e}\left(\frac{h}{k} Q(\boldsymbol{\ell} + \boldsymbol{\mu})\right) B_1\left(\frac{\ell_1 + \mu_1}{k\bar{m}}\right) \\
&= \int_0^\infty dx \mathcal{F}_1(0, x) \sum_{w \in W_+} \sum_{\boldsymbol{\mu} \in \{\boldsymbol{\mu}_w^{(1)}, \boldsymbol{\mu}_w^{(2)}\}} \sum_{0 \le \ell_1, \ell_2 < k\bar{m}} \mathrm{e}\left(\frac{h}{k} Q(\boldsymbol{\ell} + \boldsymbol{\mu})\right) B_1\left(\frac{\ell_1 + \mu_1}{k\bar{m}}\right) \\
&= 0
\end{aligned} \tag{4.2.27}$$

which is true by Lemma 4.2.0.4. □

4.2.3 Companions

Having established the asymptotic expansions of the functions $F_\nu^{(\varrho)}$ in the limit $\tau \to \frac{h}{k} \in \mathcal{Q}$, in this subsection we will show that certain functions $\mathbb{E}_\nu^{*(\varrho)}(\tau)$, consisting of generalised complementary error functions, are their companion functions in the sense that they have compatible asymptotic behaviour.

Definition 4.2.1. We say two functions $\widehat{F}$ and $\check{F}$ on the upper-half plane are *companions* of each other if their asymptotic expansions near the rationals satisfy

$$\widehat{F}\left(\tfrac{h}{k} + \tfrac{it}{2\pi}\right) \sim \sum_{\ell \ge 0} a_{h,k}(m) t^m, \tag{4.2.28}$$

and

$$\check{F}\left(\tfrac{h}{k}+\tfrac{it}{2\pi}\right) \sim \sum_{\ell\geq 0} a_{-h,k}(m)(-t)m, \tag{4.2.29}$$

for all coprime integers h, k with $k>0$.

Importantly, given a function on the upper-half-plane, its companion is anything but unique; the definition of the companion function is insensitive to the addition of functions vanishing at all rationals.

To establish companions of the generalised A_2 false theta functions, we define for $\nu = 0, 1$

$$\mathbb{E}_\nu^{*(\varrho)}(\tau) := \frac{1}{2}\sum_{\boldsymbol{\mu}\in\mathcal{S}} \eta_\nu(\boldsymbol{\mu}) \left(\sum_{\boldsymbol{n}\in\boldsymbol{\mu}+\mathbb{N}_0^2} g_\nu(n_1,n_2) + \sum_{\boldsymbol{n}\in(1-\mu_1,\mu_2)+\mathbb{N}_0^2} g_\nu(-n_1,n_2) \right) \tag{4.2.30}$$

where

$$g_\nu(n_1,n_2) := q^{-Q(\boldsymbol{n})}\left(n_2^\nu M_2^*\left(\sqrt{3};\sqrt{3v}\,(2n_1+n_2),\sqrt{v}\,n_2\right) + \delta_{\nu,1}\frac{e^{-\pi v(3n_1+2n_2)^2}}{2\pi\sqrt{v}}M^*(\sqrt{3v}\,n_1)\right)t \tag{4.2.31}$$

and $v := \Im\tau$, and show that, when writing the asymptotic expansion of $F_\nu^{(\varrho)}$ as

$$F_\nu^{(\varrho)}\left(\tfrac{h}{k}+\tfrac{it}{2\pi}\right) \sim \sum_{m\geq 0} a_{h,k}^{(\nu)}(m)t^m, \tag{4.2.32}$$

we have the following proposition.

Proposition 4.2.1.1. $\mathbb{E}_\nu^{*(\varrho)}(\tau)$ *as defined in equation* (4.2.30) *is a companion of* $F_\nu^{(\varrho)}(\tau)$*, whose asymptotic expansion in the limit* $\tau = \frac{h}{k}+\frac{it}{2\pi}$*,* $t\to 0^+$ *satisfies*

$$\mathbb{E}_\nu^{*(\varrho)}\left(\frac{h}{k}+\frac{it}{2\pi}\right) \sim \sum_{m\geq 0} a_{-h,k}^{(\nu)}(m)\,(-t)^m\ . \tag{4.2.33}$$

For the proof of the proposition, it will be convenient to define the following functions and establish the identities in Lemma 4.2.1.1. For $\nu = 0, 1$, let the functions $\mathcal{G}_\nu, \tilde{\mathcal{G}}_\nu : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$\mathcal{G}_\nu(x_1,x_2) = \frac{1}{2}\,x_2^\nu M_2^*\left(\sqrt{3};\frac{\sqrt{3}(2x_1+x_2)}{\sqrt{2\pi}},\frac{x_2}{\sqrt{2\pi}}\right)e^{Q(\boldsymbol{x})} = \tilde{\mathcal{G}}_\nu(-x_1,x_2), \tag{4.2.34}$$

where the functions M_2, M_2^* are defined in equations (C.0.9) and (C.0.11) respectively, following [86]. The following relations to $\mathcal{F}_\nu$ have been established in §7 of [96].

Lemma 4.2.1.1. *For $\nu = 0, 1$ the following identities hold for $n \in \mathbb{N}$, $n \equiv \nu+1$ (2):*

$$\begin{aligned}\int_0^\infty dx\, \mathcal{F}_\nu^{(0,n)}(x,0) &= (-1)^{\lfloor\frac{n-1}{2}\rfloor} \int_0^\infty dx \left(\mathcal{G}_\nu^{(0,n)} + \tilde{\mathcal{G}}_\nu^{(0,n)}\right)(x,0)\\ \int_0^\infty dx\, \mathcal{F}_\nu^{(n,0)}(0,x) &= (-1)^{\lfloor\frac{n-1}{2}\rfloor} \int_0^\infty dx \left(\mathcal{G}_\nu^{(n,0)} + (-1)^\nu \tilde{\mathcal{G}}_\nu^{(n,0)}\right)(0,x)\\ &\quad - \frac{1}{\sqrt{2}}\left[\frac{d^n}{dy^n} e^{\frac{3y^2}{4}}\right]_{y=0}\end{aligned} \tag{4.2.35}$$

and

$$\mathcal{F}_\nu^{(\boldsymbol{n})}(\mathbf{0}) = (-1)^{\lfloor\frac{n_1+n_2}{2}\rfloor}\left(\mathcal{G}_\nu^{(\boldsymbol{n})}(\mathbf{0}) + (-1)^{n_1+1}\tilde{\mathcal{G}}_\nu^{(\boldsymbol{n})}(\mathbf{0})\right) \tag{4.2.36}$$

for $n_1 + n_2 \equiv \nu$ (2).

Now we are ready to prove Proposition 4.2.1.1. As before, we can re-express $\mathbb{E}_\nu^{*(\varrho)}(\tau)$ when $\mathrm{Re}\tau \in \mathcal{Q}$ as

$$\begin{aligned}&\mathbb{E}_\nu^{*(\varrho)}\left(\tfrac{h}{k} + \tfrac{it}{2\pi}\right) =\\ &(\sqrt{t})^{-\nu} \sum_{\boldsymbol{\mu}\in\mathcal{S}} \eta_\nu(\boldsymbol{\mu}) \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/k\bar{m})^2} \left(\mathrm{e}\left(-\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\mu})\right) \sum_{\substack{\boldsymbol{n}\in\frac{1}{k\bar{m}}(\boldsymbol{\ell}+\boldsymbol{\mu})+\mathbb{Z}^2\\ k\bar{m}\boldsymbol{n}\in\boldsymbol{\mu}+\mathbb{N}_0^2}} \mathcal{G}'_\nu\left(k\bar{m}\sqrt{t}\,\boldsymbol{n}\right)\right.\\ &\left.+\,\mathrm{e}\left(-\frac{h}{k}Q(-1+\mu_1-\ell_1, \mu_2+\ell_2)\right) \sum_{\substack{\boldsymbol{n}\in\frac{1}{k\bar{m}}(\boldsymbol{\ell}+(1-\mu_1,\mu_2))+\mathbb{Z}^2\\ k\bar{m}\boldsymbol{n}\in(1-\mu_1,\mu_2)+\mathbb{N}_0^2}} \tilde{\mathcal{G}}'_\nu\left(k\bar{m}\sqrt{t}\,\boldsymbol{n}\right)\right)\end{aligned} \tag{4.2.37}$$

where

$$\mathcal{G}'_\nu(x_1,x_2) = \mathcal{G}_\nu(x_1,x_2) + \delta_{\nu,1}\frac{1}{2\sqrt{2\pi}}e^{-\left(\frac{3}{2}x_1^2+3x_1x_2+x_2^2\right)}M^*\left(\sqrt{\frac{3}{2\pi}}x_1\right). \tag{4.2.38}$$

Applying (4.2.17) to (4.2.37), from Lemma 4.2.0.3 we see that

$$\mathbb{E}_{\nu}^{*(\varrho)}(\tfrac{h}{k}+\tfrac{it}{2\pi}) \sim 2\sum_{\boldsymbol{\mu}\in\tilde{\mathcal{S}}}\eta_\nu(\boldsymbol{\mu})\sum_{0\le\ell_1,\ell_2<k\bar{m}} \mathrm{e}\Big(-\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\mu})\Big)$$
$$\times\Bigg(-\sum_{\substack{n>1\\ n\equiv\nu(2)}}\frac{(k\bar{m})^{n-2}\,t^{\frac{n-2-\nu}{2}}}{n!}\Bigg[B_n\left(\frac{\ell_2+\mu_2}{k\bar{m}}\right)\int_0^\infty dx\,(\mathcal{G}_\nu^{(0,n-1)}+\tilde{\mathcal{G}}_\nu^{(0,n-1)})(x,0)$$
$$+B_n\left(\frac{\ell_1+\mu_1}{k\bar{m}}\right)\int_0^\infty dx\,(\mathcal{G}_\nu^{(n-1,0)}+(-1)^\nu\tilde{\mathcal{G}}_\nu^{(n-1,0)})(0,x)\Bigg]$$
$$+\sum_{\substack{\mathbf{n}\in\mathbb{N}^2\\ n_1\equiv n_2+\nu\,(\mathrm{mod}\,2)}}\frac{(k\bar{m})^{n_1+n_2-2}\,t^{\frac{n_1+n_2-2-\nu}{2}}}{n_1!\,n_2!}B_{n_1}\left(\frac{\ell_1+\mu_1}{k\bar{m}}\right)B_{n_2}\left(\frac{\ell_2+\mu_2}{k\bar{m}}\right)$$
$$\times\left(\mathcal{G}_\nu^{(n_1-1,n_2-1)}-(-1)^{n_1+1}\tilde{\mathcal{G}}_\nu^{(n_1-1,n_2-1)}\right)(\mathbf{0})\Bigg) \tag{4.2.39}$$

holds for $\nu=0$, where we have also used (4.2.24)-(4.2.25) to identify the contribution from $\boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$. For $\nu=1$, one needs to take the additional term in (4.2.38) into account. As shown in detail in [96] (see §7), the contributions of these terms to the asymptotic expansion vanish due to Lemma 4.2.0.4.

Similarly, combining Proposition 4.2.0.1 and Lemma 4.2.1.1, and again evoking Lemma 4.2.0.4, the comparison with (4.2.39) shows that the Proposition 4.2.1.1 is true.

4.2.4 Eichler Integrals

In this subsection, we will relate the companion of the generalised A_2 false theta function $F^{(\rho)}$ to certain Eichler integrals. More precisely, we will show that the companion function $\mathbb{E}_\nu^{*(\varrho)}$ in Proposition 4.2.1.1 is an Eichler integral given in Proposition 4.2.1.2, up to one-dimensional integrals specified in Lemma 4.2.1.2.

To show this, for $\nu=0,1$ we first define the following functions

$$\mathbb{E}_\nu^{(\varrho)}(\tau) := \frac{1}{2}\sum_{\boldsymbol{\mu}\in\tilde{\mathcal{S}}}\eta_\nu(\boldsymbol{\mu})\sum_{\mathbf{n}\in\boldsymbol{\mu}+\mathbb{Z}^2} q^{-Q(\mathbf{n})}\times$$
$$\left[\left(\frac{1}{2\pi i}\frac{\partial}{\partial z}\right)^\nu\left(e^{2\pi i\nu n_2 z}M_2\left(\sqrt{3};\sqrt{3v}\,(2n_1+n_2),\sqrt{v}\left(n_2-\frac{2\mathrm{Im}(z)}{v}\right)\right)\right)\right]_{z=0} \tag{4.2.40}$$

that are closely related to the companion function $\mathbb{E}_\nu^{*(\varrho)}$. More precisely, their difference is given in terms of one-dimensional error function (see (C.0.2)) as

Lemma 4.2.1.2.

$$\mathbb{E}_\nu^{(\rho)}(\tau) = \mathbb{E}_\nu^{*(\rho)}(\tau) + \sum_{\boldsymbol{\mu}\in\tilde{\mathcal{S}}} \eta_\nu(\boldsymbol{\mu}) X_\nu(\boldsymbol{\mu}) \tag{4.2.41}$$

where X_ν are given by

$$X_0(\boldsymbol{\mu}) = \left(\sum_{\boldsymbol{n}\in\boldsymbol{\mu}+\mathbb{N}_0^2} + \sum_{\boldsymbol{n}\in(1,1)-\boldsymbol{\mu}+\mathbb{N}_0^2} - \sum_{\boldsymbol{n}\in(1-\mu_1,\mu_2)+\mathbb{N}_0^2} - \sum_{\boldsymbol{n}\in(\mu_1,1-\mu_2)+\mathbb{N}_0^2} \right)$$
$$\times \left(\delta_{n_1,0}\left(1-\delta_{n_2,0}\right) M\left(2\sqrt{v}n_2\right) + \delta_{n_2,0}\left(1-\delta_{n_1,0}\right) M\left(2\sqrt{3v}n_1\right) \right.$$
$$\left. - \delta_{n_1,0}\delta_{n_2,0} \right) q^{-Q(\boldsymbol{n})}$$
$$= \begin{cases} (-1)^{\mu_1}\left(\sum_{k=\mu_2+\mathbb{N}_0} - \sum_{k=1-\mu_2+\mathbb{N}_0}\right) M\left(2\sqrt{v}k\right) q^{-k^2} & \mu_1 \in \{0,1\} \not\ni \mu_2 \\ (-1)^{\mu_2}\left(\sum_{k=\mu_1+\mathbb{N}_0} - \sum_{k=1-\mu_1+\mathbb{N}_0}\right) M\left(2\sqrt{3v}k\right) q^{-3k^2} & \mu_1 \notin \{0,1\} \ni \mu_2 \\ (-1)^{\mu_1+\mu_2+1} & \mu_1 \in \{0,1\} \ni \mu_2 \\ 0 & \mu_1 \notin \{0,1\} \not\ni \mu_2 \end{cases}$$

for $\nu = 0$ and

$$X_1(\boldsymbol{\mu}) = \left(\sum_{\boldsymbol{n}\in\boldsymbol{\mu}+\mathbb{N}_0^2} - \sum_{\boldsymbol{n}\in(1,1)-\boldsymbol{\mu}+\mathbb{N}_0^2} - \sum_{\boldsymbol{n}\in(1-\mu_1,\mu_2)-\boldsymbol{\mu}+\mathbb{N}_0^2} + \sum_{\boldsymbol{n}\in(\mu_1,1-\mu_2)-\boldsymbol{\mu}+\mathbb{N}_0^2} \right)$$
$$\times\, \delta_{n_1,0}\left(n_2 M\left(2\sqrt{v}n_2\right) + \frac{1}{4\pi\sqrt{v}} e^{-4\pi n_2^2 v} \right) q^{-n_2^2} =$$
$$\begin{cases} (-1)^{\mu_1}\left(\sum_{k\in\mu_2+\mathbb{N}_0^2} + \sum_{k\in1-\mu_2+\mathbb{N}_0^2} \right)\left(kM\left(2\sqrt{v}k\right) + \frac{1}{4\pi\sqrt{v}} e^{-4\pi k^2 v} \right) q^{-k^2} & \mu_1 \in \{0,1\} \\ 0 & \mu_1 \notin \{0,1\} \end{cases}$$

for $\nu = 1$.

The proof can be found in Appendix 4.6. Note that in §4.2.3 we used $\mathbb{E}_\nu^{*(\rho)}$ for the application of Euler-Maclaurin formula, as $g_\nu(n_1,n_2)$ in (4.2.31) is continuous on $\mathbb{R}_{\geq 0}^2$ as a function of (n_1,n_2), unlike the counterpart in $\mathbb{E}_\nu^{(\rho)}$; the difference between the two functions then comes precisely from the cases when at least one of n_1 and n_2 vanishes. Moreover, from [4]

$$M(x\sqrt{v}) = i\frac{x}{\sqrt{2}}q^{\frac{x^2}{4}}\int_{-\bar{\tau}}^{i\infty}\frac{e^{\frac{\pi i x^2 w}{2}}}{\sqrt{-i(w+\tau)}}dw \tag{4.2.42}$$

we see that $X_\nu(\boldsymbol{\mu})$ can be written as a linear combination of non-holomorphic Eichler integrals of rank one theta functions (4.0.7), and hence

$$\mathbb{E}_\nu^{(\rho)} = \mathbb{E}_\nu^{*(\rho)} + z_{1\text{d}} \tag{4.2.43}$$

in the notation of (4.0.17).

Finally, by carefully rewriting the integrals in the rank two generalised complementary error functions M_2 in the definition of $\mathbb{E}_\nu^{(\varrho)}$, we arrive at the following relation between the companion and the Eichler integrals, as shown in Appendix 4.6.

Proposition 4.2.1.2.

$$\mathbb{E}_\nu^{(\varrho)}(\tau) = \sum_{w\in W^+}\mathbb{E}_{\nu,w}^{(\varrho)}(\tau) \tag{4.2.44}$$

where

$$\mathbb{E}_{\nu,w}^{(\varrho)}(\tau) := \frac{\sqrt{3}}{4\pi^\nu}\int_{-\bar{\tau}}^{i\infty}\int_{z_1}^{i\infty}\frac{\Theta_{\nu,w}^{(\varrho)}(\mathbf{z})}{(-i(z_1+\tau))^{1/2}(-i(z_2+\tau))^{\nu+1/2}}dz_2dz_1 \tag{4.2.45}$$

and

$$\Theta_{\nu,w}^{(\varrho)}(\mathbf{z}) = (m)^{2\nu-3}\left(3D\Delta w(\vec{s})\right)^{1-\nu}\sum_{\delta\in\mathbb{Z}/2}\theta^1_{m,m\delta+\langle\vec{\rho},w(\vec{\sigma})\rangle}\left(\frac{z_1}{m}\right)\theta^{1-\nu}_{m,m\delta+\langle\Delta\vec{\omega},w(\vec{\sigma})\rangle}\left(\frac{3z_2}{m}\right) \tag{4.2.46}$$

are given by sums of products of two theta functions of one-dimensional lattices.

Combining Lemma 4.2.1.2 and identity (C.0.3), we establish that the companions of the generalised A_2 false theta functions are given in terms of Eichler integrals of rank two and rank one theta functions.

4.2.5 Quantum Modularity

In this subsection we review the relation between the Eichler integrals discussed in the previous subsection and quantum modular forms.

Let us recall the definition of (higher depth) quantum modular forms, extended to the vector-valued case. We first recall the familiar definition of slash operators, acting on a (vector-valued) function on the compactified upper-halp plane $\hat{\mathbb{H}} := \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$: given $k \in \frac{1}{2}\mathbb{Z}$ and n-dimensional multiplier χ for $\Gamma \subset SL_2(\mathbb{Z})$, namely a group homomorphism $\Gamma \to GL_n(\mathbb{C})$, and for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we define the action of the slash operator $|_{\chi,k}\gamma$ acting on $f = (f_r) : \hat{\mathbb{H}} \to \mathbb{C}^n$ as

$$f|_{k,\chi}\gamma(\tau) := f(\gamma\tau)\chi(\gamma)(c\tau + d)^{-k}, \tag{4.2.47}$$

where we have written $\gamma\tau = \frac{a\tau+b}{c\tau+d}$ as usual.

Definition 4.2.2 (Vector-Valued Quantum Modular Form)**.** A function $f : \mathbb{Q} \to \mathbb{C}^n$, is a (vector-valued) quantum modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ with multiplier χ for $\Gamma \subset SL_2(\mathbb{Z})$ if for every $\gamma \in \Gamma$ the vector-valued function, the cocycle

$$h_\gamma(\tau) := f(\tau) - f|_{k,\chi}\gamma(\tau) \tag{4.2.48}$$

can be extended to an open subset of $\mathbb{R}$ and is real-analytic there. We will denote the vector space of such forms by $Q_k(\Gamma, \chi)$.

Definition 4.2.3 (Vector-Valued Higher Depth Quantum Modular Form [91])**.** A function $f : \mathbb{Q} \to \mathbb{C}^n$ is a quantum modular form of depth $N \in \mathbb{N}$ and weight $k \in \frac{1}{2}\mathbb{Z}$ with multiplier χ for $\Gamma \subset SL_2(\mathbb{Z})$ if for every $\gamma \in \Gamma$

$$h_\gamma := f - f|_{k,\chi}\gamma \in \bigoplus_j Q_{k_j}^{N_j}(\Gamma, \chi_j)\mathcal{O}(R) , \tag{4.2.49}$$

where j runs over a finite set, $k_j \in \frac{1}{2}\mathbb{Z}$, $N_j \in \mathbb{N}$ with $\max(N_j) = N - 1$, χ_j are multipliers, $\mathcal{O}(R)$ is the space of real-analytic functions on $R \subset \mathbb{R}$ which contains an open subset of $\mathbb{R}$. We also set $Q_k^1(\Gamma, \chi) = Q_k(\Gamma, \chi)$, $Q_k^0(\Gamma, \chi) = 1$ and $Q_k^N(\Gamma, \chi)$ denotes the space of quantum modular forms of weight k, depth N, and with n-dimensional multiplier χ for Γ.

Eichler Integrals and Quantum Modular Forms

It is known that (holomorphic and non-holomorphic) Eichler integrals furnish examples of quantum modular forms. We define the following two vector-valued functions

$$f^*(\tau) := \int_{-\bar{\tau}}^{i\infty} dw \, \frac{\overline{f(-\bar{w})}}{(-i(w+\tau))^{2-k}} \ , \ r_{f,\frac{d}{c}}(x) := \int_{\frac{d}{c}}^{i\infty} dw \, \frac{\overline{f(-\bar{w})}}{(-i(w+x))^{2-k}} \ , \tag{4.2.50}$$

for f a vector-valued cusp form with multiplier χ, and $\frac{d}{c} \in \mathbb{Q}$. We say f^* is the *non-holomorphic Eichler integral* of the cusp form f. It is easy to verify that $r_{f,\frac{d}{c}}$ is a real analytic function on $\mathbb{R}$, which captures the error of modularity of f^*:

$$(f^* - f^*|_{2-k,\bar{\chi}}\gamma)(\tau) = r_{f,\frac{d}{c}}(\tau) \tag{4.2.51}$$

for $\gamma \in \Gamma$, where $\bar{\chi}$ is the conjugate multiplier $\bar{\chi}(\gamma) = \overline{\chi(\gamma)}$. As a result, we have $f^* \in Q_{2-k}(\Gamma, \bar{\chi})$.

Similarly, for $f_i : \mathbb{H} \to \mathbb{C}^{n_i}$, $i = 1, 2$ a pair of vector-valued cusp forms (or modular form if the weight is $1/2$) with weight k_i and multiplier system χ_i, we define the following matrix-valued (valued in $\mathbb{C}^{n_1 \times n_2}$) functions:

$$(f_1, f_2)^*(\tau) := \int_{-\bar{\tau}}^{i\infty} dw_1 \int_{w_1}^{i\infty} dw_2 \, \frac{\overline{f_1(-\bar{w}_1) f_2(-\bar{w}_2)}}{(-i(w_1+\tau))^{2-k_1}(-i(w_2+\tau))^{2-k_2}} \tag{4.2.52}$$

and

$$r_{f_1,f_2,\frac{d}{c}}(x) := \int_{\frac{d}{c}}^{i\infty} dw_1 \int_{w_1}^{\frac{d}{c}} dw_2 \, \frac{\overline{f_1(-\bar{w}_1) f_2(-\bar{w}_2)}}{(-i(w_1+x))^{2-k_1}(-i(w_2+x))^{2-k_2}} \; . \tag{4.2.53}$$

The function $(f_1, f_2)^*$ is often referred to as a non-holomorphic double Eichler integral, or iterated non-holomorphic Eichler integral more generally.

One can show that for $\gamma \in \Gamma$,

$$((f_1, f_2)^* - (f_1, f_2)^*|_{4-k_1-k_2,\bar{\chi}_1,\bar{\chi}_2}\gamma)(\tau) = r_{f_1,f_2,\frac{d}{c}}(\tau) + I_{f_1}(\tau) r_{f_2,\frac{d}{c}}(\tau) \; , \tag{4.2.54}$$

where the slash operator acts in the following way in terms of the components. Write $I_{i,j} := (f_{1,i}, f_{2,j})^*$ to denote the non-holomorphic double Eichler integral of the components of the vector-valued modular forms f_1 and f_2. Then

$$(I|_{k,\bar{\chi}_1,\bar{\chi}_2}\gamma)_{i,j}(\tau) := (c\tau+d)^{-k} \sum_{i'=1}^{n_1} \sum_{j'=1}^{n_2} I_{i',j'}(\gamma\tau) \overline{(\chi_1(\gamma))_{i',i} (\chi_2(\gamma))_{j',j}}. \tag{4.2.55}$$

We have $r_{f_1,f_2,\frac{d}{c}}(\tau) \in \mathcal{O}(\mathbb{R}\backslash\{-\frac{d}{c}\})$, and $r_{f_1,f_2,\frac{d}{c}}(\tau) \in \mathcal{O}(\mathbb{R})$ if both f_i are cusp forms[96]. From the above, we see that $(f_1, f_2)^*$ is a vector-valued depth two quantum modular form valued in $\mathbb{C}^{n_1 \times n_2}$ with multiplier $\bar{\chi}$ given by

$$(\bar{\chi}(\gamma))_{(i,j),(i',j')} = \overline{(\chi_1(\gamma))_{i',i} (\chi_2(\gamma))_{j',j}}.$$

In this chapter, we will mainly encounter modular forms with real coefficients,

satisfying
$$\overline{f(-\bar{\tau})} = f(\tau),$$
and we will often use this property to simply write $f(\tau)$ in the integrand.

The above-mentioned quantum modular property of the double Eichler-integral $(f_1, f_2)^*$, together with the form of the companion of $F^{(\rho)}$ as given in Proposition 4.2.1.1, its rewriting up to one-dimensional pieces in Lemma 4.2.1.2, and the relation to double Eichler integrals shown in Proposition 4.2.1.2, leads to the following result.

Theorem 4.2.4. *The generalized A_2 false theta functions defined in* (4.2.5) *is, up to an overall rational power of q and possibly the addition of a finite polynomial in q and q^{-1}, a sum of depth two quantum modular forms.*

4.3 Properties of $\widehat{Z}^{SU(3)}$

In this section we turn to the main object of our study: $\widehat{Z}^G_{\underline{\vec{b}}}(M_3)$ for $G = SU(3)$ and the simplest interesting choice of M_3, namely negative Seifert manifolds with three exceptional fibers. In §4.3.1 we explain how they are assembled using the generalized A_2 false theta functions $F^{(\rho)}$ as building blocks. Combining with the results of the quantum modularity of the latter as established in the previous section, we are led to Theorem 4.0.1 and Theorem 4.0.20. While we do not have a proof for Conjecture 4.0.1.1, we provide evidence for it through studying numerous examples in §4.4.

4.3.1 Topology

A plumbed three-manifold M_3 can be defined as the boundary of glued disk bundles associated to its plumbing graph, which is a weighted graph (V, E, a), here taken to be a planar tree-shaped graph and no loops. The weights $a(v)$ give the Euler number of the disk bundle corresponding to the vertex $v \in V$. Gluing occurs when there is an edge connecting the two vertices v and v'. The data of the weighted graph (V, E, a) is equivalent to that of the adjacency or plumbing matrix M of the graph (V, E, a), with entries

$$M_{v,v'} = \begin{cases} a(v) & \text{if} \quad v = v' \\ 1 & \text{if} \quad v \neq v', v \text{ and } v' \text{ are connected} \\ 0 & \text{otherwise .} \end{cases} \tag{4.3.1}$$

Seifert manifolds are examples of such plumbed three-manifolds. A Seifert manifold

$$M_3 = \left(b; \left\{\frac{q_i}{p_i}\right\}_{i=1}^{n}\right)$$

with Seifert invariants $(q_1, p_1), \ldots, (q_n, p_n)$ is specified by a star-shaped plumbing graph with a unique junction vertex v_0 from which emanate n legs, which represent the exceptional fibers of M_3. As mentioned before, we say that M_3 is a negative/positive Seifert manifold depending on the sign of $M^{-1}_{v_0,v_0}$. Along the i-th leg, the vertices $v_k^{(i)}$ and the corresponding weights $a_k^{(i)}$ are given by the continued fraction expansion

$$\frac{q_i}{p_i} = -\cfrac{1}{a_1^{(i)} - \cfrac{1}{a_2^{(i)} - \cfrac{1}{a_3^{(i)} - \cdots}}}\ , \tag{4.3.2}$$

while $a(v_0) = b$ and and the orbifold Euler number $e \in \mathbb{Q}$ is given by

$$b = e - \sum_{i=1}^{n} \frac{q_i}{p_i}\ . \tag{4.3.3}$$

See the Appendex A of [31] for further useful relations between the Seifert data and the plumbing graph.

