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## Publication date <br> 2024

Document Version
Final published version

Link to publication

## Citation for published version (APA):

Ammanamanchi, S. R. C. (2024). Complexity \& wormholes in holography. [Thesis, fully internal, Universiteit van Amsterdam].

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This work has been accomplished at the Institute for Theoretical Physics (ITFA) of the University of Amsterdam (UvA) and is funded by the ERC Grant agreement ADG 834878 CanISeeQG.

ISBN: 978-94-6469-770-4
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# Complexity \& Wormholes in Holography 

## Academisch Proefschrift

ter verkrijging van de graad van doctor aan de Universiteit van Amsterdam op gezag van de Rector Magnificus<br>prof. dr. ir. P.P.C.C. Verbeek

ten overstaan van een door het College voor Promoties ingestelde commissie, in het openbaar te verdedigen in de Agnietenkapel op woensdag 7 februari 2024 , te 16:00 uur
door

Sri Ramesh Chandra Ammanamanchi<br>geboren te Hyderabad

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## Publications

## This thesis is based on the following publications:

[1] A.R. Chandra, J. de Boer, M. Flory, M.P. Heller, S. Hoertner and A. Rolph Spacetime as a quantum circuit,
JHEP 04 (2021) 207 [2101.01185]
Presented in Chapters 2 and 3.
[2] A.R. Chandra, J. de Boer, M. Flory, M.P. Heller, S. Hörtner and A. Rolph Cost of holographic path integrals,
SciPost Physics 14 (2023).
Presented in Chapters 2 and 3.
[3] A.R. Chandra, J. de Boer
A note on torus wormholes with matter
To appear, presented in Chapter 5.
Other publications by the author:
[4] A.R. Chandra, S. Mukhi
Towards a Classification of Two-Character Rational Conformal Field
Theories, JHEP 04 (2019) 153 [1810.09472]
[5] A.R. Chandra, S. Mukhi
Curiosities above $c=24$,
SciPost Phys. 6 (2019) 053 [1812.05109]

## Contribution of the author to the publications:

[1] A.R. Chandra, J. de Boer, M. Flory, M.P. Heller, S. Hoertner and A. Rolph Spacetime as a quantum circuit,
JHEP 04 (2021) 207 [2101.01185]

- Participated in all of the conceptual discussions,
- Performed all the calculations presented in section 2.1 and wrote the final version of most of this section,
- Verified all the calculations done by other authors presented in the paper.
[2] A.R. Chandra, J. de Boer, M. Flory, M.P. Heller, S. Hörtner and A. Rolph Cost of holographic path integrals, SciPost Physics 14 (2023).
- Participated in all of the conceptual discussions,
- Performed all the calculations in sections $4.3,4,4$ and 4.5 . Wrote the final version of section 4.5,
- Verified all the calculations done by other authors presented in the paper.