For such manifolds, define D to be the smallest positive integer such that $DM^{-1}_{v_0,v} \in \mathbb{Z}$ for all $v \in V$, and let $m = D^2|M^{-1}_{v_0,v_0}|^3$. Examples are given by Brieskorn spheres $M_3 = \Sigma(p_1, p_2, p_3)$, which have trivial integral homology and are determined by three coprime integers p_1, p_2, p_3 through the defining equation

$$\Sigma(p_1, p_2, p_3) = \{(x, y, z) \in \mathbb{C}^3 | x^{p_1} + y^{p_2} + z^{p_3} = 0\} \cap S^5\ . \tag{4.3.4}$$

The Seifert data that specify the plumbing diagram are related to the integers $\{p_1, p_2, p_3\}$ by the following relation

$$b + \sum_{i=1}^{3} \frac{q_i}{p_i} = -\frac{1}{p_1 p_2 p_3}. \tag{4.3.5}$$

For Brieskorn spheres, which satisfy $|\det(M)| = 1$, we have $D = 1$ and $m = |M^{-1}_{v_0,v_0}|$.

For a weakly-negative plumbed three-manifold M_3, we define the $\widehat{Z}^G$-invariants

[3] Comparison of conventions: m in this chapter is what is written as mD in [31].

for any ADE gauge group G in the following way, where we mostly adopt the same notation of as in [31]. In particular, for a given simply-laced Lie group G and a plumbing graph (V, E, a), we let Λ be the root lattice and

$$\Gamma_{M,G} := M\mathbb{Z}^{\otimes|Z|} \otimes_{\mathbb{Z}} \Lambda \ . \tag{4.3.6}$$

For $\underline{\vec{x}} \in \mathbb{R}^{\otimes|Z|} \otimes_{\mathbb{Z}} \Lambda$, we define its norm to be given by the inverse plumbing matrix in the direction along the vertices and by the Cartan matrix in the root lattice directions:

$$||\underline{\vec{x}}||^2 := \sum_{v,v' \in V} M^{-1}_{v,v'} \langle \vec{x}_v, \vec{x}_{v'} \rangle. \tag{4.3.7}$$

Definition 4.3.1 (Higher Rank $\widehat{Z}$ Invariants [25] , [31])**.** Let G be a simply-laced Lie group and M_3 a weakly negative plumbed three-manifold with plumbing matrix M. Let $\underline{\vec{b}}$ be a generalized Spin^c structure on the manifold, given by

$$\underline{\vec{b}} \in \left(\mathbb{Z}^{|V|} \otimes_{\mathbb{Z}} \Lambda + \underline{\vec{b}}_0\right) / \Gamma_{M,G}, \tag{4.3.8}$$

where $\vec{b}_{0,v} = \deg(v)\vec{\rho}$.

We define

$$\begin{aligned} \widehat{Z}^G_{\underline{\vec{b}}}(M_3;\tau) := C^G(q) \int_{\mathcal{C}} d\underline{\vec{\xi}} \left(\prod_{v \in V} \Delta(\vec{\xi}_v)^{2-\deg v} \right) \\ \times \sum_{w \in W} \sum_{\underline{\vec{\ell}} \in \Gamma_{M,G} + w(\underline{\vec{b}})} q^{-\frac{1}{2}||\underline{\vec{\ell}}||^2} \left(\prod_{v' \in V} \mathrm{e}^{\langle \vec{\ell}_{v'}, \vec{\xi}_{v'} \rangle} \right) \ , \end{aligned} \tag{4.3.9}$$

where W denotes the Weyl group of the root lattice of G, $w(\underline{\vec{b}})$ denotes the diagonal action $w(\underline{\vec{b}}) = (w(\vec{b}_v), w(\vec{b}_{v'}), \dots)$ and the integration measure is given by

$$\int_{\mathcal{C}} d\underline{\vec{\xi}} := \text{p.v.} \int \prod_{v \in V} \prod_{i=1}^{\mathrm{rank} G} \frac{dz_{i,v}}{2\pi i z_{i,v}},$$

with the contour $\mathcal{C}$ given by the Cauchy principal value integral around the unique circle in the $z_{i,v}$-plane. Letting π_M be the number of positive eigenvalues of M and σ_M the signature of M, according to [25],

$$C^G(q) = (-1)^{|\Phi_+|\pi_M} q^{\frac{3\sigma_M - \mathrm{Tr} M}{2}|\vec{\rho}|^2} \ , \tag{4.3.10}$$

where Φ_+ is a set of positive roots for G and $\vec{\rho}$ is a Weyl vector for G. Lastly, Δ is the Weyl determinant.

As shown in [31], the $\widehat{Z}$-invariants for negative Seifert manifolds with three exceptional fibers and for $G = SU(3)$ can be expressed as combinations of the generalized A_2 false theta functions (4.2.5) in the following way. Given $\underline{\vec{b}}$ and a choice of $\hat{w} = (w_1, w_2, w_3) \in W^{\otimes 3}$, one of the following two statements is true. Either there does not exist any root vector $\vec{\ell_0}$ such that

$$\underline{\vec{b}} - (\vec{\ell_0}, w_1(\vec{\rho}), w_2(\vec{\rho}), w_3(\vec{\rho})) \in M\mathbb{Z}^{|V|} \otimes_{\mathbb{Z}} \Lambda \tag{4.3.11}$$

or there exists a unique $\vec{k}_{\hat{w}} \in \Lambda/D\Lambda$ such that such that (4.3.11) holds if and only if $\vec{k}_{\hat{w}} = \vec{\ell_0}/D\Lambda$.

Now, let $\mathcal{W}_{\underline{\vec{b}}} \subseteq W^{\otimes 3}$ be the subset consisting of all $\hat{w}$ for which the latter is true. For $\hat{w} \in \mathcal{W}_{\underline{\vec{b}}}$, let

$$\vec{s}_{\hat{w}} = D \sum_{v_i \in \{v_1, v_2, v_3\}} M^{-1}_{v_0, v_i} w_i(\vec{\rho}) \ . \tag{4.3.12}$$

The above defines $\vec{\sigma}_{\hat{w}} \in \Lambda/m\Lambda$ via

$$\vec{s}_{\hat{w}} = \vec{\sigma}_{\hat{w}} + \frac{m}{D}\vec{k}_{\hat{w}}.$$

The $\widehat{Z}$-invariant is then given by

$$\widehat{Z}^{SU(3)}_{\underline{\vec{b}}}(M_3; \tau) = C(q) \sum_{\hat{w} \in \mathcal{W}_{\underline{\vec{b}}}} (-1)^{\ell(\hat{w})} F^{(\rho_{\hat{w}})}(\tau) \ , \tag{4.3.13}$$

where $\ell(\hat{w}) := \sum_{i=1}^{3} \ell(w_i)$ is the total Weyl length, $\rho_{\hat{w}} = (\vec{\sigma}_{\hat{w}}, \vec{k}_{\hat{w}}, m, D)$ specifies the functions $F^{(\rho_{\hat{w}})}(\tau)$ from equation (4.2.5), and

$$C(q) = (-1)^{\pi_M} q^{3\sigma_M - \mathrm{Tr}M + \delta_M}, \quad \delta_M = \sum_{v \in V_1} \left(\frac{\left(M^{-1}_{v_0,v}\right)^2}{M^{-1}_{v_0,v_0}} - M^{-1}_{v,v} \right), \tag{4.3.14}$$

with π_M and σ_M denoting the number of positive eigenvalues resp. the signature of the adjacency matrix M. The additional power q^{δ_M} comes from performing the integral (4.3.9) along the directions corresponding to the "non-junction" vertices with v with degree less than three.

4.3.2 Companions

In this subsection we will put the results obtained so far together and derive the form of the companion function for $\widehat{Z}^{SU(3)}_{\underline{\vec{b}}}(M_3)$ for negative Seifert M_3 with three exceptional fibers, before we further specialize to the case of Brieskorn spheres.

Combining 1) (4.3.13), the expression of $\widehat{Z}^{SU(3)}_{\underline{\vec{b}}}(M_3)$ in terms of the generalized A_2 false theta function $F^{(\varrho)}$, 2) Lemma 4.2.0.2 and (4.2.6), the splitting of $F^{(\varrho)}$ into components, 3) Proposition 4.2.1.1, the companion of the components, and 4) Proposition 4.2.1.2 and Lemma 4.2.1.2, the iterated non-holomorphic Eichler integral expressions for the companions, we finally obtain the following.

Proposition 4.3.1.1. *For a negative Seifert manifold M_3 with three exceptional fibers, a companion function $\check{Z}^{SU(3)}_{\underline{\vec{b}}}(M_3)$ of the rank two homological blocks $\widehat{Z}^{SU(3)}_{\underline{\vec{b}}}(M_3)$ is, up to potential one-dimensional pieces, given by the following non-holomorphic double Eichler integral*

$$\check{Z}^{SU(3)}_{\underline{\vec{b}}}(M_3;\tau,\bar{\tau}) = z_{1\mathrm{d}} +$$
$$\frac{D}{m}\, C(q^{-1}) \sum_{\hat{w}\in\mathcal{W}_{\underline{\vec{b}}}} (-1)^{\ell(\hat{w})} \sum_{\nu=0,1} \frac{\sqrt{3}}{4\pi^{\nu}} \left(\frac{3\Delta\vec{s}_{\hat{w}}}{m}\right)^{1-\nu} \sum_{w\in W^+} \sum_{\delta\in\mathbb{Z}/2} (\vartheta'_{w,\hat{w},\delta}, \vartheta^{1-\nu}_{w,\hat{w},\delta})^*(\tau)\ , \tag{4.3.15}$$

where the non-holomorphic double Eichler integral is of the theta functions

$$\begin{aligned} \vartheta'_{w,\hat{w},\delta}(\tau) &= \theta^1_{m,m\delta+\langle\vec{\rho},w(\vec{\sigma}_{\hat{w}})\rangle}(\tau) \\ \vartheta^{1-\nu}_{w,\hat{w},\delta}(\tau) &= \theta^{1-\nu}_{m,m\delta+\langle\Delta\vec{\omega},w(\vec{\sigma}_{\hat{w}})\rangle}(3\tau). \end{aligned} \tag{4.3.16}$$

Note that the above, together with the quantum modular properties of the non-holomorphic double Eichler integrals discussed in §4.2.5, leads immediately to Theorem 4.0.1.

Weil representations

From the fact that $\theta^{\nu}_m = (\theta^{\nu}_{m,r})$ is a vector-valued modular form for $\nu = 0, 1$, we see from the discussion in §4.2.5 that, potentially up to certain one-dimensional pieces, $\check{Z}^{SU(3)}_{\underline{\vec{b}}}(M_3)$ is a linear combination of components of vector-valued quantum modular forms of depth two. In what follows, we will investigate the recursive structure relating the quantum modular properties of $\widehat{Z}^{SU(2)}_{\underline{\vec{b}}}(M_3)$ and $\widehat{Z}^{SU(3)}_{\underline{\vec{b}}}(M_3)$, or equivalently $\check{Z}^{SU(2)}_{\underline{\vec{b}}}(M_3)$ and $\check{Z}^{SU(3)}_{\underline{\vec{b}}}(M_3)$. In order to do that, we need to take a closer look at the underlying representations of the metaplectic group $\widetilde{SL_2}(\mathbb{Z})$. For this purpose, we will introduce specific Weil representations specified by a positive integer m and a subgroup K of the group of its exact divisors Ex_m, as mentioned in the introduction of this chapter.

To such a group K we associate a subrepresentation of Θ_m (4.0.7), which we write as Θ^{m+K}, in the following way. First we make use of the fact that the space of matrices commuting with the S- and T-matrices of Θ_m is spanned by [107]

$$\Omega_m(n)_{r,r'} = \begin{cases} 1 & \text{if } r \equiv -r' \mod 2n \text{ and } r \equiv r' \mod 2m/n \\ 0 & \text{otherwise}, \ r, r' \in \mathbb{Z}/2m \end{cases} \tag{4.3.17}$$

for $n|m$. Note that $\Omega_m(n)$ and $\Omega_m(n')$ commute for every pair of divisors n and n'. For instance $\Omega(1) = \mathbf{1}_m$ is the identity matrix of size $2m \times 2m$.

Now define the corresponding projection operators

$$P^{\pm}_m(n) := (\mathbf{1}_m \pm \Omega_m(n))/2 \ , \quad n \in \mathrm{Ex}_m, \tag{4.3.18}$$

satisfying $(P^{\pm}_m(n))^2 = P^{\pm}_m(n)$.

Since in our application we are mostly interested in Eichler integrals involving $\theta^1_{m,r}(\tau) = \frac{1}{2\pi i}\frac{\partial}{\partial z}\theta_{m,r}(\tau,z)|_{z=0}$ which has the property $\theta^1_{m,r} = -\theta^1_{m,-r}$, or $P^-_m(m)\theta^1_m = \theta^1_m$, we will from now on focus on the subgroups K satisfying $m \notin K$ and define the projector

$$P^{m+K} = \left(\prod_{n\in K} P^+_m(n)\right) P^-_m(m) \ , \tag{4.3.19}$$

using the notation of [23]. When K is maximal, in the sense that $\mathrm{Ex}_m = K \cup (m * K)$, $\Theta^{m+K} := P^{m+K}\Theta_m$ furnishes an irreducible representation of $\widetilde{SL_2}(\mathbb{Z})$ when m is square-free. In general, $\Theta^{m+K,\mathrm{irred}} := P^{m+K,\mathrm{irred}}\Theta_m$ with K maximal and

$$P^{m+K,\mathrm{irred}} := \left(\prod_{n\in K} P^+_m(n)\right)\left(\prod_{f^2|m}\left(\mathbf{1}_m - \tfrac{1}{f}\Omega_m(f)\right)\right) P^-_m(m) \tag{4.3.20}$$

is irreducible [108] [109].

Using the above we introduce the notation

$$\theta^{1,m+K}_r := \sum_{r'\in\mathbb{Z}/2m} P^{m+K}_{r,r'}\theta^1_{m,r'} \ . \tag{4.3.21}$$

which will be used extensively below.

In what follows, we will focus on the manifolds M_3 that are homological spheres, to obtain Theorem 4.0.2. First, we simplify the expression for the companions of $\widehat{Z}^{SU(3)}_{\underline{\vec{b}}}(M_3)$ given in Proposition 4.3.1.1 in these cases.

Lemma 4.3.1.1. *For Brieskorn spheres $\Sigma(p_1,p_2,p_3)$, we have*

$$\begin{aligned}&\check{Z}^{SU(3)}_{\underline{\vec{b}}}(M_3;\tau,\bar{\tau}) = z_{1\mathrm{d}}\\ &+\tfrac{|W|}{2m}\,C(q^{-1})\sum_{\hat{w}\in W^{\otimes 3}}(-1)^{\ell(\hat{w})}\sum_{\nu=0,1}\frac{\sqrt{3}}{4\pi^{\nu}}\left(\frac{3\Delta\vec{s}_{\hat{w}}}{m}\right)^{1-\nu}\sum_{\delta\in\mathbb{Z}/2}(\vartheta'_{\hat{w},\delta},\vartheta^{1-\nu}_{\hat{w},\delta})^*(\tau)\ ,\end{aligned} \tag{4.3.22}$$

where the non-holomorphic double Eichler integral is of the theta functions

$$\begin{aligned}\vartheta'_{\hat{w},\delta}(\tau) &= \theta^1_{m,m\delta+\langle\vec{\rho},\vec{\sigma}_{\hat{w}}\rangle}(\tau)\\ \vartheta^{1-\nu}_{\hat{w},\delta}(\tau) &= \theta^{1-\nu}_{m,m\delta+\langle\Delta\vec{\omega},\vec{\sigma}_{\hat{w}}\rangle}(3\tau),\end{aligned} \tag{4.3.23}$$

for $m=p_1p_2p_3$, $\bar{p}_i=m/p_i$ and

$$\vec{\sigma}_{\hat{w}} = \vec{s}_{\hat{w}} = -\sum_{i=1}^{3}\bar{p}_i w_i(\vec{\rho}). \tag{4.3.24}$$

The proof of this Lemma can be found in Appendix 4.6. [4]

It is know that the $SU(2)$ companion for Brieskorn spheres with three exceptional fibers is given by [23]

$$\check{Z}^{SU(2)}_{\underline{\vec{b}}}(M_3;\tau,\bar{\tau}) \doteq (\theta^{1,m+K}_r)^* \tag{4.3.25}$$

up to an overall rational power of q (and the addition of a finite polynomial in q^{-1} for the case $M_3=\Sigma(2,3,5)$), where

$$m = p_1p_2p_3,\ K=\{1,\bar{p}_1,\bar{p}_2,\bar{p}_3\}, \tag{4.3.26}$$

and $r = m-\bar{p}_1-\bar{p}_2-\bar{p}_3$. For the $SU(3)$ companions, we have the following non-holomorphic double Eichler integral.

Proposition 4.3.1.2. *For Brieskorn spheres $\Sigma(p_1,p_2,p_3)$, using the same nota-*

[4] Regarding the one-dimensional non-holomorphic Eichler integral z_{1d}, we also comment that, when all $\vec{\sigma}_{\hat{w}}$ satisfy $0\le\langle\vec{\sigma}_{\hat{w}},\vec{\omega}_i\rangle\le m$, the different contributions from X_ν in Lemma 4.2.1.2 to $\hat{Z}^{SU(3)}(M_3)$ cancel for $M_3=\Sigma(p_1,p_2,p_3)$.

tion as in Lemma 4.3.1.1 and in (4.3.26)*, we have*

$$\begin{aligned} &\check{Z}^{SU(3)}_{\underline{\vec{b}}}(M_3;\tau,\bar{\tau}) = z_{1d} \\ &+ \tfrac{3\sqrt{3}}{2m}\, C(q^{-1}) \sum_{\nu=0,1} \pi^{-\nu} \sum_{\delta\in\mathbb{Z}/2} \sum_{r\in\mathcal{R}} (\tfrac{r}{m})^{1-\nu} (\vartheta'_{r,\delta}, \vartheta^{1-\nu}_{r,\delta})^*(\tau)\ , \end{aligned} \tag{4.3.27}$$

where the non-holomorphic double Eichler integral is of the theta functions

$$\begin{aligned} \vartheta'_{r,\delta}(\tau) &= 4\,\theta^{1,m+K}_{m\delta+\sum_i \bar{p}_i c_i^{(r)}}(\tau) \\ \vartheta^{1-\nu}_{r,\delta}(\tau) &= \theta^{1-\nu}_{m,m\delta+r}(3\tau). \end{aligned} \tag{4.3.28}$$

In the above, $\mathcal{R}\subset\mathbb{Z}/2m$ *is given by*

$$\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \tag{4.3.29}$$

and

$$\begin{aligned} \mathcal{R}_0 &= \{0\} \\ \mathcal{R}_1 &= \mathbf{P}^+\{\bar{p}_1\} \\ \mathcal{R}_2 &= \mathbf{P}^+\{\bar{p}_1+\bar{p}_2, \bar{p}_1-\bar{p}_2\} \\ \mathcal{R}_3 &= \mathbf{P}^+\{\bar{p}_1+\bar{p}_2-\bar{p}_3, -\bar{p}_1-\bar{p}_2+\bar{p}_3\}\ . \end{aligned} \tag{4.3.30}$$

where we denote by $\mathbf{P}^+$ *by the group of even permutations of* (p_1,p_2,p_3)*. For each* $r\in\mathcal{R}$*, we set* $c_i^{(r)} := 2-|r_i|$ *if* $r=\sum_i r_i\bar{p}_i$*.*

From the above, we see that (4.0.20) holds, and in particular Theorem 4.0.2 holds. That is, up to possible one-dimensional terms, the same $\widetilde{SL_2(\mathbb{Z})}$ representation Θ^{m+K} governs not just $\widehat{Z}^{SU(2)}$ but also the $SU(3)$ quantum modularity. Note, when all p_is are square free, the underlying representation Θ^{m+K} is irreducible. Furthermore, when $2^2 \nmid m$, one can replace $\theta^{1,m+K}_r$ in Proposition 4.3.1.2 with the irreducible representation $\Theta^{m+K,\text{irred}}$ (cf. (4.3.20)). The proof of the above Proposition is given in Appendix 4.6.

4.4 Examples

In this section we present in detail the structure of $\widehat{Z}$ invariants discussed in §4.3. We further show the recursive structure, proven in §4.3 for homological spheres, is also present for other non-spherical negative Seifert manifolds with three exceptional fibers. In particular, we compute explicitly the underlying $\widetilde{SL}_2(\mathbb{Z})$ Weil representations.

4.4.1 Example: $M\left(-1; \frac{1}{4}, \frac{3}{5}, \frac{1}{7}\right)$

We begin with the spherical Seifert manifold $X = M\left(-1, \frac{1}{4}, \frac{3}{5}, \frac{1}{7}\right) \cong \Sigma(4,5,7)$. To determine the plumbing matrix M we compute continued fraction expansions of the Seifert data 4.3.2. From

$$\frac{3}{5} = \frac{-1}{-2 - \cfrac{1}{-3}}\,, \tag{4.4.1}$$

we have

$$M = \begin{pmatrix} -1 & 1 & 0 & 1 & 1 \\ 1 & -4 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 1 \\ 1 & 0 & 0 & -7 & 0 \\ 1 & 0 & 1 & 0 & -2 \end{pmatrix}. \tag{4.4.2}$$

The corresponding plumbing graph has one junction vertex connecting to three legs. Since X is a homological sphere, the adjacency matrix M is unimodular and consequently the only inequivalent generalized Spin^c structure is

$$\vec{\underline{b}}_0 = \left(\vec{\rho}, -\vec{\rho}, -\vec{\rho}, -\vec{\rho}, \vec{0}\right). \tag{4.4.3}$$

The unimodularity also leads to the parameters $D = 1$ and $m = -M^{-1}_{v_0,v_0} = 140$.

Since X is a spherical Seifert manifold the condition (4.3.11) is always satisfied so $\mathcal{W}_{\vec{\underline{b}}}$ in equation (4.3.13) is equal to $W^{\otimes 3}$. Because

$$(-1)^{\ell(w\hat{w})} F^{\varrho_{w\hat{w}}}(\tau) = (-1)^{\ell(\hat{w})} F^{\varrho_{\hat{w}}}(\tau)\,, \tag{4.4.4}$$

where $w\hat{w} = (ww_1, ww_2, ww_3)$, we may simplify the sum over $\mathcal{W}_{\vec{\underline{b}}}$ in (4.3.13) to a sum over representatives $\hat{w}$ in the conjugacy classes of $W^{\otimes 3}/W$

$$\widehat{Z}^{SU(3)}_{\vec{\underline{b}}}(M_3;\tau) = |W|\, C(q) \sum_{\hat{w} \in W^{\otimes 3}/W} (-1)^{\ell(\hat{w})} F^{(\rho_{\hat{w}})}(\tau)\,. \tag{4.4.5}$$

For this manifold, we can choose the representatives $\hat{w}$ such that $\vec{s}_{\hat{w}} = (s_1, s_2)$ have components $s_i \in \{1, ..., m\}$. These parameters and their associated total Weyl length $(-1)^{\ell(\hat{w})}$ in (4.3.13) are collected in Table 4.2.

Since $D = 1$ and we can set $\vec{k}_{\hat{w}} = \vec{0}$, and therefore $\vec{\sigma}_{\hat{w}} = \vec{s}_{\hat{w}}$, whereby the $\widehat{Z}$

$\vec{s}_{\hat{w}} = (s_1, s_2)\,,\ (-1)^{\ell(\hat{w})} = 1$		
(43, 103)	(103, 43)	(43, 43)
(27, 111)	(27, 51)	(111, 27)
(51, 27)	(43, 19)	(19, 43)
(27, 27)	(13, 118)	(13, 58)
(1, 82)	(61, 22)	(1, 22)
(13, 34)	(118, 13)	(58, 13)
(22, 61)	(82, 1)	(22, 1)
(34, 13)	(27, 6)	(6, 27)
	(13, 13)	

$\vec{s}_{\hat{w}} = (s_1, s_2)\,,\ (-1)^{\ell(\hat{w})} = -1$		
(83, 83)	(83, 23)	(23, 83)
(47, 71)	(71, 47)	(33, 78)
(41, 62)	(41, 2)	(78, 33)
(62, 41)	(2, 41)	

Table 4.2: $\vec{s}_{\hat{w}}$ and its parity $(-1)^{\ell(\hat{w})}$ for the 36 inequivalent representatives of $W^{\otimes 3}/W$ for $M_3 = M\left(-1; \frac{1}{4}, \frac{3}{5}, \frac{1}{7}\right)$

invariant in equation (4.3.13) is

$$C(q) \sum_{\hat{w} \in \mathcal{W}_{\underline{\tilde{b}}}} (-1)^{\ell(\hat{w})} F^{(\varrho_{\hat{w}})}(\tau) = 6q^{26} - 12q^{37} - 12q^{43} - 12q^{49} + \mathcal{O}\left(q^{50}\right). \quad (4.4.6)$$

For each $\vec{s}$ we can then compute the set $\tilde{\mathcal{S}}$ (4.2.14). These values are collected in Table B.1. As a selected example, consider $(s_1, s_2) = (83, 83)$ which correspond to $\hat{w} = (aba, aba, aba)$, where a, b are Weyl group elements given as in (4.6.1). Using equation (4.2.9) we find the set $\tilde{\mathcal{S}}$ contains

$\boldsymbol{\alpha}_w^{(1)}$	$\boldsymbol{\alpha}_w^{(2)}$
$\left(0, -\frac{83}{140}\right)$	$\left(1, -\frac{83}{140}\right)$
$\left(\frac{83}{140}, -\frac{83}{140}\right)$	$\left(\frac{57}{140}, \frac{83}{70}\right)$
$\left(-\frac{83}{140}, \frac{83}{70}\right)$	$\left(\frac{223}{140}, -\frac{83}{140}\right)$

.

The $\boldsymbol{\alpha}$ for all choices of $\vec{s}$ are collected in Table B.1.