## Contents

I Holographic Complexity ..... 1
1 Setting the stage ..... 3
1.1 Holography ..... 4
1.1.1 The AdS/CFT correspondence ..... 5
1.2 Gravity from Quantum Information? ..... 7
1.2.1 Entanglement entropy is not enough ..... 10
1.3 Quantum Complexity ..... 11
1.4 Holographic Complexity ..... 14
2 Cost Proposals for Holographic Path Integrals ..... 19
2.1 Introduction ..... 19
2.2 Path integrals between geometric states ..... 20
2.2.1 Holography at a finite cutoff and $T \bar{T}$ ..... 20
2.2.2 Bulk path integrals ..... 24
2.2.3 Path integral cost and holographic state complexity ..... 28
2.3 Holographic path integral cost proposals ..... 30
2.3.1 Path integral cost ..... 31
2.3.2 Physical properties of path integral cost ..... 33
2.3.3 The space of all proposals: from boundary path integrals to functions on bulk subregions ..... 35
2.3.4 Reducing the space of cost proposals ..... 37
3 Holographic Complexity from Optimising Costs ..... 43
3.1 CV from optimising boundary volume ..... 44
3.2 Towards CV2.0 from optimising bulk volume ..... 46
3.3 CV from optimising Euclidean gravitational action ..... 48
3.4 Bulk action and $T \bar{T}$ ..... 59
3.5 Relation to kinematic space ..... 63
3.6 Obstacles to obtain the CA proposal from a cost ..... 66
3.7 Linear growth at late times for BTZ black hole ..... 68
3.8 General methods for gravitational action proposals ..... 70
3.8.1 Equations of motion ..... 71
3.8.2 Solution method, totally geodesic foliations ..... 72
3.8.3 Examples ..... 75
3.8.4 Implications of the Gauss-Bonnet theorem ..... 77
3.8.5 Lemons in Lorentzian $\mathrm{AdS}_{3}$ ..... 79
3.9 Discussion ..... 85
II Euclidean Wormholes ..... 89
4 Recent Developments ..... 91
4.1 Introduction ..... 91
4.2 Black hole evaporation and replica wormholes ..... 92
4.3 Spectral statistics and wormholes ..... 93
4.3.1 Wormholes in JT gravity ..... 94
4.3.2 Wormholes in AdS/CFT ..... 97
4.4 Operator statistics and wormholes ..... 100
5 Wormholes with Matter ..... 103
5.1 Brief Introduction ..... 103
5.2 General strategy ..... 104
5.3 Explicit examples ..... 106
5.3.1 Free field ..... 106
5.3.2 Interacting field: Cubic superpotential ..... 110
5.4 More general wormholes ..... 111
5.4.1 Backreacted double cone ..... 113
5.5 Constrained saddles ..... 115
5.6 Conclusions ..... 120
6 Summary \& Outlook ..... 123
A Cost as Lorentzian gravitational action ..... 127
B Conical singularities in the Gauss-Bonnet formula ..... 129
B.0.1 Euclidean case ..... 129
B.0.2 Lorentzian case ..... 132
B.0.3 Application to Lemons ..... 132
C Lemons in higher dimensions ..... 135
Bibliography ..... 137
Samenvatting ..... 151
Summary ..... 155
Acknowledgements ..... 157

## Part I

## Holographic Complexity

## Setting the stage

There is a remarkable interplay of ideas in modern theoretical physics. Tools and techniques developed in one field can be useful in a completely distinct field. Nowhere is this interplay of ideas more apparent than in our quest to understand the quantum nature of gravity. Quantum mechanics and general relativity are the foundational pillars upon which much of our comprehension of the universe rests. These theories explain diverse phenomenon over a vast range of scales.

Quantum mechanics is essential to grasp the fundamental nature of matter. This encompasses phenomena at the microscopic level, such as elementary particles governed by the standard model which is a quantum field theory. Moreover, it also extends to applications in diverse areas, from superconductors to quantum computers, and many more.

General relativity on the other hand is Einstein's theory describing gravity as a manifestation of the geometry of space and time. It is required to describe the universe at the largest scales. Its most spectacular predictions are gravitational waves and black holes, both of which have been experimentally verified. Despite being an extremely successful theory, there is ample evidence that it is insufficient to describe all natural phenomenon. For instance, to fully describe the dramatic physics of black holes, we need a theory of quantum gravity that unifies both of these pillars into a single framework.

Black holes present us with the best hints and also at the same time the biggest puzzles in developing a theory of quantum gravity. One of the most significant hints is the holographic principle, which states that gravity emerges from a quantum theory in a lower dimension. This has in turn uncovered the fact that quantum mechanics and gravity are deeply intertwined. In particular, tools from quantum
information and computation have been invaluable to understand quantum gravity. After a general introduction, this and the next couple of chapters explore the notion of quantum complexity in holography and how it could essential to solve some puzzles of quantum gravity.