For $\boldsymbol{\alpha} \in \tilde{\mathcal{S}}$, let $\boldsymbol{\beta}$ be the unique vector satisfying $\boldsymbol{\alpha} \equiv \boldsymbol{\beta}$ $(\mathbb{Z}^2)$ and $\beta_1, \beta_2 \in [0, 1)$. Lemma 4.2.0.2 justifies the splitting of the generalized A_2 false theta function into

1D and 2D contributions

$$
\begin{aligned}
F_\nu^{1\mathrm{D}}(\tau) &:= \sum_{w\in\mathcal{W}_{\underline{\vec{b}}}} (-1)^{\ell(\hat{w})} F_\nu^{(\varrho_{\hat{w}}),1D}(\tau)\,,\\
F_\nu^{(\varrho),1\mathrm{D}}(\tau) &:= \sum_{\boldsymbol{\alpha}\in\tilde{\mathcal{S}}} \eta_\nu(\boldsymbol{\alpha})(F^{(\varrho)}_{\nu,\boldsymbol{\alpha}}(\tau) - F^{(\varrho)}_{\nu,\boldsymbol{\beta}}(\tau))\\
F_\nu^{2\mathrm{D}}(\tau) &:= \sum_{w\in\mathcal{W}_{\underline{\vec{b}}}} (-1)^{\ell(\hat{w})} F_\nu^{(\varrho_{\hat{w}}),2\mathrm{D}}(\tau)\,,\\
F_\nu^{(\varrho),2\mathrm{D}}(\tau) &:= \sum_{\boldsymbol{\alpha}\in\tilde{\mathcal{S}}} \eta_\nu(\boldsymbol{\alpha}) F^{(\varrho)}_{\nu,\boldsymbol{\beta}}(\tau)\,.
\end{aligned}
$$

For X this splitting gives

$$
\begin{aligned}
\tilde{F} &= -\frac{9}{14}q^4 - \frac{18}{35}q^5 - \frac{33}{35}q^7 - \frac{81}{70}q^8 - \frac{57}{35}q^{10} - \frac{39}{35}q^{13} - \frac{81}{35}q^{14}\\
&\quad - \frac{261}{70}q^{16} - \frac{3}{35}q^{19} + \frac{123}{35}q^{22} + \frac{69}{35}q^{25}\\
C(q)F_0^{1\mathrm{D}}(m\tau) &= -\tilde{F} - \frac{99}{35}q^{26} - \frac{141}{35}q^{28} + \frac{18}{7}q^{37} + \frac{39}{35}q^{40} + \frac{81}{35}q^{41} + \mathcal{O}\left(q^{42}\right)\\
C(q)F_0^{2\mathrm{D}}(m\tau) &= \tilde{F} + \frac{447}{70}q^{26} + \frac{141}{35}q^{28} - \frac{309}{35}q^{37} - \frac{39}{35}q^{40} - \frac{81}{35}q^{41} + \mathcal{O}\left(q^{42}\right)\\
C(q)F_1^{1\mathrm{D}}(m\tau) &= \tilde{F} + \frac{99}{35}q^{26} + \frac{141}{35}q^{28} - \frac{18}{7}q^{37} - \frac{39}{35}q^{40} - \frac{81}{35}q^{41} + \mathcal{O}\left(q^{42}\right)\\
C(q)F_1^{2\mathrm{D}}(m\tau) &= -\tilde{F} - \frac{27}{70}q^{26} - \frac{141}{35}q^{28} - \frac{111}{35}q^{37} + \frac{39}{35}q^{40} + \frac{81}{35}q^{41} + \mathcal{O}\left(q^{42}\right)\ .
\end{aligned}
\tag{4.4.7}
$$

Here the one dimensional contribution is

$$
C(q)\left(F_0^{1\mathrm{D}}(m\tau) + F_1^{1\mathrm{D}}(m\tau)\right) = 6q^{109} - 6q^{113} - 6q^{121} - 6q^{131} + 6q^{157} + \mathcal{O}\left(q^{160}\right)
\tag{4.4.8}
$$

and the total $\widehat{Z}^{SU(3)}_{\underline{\vec{b}}}(X)$ has integral coefficients

$$
\widehat{Z}^{SU(3)}_{\underline{\vec{b}}}(X;\tau) = -6q^{26} + 12q^{37} + 12q^{43} + 12q^{49} + \mathcal{O}\left(q^{50}\right)\,.
\tag{4.4.9}
$$

The companion functions to the 2D contributions $F_\nu^{2\mathrm{D}}(m\tau)$ are double Eichler integrals, whose integrands computed using Lemma 4.3.1.1 contain

$$\frac{1}{4}\sum_{\hat{w}\in W^{\otimes 3}}(-1)^{\ell(\hat{w})}\Theta^{(\varrho_{\hat{w}})}_{\nu,e}(\mathbf{z}) = \tag{4.4.10}$$

$$\begin{aligned}
&\big(63\,\theta^1_{140,63}+7\,\theta^1_{140,7}\big)\theta^{1,140+K}_{23}\\
&+\big(15\,\theta^1_{140,15}+55\,\theta^1_{140,55}\big)\theta^{1,140+K}_{1}\\
&-\big(7\theta^1_{140,133}+63\,\theta^1_{140,77}\big)\theta^{1,140+K}_{37}\\
&+\big(8\theta^1_{140,132}+48\,\theta^1_{140,48}+8\theta^1_{140,8}+48\,\theta^1_{140,92}\big)\,\theta^{1,140+K}_{118}\\
&+\big(27\,\theta^1_{140,113}+13\,\theta^1_{140,127}+83\,\theta^1_{140,57}+43\,\theta^1_{140,97}\big)\,\theta^{1,140+K}_{57}\\
&+\big(20\,\theta^1_{140,120}+20\,\theta^1_{140,20}\big)\theta^{1,140+K}_{6}\\
&-\big(28\theta^1_{140,112}+28\theta^1_{140,28}\big)\,\theta^{1,140+K}_{2}\\
&+\big(15\,\theta^1_{140,125}+55\,\theta^1_{140,85}\big)\theta^{1,140+K}_{29}\\
&+35\,\theta^1_{140,105}\theta^{1,140+K}_{9}-35\,\theta^1_{140,35}\theta^{1,140+K}_{19}\\
&-\big(13\,\theta^1_{140,13}+27\,\theta^1_{140,27}+43\,\theta^1_{140,43}+83\,\theta^1_{140,83}\big)\,\theta^{1,140+K}_{13}
\end{aligned}\tag{4.4.11}$$

using the shorthand notation

$$\theta^1_{m,r}\theta^{1,140+K}_{r'}\equiv\theta^1_{m,r}\left(3z_2\right)\theta^{1,140+K}_{r'}\left(z_1\right)\ , \tag{4.4.12}$$

and similarly

$$\frac{1}{4}\sum_{\hat{w}\in W^{\otimes 3}}(-1)^{\ell(\hat{w})}\Theta^{(\varrho_{\hat{w}})}_{1,e}(\mathbf{z}) = \tag{4.4.13}$$

$$\begin{aligned}
&\big(\theta^0_{140,63}+\theta^0_{140,7}\big)\theta^{1,140+K}_{23}\\
&+\big(\theta^0_{140,15}+\theta^0_{140,55}\big)\theta^{1,140+K}_{1}\\
&-\big(\theta^0_{140,0}-\theta^0_{140,140}\big)\theta^{1,140+K}_{26}\\
&+\big(\theta^0_{140,133}+\theta^0_{140,77}\big)\theta^{1,140+K}_{37}\\
&-\big(\theta^0_{140,132}-\theta^0_{140,48}-\theta^0_{140,8}+\theta^0_{140,92}\big)\,\theta^{1,140+K}_{22}-\theta^0_{140,105}\theta^{1,140+K}_{9}\\
&-\big(\theta^0_{140,113}+\theta^0_{140,127}+\theta^0_{140,57}+\theta^0_{140,97}\big)\,\theta^{1,140+K}_{57}-\theta^0_{140,35}\theta^{1,140+K}_{19}\\
&-\big(\theta^0_{140,120}-\theta^0_{140,20}\big)\theta^{1,140+K}_{6}\\
&+\big(\theta^0_{140,112}-\theta^0_{140,2}\big)\theta^{1,140+K}_{2}\\
&-\big(\theta^0_{140,125}+\theta^0_{140,85}\big)\theta^{1,140+K}_{29}\\
&-\big(\theta^0_{140,13}+\theta^0_{140,27}+\theta^0_{140,43}+\theta^0_{140,83}\big)\,\theta^{1,140+K}_{13}\ ,
\end{aligned}\tag{4.4.14}$$

where

$$\theta^0_{m,r}\theta^{1,140+K}_{r'} \equiv \theta^0_{m,r}(3z_2)\,\theta^{1,140+K}_{r'}(z_1) \tag{4.4.15}$$

and $K = \{1, p_1p_2, p_1p_3, p_2p_3\} = \{1, 20, 28, 35\}$. This example makes manifest the recursive structure described in Proposition 4.3.1.2.

4.4.2 Further examples

The above subsections have shown in great detail that the $\widehat{Z}$-invariant of Seifert manifolds, including non-spherical manifolds for which Conjecture 4.0.1.1 is not yet proven, display a recursion relation across different ranks. In particular the companions for $\widehat{Z}^{SU(2)}$ and $\widehat{Z}^{SU(3)}$ are carefully analyzed.

In the following Table we provide further evidence of this phenomenon. We organize examples in blocks. In each block the data is organized as follows: where

Seifert data	σ^{m+K}
m, D	$\sigma^{m+K}_{A_1}$
$m+K$	$\sigma^{m+K}_{A_2}$ or $\bar{\sigma}^{m+K}_{A_2}$

σ^{m+K} is the set of r giving inequivalent $\theta^{1,m+K}_r$ (4.3.21), $\sigma^{m+K}_{A_1}$ is the minimal subset of σ^{m+K} such that (4.0.13) and (4.0.14) hold also when σ^{m+K} is replaced by $\sigma^{m+K}_{A_1}$, for all inequivalently choices of boundary conditions $\underline{\vec{b}}$. Similarly, $\sigma^{m+K}_{A_2}$ is the minimal subset of σ^{m+K} such that (4.0.16) holds also when σ^{m+K} is replaced by $\sigma^{m+K}_{A_1}$, for all inequivalently choices of boundary conditions $\underline{\vec{b}}$. Note that we have $\sigma^{m+K} \subset \sigma^{m+K}_{A_2} \subset \sigma^{m+K}_{A_1}$ in all cases we study.

Table 4.3:

$(-2; 1/2,\ 1/2,\ 3/5)$	$\sigma^{40} = \{1, \text{-}, 39\}$
40, 4	$\sigma^{40}_{A_1} = \{28, 32, 38\}$
40	$\bar{\sigma}^{40}_{A_2} = \{5, 10, 15, 16, 20, 24, 25, 30, 35\}$
$(-1; 1/2,\ 1/3,\ 1/8)$	$\sigma^{24+8} = \{1, 2, 4, 5, 7, 8, 10, 13, 16\}$
24, 1	$\sigma^{24+8}_{A_1} = \{1, 7\}$
$24+8$	$\sigma^{24+8}_{A_2} = \{1, 2, 7, 10\}$
$(-1; 1/2,\ 1/7,\ 2/7)$	$\sigma^{14+7} = \{1, 3, 5, 7\}$
14, 1	$\sigma^{14+7}_{A_1} = \{3\}$
$14+7$	$\sigma^{14+7}_{A_2} = \{1, 3, 5\}$

Continued on next page

Table 4.3: (Continued)

$(-1; 1/4,\ 1/7,\ 4/7)$	$\sigma^{28+7} = \{1,2,3,5,6,7,9,10,13,14,17,21\}$
$28,\ 1$	$\sigma^{28+7}_{A_1} = \{13,21\}$
$28+7$	$\sigma^{28+7}_{A_2} = \{1,2,3,5,6,7,9,10,13,14,17,21\}$
$(-1; 1/3,\ 1/5,\ 2/5)$	$\sigma^{15+5} = \{1,2,4,5,7,10\}$
$15,\ 1$	$\sigma^{15+5}_{A_1} = \{4\}$
$15+5$	$\sigma^{15+5}_{A_2} = \{1,2,4,7\}$
$(-1; 1/3,\ 1/3,\ 1/4)$	$\sigma^{12+3} = \{1,2,3,5,6,9\}$
$12,\ 1$	$\sigma^{12+3}_{A_1} = \{1,9\}$
$12+3$	$\bar{\sigma}^{12+3}_{A_2} = \{4,7,8,10,11\}$
$(-2; 1/2,\ 1/2,\ 12/13)$	$\sigma^{52} = \{1,\text{-},51\}$
$52,\ 2$	$\sigma^{52}_{A_1} = \{24,28,50\}$
52	$\sigma^{52}_{A_2} = \{2,4,9,11,15,17,22,24,28,30,35,37,41,43,48,50\}$
$(-1; 1/3,\ 1/11,\ 6/11)$	$\sigma^{33} = \{1,\text{-},32\}$
$33,\ 1$	$\sigma^{33}_{A_1} = \{16,22,28\}$
33	$\bar{\sigma}^{33}_{A_2} = \{3,4,6,7,9,12,15,18,21,24,26,27,29,30\}$
$(-2; 1/2,\ 2/3,\ 2/3)$	$\sigma^{6+3} = \{1,3\}$
$6,\ 1$	$\sigma^{6+3}_{A_1} = \{1,3\}$
$6+3$	$\sigma^{6+3}_{A_2} = \{1,3\}$
$(-2; 1/2,\ 1/2,\ 8/9)$	$\sigma^{36} = \{1,\text{-},35\}$
$36,\ 2$	$\sigma^{36}_{A_1} = \{16,20,34\}$
36	$\sigma^{36}_{A_2} = \{2,4,5,7,11,13,14,16,20,22,23,25,29,31,32,34\}$
$(-2; 1/2,\ 1/2,\ 4/5)$	$\sigma^{20} = \{1,\text{-},19\}$
$20,\ 2$	$\sigma^{20}_{A_1} = \{8,12,18\}$
20	$\bar{\sigma}^{20}_{A_2} = \{5,10,15\}$
$(-2; 1/2,\ 2/3,\ 3/4)$	$\sigma^{12+4} = \{1,2,4,5,8\}$
$12,\ 1$	$\sigma^{12+4}_{A_1} = \{1,5\}$
$12+4$	$\sigma^{12+4}_{A_2} = \{1,4,5,8\}$
$(-1; 1/2,\ 1/3,\ 1/9)$	$\sigma^{18+9} = \{1,3,5,7,9\}$
$18,\ 1$	$\sigma^{18+9}_{A_1} = \{1,5\}$

Continued on next page

Table 4.3: (Continued)

$18+9$	$\sigma_{A_2}^{18+9} = \{1,5,7\}$
$(-1; 1/2,\ 1/5,\ 1/5)$	$\sigma^{10+5} = \{1,3,5\}$
$10,\ 1$	$\sigma_{A_1}^{10+5} = \{1,5\}$
$10+5$	$\sigma_{A_2}^{10+5} = \{1,3,5\}$
$(-1; 1/2,\ 2/5,\ 1/15)$	$\sigma^{30+15} = \{1,3,5,7,9,11,13,15\}$
$30,\ 1$	$\sigma_{A_1}^{30+15} = \{7,11\}$
$30+15$	$\sigma_{A_2}^{30+15} = \{1,5,7,11\}$
$(-1; 1/2,\ 1/11,\ 4/11)$	$\sigma^{22} = \{1, \text{-}, 21\}$
$22,\ 1$	$\sigma_{A_1}^{22} = \{7,11,15\}$
22	$\sigma_{A_2}^{22} = \{3,5,7,11,15,17,19\}$
$(-2; 1/2,\ 1/2,\ 6/7)$	$\sigma^{28} = \{1, \text{-}, 27\}$
$28,\ 2$	$\sigma_{A_1}^{28} = \{12,16,26\}$
28	$\bar{\sigma}_{A_2}^{28} = \{1,6,7,8,13,14,15,20,21,22,27\}$
$(-1; 1/2,\ 1/4,\ 1/5)$	$\sigma^{20+4} = \{1,2,3,4,6,7,8,11,12,16\}$
$20,\ 1$	$\sigma_{A_1}^{20+4} = \{1,11\}$
$20+4$	$\sigma_{A_2}^{20+4} = \{1,3,4,7,8,11,12,16\}$
$(-2; 1/2,\ 1/3,\ 1/2)$	$\sigma^{24} = \{1, \text{-}, 23\}$
$24,\ 4$	$\sigma_{A_1}^{24} = \{16,20,22\}$
24	$\bar{\sigma}_{A_2}^{24} = \{3,6,9,12,15,18,21\}$

4.5 Conclusion

In the chapter we continued to study of quantum modular properties of $\widehat{Z}^G$-invariants, extending the analysis to higher rank G. The results and conjectures of the work presented in this chapter lead to many further research questions and open questions, which we list below.

- Conjecture 4.0.0.1 is plausible. Starting from the Definition 4.3.1 of the rank-r $\widehat{Z}$-invariants, after straightforwardly performing the contour integration in the directions spanned by all non-junction vertices, we are left with a rank $N = r \times n$ lattice sum in the integrand of the remaining contour integral. In the weakly-negative/positive case, the signature of the lattice is purely positive/negative. In particular, in the weakly-negative case we obtain a sum over (derivatives of) rank N false-theta-like function. It should be interesting to prove their quantum modularity explicitly. Similarly, for the weakly-positive case we expect to obtain a close cousin of higher depth mock modular form, though at present we do not have a universal recipe for defining $\widehat{Z}$-invariants for these cases.

- Beyond Conjecture 4.0.0.1, it would be very interesting to analyse quantum modularity of $\widehat{Z}$-invariants when the plumbed manifold is neither weakly-negative now weakly-positive, in other words when the space spanned by junction vertices has signature $(k, N-k)$ when $k \neq 0$. For this purpose, it should be interesting to generalize the generalized error function [86] , [97] to accommodate both the "false" as well as the "mock" directions.

- As mentioned in the introduction, Rademacher sum expressions are interesting for many purposes and are often available for holomorphic quantum modular forms of the kind we study here. It would be interesting to systematically develop the Rademacher sum techniques for general quantum modular forms. In terms of the physics on the field theory side, we wish to compare the $S^2 \times S^1$ superconformal indices of the 3d theory $T[M_3]$, conjectured to be related to $\widehat{Z}$ by
$$I^G(\tau) \sim \sum_b \widehat{Z}_b^G(\tau)\widehat{Z}_b^G(-\tau), \tag{4.5.1}$$
with a summing over saddle point contributions from different gravity solutions. As argued in [23], it is tempting to define $\widehat{Z}_b^G(M_3; -\tau)$ by identifying it with $\widehat{Z}_b^G(-M_3; \tau)$. On the gravity side, while we do not yet have a complete catalogue of supergravity solutions, the solutions described recently [110] in the $AdS_4 \times S^7$ context encouragingly take the form as geometries that might be matched with the different Rademacher contributions.

- Often, $\widehat{Z}$-invariants admit totally different expressions, arising from realizing M_3 not by plumbing but by surgery along knots [58], [25] , [26] , [111] , or from alternative ways of expressing characters of logarithmic vertex algebras [31], leading to interesting q-series identities. While so far the analysis of quantum modularity relies mostly on the connection to lattice theta functions, it will be very interesting if modular properties can also be analyzed directly through these other expressions as well, as they are connected to yet different areas of mathematics and will lead to different applications.

- It will be very interesting to understand the nature of the recursive relation we observed in more concrete terms. We can think of the following routes for exploration. 1) Work out the recursion at higher rank in order to gain a more complete understanding of the recursive structure. 2) We already mentioned the analogy to the structure in higher rank Vafa-Witten theory (4.0.3). It would be helpful to develop a similar interpretation for the 3d case. 3) Apart from the geometrical M-theory perspective, the Vafa-Witten recursion also admits an interpretation in terms of the reducible connections of the higher rank gauge group. From the $SL(N,\mathbb{C})$ Chern-Simons point of view, we believe it would be illuminating to work out the higher rank/higher depth analogue of (4.0.2), from which we should be able to see explicitly the role played by the lower rank flat connections. It is also desirable to compare with the resurgence analysis analogous to [112]. It will be particularly interesting to see what it means for the proposal in [23] to view the orbits of Weil representation as corresponding to the non-Abelian $SU(2)$ flat connections on M_3, or relatedly to the different Wilson line insertions [104].

- According to the false-mock conjecture [23] and its higher rank generalization, the recursion relation reported in Conjecture 4.0.1.1 and Theorem 4.0.2 should hold for $-M_3$, the orientation-flipped cousin of M_3, in a completely analogous way. It would be interesting to compute $\hat{Z}^G(-M_3)$ for higher rank G and check it.

4.6 Proofs

Proof for Lemma 4.2.0.1

Let $a,\ b \in W$ be the Weyl group elements whose action on a root $\vec{k} = \sum_{i=1,2} k_i^r \vec{\alpha}_i$ reads

$$\begin{aligned} a &: \vec{k} \mapsto (k_2^r - k_1^r)\vec{\alpha}_1 + k_2^r\vec{\alpha}_2 \ , \\ b &: \vec{k} \mapsto k_1^r\vec{\alpha}_1 + (k_1^r - k_2^r)\vec{\alpha}_2 \ . \end{aligned} \tag{4.6.1}$$

They represent reflections with respect to the planes orthogonal to the simple roots $\vec{\alpha}_1$ resp. $\vec{\alpha}_2$. In terms of these, we have $W = \{\mathbf{1}, a, b, ab, ba, aba = bab\}$ and $W_+ = \{\mathbf{1}, ab, ba\}$.

From $0 \leq \langle \vec{k}, \vec{\omega}_i \rangle$ for $i = 1, 2$ we conclude that at least one of the triple $\vec{k}$, $a(\vec{k})$ and $b(\vec{k})$ is in $\bar{P}^+$. Evoking the identity

$$(-1)^{\ell(w')} F^{(\varrho')}(\tau) = F^{(\varrho)}(\tau) \tag{4.6.2}$$

for $\varrho = (\vec{s}, \vec{k}, m, D)$ and $\varrho' = (w'(\vec{s}), w'(\vec{k}), m, D)$, from now on we assume that $\vec{k} \in \bar{P}^+$ without loss of generality.

In the sum over $\vec{n}$ in (4.2.5), write $\vec{n} = D\vec{m} + w(\vec{k})$ for $\vec{m} \in \Lambda$. We have

$$\vec{m} + w(\vec{k}) \in \bar{P}^+ \quad \Leftrightarrow \quad m_i \geq \xi_{w,i} \ , \qquad i = 1, 2 \ , \tag{4.6.3}$$

where $\xi_{w,i}$ are defined by

$$\xi_{w,i} := \left\lceil -\frac{w(\vec{k})|_i}{D} \right\rceil \ . \tag{4.6.4}$$

Since $0 \leq \langle \vec{k}, \vec{\omega}_i \rangle < D$, we have

$$|(w(\vec{k})|_1 - w(\vec{k})|_2)| \leq \max(2k_1 + k_2, k_1 + 2k_2) < 3D \ , \tag{4.6.5}$$

and hence $|\xi_{w,1} - \xi_{w,2}| \leq 3$. The function $\min(n_i)$ is then given by

$$\min(Dm_i + w(\vec{k})|_i) = \begin{cases} Dm_i + w(\vec{k})|_i \ , & \text{for } m_i < m_j \ , \ i, j \in \{1, 2\} \\ Dn + \min(w(\vec{k})|_i) \ , & \text{for } m_1 = m_2 = n \ . \end{cases} \tag{4.6.6}$$

For a given w, write the sum in (4.2.5) as $F^{(\varrho)} = \sum_w (-1)^{\ell(w)} F_w^{(\varrho)}$. We now discuss $F_w^{(\varrho)}$ in the following two cases.

- Case 1: $w(\vec{k})|_2 \geq w(\vec{k})|_1$.
  In this case $\xi_{w,1} \geq \xi_{w,2}$ and

$$\begin{aligned} F_w^{(\varrho)}(\tau) = & \sum_{\substack{m_2 \geq m_1 \geq \xi_{w,1} \\ m_1 \equiv m_2 \ (\mathrm{mod}\ 3)}} (Dm_1 + w(\vec{k})|_1) q^{p_{w,\vec{m}}} \\ & + \sum_{\substack{m_1 > m_2 \geq \xi_{w,2} \\ m_1 \equiv m_2 \ (\mathrm{mod}\ 3)}} (Dm_2 + w(\vec{k})|_2) q^{p_{w,\vec{m}}} \ , \end{aligned} \tag{4.6.7}$$

  where $p_{w,\vec{m}} = \frac{1}{2m} | - w(\vec{\sigma}) + m(\vec{m})|^2$. By redefining the summation indices in the above equation in the following way

$$(n_1, n_2) := \begin{cases} \left(\frac{1}{3}(m_1 - m_2), m_2\right) & m_1 > m_2 \\ \left(\frac{1}{3}(m_2 - m_1), m_1\right) & m_2 \geq m_1, \end{cases} \tag{4.6.8}$$

  and shifting the summation ranges by $\xi_{w,1}$ resp $\xi_{w,2}$, $F_w^{(\varrho)}(\tau)$ can be rewritten as

$$F_w^{(\varrho)}(q) = \sum_{n_1,n_2\geq 0} (D(n_2+\xi_{w,1}) + w(\vec{k})|_1) q^{p_{w,(n_2+\xi_{w,1},3n_1+n_2+\xi_{w,1})}} + (D(n_2+\xi_{w,2}) + w(\vec{k})|_2) q^{p_{w,(3n_1+3+n_2+\xi_{w,2},n_2+\xi_{w,2})}} \ . \quad (4.6.9)$$

Introducing then the quadratic form

$$Q(\boldsymbol{n}) := Q(n_1,n_2) = \frac{1}{2}|(n_2, 3n_1+n_2)|^2 = (3n_1^2 + 3n_1n_2 + n_2^2) \ , \quad (4.6.10)$$

we can write the function $F_w^{(\varrho)}(q)$ in terms of this notation

$$F_w^{(\varrho)}(q) = \sum_{i=1,2} \sum_{n_1,n_2\geq 0} (D(n_2+\xi_{w,i}) + w(\vec{k})|_i) q^{mQ(\boldsymbol{n}+\boldsymbol{\alpha}_w^{(i)})} \quad (4.6.11)$$

with $\boldsymbol{\alpha}_w^{(i)}$ given by (4.2.9) with $x=0$.

- Case 2: $w(\vec{k})|_2 < w(\vec{k})|_1$.
  This case can be treated analogously to Case 1, and gives (4.6.11) with $\boldsymbol{\alpha}_w^{(i)}$ given by (4.2.9) with $x=1$.

The pairs of Weyl group elements $w, w' \in W$

$$(w,w') = (1,aba), (a,ba), (b,ab) \quad (4.6.12)$$

satisfy $w(\vec{k})|_i = -w'(\vec{k})|_j$ for $i \neq j$.

Since the condition $w(\vec{k})|_2 \geq w(\vec{k})|_1$ is satisfied if and only if $w'(\vec{k})|_2 \geq w'(\vec{k})|_1$, we have the relations

$$\boldsymbol{\alpha}_{w'}^{(1)} = \mathbf{1} - \boldsymbol{\alpha}_w^{(2)} \equiv \bar{\boldsymbol{\alpha}}_w^{(2)} \ , \qquad \boldsymbol{\alpha}_{w'}^{(2)} = \mathbf{1} - \boldsymbol{\alpha}_w^{(1)} \equiv \bar{\boldsymbol{\alpha}}_w^{(1)}. \quad (4.6.13)$$

Moreover, by shifting the summand we obtain

$$F_w^{(\varrho)}(\tau) = \sum_{i=1,2} \sum_{\boldsymbol{n}\in\mathbb{N}_0^2+\boldsymbol{\alpha}_w^{(i)}} \left(Dn_2 + \frac{w(\vec{s})|_i D}{m}\right) q^{mQ(\boldsymbol{n})} \ . \quad (4.6.14)$$

Summing over all w, we arrive at the expressions in Lemma 4.2.0.1.

Proof for Lemma 4.2.0.2

In the first step we consider $\boldsymbol{\alpha}' = \boldsymbol{\alpha} + (\delta\alpha_1, 0)$. Then a routine computation shows that

$$\begin{aligned} F_{0,\boldsymbol{\alpha}'} - F_{0,\boldsymbol{\alpha}} &= \sum_{0\le k\le \delta\alpha_1 - 1} q^{\frac{3}{4}(\alpha_1+k)^2} \sum_{n\in\mathbb{Z}} \operatorname{sgn}(n+\tfrac{1}{2}) q^{(n+\alpha_2+\frac{3}{2}(\alpha_1+k))^2} \\ F_{1,\boldsymbol{\alpha}'} - F_{1,\boldsymbol{\alpha}} &= -\sum_{0\le k\le \delta\alpha_1 - 1} q^{\frac{3}{4}(\alpha_1+k)^2} \sum_{n\in\mathbb{Z}} \operatorname{sgn}(n+\tfrac{1}{2})(n+\alpha_2)\, q^{(n+\alpha_2+\frac{3}{2}(\alpha_1+k))^2} \end{aligned} \tag{4.6.15}$$

while

$$\sum_{n\in\mathbb{Z}} \operatorname{sgn}(n+\tfrac{1}{2}) q^{(n+\alpha_2+\frac{3}{2}(\alpha_1+k))^2} - \tilde{\theta}^1[1, \alpha_2 + \frac{3}{2}(\alpha_1+k)](\tau) \in \mathbb{Z}[q]$$

since $\operatorname{sgn}(n+\frac{1}{2}) - \operatorname{sgn}(n+\alpha_2+\frac{3}{2}(\alpha_1+k))$ has finite support. Similarly,

$$\left[\sum_{n\in\mathbb{Z}} \operatorname{sgn}(n+\tfrac{1}{2})(n+\alpha_2)\, q^{(n+\alpha_2+\frac{3}{2}(\alpha_1+k))^2} + \frac{3}{2}(\alpha_1+k)\tilde{\theta}^1[1,\alpha_2+\tfrac{3}{2}(\alpha_1+k)](\tau) \right.$$
$$\left. - \tilde{\theta}[1,\alpha_2+\tfrac{3}{2}(\alpha_1+k)](\tau)\right] \in \mathbb{Z}[q].$$

Second, we consider $\boldsymbol{\beta} = \boldsymbol{\alpha}' + (0, \delta\alpha_2) = \boldsymbol{\alpha} + (\delta\alpha_1, \delta\alpha_2)$. We have

$$\begin{aligned} F_{0,\boldsymbol{\beta}} - F_{0,\boldsymbol{\alpha}'} &= \sum_{0\le k\le \delta\alpha_2 - 1} q^{\frac{1}{4}(\alpha_2+k)^2} \sum_{n\in\mathbb{Z}} \operatorname{sgn}(n+\tfrac{1}{2})\, q^{3(n+\alpha_1'+\frac{1}{2}(\alpha_2+k))^2} \\ F_{1,\boldsymbol{\beta}} - F_{1,\boldsymbol{\alpha}'} &= -\sum_{0\le k\le \delta\alpha_2 - 1} q^{\frac{1}{4}(\alpha_2+k)^2}(k+\alpha_2) \sum_{n\in\mathbb{Z}} \operatorname{sgn}(n+\tfrac{1}{2})\, q^{3(n+\alpha_1'+\frac{1}{2}(\alpha_2+k))^2} \end{aligned} \tag{4.6.16}$$

and

$$\sum_{n\in\mathbb{Z}} \operatorname{sgn}(n+\tfrac{1}{2})\, q^{3(n+\alpha_1'+\frac{1}{2}(\alpha_2+k))^2} - \tilde{\theta}^1[3, \alpha_1' + \tfrac{1}{2}(\alpha_2+k)] \in \mathbb{Z}[q].$$

Combining the above two steps proves the statement.