### 1.1 Holography

The idea of holography begins ${ }^{1}$ with the Bekenstein-Hawking formula for the black hole entropy

$$
\begin{equation*}
S_{\mathrm{BH}}=\frac{A}{4 G_{N}} \tag{1.1}
\end{equation*}
$$

where $A$ is the surface area of the event horizon of the black hole, and $G_{N}$ is Newton's constant. The remarkable feature of this black hole entropy is that it scales with the area rather than the volume. In ordinary physical systems described by local quantum field theory, we are familiar with entropies that scale extensively with volume since there are degrees of freedom present at every point in space. The Bekenstein-Hawking entropy instead is suggesting that the microscopic degrees of freedom of the black hole, unlike the ones in local quantum field theory, live in one less dimension.

This feature of black holes has interesting implications not just for black holes but even for the rest of the universe. The Bekenstein-Hawking entropy sets a upper bound on the entropy contained inside any spherical region of spacetime with a given surface area $A$. If the given region contained stuff with more entropy, then by collapsing that stuff into a black hole one would violate the second law of thermodynamics

$$
\begin{equation*}
S \leq S_{\max }=\frac{A}{4 G_{N}} \tag{1.2}
\end{equation*}
$$

Taking inspiration from such entropic bounds, it was postulated by 't Hooft and Susskind that any theory of quantum gravity must be holographic, repackaging its degrees of freedom into one lower dimension. This is often known as the holographic principle. The most precise realisation of the holographic principle occurs in what is called the "AdS/CFT" correspondence discovered by Maldacena [8] in 1998. This entails several dualities where the quantum gravitational theory in certain $(d+1)$ dimensional spacetimes is equivalent to a non-gravitational theory in $d$ dimensions.

[^0]

Figure 1.1: The $A d S_{d+1}$ spacetime in global coordinates. Time runs vertically and spatial infinity is approached as we reach the boundary of the cylinder, where the $C F T_{d}$ lives. A particular bulk time slice at $t=0$ is represented in grey.

### 1.1.1 The AdS/CFT correspondence

The Anti-de Sitter (AdS)/ Conformal Field Theory (CFT) correspondence is a duality between two theories: quantum gravity in asymptotically AdS spacetimes in $(d+1)$ dimensions and conformal field theories in $d$ dimensions. It has its origins in string theory, where several examples of such dual pairs of equivalent theories were observed. The most famous among these is the duality between type IIB string theory on $\mathrm{AdS}_{5} \times \mathrm{S}^{5}$ and the four dimensional $\mathcal{N}=4$ supersymmetric YangMills CFT with gauge group $S U(N)$. We can use this example to illustrate some immediate features of this duality. The $\mathrm{AdS}_{5}$ gravity description is referred to as the bulk theory and the CFT description is referred to as the boundary theory. We should really view the boundary theory as giving us a non-perturbative definition of the bulk quantum gravity. When the boundary CFT is strongly coupled and we take the $N \rightarrow \infty$ limit, the bulk gravitational theory is weakly coupled and reduces to classical Einstein gravity.

Over the last twenty five years there have been enormous efforts to understand how this duality exactly works. A major theme is understand how interesting quantities on one side of the duality translate into the other. This translation is sometimes called as the AdS/CFT dictionary. The Hilbert space of the bulk theory is by definition given by the Hilbert space of the boundary CFT, and similarly both the bulk and boundary Hamiltonians are equal

$$
\begin{equation*}
\mathcal{H}_{\mathrm{CFT}}=\mathcal{H}_{\text {gravity }} \tag{1.3}
\end{equation*}
$$

The conformal symmetry $S O(d, 2)$ of the Lorentzian $\mathrm{CFT}_{d}$ are matched with the
bulk isometries of the $\mathrm{AdS}_{d+1}$ spacetime. The CFT states encode all the information about the state of the bulk gravity. The vacuum state is invariant under all of the conformal symmetries on the CFT and corresponds to the empty AdS spacetime. Low lying states in the boundary theory would correspond to small perturbations around empty AdS. High energy thermal states in the CFT are identified with massive AdS black holes in the bulk. Operators in the CFT are related to local fields in the bulk that are close to the boundary of the spacetime, using the so-called extrapolate dictionary [9,10]. For example, a bulk scalar field $\phi(r, x)$ is related to a CFT primary operators $\mathcal{O}(x)$ as