Proof of Lemma 4.2.0.3

First note

$$\sum_{\boldsymbol{\ell}\in(\mathbb{Z}/k\bar{m})^2} \mathrm{e}\left(\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})\right) = \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/k\bar{m})^2} \mathrm{e}\left(\frac{h}{k}Q(\boldsymbol{\ell}+\mathbf{1}-\boldsymbol{\alpha})\right)$$

and the statement for $\nu = 1$ immediately follows since $\eta_1(\boldsymbol{\alpha}) + \eta_1(\mathbf{1} - \boldsymbol{\alpha}) = 0$ for all $\boldsymbol{\alpha} \in \mathcal{S}$. Similarly, from the above identity we have

$$\sum_{\boldsymbol{\alpha}\in\mathcal{S}} \eta_0(\boldsymbol{\alpha}) \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/k\bar{m})^2} \mathrm{e}\left(\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})\right) = 2\sum_{\boldsymbol{\alpha}\in\tilde{\mathcal{S}}} \eta_0(\boldsymbol{\alpha}) \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/k\bar{m})^2} \mathrm{e}\left(\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})\right) \tag{4.6.17}$$

since $\eta_0(\boldsymbol{\alpha}) = \eta_0(\mathbf{1} - \boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \mathcal{S}$.

More generally, the sum over $\boldsymbol{\ell}$ is invariant if one replaces the $\boldsymbol{\alpha}$ in summand with any $\boldsymbol{\alpha}'$ as long as $\boldsymbol{\alpha} + \boldsymbol{\alpha}' \in \mathbb{Z}^2$ or $\boldsymbol{\alpha} - \boldsymbol{\alpha}' \in \mathbb{Z}^2$, as one can simultaneously shift $\boldsymbol{\ell}$. Here we choose $\boldsymbol{\alpha}_w^{(i)'} = \frac{1}{m}\mathbf{a}_w^{(i)}$, where

$$\mathbf{a}_w^{(1)} = (\Delta w(\vec{\sigma}), -w(\vec{\sigma})|_1) \ , \quad \mathbf{a}_w^{(2)} = (-\Delta w(\vec{\sigma}), -w(\vec{\sigma})|_2) \tag{4.6.18}$$

satisfy $\mathbf{a} \in \mathbb{Z}^2$ and

$$Q(\mathbf{a}) = \frac{1}{3}\left(\sigma_1^2 + \sigma_2\sigma_1 + \sigma_2^2\right) = \frac{1}{2}|\boldsymbol{\sigma}|^2. \tag{4.6.19}$$

Let $\langle\cdot,\cdot\rangle_Q$ be twice the inner product induced by the quadratic form Q

$$\langle \mathbf{v}, \mathbf{w}\rangle_Q := Q(\mathbf{v}+\mathbf{w}) - Q(\mathbf{v}) - Q(\mathbf{w}) = 3\,(2v_1 + v_2)\,w_1 + (3v_1 + 2v_2)\,w_2 = \langle \mathbf{w}, \mathbf{v}\rangle_Q\,. \tag{4.6.20}$$

Splitting the sum over $\boldsymbol{\ell}$ into a sum over $\mathbf{N}$ and $\boldsymbol{\nu}$ by writing $\boldsymbol{\ell} = \mathbf{N} + k\boldsymbol{\nu}$, we arrive at

$$\begin{aligned}\sum_{\boldsymbol{\ell}\in(\mathbb{Z}/k\bar{m})^2} \mathrm{e}\left(\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})\right) &= \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/k\bar{m})^2} \mathrm{e}\left(\frac{h}{k}Q(\boldsymbol{\ell}+\frac{\mathbf{a}}{m})\right)\\ &= \sum_{\boldsymbol{\nu}\in(\mathbb{Z}/\bar{m})^2}\sum_{\boldsymbol{N}\in(\mathbb{Z}/k)^2} \mathrm{e}\left(\frac{h}{k}Q(\mathbf{N}+k\boldsymbol{\nu}+\frac{\mathbf{a}}{m})\right)\\ &= \mathrm{e}\left(\frac{h}{km^2}Q(\mathbf{a})\right)\sum_{\boldsymbol{\nu}\in(\mathbb{Z}/\bar{m})^2} \mathrm{e}\left(\frac{h/\delta}{\bar{m}}\langle\boldsymbol{\nu},\mathbf{a}\rangle_Q\right)\\ &\times \sum_{\boldsymbol{N}\in(\mathbb{Z}/k)^2} \mathrm{e}\left(\frac{h}{k}Q(\mathbf{N})\right)\mathrm{e}\left(\frac{h}{km}\langle\mathbf{N},\mathbf{a}\rangle_Q\right).\end{aligned} \tag{4.6.21}$$

Focus on the factor $\sum_{\boldsymbol{\nu}\in(\mathbb{Z}/\bar{m})^2} \mathrm{e}\left(\frac{h/\delta}{\bar{m}}\langle\boldsymbol{\nu},\mathbf{a}\rangle_Q\right)$, we see that the sum vanishes unless $\bar{m}|3a_1, a_2$, which is equivalent to $\vec{\sigma} \in \bar{m}\Lambda$, in which case

$$\sum_{\boldsymbol{\nu}\in(\mathbb{Z}/\bar{m})^2} \mathrm{e}\left(\frac{h/\delta}{\bar{m}}\langle\boldsymbol{\nu},\mathbf{a}\rangle_Q\right) = \bar{m}^2.$$

As a result, next we study the factor $\sum_{\boldsymbol{N}\in(\mathbb{Z}/k)^2} \mathrm{e}\left(\frac{h}{k}Q(\mathbf{N})\right) \mathrm{e}\left(\frac{h}{km}\left\langle \mathbf{N},\mathbf{a}\right\rangle_Q\right)$ when $\vec{\sigma}\in\bar{m}\Lambda$. Let δ^* be the modular inverse of δ mod k. This exists because δ is a divisor of h, which is coprime with k. Shifting the summation over $\boldsymbol{N}$ to $\boldsymbol{N}-\delta^*\frac{\mathbf{a}}{\bar{m}}$, we can cancel the $\langle\mathbf{N},\mathbf{a}\rangle_Q$ term and arrive at the result that

$$\sum_{\boldsymbol{\ell}\in(\mathbb{Z}/k\bar{m})^2} \mathrm{e}\left(\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})\right) = \bar{m}^2\,\mathrm{e}\left(\frac{h/\delta}{k}(\delta^*+\frac{1}{\delta})Q(\mathbf{a})\right)$$

where c and c' are $\mathbf{a}$-independent constants that only depend on h,k and m,D. Using the fact $Q(\mathbf{a})$ is a constant over $\mathcal{S}$ (see (4.6.19)), we obtain from (4.6.17)

$$\sum_{\boldsymbol{\alpha}\in\mathcal{S}}\eta_0(\boldsymbol{\alpha})\sum_{\boldsymbol{\ell}\in(\mathbb{Z}/k\bar{m})^2}\mathrm{e}\left(\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})\right) = 2c\,\mathrm{e}\left(\frac{c'}{2}|\boldsymbol{\sigma}|^2\right)\sum_{\boldsymbol{\alpha}\in\tilde{\mathcal{S}}}\eta_0(\boldsymbol{\alpha}) \tag{4.6.22}$$

which vanishes as a result of $\sum_{w\in W_+}w(\vec{v})=0$ for any $\vec{v}$, and hence $\sum_{\boldsymbol{\alpha}\in\tilde{\mathcal{S}}}\eta_0(\boldsymbol{\alpha})=0$.

Proof for Lemma 4.2.0.4

Proof. We first rewrite, using $a_i := m\boldsymbol{\alpha}_i$ and writing $\boldsymbol{\ell}=\mathbf{N}+k\boldsymbol{\nu}$

$$\begin{aligned}
&\sum_{0\le\ell_1,\ell_2<k\bar{m}} B_n\left(\frac{\ell_1+\alpha_1}{k\bar{m}}\right)\mathrm{e}\left(\frac{h}{k}Q\left(\boldsymbol{\ell}+\boldsymbol{\alpha}\right)\right)\\
&=\mathrm{e}\left(\frac{h}{k}Q\left(\boldsymbol{\alpha}\right)\right)\sum_{0\le\mathbf{N}<k}\mathrm{e}\left(\frac{h/\delta}{k\bar{m}}\left(\bar{m}\delta Q\left(\mathbf{N}\right)+3N_1\left(2a_1+a_2\right)+N_2\left(3a_1+2a_2\right)\right)\right)\times\\
&\sum_{0\le\boldsymbol{\nu}<\bar{m}} B_n\left(\frac{N_1+k\nu_1+\alpha_1}{\bar{m}}\right)\mathrm{e}\left(\frac{h/\delta}{\bar{m}}\left(3\nu_1\left(2a_1+a_2\right)+\nu_2\left(3a_1+2a_2\right)\right)\right)\ .
\end{aligned} \tag{4.6.23}$$

The sum over ν_2 shows that the quantity vanishes when $3a_1+2a_2$ is not divisible by $\bar{m}$. For both $\boldsymbol{\alpha}=\boldsymbol{\alpha}_w^{(1)}$ or $\boldsymbol{\alpha}=\boldsymbol{\alpha}_w^{(2)}$, the condition is equivalent to the condition $\bar{m}|\sum_{i=1,2}w(\vec{\sigma})|_i$. Writing $\sum_{i=1,2}w(\vec{\sigma})|_i=\bar{m}y$, we write

$$\begin{aligned}
\boldsymbol{\alpha}_w^{(1)} &= (\alpha_1,\alpha_2) \bmod (0,1)\\
\boldsymbol{\alpha}_w^{(2)} &= (1-\alpha_1, 1-\alpha_2-\frac{y}{\delta}) \bmod (0,1)
\end{aligned} \tag{4.6.24}$$

Invoking the reflection property (4.2.18) of the Bernoulli polynomials, we have for

$\boldsymbol{\alpha} = \boldsymbol{\alpha}_w^{(2)}$

$$\begin{aligned}
&\sum_{\substack{0\le \ell_1<k\bar{m}\\ \ell_2\in\mathbb{Z}/k\bar{m}}} B_n\left(\frac{\ell_1+1-\alpha_1}{k\bar{m}}\right)\mathrm{e}\left(\frac{h}{k}Q\left(\boldsymbol{\ell}+\mathbf{1}-\boldsymbol{\alpha}-\tfrac{y}{\delta}(0,1)\right)\right)\\
&=\sum_{\substack{0\le \ell_1<k\bar{m}\\ \ell_2\in\mathbb{Z}/k\bar{m}}} B_n\left(1-\frac{\ell_1+\alpha_1}{k\bar{m}}\right)\mathrm{e}\left(\frac{h}{k}Q\left(\boldsymbol{\ell}+\boldsymbol{\alpha}+\tfrac{y}{\delta}(0,1)\right)\right)\\
&=-\sum_{\substack{0\le \ell_1<k\bar{m}\\ \ell_2\in\mathbb{Z}/k\bar{m}}} B_n\left(\frac{\ell_1+\alpha_1}{k\bar{m}}\right)\mathrm{e}\left(\frac{h}{k}Q\left(\boldsymbol{\ell}+\boldsymbol{\alpha}+y(\tfrac{1}{\delta}-\delta^*)(0,1)\right)\right)\\
&=-\sum_{\substack{0\le \ell_1<k\bar{m}\\ \ell_2\in\mathbb{Z}/k\bar{m}}} B_n\left(\frac{\ell_1+\alpha_1}{k\bar{m}}\right)\mathrm{e}\left(\frac{h}{k}Q\left(\boldsymbol{\ell}+\boldsymbol{\alpha}\right)\right)
\end{aligned} \tag{4.6.25}$$

Going from the first to the second line, we have relabeled $\boldsymbol{\ell}$ by $(k\bar{m}-1)\mathbf{1}-\boldsymbol{\ell}$. Going to the third line, we have invoked the reflection property (4.2.18) of the Bernoulli polynomials, and shifted ℓ_2 in the sum by $\delta^* y$, where $\delta^*\delta \equiv 1 \ \ (k)$. In the last step, we used that $\langle \boldsymbol{\alpha}, (0,1)\rangle_Q = -\frac{1}{\delta}y$. From this we immediately see that the contributions from $\boldsymbol{\alpha}_w^{(1)}$ and $\boldsymbol{\alpha}_w^{(2)}$ cancel.

Proof for Lemma 4.2.1.2

For $\nu = 0$ and $\varepsilon \in \{1, -1\}$ we have:

$$
\begin{aligned}
&\mathbb{E}_0^{(\varrho)}(\tau) = \frac{1}{2} \sum_{\varepsilon \in \{1,-1\}} \sum_{\boldsymbol{\mu} \in \tilde{\mathcal{S}}} \eta_0(\boldsymbol{\mu}) \sum_{\boldsymbol{n} \in \boldsymbol{\mu} + \mathbb{Z}} q^{-Q(\boldsymbol{n})} M_2\left(\sqrt{3}; \sqrt{3v}\left(2\varepsilon n_1 + n_2\right), \sqrt{v} n_2\right) \\
&= \frac{1}{2} \sum_{\varepsilon \in \{1,-1\}} \sum_{\boldsymbol{\mu} \in \mathcal{S}} \eta_0(\boldsymbol{\mu}) \left(\sum_{\boldsymbol{n} \in \boldsymbol{\mu} + \mathbb{N}_0^2} q^{-Q(\boldsymbol{n})} M_2\left(\sqrt{3}; \sqrt{3v}\left(2\varepsilon n_1 + n_2\right), \sqrt{v} n_2\right) \right. \\
&+ \left. \sum_{\boldsymbol{n} \in (1-\mu_1, \mu_2) + \mathbb{N}_0^2} q^{-Q(-n_1, n_2)} M_2\left(\sqrt{3}; \sqrt{3v}\left(2\varepsilon n_1 + n_2\right), \sqrt{v} n_2\right) \right) \\
&= \frac{1}{2} \sum_{\boldsymbol{\mu} \in \mathcal{S}} \eta_0(\boldsymbol{\mu}) \left(\sum_{\mathbf{n} \in \boldsymbol{\mu} + \mathbb{N}_0^2} q^{-Q(\boldsymbol{n})} \left(M_2^*\left(\sqrt{3}; \sqrt{3v}\left(2 n_1 + n_2\right), \sqrt{v} n_2\right) \right. \right. \\
&\left. + \delta_{n_1,0}\left(1 - \delta_{n_2,0}\right) M\left(2\sqrt{v} n_2\right) + \delta_{n_2,0}\left(1 - \delta_{n_1,0}\right) M\left(2\sqrt{3v} n_1\right) - \delta_{n_1,0}\delta_{n_2,0} \right) \\
&+ \sum_{\mathbf{n} \in (1-\mu_1, \mu_2) + \mathbb{N}_0^2} q^{-Q(-n_1, n_2)} \left(M_2^*\left(\sqrt{3}; \sqrt{3v}\left(-2 n_1 + n_2\right), \sqrt{v} n_2\right) \right. \\
&- \delta_{n_1,0}\left(1 - \delta_{n_2,0}\right) M\left(2\sqrt{v} n_2\right) - \delta_{n_2,0}\left(1 - \delta_{n_1,0}\right) M\left(2\sqrt{3v} n_1\right) \\
&\left. \left. + \delta_{n_1,0}\delta_{n_2,0} \right) \right) \\
&= \mathbb{E}_0^{*(\varrho)}(\tau) + \\
&\frac{1}{2} \sum_{\boldsymbol{\mu} \in \tilde{\mathcal{S}}} \eta(\boldsymbol{\mu}) \left(\sum_{\boldsymbol{n} \in \boldsymbol{\mu} + \mathbb{N}_0^2} + \sum_{\boldsymbol{n} \in (1,1) - \boldsymbol{\mu} + \mathbb{N}_0^2} - \sum_{\boldsymbol{n} \in (1-\mu_1, \mu_2) + \mathbb{N}_0^2} - \sum_{\boldsymbol{n} \in (\mu_1, 1-\mu_2) + \mathbb{N}_0^2} \right) \\
&\times \left(\delta_{n_1,0}\left(1 - \delta_{n_2,0}\right) M\left(2\sqrt{v} n_2\right) + \delta_{n_2,0}\left(1 - \delta_{n_1,0}\right) M\left(2\sqrt{3v} n_1\right) \right. \\
&\qquad \left. - \delta_{n_1,0}\delta_{n_2,0} \right) q^{-Q(\boldsymbol{n})} \\
&= \mathbb{E}_0^{*(\varrho)}(\tau) + \frac{1}{2} \sum_{\boldsymbol{\mu} \in \tilde{\mathcal{S}}} \eta_0(\boldsymbol{\mu}) X_0(\boldsymbol{\mu})
\end{aligned}
\tag{4.6.26}
$$

For $\nu = 1$ we have:

$$\begin{aligned}
\mathbb{E}_1^{(\varrho)}(\tau) &= \frac{1}{2}\sum_{\boldsymbol{\mu}\in\mathcal{S}}\eta_1(\boldsymbol{\mu})\sum_{\boldsymbol{n}\in\boldsymbol{\mu}+\mathbb{N}_0^2} n_2 M_2\left(\sqrt{3};\sqrt{3v}\left(2n_1+n_2\right)\right)q^{-Q(\boldsymbol{n})} \\
&+\frac{1}{2}\sum_{\boldsymbol{\mu}\in\bar{\mathcal{S}}}\bar{\eta}_1(\boldsymbol{\mu})\sum_{\boldsymbol{n}\in\boldsymbol{\mu}+\mathbb{N}_0^2} n_2 M_2\left(\sqrt{3};\sqrt{3v}\left(-2n_1+n_2\right)\right)q^{-Q(-n_1,n_2)} \\
&+\frac{1}{4\pi\sqrt{v}}\sum_{\boldsymbol{\mu}\in\mathcal{S}}\eta_1(\boldsymbol{\mu})\sum_{\boldsymbol{n}\in\boldsymbol{\mu}+\mathbb{N}_0^2} e^{-\pi(3n_1+2n_2)^2 v}M\left(\sqrt{3v}n_1\right)q^{-Q(\boldsymbol{n})} \\
&+\frac{1}{4\pi\sqrt{v}}\sum_{\boldsymbol{\mu}\in\bar{\mathcal{S}}}\bar{\eta}_1(\boldsymbol{\mu})\sum_{\boldsymbol{n}\in\boldsymbol{\mu}+\mathbb{N}_0^2} e^{-\pi(-3n_1+2n_2)^2 v}M\left(-\sqrt{3v}n_1\right)q^{-Q(-n_1,n_2)} \\
&=\frac{1}{2}\sum_{\boldsymbol{\mu}\in\mathcal{S}}\eta_1(\boldsymbol{\mu})\Bigg[\sum_{\boldsymbol{n}\in\boldsymbol{\mu}+\mathbb{N}_0^2}\Big(n_2 M_2\left(\sqrt{3};\sqrt{3v}\left(2n_1+n_2\right),\sqrt{v}n_2\right) \\
&\qquad+\frac{1}{4\pi\sqrt{v}}e^{-\pi(3n_1+2n_2)^2 v}M\left(\sqrt{3v}n_1\right)\Big)q^{-Q(\boldsymbol{n})} \\
&+\sum_{\boldsymbol{n}\in(1-\mu_1,\mu_2)+\mathbb{N}_0^2}\Big(n_2 M_2\left(\sqrt{3};\sqrt{3v}\left(-2n_1+n_2\right),\sqrt{v}n_2\right) \\
&\qquad+\frac{1}{4\pi\sqrt{v}}e^{-\pi(-3n_1+2n_2)^2 v}M\left(-\sqrt{3v}n_1\right)\Big)q^{-Q(-n_1,n_2)}\Bigg] \\
&=\mathbb{E}_1^{*(\varrho)}(\tau)+\frac{1}{2}\sum_{\boldsymbol{\mu}\in\tilde{\mathcal{S}}}\eta_1(\boldsymbol{\mu})X_1(\boldsymbol{\mu})
\end{aligned}$$

Proof for Proposition 4.2.1.2

Following [96], we can rewrite $\mathbb{E}_0^{(\varrho)}$ and $\mathbb{E}_1^{(\varrho)}$ as

$$\mathbb{E}_0^{(\varrho)}(\tau) = -\frac{\sqrt{3}}{4}\sum_{\boldsymbol{\alpha}\in\tilde{\mathcal{S}}}\eta_0(\boldsymbol{\alpha})\int_{-\bar{\tau}}^{i\infty}\int_{z_1}^{i\infty}\frac{\theta_1(\boldsymbol{\alpha},\mathbf{z})+\theta_2(\boldsymbol{\alpha},\mathbf{z})}{\sqrt{-i(z_1+\tau)}\sqrt{-i(z_2+\tau)}}dz_2dz_1 \tag{4.6.27}$$

and

$$\begin{aligned}
\mathbb{E}_1^{(\varrho)}(\tau) &= \frac{\sqrt{3}}{8\pi}\sum_{\boldsymbol{\alpha}\in\tilde{\mathcal{S}}}\int_{-\bar{\tau}}^{i\infty}\int_{z_1}^{i\infty}\frac{2\theta_3(\boldsymbol{\alpha},\mathbf{z})-\theta_4(\boldsymbol{\alpha},\mathbf{z})}{\sqrt{-i(z_1+\tau)}(-i(z_2+\tau))^{\frac{3}{2}}}dz_2dz_1 \\
&+\frac{\sqrt{3}}{8\pi}\sum_{\boldsymbol{\alpha}\in\tilde{\mathcal{S}}}\int_{-\bar{\tau}}^{i\infty}\int_{z_1}^{i\infty}\frac{\theta_5(\boldsymbol{\alpha},\mathbf{z})}{(-i(z_1+\tau))^{\frac{3}{2}}\sqrt{-i(z_2+\tau)}}dz_2dz_1\ .
\end{aligned} \tag{4.6.28}$$

The functions $\theta_\ell(\boldsymbol{\alpha}, \mathbf{z})$ are defined in equations (C.0.14) and can be equivalently written as

$$\begin{aligned}
\theta_1(\boldsymbol{\alpha},\mathbf{z}) &= \frac{1}{m^2}\sum_{\delta\in\mathbb{Z}/2}\theta^1_{m,m(2\alpha_1+\alpha_2+\delta)}\left(\frac{3z_1}{m}\right)\theta^1_{m,m(\alpha_2+\delta)}\left(\frac{z_2}{m}\right) ,\\
\theta_2(\boldsymbol{\alpha},\mathbf{z}) &= \frac{1}{m^2}\sum_{\delta\in\mathbb{Z}/2}\theta^1_{m,m(3\alpha_1+2\alpha_2+\delta)}\left(\frac{z_1}{m}\right)\theta^1_{m,m(\alpha_1+\delta)}\left(\frac{3z_2}{m}\right) ,\\
\theta_3(\boldsymbol{\alpha},\mathbf{z}) &= \frac{1}{m}\sum_{\delta\in\mathbb{Z}/2}\theta^1_{m,m(2\alpha_1+\alpha_2+\delta)}\left(\frac{3z_1}{m}\right)\theta^0_{m,m(\alpha_2+\delta)}\left(\frac{z_2}{m}\right) ,\\
\theta_4(\boldsymbol{\alpha},\mathbf{z}) &= \frac{1}{m}\sum_{\delta\in\mathbb{Z}/2}\theta^1_{m,m(3\alpha_1+2\alpha_2+\delta)}\left(\frac{z_1}{m}\right)\theta^0_{m,m(\alpha_1+\delta)}\left(\frac{3z_2}{m}\right) ,\\
\theta_5(\boldsymbol{\alpha},\mathbf{z}) &= \frac{1}{m}\sum_{\delta\in\mathbb{Z}/2}\theta^0_{m,m(3\alpha_1+2\alpha_2+\delta)}\left(\frac{z_1}{m}\right)\theta^1_{m,m(\alpha_1+\delta)}\left(\frac{3z_2}{m}\right) .
\end{aligned} \tag{4.6.29}$$

Most of these terms however sum to zero as proved in the following Lemma.

Lemma 4.6.0.1. *Using the definitions above:*

$$\begin{aligned}
\sum_{\boldsymbol{\alpha}\in\tilde{\mathcal{S}}}\eta_0(\boldsymbol{\alpha})\,\theta_1(\boldsymbol{\alpha},\mathbf{z}) &= 0\\
\sum_{\boldsymbol{\alpha}\in\tilde{\mathcal{S}}}\theta_3(\boldsymbol{\alpha},\mathbf{z}) &= 0\\
\sum_{\boldsymbol{\alpha}\in\tilde{\mathcal{S}}}\theta_5(\boldsymbol{\alpha},\mathbf{z}) &= 0
\end{aligned}$$

Proof. Due to the symmetries of the theta series and the sum over δ we only need to focus on the non-integer part of the $\boldsymbol{\alpha}$ defined in equation (4.2.9). By direct computation one can see that:

$$\begin{aligned}
\eta_0\left(\boldsymbol{\alpha}^{(1)}_w\right)\theta_1\left(\boldsymbol{\alpha}^{(1)}_w,\mathbf{z}\right) &= -\eta_0\left(\boldsymbol{\alpha}^{(1)}_{baw}\right)\theta_1\left(\boldsymbol{\alpha}^{(2)}_{baw},\mathbf{z}\right)\\
\theta_3\left(\boldsymbol{\alpha}^{(1)}_w,\mathbf{z}\right) &= -\theta_3\left(\boldsymbol{\alpha}^{(2)}_{baw},\mathbf{z}\right)\\
\theta_5\left(\boldsymbol{\alpha}^{(1)}_w,\mathbf{z}\right) &= -\theta_5\left(\boldsymbol{\alpha}^{(2)}_{w},\mathbf{z}\right).
\end{aligned}$$

The result follows from the fact that ba and $\mathbf{1}$ are in W^+. □

This yields

$$\mathbb{E}_\nu^{(\varrho)}(\tau) = \\ -\frac{\sqrt{3}}{4}(2\pi)^{-\nu}\sum_{\boldsymbol{\mu}\in\tilde{\mathcal{S}}}\eta_\nu(\boldsymbol{\mu})\int_{-\bar{\tau}}^{i\infty}\int_{z_1}^{i\infty}\frac{\Theta_\nu(\boldsymbol{\mu};\mathbf{z})}{(-i(z_1+\tau))^{1/2}(-i(z_2+\tau))^{\nu+1/2}}dz_2dz_1 \tag{4.6.30}$$

where

$$\Theta_\nu(\boldsymbol{\mu},\mathbf{z}) = (m)^{-2+\nu}\sum_{\delta\in\mathbb{Z}/2}\theta^1_{m,m(3\mu_1+2\mu_2+\delta)}\left(\frac{z_1}{m}\right)\theta^{1-\nu}_{m,m(\mu_1+\delta)}\left(\frac{3z_2}{m}\right) . \tag{4.6.31}$$

Substituting then the elements $\boldsymbol{\alpha}_w^{(1)}$ and $\boldsymbol{\alpha}_w^{(2)}$ of the set $\tilde{\mathcal{S}}$ for each $w \in W^+$ and using the shift and symmetry properties of theta functions $\theta^\nu_{m,r}$ for $\nu = 0,1$ allows to reduce the summation over $\boldsymbol{\mu}\in\tilde{\mathcal{S}}$ to a summation over $\boldsymbol{w}\in W^+$ in the expression for $\mathbb{E}_\nu^{(\varrho)}(\tau)$ in terms of $\Theta^{(\varrho)}_{\nu,w}(\mathbf{z})$.

Proof for Lemma 4.3.1.1

Proof. When $M_3 = \Sigma(p_1,p_2,p_3)$ we have the unique $\underline{\vec{b}} = \underline{\vec{b}}_0$, $D = 1$, and $m = p_1p_2p_3$. Using equation (4.3.13), Proposition 4.2.1.1 and Lemma 4.2.1.2, we can express the rank two part of companion of the $\widehat{Z}$-invariant in terms of the functions $\mathbb{E}_\nu^{(\varrho)}(\tau)$ defined in (4.2.40) as

$$\check{Z}^{SU(3)}_{\underline{\vec{b}}_0}(\tau) = C(q^{-1})\sum_{\hat{w}\in W^{\otimes 3}}(-1)^{\ell(\hat{w})}\left(\mathbb{E}_0^{(\varrho_{\hat{w}})}(m\tau)+\mathbb{E}_1^{(\varrho_{\hat{w}})}(m\tau)\right) , \tag{4.6.32}$$

up to a one-dimensional piece, where we have $\rho_{\hat{w}} = (\vec{\sigma}_{\hat{w}},\vec{k}_{\hat{w}},m,D) = (\vec{s}_{\hat{w}},0,p_1p_2p_3,1)$.

Together with

$$\mathbb{E}^{(\varrho)}_{\nu,\tilde{w}} = -\mathbb{E}^{(\varrho)}_{\nu,\tilde{w}(aba)} \tag{4.6.33}$$

in the notation of (4.6.1), which can easily be seen from

$$aba\,\vec{\rho} = -\vec{\rho},\ aba\,\Delta\vec{\omega} = \Delta\vec{\omega},$$

we can extend the sum in Proposition 4.2.1.2 to write

$$\mathbb{E}_\nu^{(\varrho)}(\tau) = \frac{1}{2}\sum_{w\in W}(-1)^{\ell(\hat{w})}\mathbb{E}^{(\varrho)}_{\nu,w}(\tau) \tag{4.6.34}$$

We then have the following identity

$$\begin{aligned}
&C(q^{-1}) \sum_{\nu=0,1} \sum_{\hat{w}\in W^{\otimes 3}} (-1)^{\ell(\hat{w})} \mathbb{E}_{\nu}^{(\varrho_{\hat{w}})}(m\tau) \\
&\quad= \frac{1}{2} C(q^{-1}) \sum_{\nu=0,1} \sum_{\hat{w}\in W^{\otimes 3}} \sum_{w\in W} (-1)^{\ell(\hat{w})}(-1)^{\ell(w)} \mathbb{E}_{\nu,w}^{(\varrho_{\hat{w}})}(m\tau) \\
&\quad= \frac{1}{2} C(q^{-1}) \sum_{\nu=0,1} \sum_{\hat{w}\in W^{\otimes 3}} \sum_{w\in W} (-1)^{\ell(w\hat{w})} \mathbb{E}_{\nu,e}^{(\varrho_{w\hat{w}})}(m\tau) \\
&\quad= \frac{1}{2} |W|\, C(q^{-1}) \sum_{\nu=0,1} \sum_{\hat{w}\in W^{\otimes 3}} (-1)^{\ell(\hat{w})}\, \mathbb{E}_{\nu,e}^{(\varrho_{\hat{w}})}(m\tau)
\end{aligned} \tag{4.6.35}$$

where we have used

$$\mathbb{E}_{\nu,\tilde{w}}^{(\varrho_{\hat{w}})} = \mathbb{E}_{\nu,e}^{(\varrho_{\tilde{w}\hat{w}})} \tag{4.6.36}$$

in the third line, which is manifest from (4.2.46). Combining the above with the double Eichler integral expression in (4.2.45) for $\mathbb{E}_{\nu,\tilde{w}}^{(\varrho_{\hat{w}})}$ leads to the statement of the Lemma.

□

Proof of Proposition 4.3.1.2

We take as our starting point Lemma 4.3.1.1, which states

$$\begin{aligned}
&\check{Z}_{\underline{\vec{b}}_0}^{SU(3)}(M_3;\tau) = z_{1d} \\
&+ \tfrac{|W|}{2m}\, C(q^{-1}) \sum_{\hat{w}\in W^{\otimes 3}} (-1)^{\ell(\hat{w})} \sum_{\nu=0,1} \frac{\sqrt{3}}{4\pi^{\nu}} \left(\frac{3\Delta\vec{s}_{\hat{w}}}{m}\right)^{1-\nu} \sum_{\delta\in\mathbb{Z}/2} (\vartheta'_{w,\delta}, \vartheta_{w,\delta}^{1-\nu})^*(\tau)\ ,
\end{aligned} \tag{4.6.37}$$

where the non-holomorphic double Eichler integral is of the theta functions

$$\begin{aligned}
\vartheta'_{w,\delta}(\tau) &= \theta^{1}_{m,m\delta+\langle\vec{\rho},\vec{\sigma}\rangle}(\tau) \\
\vartheta_{w,\delta}^{1-\nu}(\tau) &= \theta^{1-\nu}_{m,m\delta+\langle\Delta\vec{\omega},\vec{\sigma}\rangle}(3\tau).
\end{aligned} \tag{4.6.38}$$

for

$$\vec{\sigma}_{\hat{w}} = \vec{s}_{\hat{w}} = -\sum_{i=1}^{3} \bar{p}_i w_i(\vec{\rho}). \tag{4.6.39}$$

From $(aba)\vec{\rho} = -\vec{\rho}$, we have

$$\vec{\sigma}_{(w_1(aba)^{\varepsilon_1}, w_2(aba)^{\varepsilon_2}, w_3(aba)^{\varepsilon_3})} = -\sum_{i=1}^{3}(-1)^{\varepsilon_i}\bar{p}_i w_i(\vec{\rho}) \tag{4.6.40}$$

for $\varepsilon_i \in \mathbb{Z}/2$. Then

$$\check{Z}^{SU(3)}_{\underline{\vec{b}}_0}(M_3;\tau) = z_{1d} + \tfrac{|W|}{2m}\, C(q^{-1}) \sum_{\hat{w}\in W_+^{\otimes 3}} \sum_{\nu=0,1} \frac{\sqrt{3}}{4\pi^\nu} \sum_{\delta\in\mathbb{Z}/2} \tilde{\mathbb{E}}^{(\varrho_{\hat{w}})}_{\nu,\delta}(\tau) \tag{4.6.41}$$

where $\tilde{\mathbb{E}}_{\nu,\delta}$ is the integral

$$\tilde{\mathbb{E}}^{(\rho_{\hat{w}})}_{\nu,\delta}(\tau) := -\int_{-\bar{\tau}}^{i\infty}\int_{z_1}^{i\infty} \frac{\tilde{\Theta}^{(\varrho_{\hat{w}})}_{\nu,\delta}(\mathbf{z})}{(-i(z_1+\tau))^{1/2}(-i(z_2+\tau))^{\nu+1/2}} dz_2 dz_1 \tag{4.6.42}$$

of

$$\begin{aligned}\tilde{\Theta}^{(\varrho_{\hat{w}})}_{\nu,\delta}(\mathbf{z}) = &\sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3\in\mathbb{Z}/2} (-1)^{\sum_i \varepsilon_i} \left(\sum_i \tfrac{1}{p_i}(-1)^{\varepsilon_i}\langle \Delta\vec{\omega}, w_i(\vec{\rho})\rangle\right)^{1-\nu} \\ &\times \theta^1_{m,m\delta+\sum_i(-1)^{\varepsilon_i}\bar{p}_i\langle\vec{\rho},w_i(\vec{\rho})\rangle}(z_1)\, \theta^{1-\nu}_{m,m\delta-\sum_i(-1)^{\varepsilon_i}\bar{p}_i\langle\Delta\vec{\omega},w_i(\vec{\rho})\rangle}(3z_2 m) \\ &= 2\sum_{\varepsilon_1,\varepsilon_2\in\mathbb{Z}/2} \tilde{\Theta}^{(\varrho_{\hat{w}})}_{\nu,\delta,(\varepsilon_1,\varepsilon_2)}(\mathbf{z})\ ,\end{aligned} \tag{4.6.43}$$

where we have used that the summand is invariant under $(\varepsilon_1,\varepsilon_2,\varepsilon_3) \mapsto (1,1,1) + (\varepsilon_1,\varepsilon_2,\varepsilon_3)$, and we write

$$\tilde{\Theta}^{(\varrho_{\hat{w}})}_{\nu,\delta,(\varepsilon_1,\varepsilon_2)}(\mathbf{z}) := \left((-1)^{\sum_i \varepsilon_i}\left(\sum_i \tfrac{1}{p_i}(-1)^{\varepsilon_i}\langle\Delta\vec{\omega}, w_i(\vec{\rho})\rangle\right)^{1-\nu}\right. \tag{4.6.44}$$

$$\left.\times\, \theta^1_{m,m\delta+\sum_i(-1)^{\varepsilon_i}\bar{p}_i\langle\vec{\rho},w_i(\vec{\rho})\rangle}(z_1)\, \theta^{1-\nu}_{m,m\delta-\sum_i(-1)^{\varepsilon_i}\bar{p}_i\langle\Delta\vec{\omega},w_i(\vec{\rho})\rangle}(3z_2)\right)\Big|_{\varepsilon_3=0}\ . \tag{4.6.45}$$

To simplify notation, in this appendix we will often skip writing the arguments of the functions, with the understanding that $\theta^{1-\nu}_r = \theta^{1-\nu}_r(3z_2)$ and $\theta^1_r = \theta^1_r(z_1)$.

Using

$$\langle ab\vec{\rho},\vec{\rho}\rangle = \langle ba\vec{\rho},\vec{\rho}\rangle = \langle ab\vec{\rho},\Delta\vec{\omega}\rangle = -\langle ba\vec{\rho},\Delta\vec{\omega}\rangle = -1 \tag{4.6.46}$$

and

$$\langle\vec{\rho},\vec{\rho}\rangle = 2,\ \langle\vec{\rho},\Delta\vec{\omega}\rangle = 0\ , \tag{4.6.47}$$

as well as

$$\theta^{1-\nu}_{m,m\delta+r} = (-1)^{\nu-1}\theta^{1-\nu}_{m,m\delta-r}\quad \text{for all } \delta\in\mathbb{Z}/2, \nu\in\{0,1\}, r\in\mathbb{Z}/2m\ , \tag{4.6.48}$$

it is straightforward to discuss the separate contributions individually.

Case 1: $\hat{w} = (e,e,e)$

From $\langle\vec{\rho},\Delta\vec{\omega}\rangle = 0$ we see $\tilde{\Theta}^{(\varrho_{\hat{w}})}_{\nu,\delta}(\mathbf{z}) = 0$ for $\nu = 0$, and

$$\begin{aligned}\tilde{\Theta}^{(\varrho_{(e,e,e)})}_{\nu,\delta}(\mathbf{z}) &= 2\theta^{1-\nu}_{m,m\delta}\sum_{\varepsilon_1,\varepsilon_2\in\mathbb{Z}/2}(-1)^{\sum_i\varepsilon_i}\theta^1_{m,m\delta+2\sum_i(-1)^{\varepsilon_i}\bar{p}_i+2\bar{p}_3} &(4.6.49)\\ &= 2\theta^{1-\nu}_{m,m\delta}\sum_{\epsilon,\epsilon'}\epsilon\epsilon'\theta^1_{m,m\delta+2\epsilon\bar{p}_1+2\epsilon'\bar{p}_2+2\bar{p}_3} &(4.6.50)\end{aligned}$$

for $\nu = 1$.

Case 2: $\hat{w} = (ab,e,e)$, $\hat{w} = (ba,e,e)$ and permutations

In the case of $\hat{w} = (ab,e,e)$, we have

$$\Theta^{(\varrho_{(ab,e,e)})}_{\nu,\delta}(\mathbf{z}) = (\tfrac{1}{p_1})^{1-\nu}\theta^{1-\nu}_{m,m\delta+\bar{p}_1}\sum_{\epsilon,\epsilon'\in\{1,-1\}}\epsilon\epsilon'\theta^1_{m,m\delta+\epsilon\bar{p}_1+2\epsilon'\bar{p}_2+2\bar{p}_3}\ . \tag{4.6.51}$$

Similarly, $\hat{w} = (ba,e,e)$ renders the same answer and we get

$$\tilde{\Theta}^{(\varrho_{(ba,e,e)})}_{\nu,\delta}(\mathbf{z}) = (\tfrac{1}{p_1})^{1-\nu}\theta^{1-\nu}_{m,m\delta+\bar{p}_1}\sum_{\epsilon,\epsilon'\in\{1,-1\}}\epsilon\epsilon'\theta^1_{m,m\delta+\epsilon\bar{p}_1+2\epsilon\bar{p}_2+2\bar{p}_3}\ . \tag{4.6.52}$$

All other six choices of $\hat{w}\in W_+^{\otimes 3}$ where only one of the three elements is different from $e\in W$ can be treated in exactly the same way, and we get the sum

$$\sum_{\substack{\hat{w}=(w_1,w_2,w_3)\\ \text{one of the } w_i\neq e}}\tilde{\Theta}^{(\varrho_{\hat{w}})}_{\nu,\delta} = 2\mathbf{P}^+\left((\tfrac{1}{p_1})^{1-\nu}\theta^{1-\nu}_{m,m\delta+\bar{p}_1}\sum_{\epsilon,\epsilon'\in\{1,-1\}}\epsilon\epsilon'\theta^1_{m,m\delta+\epsilon\bar{p}_1+2\epsilon'\bar{p}_2+2\bar{p}_3}\right) \tag{4.6.53}$$

where we denote by $\mathbf{P}^+$ by the group of even permutations of (p_1,p_2,p_3).

Case 3: $\hat{w} = (ab, ab, e)$, $\hat{w} = (ba, ba, e)$, $\hat{w} = (ab, ba, e)$, $\hat{w} = (ba, ab, e)$ *and permutations*

We observe that $(-1)^{\varepsilon}\langle\Delta\vec{\omega},\ w(\vec{\rho})\rangle$ is invariant under $\varepsilon \leftrightarrow \varepsilon+1$, $ab \leftrightarrow ba$.

From this we obtain

$$\begin{aligned}&\sum_{\varepsilon\in\mathbb{Z}/2}\Theta^{(\varrho_{(ab,ab,e)})}_{\nu,\delta,(\varepsilon,\varepsilon)}+\Theta^{(\varrho_{(ba,ba,e)})}_{\nu,\delta,(\varepsilon,\varepsilon)}+\Theta^{(\varrho_{(ab,ba,e)})}_{\nu,\delta,(\varepsilon,1+\varepsilon)}+\Theta^{(\varrho_{(ba,ab,e)})}_{\nu,\delta,(1+\varepsilon,\varepsilon)}\\&=2(\tfrac{1}{p_1}+\tfrac{1}{p_2})^{1-\nu}\theta^{1-\nu}_{m,m\delta+\bar{p}_1+\bar{p}_2}\sum_{\epsilon,\epsilon'\in\{1,-1\}}\epsilon\epsilon'\theta^{1}_{m,m\delta+\epsilon\bar{p}_1+\epsilon'\bar{p}_2+2\bar{p}_3}\end{aligned} \tag{4.6.54}$$

and similarly

$$\begin{aligned}&\sum_{\varepsilon\in\mathbb{Z}/2}\Theta^{(\varrho_{(ab,ab,e)})}_{\nu,\delta,(\varepsilon,1+\varepsilon)}+\Theta^{(\varrho_{(ba,ba,e)})}_{\nu,\delta,(\varepsilon,1+\varepsilon)}+\Theta^{(\varrho_{(ab,ba,e)})}_{\nu,\delta,(\varepsilon,\varepsilon)}+\Theta^{(\varrho_{(ba,ab,e)})}_{\nu,\delta,(\varepsilon,\varepsilon)}\\&=2(\tfrac{1}{p_1}-\tfrac{1}{p_2})^{1-\nu}\theta^{1-\nu}_{m,m\delta+\bar{p}_1-\bar{p}_2}\sum_{\epsilon,\epsilon'\in\{1,-1\}}\epsilon\epsilon'\theta^{1}_{m,m\delta+\epsilon\bar{p}_1+\epsilon'\bar{p}_2+2\bar{p}_3}\ .\end{aligned} \tag{4.6.55}$$

We also have images of the above under even permutations, corresponding to the cases where $w_1 = e$ or $w_2 = e$.
Case 4: $\hat{w} = (ab, ab, ab)$, $\hat{w} = (ba, ba, ba)$

Similarly as before, we have

$$\begin{aligned}
\Theta^{(\varrho_{(ab,ab,ab)})}_{\nu,\delta,(\varepsilon,\varepsilon)} &= \Theta^{(\varrho_{(ba,ba,ba)})}_{\nu,\delta,(\varepsilon,\varepsilon)}(\tfrac{(-1)^{\varepsilon+1}(\bar p_1+\bar p_2)-\bar p_3}{m}))^{1-\nu} \\
&\times\theta^{1-\nu}_{m,m\delta+(-1)^{\varepsilon}(\bar p_1+\bar p_2)+\bar p_3}\theta^{1}_{m,m\delta+(-1)^{\varepsilon+1}(\bar p_1+\bar p_2)-\bar p_3} && (4.6.56)\\
\Theta^{(\varrho_{(ab,ba,ab)})}_{\nu,\delta,(\varepsilon,\varepsilon+1)} &= \Theta^{(\varrho_{(ba,ab,ba)})}_{\nu,\delta,(\varepsilon,\varepsilon+1)} = -(\tfrac{(-1)^{\varepsilon+1}(\bar p_1+\bar p_2)-\bar p_3}{m}))^{1-\nu} \\
&\times\theta^{1-\nu}_{m,m\delta+(-1)^{\varepsilon}(\bar p_1+\bar p_2)+\bar p_3}\theta^{1}_{m,m\delta+(-1)^{\varepsilon+1}(\bar p_1-\bar p_2)-\bar p_3} && (4.6.57)\\
\Theta^{(\varrho_{(ba,ab,ab)})}_{\nu,\delta,(\varepsilon+1,\varepsilon)} &= \Theta^{(\varrho_{(ab,ba,ba)})}_{\nu,\delta,(\varepsilon+1,\varepsilon)} = -(\tfrac{(-1)^{\varepsilon+1}(\bar p_1+\bar p_2)-\bar p_3}{m}))^{1-\nu} \\
&\times\theta^{1-\nu}_{m,m\delta+(-1)^{\varepsilon}(\bar p_1+\bar p_2)+\bar p_3}\theta^{1}_{m,m\delta+(-1)^{\varepsilon+1}(-\bar p_1+\bar p_2)-\bar p_3} && (4.6.58)\\
\Theta^{(\varrho_{(ab,ab,ba)})}_{\nu,\delta,(\varepsilon+1,\varepsilon)} &= \Theta^{(\varrho_{(ba,ba,ab)})}_{\nu,\delta,(\varepsilon+1,\varepsilon)} = -(\tfrac{(-1)^{\varepsilon+1}(\bar p_1-\bar p_2)-\bar p_3}{m}))^{1-\nu} \\
&\times\theta^{1-\nu}_{m,m\delta+(-1)^{\varepsilon}(\bar p_1-\bar p_2)+\bar p_3}\theta^{1}_{m,m\delta+(-1)^{\varepsilon}(\bar p_1-\bar p_2)-\bar p_3} && (4.6.59)\\
\Theta^{(\varrho_{(ab,ab,ab)})}_{\nu,\delta,(\varepsilon,\varepsilon+1)} &= \Theta^{(\varrho_{(ba,ba,ba)})}_{\nu,\delta,(\varepsilon,\varepsilon+1)} = -(\tfrac{(-1)^{\varepsilon+1}(\bar p_1-\bar p_2)-\bar p_3}{m}))^{1-\nu} \\
&\times\theta^{1-\nu}_{m,m\delta+(-1)^{\varepsilon}(\bar p_1-\bar p_2)+\bar p_3}\theta^{1}_{m,m\delta+(-1)^{\varepsilon+1}(\bar p_1-\bar p_2)-\bar p_3} && (4.6.60)\\
\Theta^{(\varrho_{(ba,ab,ab)})}_{\nu,\delta,(\varepsilon,\varepsilon)} &= \Theta^{(\varrho_{(ab,ba,ba)})}_{\nu,\delta,(\varepsilon,\varepsilon)} = -(\tfrac{(-1)^{\varepsilon}(\bar p_1-\bar p_2)-\bar p_3}{m}))^{1-\nu} \\
&\times\theta^{1-\nu}_{m,m\delta+(-1)^{\varepsilon+1}(\bar p_1-\bar p_2)+\bar p_3}\theta^{1}_{m,m\delta+(-1)^{\varepsilon+1}(\bar p_1+\bar p_2)-\bar p_3} && (4.6.61)\\
\Theta^{(\varrho_{(ab,ab,ba)})}_{\nu,\delta,(\varepsilon,\varepsilon)} &= \Theta^{(\varrho_{(ba,ba,ab)})}_{\nu,\delta,(\varepsilon,\varepsilon)} = (\tfrac{(-1)^{\varepsilon}(\bar p_1+\bar p_2)-\bar p_3}{m}))^{1-\nu} \\
&\times\theta^{1-\nu}_{m,m\delta+(-1)^{\varepsilon+1}(\bar p_1+\bar p_2)+\bar p_3}\theta^{1}_{m,m\delta+(-1)^{\varepsilon+1}(\bar p_1+\bar p_2)-\bar p_3}\ . && (4.6.62)
\end{aligned}$$

Summing up, we get

$$\begin{aligned}
&\Theta^{(\varrho_{(ab,ab,ab)})}_{\nu,\delta,(\varepsilon,\varepsilon)} + \Theta^{(\varrho_{(ba,ab,ab)})}_{\nu,\delta,(\varepsilon+1,\varepsilon)} + \Theta^{(\varrho_{(ab,ba,ab)})}_{\nu,\delta,(\varepsilon,\varepsilon+1)} + \Theta^{(\varrho_{(ab,ab,ba)})}_{\nu,\delta,(\varepsilon+1,\varepsilon+1)} + (a\leftrightarrow b) = \\
&\qquad 2(\tfrac{1}{p_1}+\tfrac{1}{p_2}+(-1)^{\varepsilon}\tfrac{1}{p_3})^{1-\nu}\theta^{1-\nu}_{m,m\delta+(\bar p_1+\bar p_2)+(-1)^{\varepsilon}\bar p_3}\sum_{\epsilon,\epsilon'}\epsilon\epsilon'\theta^{1}_{m,m\delta+\epsilon\bar p_1+\epsilon'\bar p_2-\bar p_3} \\
&\Theta^{(\varrho_{(ab,ba,ab)})}_{\nu,\delta,(\varepsilon,\varepsilon)} + \Theta^{(\varrho_{(ba,ba,ab)})}_{\nu,\delta,(\varepsilon+1,\varepsilon)} + \Theta^{(\varrho_{(ab,ab,ab)})}_{\nu,\delta,(\varepsilon,\varepsilon+1)} + \Theta^{(\varrho_{(ab,ba,ba)})}_{\nu,\delta,(\varepsilon+1,\varepsilon+1)} + (a\leftrightarrow b) = \\
&\qquad 2(\tfrac{1}{p_1}-\tfrac{1}{p_2}+(-1)^{\varepsilon}\tfrac{1}{p_3})^{1-\nu}\theta^{1-\nu}_{m,m\delta+(\bar p_1-\bar p_2)+(-1)^{\varepsilon}\bar p_3}\sum_{\epsilon,\epsilon'}\epsilon\epsilon'\theta^{1}_{m,m\delta+\epsilon\bar p_1+\epsilon'\bar p_2-\bar p_3}\ .
\end{aligned} \tag{4.6.63}$$

Finally, summing up the contributions from all the above four cases, we define a set $\mathcal{R}\subset\mathbb{Z}/2m$, with

$$\mathcal{R} = \mathcal{R}_0+\mathcal{R}_1+\mathcal{R}_2+\mathcal{R}_3 \tag{4.6.64}$$

and

$$\begin{aligned}\mathcal{R}_0 &= \{0\}\\ \mathcal{R}_1 &= \mathbf{P}^+\{\bar p_1\}\\ \mathcal{R}_2 &= \mathbf{P}^+\{\bar p_1+\bar p_2, \bar p_1-\bar p_2\}\\ \mathcal{R}_3 &= \mathbf{P}^+\{\bar p_1+\bar p_2-\bar p_3, -\bar p_1-\bar p_2+\bar p_3\}\ .\end{aligned} \tag{4.6.65}$$

For each $r\in\mathcal{R}$, we set $a_i^{(r)} := 2-|r_i|$ if $r=\sum_i r_i\bar p_i$. For instance, we have

$$(a_1^{(\bar p_1+\bar p_2)}, a_2^{(\bar p_1+\bar p_2)}, a_3^{(\bar p_1+\bar p_2)}) = (a_1^{(\bar p_1-\bar p_2)}, a_2^{(\bar p_1-\bar p_2)}, a_3^{(\bar p_1+\bar p_2)}) = (1,1,2)\ .$$

Using the above definition, we can write

$$\begin{aligned}\sum_{\hat w\in W_+^{\otimes 3}} \tilde\Theta^{(\varrho_{\hat w})}_{\nu,\delta} &= 2\sum_{r\in\mathcal{R}}(\tfrac{r}{m})^{1-\nu}\theta^{1-\nu}_{m,m\delta+r}\sum_{\epsilon,\epsilon}\epsilon\epsilon'\theta^1_{m,m\delta+\epsilon\bar p_1 a_1^{(r)}+\epsilon'\bar p_2 a_2^{(r)}+\bar p_3 a_3^{(r)}}\\ &= \sum_{r\in\mathcal{R}}(\tfrac{r}{m})^{1-\nu}\theta^{1-\nu}_{m,m\delta+r}\sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3\in\mathbb{Z}/2}(-1)^{\sum_i\varepsilon_i}\theta^1_{m,m\delta+\sum_i(-1)^{\varepsilon_i}\bar p_i a_i^{(r)}}\ .\end{aligned} \tag{4.6.66}$$

From $(p_i,p_j)=1$, we see that $\Omega_m(\bar p_i)$ has precisely one zero entry in each row, since

$$r+r'\equiv 0\ (2\bar p_i)\ ,\quad r-r'\equiv 0\ (2p_i) \tag{4.6.67}$$

has a unique solution in $\mathbb{Z}/2m$ for r' for any given $r\in\mathbb{Z}/2m$.

In particular, one can show that

$$\sum_{r'\in\mathbb{Z}/2m}(\Omega_m(\bar p_i))_{m\delta+\sum_j(-1)^{\varepsilon_j}\bar p_j a_j^{(r)},r'}X_{r'} = X_{m\delta+\sum_j(-1)^{\varepsilon_j+\delta_{i,j}+1}\bar p_j a_j^{(r)}} \tag{4.6.68}$$

for all $\delta, a_j^{(r)}\in\mathbb{Z}$. Consider the representation of the metapletic group $\widetilde{SL}_2(\mathbb{Z})$ corresponding to the subgroup $K=\{1,\bar p_1,\bar p_2,\bar p_3\}$ of the group of exact divisors. This representation is irreducible when all three p_i are square-free. We have

$$\begin{aligned}&(P^{m+K}\theta_m)_{m\delta+\bar p_1a_1+\bar p_2a_2+\bar p_3a_3} =\\ &\qquad\frac14\,(\theta_{m,m\delta+\bar p_1a_1+\bar p_2a_2+\bar p_3a_3}+\theta_{m,m\delta+\bar p_1a_1-\bar p_2a_2-\bar p_3a_3}+\\ &\qquad\qquad\theta_{m,m\delta-\bar p_1a_1+\bar p_2a_2-\bar p_3a_3}+\theta_{m,m\delta-\bar p_1a_1-\bar p_2a_2+\bar p_3a_3})\ .\end{aligned} \tag{4.6.69}$$

Again using $\theta^1_{m,r} = -\theta^1_{m,-r}$, we see that

$$\begin{aligned}\theta^{1,m+K}_{m\delta+\sum_i \bar{p}_i a_i^{(r)}} &:= (P^{m+K}\theta^1_m)_{m\delta+\sum_i \bar{p}_i a_i^{(r)}} \\ &= \frac{1}{4}\sum_{\epsilon,\epsilon} \epsilon\epsilon' \theta^1_{m,m\delta+\epsilon\bar{p}_1 a_1^{(r)}+\epsilon'\bar{p}_2 a_2^{(r)}+\bar{p}_3 a_3^{(r)}} \ .\end{aligned} \tag{4.6.70}$$

As a result, we obtain the following expression

$$\sum_{\hat{w}\in W_+^{\otimes 3}} \tilde{\Theta}^{(\varrho_{\hat{w}})}_{\nu,\delta} = 8\sum_{r\in\mathcal{R}} (\tfrac{r}{m})^{1-\nu} \theta^{1-\nu}_{m,m\delta+r} \theta^{1,m+K}_{m,m\delta+\sum_i \bar{p}_i a_i^{(r)}} \ . \tag{4.6.71}$$

A Indefinite Theta Representations of Mock Theta Functions

For completeness, we include expressions for the mock theta functions used in this work in terms of indefinite theta functions. A more extensive list of expressions including all Ramanujan's mock theta functions can be found in [38]. We have[1]

Order 2

$$A(q) = \frac{q^2\eta(4\tau)}{2\eta(2\tau)^2}\Theta^+_{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)}\left(3\tau, \tau+\frac{1}{2}, 4\tau\right),$$

$$B(q) = \frac{q^{\frac{17}{8}}\eta(2\tau)}{2\eta(\tau)\eta(4\tau)}\Theta^+_{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)}\left(3\tau, 2\tau+\frac{1}{2}, 4\tau\right).$$

Order 3

$$f(q) = -2\frac{q^{\frac{25}{24}}}{\eta(\tau)}\Theta^+_{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)}\left(2\tau+\frac{1}{2}, \frac{1}{2}\tau, 3\tau\right) + q^{\frac{1}{24}}\frac{\eta(3\tau)^4}{\eta(\tau)\eta(6\tau)^2},$$

$$\omega(q) = \frac{q^{\frac{13}{12}}}{\eta(\tau)}\Theta^+_{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)}\left(3\tau, 2\tau+\frac{1}{2}, 6\tau\right) + q^{-\frac{2}{3}}\frac{\eta(6\tau)^4}{\eta(2\tau)\eta(3\tau)^2}.$$

Order 6

$$\sigma(q) = q^{\frac{4}{3}}\frac{\eta(2\tau)\eta(3\tau)}{2\eta(\tau)\eta(6\tau)^2}\Theta^+_{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)}\left(3\tau, \tau+\frac{1}{2}, 6\tau\right),$$

$$\psi_6(q) = q^{\frac{25}{24}}\frac{\eta(\tau)\eta(6\tau)}{2\eta(2\tau)\eta(3\tau)^2}\Theta^+_{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)}\left(\tau+\frac{1}{2}, \frac{3}{2}\tau+\frac{1}{2}, 3\tau\right).$$

[1] Notice that in our notation $\theta_1(z,\tau) = \theta(-z,\tau)$ with $\theta(z,\tau)$ defined as in [38].

Order 8

$$\begin{aligned}
T_0(q) &= \frac{q^{\frac{9}{4}}\eta(4\tau)}{2\eta(2\tau)\eta(8\tau)}\Theta^+_{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)}\left(5\tau+\frac{1}{2},\tau,8\tau\right),\\
T_1(q) &= -\frac{q^{\frac{21}{4}}\eta(4\tau)}{2\eta(2\tau)\eta(8\tau)}\Theta^+_{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)}\left(7\tau+\frac{1}{2},3\tau,8\tau\right),\\
U_0(q) &= \frac{q^{\frac{1}{2}}\eta(4\tau)}{2\eta(8\tau)^2}\Theta^+_{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)}(\tau,\tau,4\tau),\\
V_0(q) &= -\frac{iq^{\frac{1}{2}}}{\theta_1(-\tau,8\tau)}\Theta^+_{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)}\left(\tau,4\tau+\frac{1}{2},8\tau\right)-\frac{\eta(2\tau)^3\eta(4\tau)}{\eta(\tau)^2\eta(8\tau)},\\
V_1(q) &= -\frac{iq^{\frac{3}{2}}}{2\theta_1(-\tau,8\tau)}\Theta^+_{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)}\left(3\tau,2\tau+\frac{1}{2},8\tau\right).
\end{aligned}$$

To make contact with the notation used in section 2.3, we write the function $\Theta^+_{A,\mathbf{c},\mathbf{c}'}(\mathbf{z},\tau)$ in terms of indefinite thetas functions (2.14) through relation (1.57). We can thus rewrite

Order 2

$$\begin{aligned}
A(q) &= e^{-\frac{3\pi i}{4}}q^{\frac{1}{8}}\frac{\eta(4\tau)}{2\eta(2\tau)^2}\Theta_{\left(\frac{3}{4},\frac{1}{4}\right),\left(0,\frac{1}{2}\right)}(4\tau),\\
B(q) &= e^{-\frac{3\pi i}{4}}q^{-\frac{1}{2}}\frac{\eta(2\tau)}{2\eta(\tau)\eta(4\tau)}\Theta_{\left(\frac{3}{4},\frac{1}{2}\right),\left(0,\frac{1}{2}\right)}(4\tau).
\end{aligned}$$

Order 3

$$\begin{aligned}
f(q) &= -2e^{-\frac{5\pi i}{6}}\frac{q^{\frac{1}{24}}}{\eta(\tau)}\Theta_{\left(\frac{2}{3},\frac{1}{6}\right),\left(\frac{1}{2},0\right)}(3\tau)+q^{\frac{1}{24}}\frac{\eta(3\tau)^4}{\eta(\tau)\eta(6\tau)^2},\\
\omega(q) &= e^{-\frac{\pi i}{2}}\frac{q^{-\frac{2}{3}}}{\eta(2\tau)}\Theta_{\left(\frac{1}{2},\frac{1}{3}\right),\left(0,\frac{1}{2}\right)}(6\tau)+q^{-\frac{2}{3}}\frac{\eta(6\tau)^4}{\eta(2\tau)\eta(3\tau)^2}.
\end{aligned}$$

Order 6

$$\begin{aligned}
\sigma(q) &= e^{-\frac{\pi i}{2}}q^{\frac{1}{12}}\frac{\eta(2\tau)\eta(3\tau)}{2\eta(\tau)\eta(6\tau)^2}\Theta_{\left(\frac{1}{2},\frac{1}{6}\right),\left(0,\frac{1}{2}\right)}(6\tau),\\
\psi_6(q) &= e^{-\frac{7\pi i}{6}}q^{\frac{3}{8}}\frac{\eta(\tau)\eta(6\tau)}{2\eta(2\tau)\eta(3\tau)^2}\Theta_{\left(\frac{1}{3},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right)}(3\tau).
\end{aligned}$$

Order 8

$$T_0(q) = e^{-\frac{3\pi i}{4}} q^{\frac{1}{16}} \frac{\eta(4\tau)}{2\eta(2\tau)\eta(8\tau)} \Theta_{(\frac{5}{8},\frac{1}{8}),(\frac{1}{2},0)}(8\tau),$$

$$T_1(q) = -e^{-\frac{5\pi i}{4}} q^{-\frac{7}{16}} \frac{\eta(4\tau)}{2\eta(2\tau)\eta(8\tau)} \Theta_{(\frac{7}{8},\frac{3}{8}),(\frac{1}{2},0)}(8\tau),$$

$$U_0(q) = q^{\frac{1}{8}} \frac{\eta(4\tau)}{2\eta(8\tau)^2} \Theta_{(\frac{1}{4},\frac{1}{4}),(0,0)}(4\tau),$$

$$V_0(q) = -ie^{-\frac{\pi i}{8}} \frac{q^{-\frac{1}{16}}}{\theta_1(-\tau,8\tau)} \Theta_{(\frac{1}{8},\frac{1}{2}),(0,\frac{1}{2})}(8\tau) - \frac{\eta(2\tau)^3\eta(4\tau)}{\eta(\tau)^2\eta(8\tau)},$$

$$V_1(q) = -ie^{-\frac{3\pi i}{8}} \frac{q^{\frac{3}{16}}}{2\theta_1(-\tau,8\tau)} \Theta_{(\frac{3}{8},\frac{1}{4}),(0,\frac{1}{2})}(8\tau).$$

B Tables

In this appendix we collect tables with computational data for the examples presented in Section 4.4. Each of these tables is organized following in blocks with the same format, where each block specifies the contribution to the function $\check{Z}$ which comes from a generalised A_2 false theta function. We remind the reader that the definition of a generalised A_2 false theta function can be found in Section 4.2; this function is a building block for the companion function $\widehat{Z}$ and is associated to a set $\mathcal{S} = \tilde{\mathcal{S}}_{\hat{w}}$, which is in turn determined with respect to $\widehat{Z}$ by a triplet of Weyl group elements $\hat{w}$.

Symbolically, each block is organized in the following way

(w_1, w_2, w_3)	$\boldsymbol{\alpha}_1^{(1)}$	$\boldsymbol{\alpha}_1^{(2)}$
(s_1, s_2)	$\boldsymbol{\alpha}_{ab}^{(1)}$	$\boldsymbol{\alpha}_{ab}^{(2)}$
(k_1, k_2)	$\boldsymbol{\alpha}_{ba}^{(1)}$	$\boldsymbol{\alpha}_{ba}^{(2)}$

using again the same notation (4.6.1) for Weyl group elements. The first column contains the triplet of Weyl elements $\hat{w}$ and the vectors $\vec{s}$ and $\vec{k}$, while the second and third columns contain the values of $\boldsymbol{\alpha}_w^{(1)}$, $\boldsymbol{\alpha}_w^{(2)}$, with w restricted to elements of the rotation subgroup $W_+ \subset W$.

Table B.1: $\boldsymbol{\alpha}$ of $M\left(-1; \frac{1}{4}, \frac{3}{5}, \frac{1}{7}\right)$ for values of $\vec{s}$ with components in $\{1, \ldots, 139\}$.

(e,e,e)	$\left(0, -\frac{83}{140}\right)$	$\left(1, -\frac{83}{140}\right)$	(e,e,a)	$\left(-\frac{1}{7}, -\frac{43}{140}\right)$	$\left(\frac{8}{7}, -\frac{103}{140}\right)$	(e,e,b)	$\left(\frac{1}{7}, -\frac{103}{140}\right)$	$\left(\frac{6}{7}, -\frac{43}{140}\right)$
$(83, 83)$	$\left(\frac{83}{140}, -\frac{83}{140}\right)$	$\left(\frac{57}{140}, \frac{83}{70}\right)$	$(43, 103)$	$\left(\frac{83}{140}, -\frac{103}{140}\right)$	$\left(\frac{57}{140}, \frac{73}{70}\right)$	$(103, 43)$	$\left(\frac{9}{20}, -\frac{43}{140}\right)$	$\left(\frac{11}{20}, \frac{73}{70}\right)$
$(0, 0)$	$\left(-\frac{83}{140}, \frac{83}{70}\right)$	$\left(\frac{223}{140}, -\frac{83}{140}\right)$	$(0, 0)$	$\left(-\frac{9}{20}, \frac{73}{70}\right)$	$\left(\frac{29}{20}, -\frac{43}{140}\right)$	$(0, 0)$	$\left(-\frac{83}{140}, \frac{73}{70}\right)$	$\left(\frac{223}{140}, -\frac{103}{140}\right)$
(e,e,ab)	$\left(\frac{1}{7}, -\frac{83}{140}\right)$	$\left(\frac{6}{7}, -\frac{23}{140}\right)$	(e,e,ba)	$\left(-\frac{1}{7}, -\frac{23}{140}\right)$	$\left(\frac{8}{7}, -\frac{83}{140}\right)$	(e,e,aba)	$\left(0, -\frac{43}{140}\right)$	$\left(1, -\frac{43}{140}\right)$
$(83, 23)$	$\left(\frac{43}{140}, -\frac{23}{140}\right)$	$\left(\frac{97}{140}, \frac{53}{70}\right)$	$(23, 83)$	$\left(\frac{9}{20}, -\frac{83}{140}\right)$	$\left(\frac{11}{20}, \frac{53}{70}\right)$	$(43, 43)$	$\left(\frac{43}{140}, -\frac{43}{140}\right)$	$\left(\frac{97}{140}, \frac{43}{70}\right)$
$(0, 0)$	$\left(-\frac{9}{20}, \frac{53}{70}\right)$	$\left(\frac{29}{20}, -\frac{83}{140}\right)$	$(0, 0)$	$\left(-\frac{43}{140}, \frac{53}{70}\right)$	$\left(\frac{183}{140}, -\frac{23}{140}\right)$	$(0, 0)$	$\left(-\frac{43}{140}, \frac{43}{70}\right)$	$\left(\frac{183}{140}, -\frac{43}{140}\right)$
(e,a,e)	$\left(-\frac{1}{5}, -\frac{27}{140}\right)$	$\left(\frac{6}{5}, -\frac{111}{140}\right)$	(e,a,b)	$\left(-\frac{2}{35}, -\frac{47}{140}\right)$	$\left(\frac{37}{35}, -\frac{71}{140}\right)$	(e,a,ab)	$\left(-\frac{2}{35}, -\frac{27}{140}\right)$	$\left(\frac{37}{35}, -\frac{51}{140}\right)$
$(27, 111)$	$\left(\frac{83}{140}, -\frac{111}{140}\right)$	$\left(\frac{57}{140}, \frac{69}{70}\right)$	$(47, 71)$	$\left(\frac{9}{20}, -\frac{71}{140}\right)$	$\left(\frac{11}{20}, \frac{59}{70}\right)$	$(27, 51)$	$\left(\frac{43}{140}, -\frac{51}{140}\right)$	$\left(\frac{97}{140}, \frac{39}{70}\right)$
$(0, 0)$	$\left(-\frac{11}{28}, \frac{69}{70}\right)$	$\left(\frac{39}{28}, -\frac{27}{140}\right)$	$(0, 0)$	$\left(-\frac{11}{28}, \frac{59}{70}\right)$	$\left(\frac{39}{28}, -\frac{47}{140}\right)$	$(0, 0)$	$\left(-\frac{1}{4}, \frac{39}{70}\right)$	$\left(\frac{5}{4}, -\frac{27}{140}\right)$
(e,b,e)	$\left(\frac{1}{5}, -\frac{111}{140}\right)$	$\left(\frac{4}{5}, -\frac{27}{140}\right)$	(e,b,a)	$\left(\frac{2}{35}, -\frac{71}{140}\right)$	$\left(\frac{33}{35}, -\frac{47}{140}\right)$	(e,b,ba)	$\left(\frac{2}{35}, -\frac{51}{140}\right)$	$\left(\frac{33}{35}, -\frac{27}{140}\right)$
$(111, 27)$	$\left(\frac{11}{28}, -\frac{27}{140}\right)$	$\left(\frac{17}{28}, \frac{69}{70}\right)$	$(71, 47)$	$\left(\frac{11}{28}, -\frac{47}{140}\right)$	$\left(\frac{17}{28}, \frac{59}{70}\right)$	$(51, 27)$	$\left(\frac{1}{4}, -\frac{27}{140}\right)$	$\left(\frac{3}{4}, \frac{39}{70}\right)$
$(0, 0)$	$\left(-\frac{83}{140}, \frac{69}{70}\right)$	$\left(\frac{223}{140}, -\frac{111}{140}\right)$	$(0, 0)$	$\left(-\frac{9}{20}, \frac{59}{70}\right)$	$\left(\frac{29}{20}, -\frac{71}{140}\right)$	$(0, 0)$	$\left(-\frac{43}{140}, \frac{39}{70}\right)$	$\left(\frac{183}{140}, -\frac{51}{140}\right)$
(e,ab,a)	$\left(\frac{2}{35}, -\frac{43}{140}\right)$	$\left(\frac{33}{35}, -\frac{19}{140}\right)$	(e,ba,b)	$\left(-\frac{2}{35}, -\frac{19}{140}\right)$	$\left(\frac{37}{35}, -\frac{43}{140}\right)$	(e,aba,e)	$\left(0, -\frac{27}{140}\right)$	$\left(1, -\frac{27}{140}\right)$
$(43, 19)$	$\left(\frac{27}{140}, -\frac{19}{140}\right)$	$\left(\frac{113}{140}, \frac{31}{70}\right)$	$(19, 43)$	$\left(\frac{1}{4}, -\frac{43}{140}\right)$	$\left(\frac{3}{4}, \frac{31}{70}\right)$	$(27, 27)$	$\left(\frac{27}{140}, -\frac{27}{140}\right)$	$\left(\frac{113}{140}, \frac{27}{70}\right)$
$(0, 0)$	$\left(-\frac{1}{4}, \frac{31}{70}\right)$	$\left(\frac{5}{4}, -\frac{43}{140}\right)$	$(0, 0)$	$\left(-\frac{27}{140}, \frac{31}{70}\right)$	$\left(\frac{167}{140}, -\frac{19}{140}\right)$	$(0, 0)$	$\left(-\frac{27}{140}, \frac{27}{70}\right)$	$\left(\frac{167}{140}, -\frac{27}{140}\right)$
(a,e,e)	$\left(-\frac{1}{4}, -\frac{13}{140}\right)$	$\left(\frac{5}{4}, -\frac{59}{70}\right)$	(a,e,b)	$\left(-\frac{3}{28}, -\frac{33}{140}\right)$	$\left(\frac{31}{28}, -\frac{39}{70}\right)$	(a,e,ab)	$\left(-\frac{3}{28}, -\frac{13}{140}\right)$	$\left(\frac{31}{28}, -\frac{29}{70}\right)$
$(13, 118)$	$\left(\frac{83}{140}, -\frac{59}{70}\right)$	$\left(\frac{57}{140}, \frac{131}{140}\right)$	$(33, 78)$	$\left(\frac{9}{20}, -\frac{39}{70}\right)$	$\left(\frac{11}{20}, \frac{111}{140}\right)$	$(13, 58)$	$\left(\frac{43}{140}, -\frac{29}{70}\right)$	$\left(\frac{97}{140}, \frac{71}{140}\right)$
$(0, 0)$	$\left(-\frac{12}{35}, \frac{131}{140}\right)$	$\left(\frac{47}{35}, -\frac{13}{140}\right)$	$(0, 0)$	$\left(-\frac{12}{35}, \frac{111}{140}\right)$	$\left(\frac{47}{35}, -\frac{33}{140}\right)$	$(0, 0)$	$\left(-\frac{1}{5}, \frac{71}{140}\right)$	$\left(\frac{6}{5}, -\frac{13}{140}\right)$

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Table B.1: $\boldsymbol{\alpha}$ of $M\left(-1; \frac{1}{4}, \frac{3}{5}, \frac{1}{7}\right)$ for values of $\vec{s}$ with components in $\{1, \ldots, 139\}$. (Continued)

(a,b,e)	$\left(-\frac{1}{20}, -\frac{41}{140}\right)$	$\left(\frac{21}{20}, -\frac{31}{70}\right)$	(a,b,a)	$\left(-\frac{27}{140}, -\frac{1}{140}\right)$	$\left(\frac{167}{140}, -\frac{41}{70}\right)$	(a,b,b)	$\left(\frac{13}{140}, -\frac{61}{140}\right)$	$\left(\frac{127}{140}, -\frac{11}{70}\right)$
$(41, 62)$	$\left(\frac{11}{28}, -\frac{31}{70}\right)$	$\left(\frac{17}{28}, \frac{103}{140}\right)$	$(1, 82)$	$\left(\frac{11}{28}, -\frac{41}{70}\right)$	$\left(\frac{17}{28}, \frac{83}{140}\right)$	$(61, 22)$	$\left(\frac{1}{4}, -\frac{11}{70}\right)$	$\left(\frac{3}{4}, \frac{83}{140}\right)$
$(0, 0)$	$\left(-\frac{12}{35}, \frac{103}{140}\right)$	$\left(\frac{47}{35}, -\frac{41}{140}\right)$	$(0, 0)$	$\left(-\frac{1}{5}, \frac{83}{140}\right)$	$\left(\frac{6}{5}, -\frac{1}{140}\right)$	$(0, 0)$	$\left(-\frac{12}{35}, \frac{83}{140}\right)$	$\left(\frac{47}{35}, -\frac{61}{140}\right)$
(a,b,ab)	$\left(\frac{13}{140}, -\frac{41}{140}\right)$	$\left(\frac{127}{140}, -\frac{1}{70}\right)$	(a,b,aba)	$\left(-\frac{1}{20}, -\frac{1}{140}\right)$	$\left(\frac{21}{20}, -\frac{11}{70}\right)$	(a,ab,e)	$\left(-\frac{1}{20}, -\frac{13}{140}\right)$	$\left(\frac{21}{20}, -\frac{17}{70}\right)$
$(41, 2)$	$\left(\frac{3}{28}, -\frac{1}{70}\right)$	$\left(\frac{25}{28}, \frac{43}{140}\right)$	$(1, 22)$	$\left(\frac{3}{28}, -\frac{11}{70}\right)$	$\left(\frac{25}{28}, \frac{23}{140}\right)$	$(13, 34)$	$\left(\frac{27}{140}, -\frac{17}{70}\right)$	$\left(\frac{113}{140}, \frac{47}{140}\right)$
$(0, 0)$	$\left(-\frac{1}{5}, \frac{43}{140}\right)$	$\left(\frac{6}{5}, -\frac{41}{140}\right)$	$(0, 0)$	$\left(-\frac{2}{35}, \frac{23}{140}\right)$	$\left(\frac{37}{35}, -\frac{1}{140}\right)$	$(0, 0)$	$\left(-\frac{1}{7}, \frac{47}{140}\right)$	$\left(\frac{8}{7}, -\frac{13}{140}\right)$
(b,e,e)	$\left(\frac{1}{4}, -\frac{59}{70}\right)$	$\left(\frac{3}{4}, -\frac{13}{140}\right)$	(b,e,a)	$\left(\frac{3}{28}, -\frac{39}{70}\right)$	$\left(\frac{25}{28}, -\frac{33}{140}\right)$	(b,e,ba)	$\left(\frac{3}{28}, -\frac{29}{70}\right)$	$\left(\frac{25}{28}, -\frac{13}{140}\right)$
$(118, 13)$	$\left(\frac{12}{35}, -\frac{13}{140}\right)$	$\left(\frac{23}{35}, \frac{131}{140}\right)$	$(78, 33)$	$\left(\frac{12}{35}, -\frac{33}{140}\right)$	$\left(\frac{23}{35}, \frac{111}{140}\right)$	$(58, 13)$	$\left(\frac{1}{5}, -\frac{13}{140}\right)$	$\left(\frac{4}{5}, \frac{71}{140}\right)$
$(0, 0)$	$\left(-\frac{83}{140}, \frac{131}{140}\right)$	$\left(\frac{223}{140}, -\frac{59}{70}\right)$	$(0, 0)$	$\left(-\frac{9}{20}, \frac{111}{140}\right)$	$\left(\frac{29}{20}, -\frac{39}{70}\right)$	$(0, 0)$	$\left(-\frac{43}{140}, \frac{71}{140}\right)$	$\left(\frac{183}{140}, -\frac{29}{70}\right)$
(b,a,e)	$\left(\frac{1}{20}, -\frac{31}{70}\right)$	$\left(\frac{19}{20}, -\frac{41}{140}\right)$	(b,a,a)	$\left(-\frac{13}{140}, -\frac{11}{70}\right)$	$\left(\frac{153}{140}, -\frac{61}{140}\right)$	(b,a,b)	$\left(\frac{27}{140}, -\frac{41}{70}\right)$	$\left(\frac{113}{140}, -\frac{1}{140}\right)$
$(62, 41)$	$\left(\frac{12}{35}, -\frac{41}{140}\right)$	$\left(\frac{23}{35}, \frac{103}{140}\right)$	$(22, 61)$	$\left(\frac{12}{35}, -\frac{61}{140}\right)$	$\left(\frac{23}{35}, \frac{83}{140}\right)$	$(82, 1)$	$\left(\frac{1}{5}, -\frac{1}{140}\right)$	$\left(\frac{4}{5}, \frac{83}{140}\right)$
$(0, 0)$	$\left(-\frac{11}{28}, \frac{103}{140}\right)$	$\left(\frac{39}{28}, -\frac{31}{70}\right)$	$(0, 0)$	$\left(-\frac{1}{4}, \frac{83}{140}\right)$	$\left(\frac{5}{4}, -\frac{11}{70}\right)$	$(0, 0)$	$\left(-\frac{11}{28}, \frac{83}{140}\right)$	$\left(\frac{39}{28}, -\frac{41}{70}\right)$
(b,a,ba)	$\left(-\frac{13}{140}, -\frac{1}{70}\right)$	$\left(\frac{153}{140}, -\frac{41}{140}\right)$	(b,a,aba)	$\left(\frac{1}{20}, -\frac{11}{70}\right)$	$\left(\frac{19}{20}, -\frac{1}{140}\right)$	(b,ba,e)	$\left(\frac{1}{20}, -\frac{17}{70}\right)$	$\left(\frac{19}{20}, -\frac{13}{140}\right)$
$(2, 41)$	$\left(\frac{1}{5}, -\frac{41}{140}\right)$	$\left(\frac{4}{5}, \frac{43}{140}\right)$	$(22, 1)$	$\left(\frac{2}{35}, -\frac{1}{140}\right)$	$\left(\frac{33}{35}, \frac{23}{140}\right)$	$(34, 13)$	$\left(\frac{1}{7}, -\frac{13}{140}\right)$	$\left(\frac{6}{7}, \frac{47}{140}\right)$
$(0, 0)$	$\left(-\frac{3}{28}, \frac{43}{140}\right)$	$\left(\frac{31}{28}, -\frac{1}{70}\right)$	$(0, 0)$	$\left(-\frac{3}{28}, \frac{23}{140}\right)$	$\left(\frac{31}{28}, -\frac{11}{70}\right)$	$(0, 0)$	$\left(-\frac{27}{140}, \frac{47}{140}\right)$	$\left(\frac{167}{140}, -\frac{17}{70}\right)$
(ab,a,e)	$\left(\frac{1}{20}, -\frac{27}{140}\right)$	$\left(\frac{19}{20}, -\frac{3}{70}\right)$	(ba,b,e)	$\left(-\frac{1}{20}, -\frac{3}{70}\right)$	$\left(\frac{21}{20}, -\frac{27}{140}\right)$	(aba,e,e)	$\left(0, -\frac{13}{140}\right)$	$\left(1, -\frac{13}{140}\right)$
$(27, 6)$	$\left(\frac{13}{140}, -\frac{3}{70}\right)$	$\left(\frac{127}{140}, \frac{33}{140}\right)$	$(6, 27)$	$\left(\frac{1}{7}, -\frac{27}{140}\right)$	$\left(\frac{6}{7}, \frac{33}{140}\right)$	$(13, 13)$	$\left(\frac{13}{140}, -\frac{13}{140}\right)$	$\left(\frac{127}{140}, \frac{13}{70}\right)$
$(0, 0)$	$\left(-\frac{1}{7}, \frac{33}{140}\right)$	$\left(\frac{8}{7}, -\frac{27}{140}\right)$	$(0, 0)$	$\left(-\frac{13}{140}, \frac{33}{140}\right)$	$\left(\frac{153}{140}, -\frac{3}{70}\right)$	$(0, 0)$	$\left(-\frac{13}{140}, \frac{13}{70}\right)$	$\left(\frac{153}{140}, -\frac{13}{140}\right)$

Table B.2: $\boldsymbol{\alpha}$ of $M\left(-1; \frac{1}{3}, \frac{1}{2}, \frac{1}{4}\right)$ for each $\vec{s}$.

(e, e, e)	$(0, -\frac{19}{20})$	$(1, -\frac{19}{20})$	(e, e, aba)	$(0, -\frac{9}{20})$	$(1, -\frac{9}{20})$	(e, a, a)	$(-\frac{3}{4}, \frac{11}{20})$	$(\frac{7}{4}, -\frac{17}{10})$
$(19, 19)$	$(\frac{19}{20}, -\frac{19}{20})$	$(\frac{1}{20}, \frac{19}{10})$	$(9, 9)$	$(\frac{9}{20}, -\frac{9}{20})$	$(\frac{11}{20}, \frac{9}{10})$	$(-11, 34)$	$(\frac{19}{20}, -\frac{17}{10})$	$(\frac{1}{20}, \frac{23}{20})$
$(0, 0)$	$(-\frac{19}{20}, \frac{19}{10})$	$(\frac{39}{20}, -\frac{19}{20})$	$(0, 0)$	$(-\frac{9}{20}, \frac{9}{10})$	$(\frac{29}{20}, -\frac{9}{20})$	$(0, 0)$	$(-\frac{1}{5}, \frac{23}{20})$	$(\frac{6}{5}, \frac{11}{20})$
(e, a, ab)	$(-\frac{1}{4}, \frac{1}{20})$	$(\frac{5}{4}, -\frac{7}{10})$	(e, b, b)	$(\frac{3}{4}, -\frac{17}{10})$	$(\frac{1}{4}, \frac{11}{20})$	(e, b, ba)	$(\frac{1}{4}, -\frac{7}{10})$	$(\frac{3}{4}, \frac{1}{20})$
$(-1, 14)$	$(\frac{9}{20}, -\frac{7}{10})$	$(\frac{11}{20}, \frac{13}{20})$	$(34, -11)$	$(\frac{1}{5}, \frac{11}{20})$	$(\frac{4}{5}, \frac{23}{20})$	$(14, -1)$	$(\frac{1}{5}, \frac{1}{20})$	$(\frac{4}{5}, \frac{13}{20})$
$(0, 0)$	$(-\frac{1}{5}, \frac{13}{20})$	$(\frac{6}{5}, \frac{1}{20})$	$(0, 0)$	$(-\frac{19}{20}, \frac{23}{20})$	$(\frac{39}{20}, -\frac{17}{10})$	$(0, 0)$	$(-\frac{9}{20}, \frac{13}{20})$	$(\frac{29}{20}, -\frac{7}{10})$
(e, ab, a)	$(\frac{1}{4}, -\frac{9}{20})$	$(\frac{3}{4}, \frac{3}{10})$	(e, ab, ab)	$(\frac{3}{4}, -\frac{19}{20})$	$(\frac{1}{4}, \frac{13}{10})$	(e, ba, b)	$(-\frac{1}{4}, \frac{3}{10})$	$(\frac{5}{4}, -\frac{9}{20})$
$(9, -6)$	$(-\frac{1}{20}, \frac{3}{10})$	$(\frac{21}{20}, \frac{3}{20})$	$(19, -26)$	$(-\frac{11}{20}, \frac{13}{10})$	$(\frac{31}{20}, -\frac{7}{20})$	$(-6, 9)$	$(\frac{1}{5}, -\frac{9}{20})$	$(\frac{4}{5}, \frac{3}{20})$
$(0, 0)$	$(-\frac{1}{5}, \frac{3}{20})$	$(\frac{6}{5}, -\frac{9}{20})$	$(0, 0)$	$(-\frac{1}{5}, -\frac{7}{20})$	$(\frac{6}{5}, -\frac{19}{20})$	$(0, 0)$	$(\frac{1}{20}, \frac{3}{20})$	$(\frac{19}{20}, \frac{3}{10})$
(e, ba, ba)	$(-\frac{3}{4}, \frac{13}{10})$	$(\frac{7}{4}, -\frac{19}{20})$	(e, aba, e)	$(0, \frac{1}{20})$	$(1, \frac{1}{20})$	(e, aba, aba)	$(0, \frac{11}{20})$	$(1, \frac{11}{20})$
$(-26, 19)$	$(\frac{1}{5}, -\frac{19}{20})$	$(\frac{4}{5}, -\frac{7}{20})$	$(-1, -1)$	$(-\frac{1}{20}, \frac{1}{20})$	$(\frac{21}{20}, -\frac{1}{10})$	$(-11, -11)$	$(-\frac{11}{20}, \frac{11}{20})$	$(\frac{31}{20}, -\frac{11}{10})$
$(0, 0)$	$(\frac{11}{20}, -\frac{7}{20})$	$(\frac{9}{20}, \frac{13}{10})$	$(0, 0)$	$(\frac{1}{20}, -\frac{1}{10})$	$(\frac{19}{20}, \frac{1}{20})$	$(0, 0)$	$(\frac{11}{20}, -\frac{11}{10})$	$(\frac{9}{20}, \frac{11}{20})$
(a, e, e)	$(-\frac{1}{5}, -\frac{11}{20})$	$(\frac{6}{5}, -\frac{23}{20})$	(a, e, aba)	$(-\frac{1}{5}, -\frac{1}{20})$	$(\frac{6}{5}, -\frac{13}{20})$	(a, a, a)	$(-\frac{19}{20}, \frac{19}{20})$	$(\frac{39}{20}, -\frac{19}{10})$
$(11, 23)$	$(\frac{19}{20}, -\frac{23}{20})$	$(\frac{1}{20}, \frac{17}{10})$	$(1, 13)$	$(\frac{9}{20}, -\frac{13}{20})$	$(\frac{11}{20}, \frac{7}{10})$	$(-19, 38)$	$(\frac{19}{20}, -\frac{19}{10})$	$(\frac{1}{20}, \frac{19}{20})$
$(0, 0)$	$(-\frac{3}{4}, \frac{17}{10})$	$(\frac{7}{4}, -\frac{11}{20})$	$(0, 0)$	$(-\frac{1}{4}, \frac{7}{10})$	$(\frac{5}{4}, -\frac{1}{20})$	$(0, 0)$	$(0, \frac{19}{20})$	$(1, \frac{19}{20})$
(a, a, ab)	$(-\frac{9}{20}, \frac{9}{20})$	$(\frac{29}{20}, -\frac{9}{10})$	(a, b, b)	$(\frac{11}{20}, -\frac{13}{10})$	$(\frac{9}{20}, \frac{7}{20})$	(a, b, ba)	$(\frac{1}{20}, -\frac{3}{10})$	$(\frac{19}{20}, -\frac{3}{20})$
$(-9, 18)$	$(\frac{9}{20}, -\frac{9}{10})$	$(\frac{11}{20}, \frac{9}{20})$	$(26, -7)$	$(\frac{1}{5}, \frac{7}{20})$	$(\frac{4}{5}, \frac{19}{20})$	$(6, 3)$	$(\frac{1}{5}, -\frac{3}{20})$	$(\frac{4}{5}, \frac{9}{20})$
$(0, 0)$	$(0, \frac{9}{20})$	$(1, \frac{9}{20})$	$(0, 0)$	$(-\frac{3}{4}, \frac{19}{20})$	$(\frac{7}{4}, -\frac{13}{10})$	$(0, 0)$	$(-\frac{1}{4}, \frac{9}{20})$	$(\frac{5}{4}, -\frac{3}{10})$

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Table B.2: $\boldsymbol{\alpha}$ of $M\left(-1; \frac{1}{3}, \frac{1}{2}, \frac{1}{4}\right)$ for each $\vec{s}$. (Continued)

(a, ab, a)	$(\frac{1}{20}, -\frac{1}{20})$	$(\frac{19}{20}, \frac{1}{10})$	(a, ab, ab)	$(\frac{11}{20}, -\frac{11}{20})$	$(\frac{9}{20}, \frac{11}{10})$	(a, ba, b)	$(-\frac{9}{20}, \frac{7}{10})$	$(\frac{29}{20}, -\frac{13}{20})$
$(1, -2)$	$(-\frac{1}{20}, \frac{1}{10})$	$(\frac{21}{20}, -\frac{1}{20})$	$(11, -22)$	$(-\frac{11}{20}, \frac{11}{10})$	$(\frac{31}{20}, -\frac{11}{20})$	$(-14, 13)$	$(\frac{1}{5}, -\frac{13}{20})$	$(\frac{4}{5}, -\frac{1}{20})$
$(0, 0)$	$(0, -\frac{1}{20})$	$(1, -\frac{1}{20})$	$(0, 0)$	$(0, -\frac{11}{20})$	$(1, -\frac{11}{20})$	$(0, 0)$	$(\frac{1}{4}, -\frac{1}{20})$	$(\frac{3}{4}, \frac{7}{10})$
(a, ba, ba)	$(-\frac{19}{20}, \frac{17}{10})$	$(\frac{39}{20}, -\frac{23}{20})$	(a, aba, e)	$(-\frac{1}{5}, \frac{9}{20})$	$(\frac{6}{5}, -\frac{3}{20})$	(a, aba, aba)	$(-\frac{1}{5}, \frac{19}{20})$	$(\frac{6}{5}, \frac{7}{20})$
$(-34, 23)$	$(\frac{1}{5}, -\frac{23}{20})$	$(\frac{4}{5}, -\frac{11}{20})$	$(-9, 3)$	$(-\frac{1}{20}, -\frac{3}{20})$	$(\frac{21}{20}, -\frac{3}{10})$	$(-19, -7)$	$(-\frac{11}{20}, \frac{7}{20})$	$(\frac{31}{20}, -\frac{13}{10})$
$(0, 0)$	$(\frac{3}{4}, -\frac{11}{20})$	$(\frac{1}{4}, \frac{17}{10})$	$(0, 0)$	$(\frac{1}{4}, -\frac{3}{10})$	$(\frac{3}{4}, \frac{9}{20})$	$(0, 0)$	$(\frac{3}{4}, -\frac{13}{10})$	$(\frac{1}{4}, \frac{19}{20})$
(b, e, e)	$(\frac{1}{5}, -\frac{23}{20})$	$(\frac{4}{5}, -\frac{11}{20})$	(b, e, aba)	$(\frac{1}{5}, -\frac{13}{20})$	$(\frac{4}{5}, -\frac{1}{20})$	(b, a, a)	$(-\frac{11}{20}, \frac{7}{20})$	$(\frac{31}{20}, -\frac{13}{10})$
$(23, 11)$	$(\frac{3}{4}, -\frac{11}{20})$	$(\frac{1}{4}, \frac{17}{10})$	$(13, 1)$	$(\frac{1}{4}, -\frac{1}{20})$	$(\frac{3}{4}, \frac{7}{10})$	$(-7, 26)$	$(\frac{3}{4}, -\frac{13}{10})$	$(\frac{1}{4}, \frac{19}{20})$
$(0, 0)$	$(-\frac{19}{20}, \frac{17}{10})$	$(\frac{39}{20}, -\frac{23}{20})$	$(0, 0)$	$(-\frac{9}{20}, \frac{7}{10})$	$(\frac{29}{20}, -\frac{13}{20})$	$(0, 0)$	$(-\frac{1}{5}, \frac{19}{20})$	$(\frac{6}{5}, \frac{7}{20})$
(b, a, ab)	$(-\frac{1}{20}, -\frac{3}{20})$	$(\frac{21}{20}, -\frac{3}{10})$	(b, b, b)	$(\frac{19}{20}, -\frac{19}{10})$	$(\frac{1}{20}, \frac{19}{20})$	(b, b, ba)	$(\frac{9}{20}, -\frac{9}{10})$	$(\frac{11}{20}, \frac{9}{20})$
$(3, 6)$	$(\frac{1}{4}, -\frac{3}{10})$	$(\frac{3}{4}, \frac{9}{20})$	$(38, -19)$	$(0, \frac{19}{20})$	$(1, \frac{19}{20})$	$(18, -9)$	$(0, \frac{9}{20})$	$(1, \frac{9}{20})$
$(0, 0)$	$(-\frac{1}{5}, \frac{9}{20})$	$(\frac{6}{5}, -\frac{3}{20})$	$(0, 0)$	$(-\frac{19}{20}, \frac{19}{20})$	$(\frac{39}{20}, -\frac{19}{10})$	$(0, 0)$	$(-\frac{9}{20}, \frac{9}{20})$	$(\frac{29}{20}, -\frac{9}{10})$
(b, ab, a)	$(\frac{9}{20}, -\frac{13}{20})$	$(\frac{11}{20}, \frac{7}{10})$	(b, ab, ab)	$(\frac{19}{20}, -\frac{23}{20})$	$(\frac{1}{20}, \frac{17}{10})$	(b, ba, b)	$(-\frac{1}{20}, \frac{1}{10})$	$(\frac{21}{20}, -\frac{1}{20})$
$(13, -14)$	$(-\frac{1}{4}, \frac{7}{10})$	$(\frac{5}{4}, -\frac{1}{20})$	$(23, -34)$	$(-\frac{3}{4}, \frac{17}{10})$	$(\frac{7}{4}, -\frac{11}{20})$	$(-2, 1)$	$(0, -\frac{1}{20})$	$(1, -\frac{1}{20})$
$(0, 0)$	$(-\frac{1}{5}, -\frac{1}{20})$	$(\frac{6}{5}, -\frac{13}{20})$	$(0, 0)$	$(-\frac{1}{5}, -\frac{11}{20})$	$(\frac{6}{5}, -\frac{23}{20})$	$(0, 0)$	$(\frac{1}{20}, -\frac{1}{20})$	$(\frac{19}{20}, \frac{1}{10})$
(b, ba, ba)	$(-\frac{11}{20}, \frac{11}{10})$	$(\frac{31}{20}, -\frac{11}{20})$	(b, aba, e)	$(\frac{1}{5}, -\frac{3}{20})$	$(\frac{4}{5}, \frac{9}{20})$	(b, aba, aba)	$(\frac{1}{5}, \frac{7}{20})$	$(\frac{4}{5}, \frac{19}{20})$
$(-22, 11)$	$(0, -\frac{11}{20})$	$(1, -\frac{11}{20})$	$(3, -9)$	$(-\frac{1}{4}, \frac{9}{20})$	$(\frac{5}{4}, -\frac{3}{10})$	$(-7, -19)$	$(-\frac{3}{4}, \frac{19}{20})$	$(\frac{7}{4}, -\frac{13}{10})$
$(0, 0)$	$(\frac{11}{20}, -\frac{11}{20})$	$(\frac{9}{20}, \frac{11}{10})$	$(0, 0)$	$(\frac{1}{20}, -\frac{3}{10})$	$(\frac{19}{20}, -\frac{3}{20})$	$(0, 0)$	$(\frac{11}{20}, -\frac{13}{10})$	$(\frac{9}{20}, \frac{7}{20})$

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Table B.2: $\boldsymbol{\alpha}$ of $M\left(-1; \frac{1}{3}, \frac{1}{2}, \frac{1}{4}\right)$ for each $\vec{s}$. (Continued)

(ab, e, e)	$\left(\frac{1}{5}, -\frac{19}{20}\right)$	$\left(\frac{4}{5}, -\frac{7}{20}\right)$	(ab, e, aba)	$\left(\frac{1}{5}, -\frac{9}{20}\right)$	$\left(\frac{4}{5}, \frac{3}{20}\right)$	(ab, a, a)	$\left(-\frac{11}{20}, \frac{11}{20}\right)$	$\left(\frac{31}{20}, -\frac{11}{10}\right)$
$(19, 7)$	$\left(\frac{11}{20}, -\frac{7}{20}\right)$	$\left(\frac{9}{20}, \frac{13}{10}\right)$	$(9, -3)$	$\left(\frac{1}{20}, \frac{3}{20}\right)$	$\left(\frac{19}{20}, \frac{3}{10}\right)$	$(-11, 22)$	$\left(\frac{11}{20}, -\frac{11}{10}\right)$	$\left(\frac{9}{20}, \frac{11}{20}\right)$
$(0, 0)$	$\left(-\frac{3}{4}, \frac{13}{10}\right)$	$\left(\frac{7}{4}, -\frac{19}{20}\right)$	$(0, 0)$	$\left(-\frac{1}{4}, \frac{3}{10}\right)$	$\left(\frac{5}{4}, -\frac{9}{20}\right)$	$(0, 0)$	$\left(0, \frac{11}{20}\right)$	$\left(1, \frac{11}{20}\right)$
(ab, a, ab)	$\left(-\frac{1}{20}, \frac{1}{20}\right)$	$\left(\frac{21}{20}, -\frac{1}{10}\right)$	(ab, b, b)	$\left(\frac{19}{20}, -\frac{17}{10}\right)$	$\left(\frac{1}{20}, \frac{23}{20}\right)$	(ab, b, ba)	$\left(\frac{9}{20}, -\frac{7}{10}\right)$	$\left(\frac{11}{20}, \frac{13}{20}\right)$
$(-1, 2)$	$\left(\frac{1}{20}, -\frac{1}{10}\right)$	$\left(\frac{19}{20}, \frac{1}{20}\right)$	$(34, -23)$	$\left(-\frac{1}{5}, \frac{23}{20}\right)$	$\left(\frac{6}{5}, \frac{11}{20}\right)$	$(14, -13)$	$\left(-\frac{1}{5}, \frac{13}{20}\right)$	$\left(\frac{6}{5}, \frac{1}{20}\right)$
$(0, 0)$	$\left(0, \frac{1}{20}\right)$	$\left(1, \frac{1}{20}\right)$	$(0, 0)$	$\left(-\frac{3}{4}, \frac{11}{20}\right)$	$\left(\frac{7}{4}, -\frac{17}{10}\right)$	$(0, 0)$	$\left(-\frac{1}{4}, \frac{1}{20}\right)$	$\left(\frac{5}{4}, -\frac{7}{10}\right)$
(ab, ab, a)	$\left(\frac{9}{20}, -\frac{9}{20}\right)$	$\left(\frac{11}{20}, \frac{9}{10}\right)$	(ab, ab, ab)	$\left(\frac{19}{20}, -\frac{19}{20}\right)$	$\left(\frac{1}{20}, \frac{19}{10}\right)$	(ab, ba, b)	$\left(-\frac{1}{20}, \frac{3}{10}\right)$	$\left(\frac{21}{20}, \frac{3}{20}\right)$
$(9, -18)$	$\left(-\frac{9}{20}, \frac{9}{10}\right)$	$\left(\frac{29}{20}, -\frac{9}{20}\right)$	$(19, -38)$	$\left(-\frac{19}{20}, \frac{19}{10}\right)$	$\left(\frac{39}{20}, -\frac{19}{20}\right)$	$(-6, -3)$	$\left(-\frac{1}{5}, \frac{3}{20}\right)$	$\left(\frac{6}{5}, -\frac{9}{20}\right)$
$(0, 0)$	$\left(0, -\frac{9}{20}\right)$	$\left(1, -\frac{9}{20}\right)$	$(0, 0)$	$\left(0, -\frac{19}{20}\right)$	$\left(1, -\frac{19}{20}\right)$	$(0, 0)$	$\left(\frac{1}{4}, -\frac{9}{20}\right)$	$\left(\frac{3}{4}, \frac{3}{10}\right)$
(ab, ba, ba)	$\left(-\frac{11}{20}, \frac{13}{10}\right)$	$\left(\frac{31}{20}, -\frac{7}{20}\right)$	(ab, aba, e)	$\left(\frac{1}{5}, \frac{1}{20}\right)$	$\left(\frac{4}{5}, \frac{13}{20}\right)$	(ab, aba, aba)	$\left(\frac{1}{5}, \frac{11}{20}\right)$	$\left(\frac{4}{5}, \frac{23}{20}\right)$
$(-26, 7)$	$\left(-\frac{1}{5}, -\frac{7}{20}\right)$	$\left(\frac{6}{5}, -\frac{19}{20}\right)$	$(-1, -13)$	$\left(-\frac{9}{20}, \frac{13}{20}\right)$	$\left(\frac{29}{20}, -\frac{7}{10}\right)$	$(-11, -23)$	$\left(-\frac{19}{20}, \frac{23}{20}\right)$	$\left(\frac{39}{20}, -\frac{17}{10}\right)$
$(0, 0)$	$\left(\frac{3}{4}, -\frac{19}{20}\right)$	$\left(\frac{1}{4}, \frac{13}{10}\right)$	$(0, 0)$	$\left(\frac{1}{4}, -\frac{7}{10}\right)$	$\left(\frac{3}{4}, \frac{1}{20}\right)$	$(0, 0)$	$\left(\frac{3}{4}, -\frac{17}{10}\right)$	$\left(\frac{1}{4}, \frac{11}{20}\right)$
(ba, e, e)	$\left(-\frac{1}{5}, -\frac{7}{20}\right)$	$\left(\frac{6}{5}, -\frac{19}{20}\right)$	(ba, e, aba)	$\left(-\frac{1}{5}, \frac{3}{20}\right)$	$\left(\frac{6}{5}, -\frac{9}{20}\right)$	(ba, a, a)	$\left(-\frac{19}{20}, \frac{23}{20}\right)$	$\left(\frac{39}{20}, -\frac{17}{10}\right)$
$(7, 19)$	$\left(\frac{3}{4}, -\frac{19}{20}\right)$	$\left(\frac{1}{4}, \frac{13}{10}\right)$	$(-3, 9)$	$\left(\frac{1}{4}, -\frac{9}{20}\right)$	$\left(\frac{3}{4}, \frac{3}{10}\right)$	$(-23, 34)$	$\left(\frac{3}{4}, -\frac{17}{10}\right)$	$\left(\frac{1}{4}, \frac{11}{20}\right)$
$(0, 0)$	$\left(-\frac{11}{20}, \frac{13}{10}\right)$	$\left(\frac{31}{20}, -\frac{7}{20}\right)$	$(0, 0)$	$\left(-\frac{1}{20}, \frac{3}{10}\right)$	$\left(\frac{21}{20}, \frac{3}{20}\right)$	$(0, 0)$	$\left(\frac{1}{5}, \frac{11}{20}\right)$	$\left(\frac{4}{5}, \frac{23}{20}\right)$
(ba, a, ab)	$\left(-\frac{9}{20}, \frac{13}{20}\right)$	$\left(\frac{29}{20}, -\frac{7}{10}\right)$	(ba, b, b)	$\left(\frac{11}{20}, -\frac{11}{10}\right)$	$\left(\frac{9}{20}, \frac{11}{20}\right)$	(ba, b, ba)	$\left(\frac{1}{20}, -\frac{1}{10}\right)$	$\left(\frac{19}{20}, \frac{1}{20}\right)$
$(-13, 14)$	$\left(\frac{1}{4}, -\frac{7}{10}\right)$	$\left(\frac{3}{4}, \frac{1}{20}\right)$	$(22, -11)$	$\left(0, \frac{11}{20}\right)$	$\left(1, \frac{11}{20}\right)$	$(2, -1)$	$\left(0, \frac{1}{20}\right)$	$\left(1, \frac{1}{20}\right)$
$(0, 0)$	$\left(\frac{1}{5}, \frac{1}{20}\right)$	$\left(\frac{4}{5}, \frac{13}{20}\right)$	$(0, 0)$	$\left(-\frac{11}{20}, \frac{11}{20}\right)$	$\left(\frac{31}{20}, -\frac{11}{10}\right)$	$(0, 0)$	$\left(-\frac{1}{20}, \frac{1}{20}\right)$	$\left(\frac{21}{20}, -\frac{1}{10}\right)$

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Table B.2: $\boldsymbol{\alpha}$ of $M\left(-1; \frac{1}{3}, \frac{1}{2}, \frac{1}{4}\right)$ for each $\vec{s}$. (Continued)

(ba, ab, a)	$\left(\frac{1}{20}, \frac{3}{20}\right)$	$\left(\frac{19}{20}, \frac{3}{10}\right)$	(ba, ab, ab)	$\left(\frac{11}{20}, -\frac{7}{20}\right)$	$\left(\frac{9}{20}, \frac{13}{10}\right)$	(ba, ba, b)	$\left(-\frac{9}{20}, \frac{9}{10}\right)$	$\left(\frac{29}{20}, -\frac{9}{20}\right)$
$(-3, -6)$	$\left(-\frac{1}{4}, \frac{3}{10}\right)$	$\left(\frac{5}{4}, -\frac{9}{20}\right)$	$(7, -26)$	$\left(-\frac{3}{4}, \frac{13}{10}\right)$	$\left(\frac{7}{4}, -\frac{19}{20}\right)$	$(-18, 9)$	$\left(0, -\frac{9}{20}\right)$	$\left(1, -\frac{9}{20}\right)$
$(0, 0)$	$\left(\frac{1}{5}, -\frac{9}{20}\right)$	$\left(\frac{4}{5}, \frac{3}{20}\right)$	$(0, 0)$	$\left(\frac{1}{5}, -\frac{19}{20}\right)$	$\left(\frac{4}{5}, -\frac{7}{20}\right)$	$(0, 0)$	$\left(\frac{9}{20}, -\frac{9}{20}\right)$	$\left(\frac{11}{20}, \frac{9}{10}\right)$
(ba, ba, ba)	$\left(-\frac{19}{20}, \frac{19}{10}\right)$	$\left(\frac{39}{20}, -\frac{19}{20}\right)$	(ba, aba, e)	$\left(-\frac{1}{5}, \frac{13}{20}\right)$	$\left(\frac{6}{5}, \frac{1}{20}\right)$	(ba, aba, aba)	$\left(-\frac{1}{5}, \frac{23}{20}\right)$	$\left(\frac{6}{5}, \frac{11}{20}\right)$
$(-38, 19)$	$\left(0, -\frac{19}{20}\right)$	$\left(1, -\frac{19}{20}\right)$	$(-13, -1)$	$\left(-\frac{1}{4}, \frac{1}{20}\right)$	$\left(\frac{5}{4}, -\frac{7}{10}\right)$	$(-23, -11)$	$\left(-\frac{3}{4}, \frac{11}{20}\right)$	$\left(\frac{7}{4}, -\frac{17}{10}\right)$
$(0, 0)$	$\left(\frac{19}{20}, -\frac{19}{20}\right)$	$\left(\frac{1}{20}, \frac{19}{10}\right)$	$(0, 0)$	$\left(\frac{9}{20}, -\frac{7}{10}\right)$	$\left(\frac{11}{20}, \frac{13}{20}\right)$	$(0, 0)$	$\left(\frac{19}{20}, -\frac{17}{10}\right)$	$\left(\frac{1}{20}, \frac{23}{20}\right)$
(aba, e, e)	$\left(0, -\frac{11}{20}\right)$	$\left(1, -\frac{11}{20}\right)$	(aba, e, aba)	$\left(0, -\frac{1}{20}\right)$	$\left(1, -\frac{1}{20}\right)$	(aba, a, a)	$\left(-\frac{3}{4}, \frac{19}{20}\right)$	$\left(\frac{7}{4}, -\frac{13}{10}\right)$
$(11, 11)$	$\left(\frac{11}{20}, -\frac{11}{20}\right)$	$\left(\frac{9}{20}, \frac{11}{10}\right)$	$(1, 1)$	$\left(\frac{1}{20}, -\frac{1}{20}\right)$	$\left(\frac{19}{20}, \frac{1}{10}\right)$	$(-19, 26)$	$\left(\frac{11}{20}, -\frac{13}{10}\right)$	$\left(\frac{9}{20}, \frac{7}{20}\right)$
$(0, 0)$	$\left(-\frac{11}{20}, \frac{11}{10}\right)$	$\left(\frac{31}{20}, -\frac{11}{20}\right)$	$(0, 0)$	$\left(-\frac{1}{20}, \frac{1}{10}\right)$	$\left(\frac{21}{20}, -\frac{1}{20}\right)$	$(0, 0)$	$\left(\frac{1}{5}, \frac{7}{20}\right)$	$\left(\frac{4}{5}, \frac{19}{20}\right)$
(aba, a, ab)	$\left(-\frac{1}{4}, \frac{9}{20}\right)$	$\left(\frac{5}{4}, -\frac{3}{10}\right)$	(aba, b, b)	$\left(\frac{3}{4}, -\frac{13}{10}\right)$	$\left(\frac{1}{4}, \frac{19}{20}\right)$	(aba, b, ba)	$\left(\frac{1}{4}, -\frac{3}{10}\right)$	$\left(\frac{3}{4}, \frac{9}{20}\right)$
$(-9, 6)$	$\left(\frac{1}{20}, -\frac{3}{10}\right)$	$\left(\frac{19}{20}, -\frac{3}{20}\right)$	$(26, -19)$	$\left(-\frac{1}{5}, \frac{19}{20}\right)$	$\left(\frac{6}{5}, \frac{7}{20}\right)$	$(6, -9)$	$\left(-\frac{1}{5}, \frac{9}{20}\right)$	$\left(\frac{6}{5}, -\frac{3}{20}\right)$
$(0, 0)$	$\left(\frac{1}{5}, -\frac{3}{20}\right)$	$\left(\frac{4}{5}, \frac{9}{20}\right)$	$(0, 0)$	$\left(-\frac{11}{20}, \frac{7}{20}\right)$	$\left(\frac{31}{20}, -\frac{13}{10}\right)$	$(0, 0)$	$\left(-\frac{1}{20}, -\frac{3}{20}\right)$	$\left(\frac{21}{20}, -\frac{3}{10}\right)$
(aba, ab, a)	$\left(\frac{1}{4}, -\frac{1}{20}\right)$	$\left(\frac{3}{4}, \frac{7}{10}\right)$	(aba, ab, ab)	$\left(\frac{3}{4}, -\frac{11}{20}\right)$	$\left(\frac{1}{4}, \frac{17}{10}\right)$	(aba, ba, b)	$\left(-\frac{1}{4}, \frac{7}{10}\right)$	$\left(\frac{5}{4}, -\frac{1}{20}\right)$
$(1, -14)$	$\left(-\frac{9}{20}, \frac{7}{10}\right)$	$\left(\frac{29}{20}, -\frac{13}{20}\right)$	$(11, -34)$	$\left(-\frac{19}{20}, \frac{17}{10}\right)$	$\left(\frac{39}{20}, -\frac{23}{20}\right)$	$(-14, 1)$	$\left(-\frac{1}{5}, -\frac{1}{20}\right)$	$\left(\frac{6}{5}, -\frac{13}{20}\right)$
$(0, 0)$	$\left(\frac{1}{5}, -\frac{13}{20}\right)$	$\left(\frac{4}{5}, -\frac{1}{20}\right)$	$(0, 0)$	$\left(\frac{1}{5}, -\frac{23}{20}\right)$	$\left(\frac{4}{5}, -\frac{11}{20}\right)$	$(0, 0)$	$\left(\frac{9}{20}, -\frac{13}{20}\right)$	$\left(\frac{11}{20}, \frac{7}{10}\right)$
(aba, ba, ba)	$\left(-\frac{3}{4}, \frac{17}{10}\right)$	$\left(\frac{7}{4}, -\frac{11}{20}\right)$	(aba, aba, e)	$\left(0, \frac{9}{20}\right)$	$\left(1, \frac{9}{20}\right)$	(aba, aba, aba)	$\left(0, \frac{19}{20}\right)$	$\left(1, \frac{19}{20}\right)$
$(-34, 11)$	$\left(-\frac{1}{5}, -\frac{11}{20}\right)$	$\left(\frac{6}{5}, -\frac{23}{20}\right)$	$(-9, -9)$	$\left(-\frac{9}{20}, \frac{9}{20}\right)$	$\left(\frac{29}{20}, -\frac{9}{10}\right)$	$(-19, -19)$	$\left(-\frac{19}{20}, \frac{19}{20}\right)$	$\left(\frac{39}{20}, -\frac{19}{10}\right)$
$(0, 0)$	$\left(\frac{19}{20}, -\frac{23}{20}\right)$	$\left(\frac{1}{20}, \frac{17}{10}\right)$	$(0, 0)$	$\left(\frac{9}{20}, -\frac{9}{10}\right)$	$\left(\frac{11}{20}, \frac{9}{20}\right)$	$(0, 0)$	$\left(\frac{19}{20}, -\frac{19}{10}\right)$	$\left(\frac{1}{20}, \frac{19}{20}\right)$

Table B.3: $\boldsymbol{\alpha}$ of $M\left(-2; \frac{1}{2}, \frac{1}{2}, \frac{3}{4}\right)$ for each $\vec{s}$.

(e, e, e)	$\left(0, -\frac{5}{4}\right)$	$\left(1, -\frac{5}{4}\right)$	(e, e, aba)	$\left(0, -\frac{3}{4}\right)$	$\left(1, -\frac{3}{4}\right)$	(e, aba, e)	$\left(0, -\frac{1}{4}\right)$	$\left(1, -\frac{1}{4}\right)$
$(5, 5)$	$\left(\frac{5}{4}, -\frac{5}{4}\right)$	$\left(-\frac{1}{4}, \frac{5}{2}\right)$	$(3, 3)$	$\left(\frac{3}{4}, -\frac{3}{4}\right)$	$\left(\frac{1}{4}, \frac{3}{2}\right)$	$(1, 1)$	$\left(\frac{1}{4}, -\frac{1}{4}\right)$	$\left(\frac{3}{4}, \frac{1}{2}\right)$
$(0, 0)$	$\left(-\frac{5}{4}, \frac{5}{2}\right)$	$\left(\frac{9}{4}, -\frac{5}{4}\right)$	$(0, 0)$	$\left(-\frac{3}{4}, \frac{3}{2}\right)$	$\left(\frac{7}{4}, -\frac{3}{4}\right)$	$(0, 0)$	$\left(-\frac{1}{4}, \frac{1}{2}\right)$	$\left(\frac{5}{4}, -\frac{1}{4}\right)$
(e, aba, aba)	$\left(0, \frac{1}{4}\right)$	$\left(1, \frac{1}{4}\right)$	(a, a, a)	$\left(-\frac{5}{4}, \frac{5}{4}\right)$	$\left(\frac{9}{4}, -\frac{5}{2}\right)$	(a, a, ab)	$\left(-\frac{3}{4}, \frac{3}{4}\right)$	$\left(\frac{7}{4}, -\frac{3}{2}\right)$
$(-1, -1)$	$\left(-\frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{5}{4}, -\frac{1}{2}\right)$	$(-5, 10)$	$\left(\frac{5}{4}, -\frac{5}{2}\right)$	$\left(-\frac{1}{4}, \frac{5}{4}\right)$	$(-3, 6)$	$\left(\frac{3}{4}, -\frac{3}{2}\right)$	$\left(\frac{1}{4}, \frac{3}{4}\right)$
$(0, 0)$	$\left(\frac{1}{4}, -\frac{1}{2}\right)$	$\left(\frac{3}{4}, \frac{1}{4}\right)$	$(0, 0)$	$\left(0, \frac{5}{4}\right)$	$\left(1, \frac{5}{4}\right)$	$(0, 0)$	$\left(0, \frac{3}{4}\right)$	$\left(1, \frac{3}{4}\right)$
(a, ab, a)	$\left(-\frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{5}{4}, -\frac{1}{2}\right)$	(a, ab, ab)	$\left(\frac{1}{4}, -\frac{1}{4}\right)$	$\left(\frac{3}{4}, \frac{1}{2}\right)$	(b, b, b)	$\left(\frac{5}{4}, -\frac{5}{2}\right)$	$\left(-\frac{1}{4}, \frac{5}{4}\right)$
$(-1, 2)$	$\left(\frac{1}{4}, -\frac{1}{2}\right)$	$\left(\frac{3}{4}, \frac{1}{4}\right)$	$(1, -2)$	$\left(-\frac{1}{4}, \frac{1}{2}\right)$	$\left(\frac{5}{4}, -\frac{1}{4}\right)$	$(10, -5)$	$\left(0, \frac{5}{4}\right)$	$\left(1, \frac{5}{4}\right)$
$(0, 0)$	$\left(0, \frac{1}{4}\right)$	$\left(1, \frac{1}{4}\right)$	$(0, 0)$	$\left(0, -\frac{1}{4}\right)$	$\left(1, -\frac{1}{4}\right)$	$(0, 0)$	$\left(-\frac{5}{4}, \frac{5}{4}\right)$	$\left(\frac{9}{4}, -\frac{5}{2}\right)$
(b, b, ba)	$\left(\frac{3}{4}, -\frac{3}{2}\right)$	$\left(\frac{1}{4}, \frac{3}{4}\right)$	(b, ba, b)	$\left(\frac{1}{4}, -\frac{1}{2}\right)$	$\left(\frac{3}{4}, \frac{1}{4}\right)$	(b, ba, ba)	$\left(-\frac{1}{4}, \frac{1}{2}\right)$	$\left(\frac{5}{4}, -\frac{1}{4}\right)$
$(6, -3)$	$\left(0, \frac{3}{4}\right)$	$\left(1, \frac{3}{4}\right)$	$(2, -1)$	$\left(0, \frac{1}{4}\right)$	$\left(1, \frac{1}{4}\right)$	$(-2, 1)$	$\left(0, -\frac{1}{4}\right)$	$\left(1, -\frac{1}{4}\right)$
$(0, 0)$	$\left(-\frac{3}{4}, \frac{3}{4}\right)$	$\left(\frac{7}{4}, -\frac{3}{2}\right)$	$(0, 0)$	$\left(-\frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{5}{4}, -\frac{1}{2}\right)$	$(0, 0)$	$\left(\frac{1}{4}, -\frac{1}{4}\right)$	$\left(\frac{3}{4}, \frac{1}{2}\right)$
(ab, a, a)	$\left(-\frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{5}{4}, -\frac{1}{2}\right)$	(ab, a, ab)	$\left(\frac{1}{4}, -\frac{1}{4}\right)$	$\left(\frac{3}{4}, \frac{1}{2}\right)$	(ab, ab, a)	$\left(\frac{3}{4}, -\frac{3}{4}\right)$	$\left(\frac{1}{4}, \frac{3}{2}\right)$
$(-1, 2)$	$\left(\frac{1}{4}, -\frac{1}{2}\right)$	$\left(\frac{3}{4}, \frac{1}{4}\right)$	$(1, -2)$	$\left(-\frac{1}{4}, \frac{1}{2}\right)$	$\left(\frac{5}{4}, -\frac{1}{4}\right)$	$(3, -6)$	$\left(-\frac{3}{4}, \frac{3}{2}\right)$	$\left(\frac{7}{4}, -\frac{3}{4}\right)$
$(0, 0)$	$\left(0, \frac{1}{4}\right)$	$\left(1, \frac{1}{4}\right)$	$(0, 0)$	$\left(0, -\frac{1}{4}\right)$	$\left(1, -\frac{1}{4}\right)$	$(0, 0)$	$\left(0, -\frac{3}{4}\right)$	$\left(1, -\frac{3}{4}\right)$
(ab, ab, ab)	$\left(\frac{5}{4}, -\frac{5}{4}\right)$	$\left(-\frac{1}{4}, \frac{5}{2}\right)$	(ba, b, b)	$\left(\frac{1}{4}, -\frac{1}{2}\right)$	$\left(\frac{3}{4}, \frac{1}{4}\right)$	(ba, b, ba)	$\left(-\frac{1}{4}, \frac{1}{2}\right)$	$\left(\frac{5}{4}, -\frac{1}{4}\right)$
$(5, -10)$	$\left(-\frac{5}{4}, \frac{5}{2}\right)$	$\left(\frac{9}{4}, -\frac{5}{4}\right)$	$(2, -1)$	$\left(0, \frac{1}{4}\right)$	$\left(1, \frac{1}{4}\right)$	$(-2, 1)$	$\left(0, -\frac{1}{4}\right)$	$\left(1, -\frac{1}{4}\right)$
$(0, 0)$	$\left(0, -\frac{5}{4}\right)$	$\left(1, -\frac{5}{4}\right)$	$(0, 0)$	$\left(-\frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{5}{4}, -\frac{1}{2}\right)$	$(0, 0)$	$\left(\frac{1}{4}, -\frac{1}{4}\right)$	$\left(\frac{3}{4}, \frac{1}{2}\right)$
(ba, ba, b)	$\left(-\frac{3}{4}, \frac{3}{2}\right)$	$\left(\frac{7}{4}, -\frac{3}{4}\right)$	(ba, ba, ba)	$\left(-\frac{5}{4}, \frac{5}{2}\right)$	$\left(\frac{9}{4}, -\frac{5}{4}\right)$	(aba, e, e)	$\left(0, -\frac{1}{4}\right)$	$\left(1, -\frac{1}{4}\right)$

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Table B.3: $\boldsymbol{\alpha}$ of $M\left(-2; \frac{1}{2}, \frac{1}{2}, \frac{3}{4}\right)$ for each $\vec{s}$. (Continued)

$(-6, 3)$	$\left(0, -\frac{3}{4}\right)$	$\left(1, -\frac{3}{4}\right)$	$(-10, 5)$	$\left(0, -\frac{5}{4}\right)$	$\left(1, -\frac{5}{4}\right)$	$(1, 1)$	$\left(\frac{1}{4}, -\frac{1}{4}\right)$	$\left(\frac{3}{4}, \frac{1}{2}\right)$
$(0, 0)$	$\left(\frac{3}{4}, -\frac{3}{4}\right)$	$\left(\frac{1}{4}, \frac{3}{2}\right)$	$(0, 0)$	$\left(\frac{5}{4}, -\frac{5}{4}\right)$	$\left(-\frac{1}{4}, \frac{5}{2}\right)$	$(0, 0)$	$\left(-\frac{1}{4}, \frac{1}{2}\right)$	$\left(\frac{5}{4}, -\frac{1}{4}\right)$
(aba, e, aba)	$\left(0, \frac{1}{4}\right)$	$\left(1, \frac{1}{4}\right)$	(aba, aba, e)	$\left(0, \frac{3}{4}\right)$	$\left(1, \frac{3}{4}\right)$	(aba, aba, aba)	$\left(0, \frac{5}{4}\right)$	$\left(1, \frac{5}{4}\right)$
$(-1, -1)$	$\left(-\frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{5}{4}, -\frac{1}{2}\right)$	$(-3, -3)$	$\left(-\frac{3}{4}, \frac{3}{4}\right)$	$\left(\frac{7}{4}, -\frac{3}{2}\right)$	$(-5, -5)$	$\left(-\frac{5}{4}, \frac{5}{4}\right)$	$\left(\frac{9}{4}, -\frac{5}{2}\right)$
$(0, 0)$	$\left(\frac{1}{4}, -\frac{1}{2}\right)$	$\left(\frac{3}{4}, \frac{1}{4}\right)$	$(0, 0)$	$\left(\frac{3}{4}, -\frac{3}{2}\right)$	$\left(\frac{1}{4}, \frac{3}{4}\right)$	$(0, 0)$	$\left(\frac{5}{4}, -\frac{5}{2}\right)$	$\left(-\frac{1}{4}, \frac{5}{4}\right)$

C Special functions

In this appendix we collect the definitions of the special functions used in chapter 4, as well as properties and relations that they satisfy. As for notations, we use throughout $q := e^{2\pi i\tau}$, where $\tau \in \mathbb{H}$ and $v := \Im\tau$. The functions

$$E(u) := 2\int_0^u e^{-\pi w^2} dw \ , \qquad u \in \mathbb{R} \tag{C.0.1}$$

and

$$M(u) := \frac{i}{\pi}\int_{\mathbb{R}-iu} e^{-\pi w^2 - 2\pi i u w} w^{-1} dw \ , \qquad u \neq 0 \tag{C.0.2}$$

are closely related to the error and the complementary error functions. A useful rewriting of $M(u)$ is [4]

$$M(x\sqrt{v}) = i\frac{x}{\sqrt{2}} q^{\frac{x^2}{4}} \int_{-\bar{\tau}}^{i\infty} \frac{e^{\frac{\pi i x^2 w}{2}}}{\sqrt{-i(w+\tau)}} dw \ . \tag{C.0.3}$$

They satisfy the relation

$$M(u) = E(u) - \mathrm{sgn}(u) \ , \tag{C.0.4}$$

where

$$\mathrm{sgn}(u) := \begin{cases} 1 & \text{if } u > 0 \\ -1 & \text{if } u < 0 \\ 0 & \text{if } u = 0 \end{cases} \ . \tag{C.0.5}$$

We also define

$$M^*(u) = E(u) - \mathrm{sgn}^*(u) \ , \tag{C.0.6}$$

where

$$\mathrm{sgn}^*(x) := \mathrm{sgn}(x) \ \ \text{if } x \neq 0 \qquad \text{and} \qquad \mathrm{sgn}^*(0) := 1. \tag{C.0.7}$$

The generalised error function $E_2 : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$E_2(\kappa; \mathbf{u}) := \int_{\mathbb{R}^2} \mathrm{sgn}(w_1)\mathrm{sgn}(w_2 + \kappa w_1) e^{-\pi\left((w_1-u_1)^2+(w_2-u_2)^2\right)} dw_1 dw_2 \ . \quad \text{(C.0.8)}$$

For $u_2, u_1 - \kappa u_2 \neq 0$, the generalised complementary error function is

$$M_2(\kappa; \mathbf{u}) := -\frac{1}{\pi^2} \int_{\mathbb{R}-iu_2} \int_{\mathbb{R}-iu_1} \frac{e^{-\pi w_1^2 - \pi w_2^2 - 2\pi i (u_1 w_1 + u_2 w_2)}}{w_2(w_1 - \kappa w_2)} dw_1 dw_2 \ . \quad \text{(C.0.9)}$$

These functions satisfy the following relation

$$\begin{aligned} M_2(\kappa; \mathbf{u}) = {} & E_2(\kappa; u_1, u_2) + \mathrm{sgn}(u_1 - \kappa u_2)\mathrm{sgn}(u_2) \\ & - \mathrm{sgn}(u_2) E(u_1) - \mathrm{sgn}(u_1 - \kappa u_2) E\left(\frac{\kappa u_1 + u_2}{\sqrt{1+\kappa^2}}\right) \ , \end{aligned} \quad \text{(C.0.10)}$$

and

$$\begin{aligned} M_2^*(\kappa; u_1, u_2) := {} & \mathrm{sgn}^*(u_1 - \kappa u_2)\mathrm{sgn}^*(u_2) + E_2(\kappa; u_1, u_2) \\ & - \mathrm{sgn}^*(u_2) E(u_1) - \mathrm{sgn}^*(u_1 - \kappa u_2) E\left(\frac{\kappa u_1 + u_2}{\sqrt{1+\kappa^2}}\right) \ . \end{aligned} \quad \text{(C.0.11)}$$

The following identities hold for derivatives of the function $M_2(\kappa; \mathbf{u})$ [86]

$$M_2^{(0,1)}(\kappa; \mathbf{u}) = \frac{2}{\sqrt{1+\kappa^2}} e^{-\frac{\pi(u_2+\kappa u_1)^2}{1+\kappa^2}} M\left(\frac{u_1 - \kappa u_2}{\sqrt{1+\kappa^2}}\right) \ , \quad \text{(C.0.12)}$$

$$M_2^{(1,0)}(\kappa; \mathbf{u}) = 2e^{-\pi u_1^2} M(u_2) + \frac{2\kappa}{\sqrt{1+\kappa^2}} e^{-\frac{\pi(u_2+\kappa u_1)^2}{1+\kappa^2}} M\left(\frac{u_1 - \kappa u_2}{\sqrt{1+\kappa^2}}\right) \ . \quad \text{(C.0.13)}$$

Error function complements as integrals of theta functions: Let $\theta_i\left(\boldsymbol{\mu}, \mathbf{w}\right)$ be the following theta functions

$$\begin{aligned}
\theta_1(\boldsymbol{\mu};\mathbf{w}) &= \sum_{\mathbf{n}\in\boldsymbol{\mu}+\mathbb{Z}^2} (2n_1+n_2)n_2\, e^{\frac{\pi i}{2}\left(3(2n_1+n_2)^2 w_1 + n_2^2 w_2\right)} \\
\theta_2(\boldsymbol{\mu};\mathbf{w}) &= \sum_{\mathbf{n}\in\boldsymbol{\mu}+\mathbb{Z}^2} (3n_1+2n_2)n_1\, e^{\frac{\pi i}{2}\left((3n_1+2n_2)^2 w_1 + 3n_1^2 w_2\right)} \\
\theta_3(\boldsymbol{\mu};\mathbf{w}) &= \sum_{\mathbf{n}\in\boldsymbol{\mu}+\mathbb{Z}^2} (2n_1+n_2)\, e^{\frac{\pi i}{2}\left(3(2n_1+n_2)^2 w_1 + n_2^2 w_2\right)} \\
\theta_4(\boldsymbol{\mu};\mathbf{w}) &= \sum_{\mathbf{n}\in\boldsymbol{\mu}+\mathbb{Z}^2} (3n_1+2n_2)\, e^{\frac{\pi i}{2}\left((3n_1+2n_2)^2 w_1 + 3n_1^2 w_2\right)} \\
\theta_5(\boldsymbol{\mu};\mathbf{w}) &= \sum_{\mathbf{n}\in\boldsymbol{\mu}+\mathbb{Z}^2} n_1\, e^{\frac{\pi i}{2}\left((3n_1+2n_2)^2 w_1 + 3n_1^2 w_2\right)}\ .
\end{aligned} \tag{C.0.14}$$

We can rewrite the error function complement $M_2\left(\kappa, \mathbf{u}\right)$ from equation (C.0.9) as an iterated Eichler integral like in [96]

$$\begin{aligned}
&M_2\left(\sqrt{3};\sqrt{3v}(2n_1+n_2),\sqrt{v}n_2\right) = \\
&\quad -\frac{\sqrt{3}}{2}(2n_1+n_2)n_2 q^{Q(\boldsymbol{n})}\int_{-\bar{\tau}}^{i\infty} \frac{e^{\frac{3\pi i}{2}(2n_1+n_2)^2 w_1}}{\sqrt{-i(w_1+\tau)}}\int_{w_1}^{i\infty}\frac{e^{\frac{\pi i n_2^2 w_2}{2}}}{\sqrt{-i(w_2+\tau)}} dw_2 dw_1 \\
&\quad -\frac{\sqrt{3}}{2}(3n_1+2n_2)n_1 q^{Q(\boldsymbol{n})}\int_{-\bar{\tau}}^{i\infty} \frac{e^{\frac{\pi i}{2}(3n_1+2n_2)^2 w_1}}{\sqrt{-i(w_1+\tau)}}\int_{w_1}^{i\infty}\frac{e^{\frac{3\pi i n_1^2 w_2}{2}}}{\sqrt{-i(w_2+\tau)}} dw_2 dw_1\ ,
\end{aligned} \tag{C.0.15}$$

whereby

$$\begin{aligned}
&\sum_{\mathbf{n}\in\boldsymbol{\mu}+\mathbb{Z}^2} M_2\left(\sqrt{3};\sqrt{3v}(2n_1+n_2),\sqrt{v}n_2\right) \\
&\qquad = -\frac{\sqrt{3}}{2}\int_{-\bar{\tau}}^{i\infty}\int_{w_1}^{i\infty}\frac{\theta_1(\boldsymbol{\mu},\mathbf{w})+\theta_2(\boldsymbol{\mu},\mathbf{w})}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}} dw_2 dw_1
\end{aligned} \tag{C.0.16}$$

and

$$\frac{1}{2\pi i}\left[\frac{\partial}{\partial z}\left(M_2\left(\sqrt{3};\sqrt{3v}\,(2n_1+n_2)\,,\sqrt{v}\left(n_2-\frac{2\mathrm{Im}(z)}{v}\right)\right)e^{2\pi i n_2 z}\right)\right]_{z=0}$$
$$=\frac{\sqrt{3}}{2\pi}(2n_1+n_2)\int_{-\bar\tau}^{i\infty}\frac{e^{\frac{3i\pi}{2}(2n_1+n_2)^2 w_1}}{\sqrt{-i(w_1+\tau)}}\int_{w_1}^{i\infty}\frac{e^{\frac{i\pi n_2^2 w_2}{2}}}{(-i(w_2+\tau))^{\frac{3}{2}}}dw_2dw_1$$
$$-\frac{\sqrt{3}}{4\pi}(3n_1+2n_2)\int_{-\bar\tau}^{i\infty}\frac{e^{-\frac{i\pi}{2}(3n_1+2n_2)^2 w_1}}{\sqrt{-i(w_1+\tau)}}\int_{w_1}^{i\infty}\frac{e^{\frac{3i\pi n_1^2 w_2}{2}}}{(-i(w_2+\tau))^{\frac{3}{2}}}dw_2dw_1$$
$$-\frac{\sqrt{3}n_1}{4\pi}\int_{-\bar\tau}^{i\infty}\frac{e^{\frac{i\pi}{2}(3n_1+2n_2)^2 w_1}}{(-(w_1+\tau))^{\frac{3}{2}}}\int_{w_1}^{i\infty}\frac{e^{\frac{3i\pi n_1^2 w_2}{2}}}{\sqrt{-i(w_2+\tau)}}dw_2dw_1\ , \quad \text{(C.0.17)}$$

which implies

$$\frac{1}{2\pi i}\sum_{\mathbf{n}\in\boldsymbol{\mu}+\mathbb{Z}^2}\left[\frac{\partial}{\partial z}\left(M_2\left(\sqrt{3};\sqrt{3v}\,(2n_1+n_2)\,,\sqrt{v}\left(n_2-\frac{2\mathrm{Im}(z)}{v}\right)\right)e^{2\pi i n_2 z}\right)\right]_{z=0}$$
$$=\frac{\sqrt{3}}{4\pi}\int_{-\bar\tau}^{i\infty}\int_{w_1}^{i\infty}\frac{2\theta_3(\boldsymbol{\mu},\mathbf{w})-\theta_4(\boldsymbol{\mu},\mathbf{w})}{\sqrt{-i(w_1+\tau)}(-i(w_2+\tau))^{\frac{3}{2}}}dw_2dw_1$$
$$+\frac{\sqrt{3}}{4\pi}\int_{-\bar\tau}^{i\infty}\int_{w_1}^{i\infty}\frac{\theta_5(\boldsymbol{\mu},\mathbf{w})}{(-i(w_1+\tau))^{\frac{3}{2}}\sqrt{-i(w_2+\tau)}}dw_2dw_1\ . \quad \text{(C.0.18)}$$

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Summary

Physics and Mathematics, albeit very different in methods and goals, are deeply interconnected. Whether this is a result of a fundamental first principle of Nature that is indeed “written in mathematical language” or just the result of the fact that Mathematics has been designed and adapted to describe natural phenomena, it is clear that the synergy between these fields can lead to impressive results that are perhaps not achievable using the ideas and methods of each alone.

In this thesis we explored a small area of interconnection between Physics and Mathematics, studying some appearances of mock and quantum modularity in the context of String Theory. While doing so, we encountered a net of interconnected relations including other mathematical topics such as finite groups, 3d manifold invariants, and vertex operator algebras.

We have seen how mock modularity, when looked through the broader scope of quantum modularity, can provide an important tool to study the $\widehat{Z}$ invariants of plumbed 3-manifolds, and can give insights on how to extend the range of validity of some formulas. We have also noted how quantum modularity, at higher depth, can lead to interesting nested structures connecting the $\widehat{Z}$ invariants with gauge groups of different ranks. We have also explored the appearance of mock modular forms connected to umbral moonshine and provided constructions of cone vertex operator algebra modules whose graded characters reproduce this exotic modular behavior.

In the encounters of mock modularity in this thesis, indefinite theta functions have proved themselves to be invaluable tools to study and construct the relevant mock modular forms. They allowed taming these exotic objects by relating them to q-series with support on some integer cones in a controllable manner. In fact, indefinite theta functions proved themselves to be a very useful proxy to build mock modular forms.

A better understanding of the mathematical relations and objects appearing in String Theory can lead to great advancements both in Mathematics and Physics. Sometimes it is Physics to bring new ideas to Mathematics, as happened with the

invaluable role played by Conformal Field Theories in the understanding of Moonshine through the formal formulation of Vertex Operator Algebras; other times abstract and complex Mathematics can shed light on physical quantities that are not easily understandable with standard methods, as we have observed in studying the quantum modular properties of the $\widehat{Z}$ invariants. There is still a long way before we can fully understand String Theory in a formal and coherent mathematical framework, but we believe that the journey will be full of breakthroughs and that it will bring great advancements to both fields.

Samenvatting

Natuur- en wiskunde, zij ze verschillend van aard, zijn sterk met elkaar vervlochten. Dit kan een gevolg zijn van het beginprincipe der natuur als "geschreven in wiskundige taal", of van een menselijke gewoonte om natuurverschijnselen wiskundig te omschrijven. Ongeacht de reden, is het duidelijk dat de kruisbestuiving tussen de vakken indrukwekkende bevindingen oplevert; bevindingen die naar alle waarschijnlijkheid niet verkregen zouden kunnen worden zonder de vakken bijeen te voegen.
In deze thesis onderzoeken we een overlappend vakgebied van de natuur- en wiskunde, namelijk het gebruik van mock-modularitei en kwantummodulariteit in de context van snaartheorie. Bijgevolg leggen we een verzameling van verwantschappen bloot met wiskundige concepten zoals eindige groepen, 3d invariante variëteiten, en vertexoperatoralgebra's bloot. Specifiek tonen we aan dat mockmodulariteit, beschouwd door de bredere lens van kwantummodulariteit, een nuttige hulpmiddel is in de studie van $\widehat{Z}$-invarianten van plumbed 3- variëteiten, en inzicht kan bieden bij het vergroten van de geldigheidsmarges van bepaalde formules. We bemerken ook hoe kwantummodulariteit, op grotere diepte, kan leiden tot verweven structuren die de $\widehat{Z}$-invarianten verbinden met gaugegroepen van verschillende rangen. We hebben ook de verschijning van mockmodulaire vormen, verbonden aan umbral moonshine onderzocht en constructies voorgesteld van kegelvertexoperatoralgebramodules waarvan graded characters dit exotisch modulaire gedrag nabootsen. Bij het optreden van mockmodulariteit in deze thesis waren onbepaalde thetafuncties een onmisbaar hulpmiddel voor het bestuderen en construeren van de relevante mockmodulaire vormen. Deze functies staan het temmen van deze exotische voorwerpen toe, door ze te verbinden met q-reeksen met behulp van geheeltallige kegels op regelbare wijze. Sterker nog, onbepaalde thetafuncties bleken een zeer nuttige benaderingswijze bij het opbouwen van modulaire vormen. Een beter begrip van wiskundige verbanden en snaartheoretische voorwerpen kan vernieuwende inzichten bieden in zowel natuur- als wiskunde. Dit gaat over en weer: soms leidt natuurkunde tot nieuwe wiskundige inzichten, zoals de onmiskenbare

rol die conforme veldentheorie speelde in het begrijpen van Moonshine middels de formele verwoording van vertexoperatoralgebra's; soms leidt abstracte wiskundige theorie ook tot nieuw begrip van natuurkundige concepten die moeilijk te begrijpen zijn met standaardmethodes, zoals bemerkt bij het bestuderen van de $\widehat{Z}$-invatianten middels kwantummodulaire eigenschappen. Er is nog veel werk te verrichten voordat snaartheorie in een formeel en alomvattend wiskundig kader omgoten is. Desalniettemin kijken we hoopvol uit naar dit werk, dat zal leiden tot doorbraken en dat zowel de natuur- als de wiskunde zal verrijken.