$$
\begin{equation*}
\lim _{r \rightarrow \infty} r^{\Delta_{\mathcal{O}}} \phi(r, x)=\mathcal{O}(x) \tag{1.4}
\end{equation*}
$$

where $\Delta_{\mathcal{O}}$ is the operator's scaling dimension. Quantities that just depend on the Hamiltonian and the space of states, such as thermal partition functions can be immediately matched on both sides

$$
\begin{align*}
Z_{\mathrm{CFT}}(\beta) & =Z_{\mathrm{AdS}}(\beta) \\
& \approx \sum_{\text {saddles } \bar{g}} e^{-S_{E}(\bar{g})} \tag{1.5}
\end{align*}
$$

The partition function on the bulk side can usually only be evaluated by using a semi-classical path integral, in which case it would be given in terms of the Euclidean action of the classical gravitational solutions with the specified boundary conditions. The above statement can be generalised in the presence of sources that are dual to CFT operators

$$
\begin{equation*}
Z_{\mathrm{CFT}}\left(\phi_{0}\right)=Z_{\mathrm{AdS}}\left(\phi \rightarrow \phi_{0}\right) \tag{1.6}
\end{equation*}
$$

On the left hand side we have the generating function of the CFT correlation functions, with $\phi_{0}$ corresponding to the sources of the operators $\mathcal{O}$ in the CFT. On the right hand side is the gravitational path integral with bulk fields whose boundary conditions at the asymptotic boundary are specified by $\phi_{0}$. Using this we can derive local correlation functions of the boundary theory from bulk computations.

Some typically quantum field theoretic questions in the bulk gravity such as the scattering of particles on a given AdS spacetime are easily captured by the local correlation functions of the boundary CFT. But there are several other important and perhaps more interesting questions such as how does a smooth bulk geometry, and an (approximate) bulk locality emerge from an underlying boundary theory? Ideas and tools from quantum information theory have been vital to understand and answer some of these questions.

### 1.2 Gravity from Quantum Information?

The fact that black holes are dual to thermal high-energy states in a quantum theory already illustrates the link between gravity, quantum mechanics and information: the entropy of the quantum state is geometrised as the horizon area in the Bekenstein-Hawking formula. This only scratches the surface, there's far more to be said about this link between quantum information and gravity.

The most striking feature of quantum states is their entanglement. For any two quantum systems $\mathcal{H}_{A}$ and $\mathcal{H}_{B}$ we can consider states in the combined system $\mathcal{H}_{A} \otimes \mathcal{H}_{B}$. It is this feature that sets quantum information apart from classical information. We can illustrate how entanglement plays a role in the emergence of bulk spacetime by studying two-sided black holes.

In general relativity, we often work with the maximal extensions of black hole spacetimes such as the Kruskal extension. Just like in flat space, such a maximal extension in AdS gives an eternal black hole, having two asymptotic boundaries. The constant time slices (the spatial geometries) of such eternal black holes are non-traversable wormholes that join the two boundaries.


Figure 1.2: A Penrose diagram of an AdS eternal black hole. It has two exterior $L$ and $R$ regions. The dashed lines are the singularities, and the diagonal lines are correspond to the horizon. This geometry is dual to the TFD state in the CFT.

The eternal black hole geometry was shown [11] to be dual to an entangled state, called the thermofield double which lives in the Hilbert space of two copies of the CFT

$$
\begin{equation*}
|\mathrm{TFD}\rangle=\frac{1}{\sqrt{Z(\beta)}} \sum_{n} e^{-\beta E_{n} / 2}\left|E_{n}\right\rangle_{L} \otimes\left|E_{n}^{*}\right\rangle_{R} \tag{1.7}
\end{equation*}
$$

where $E_{n}$ are the energy eigenstates of the CFT, and $\beta$ the inverse temperature. This is a pure state in the product Hilbert space of two copies (usually denoted
as $L$ and $R$ ) of the original CFT living on two disconnected boundaries of the eternal black hole. This is a state with maximal entanglement between the two subsystems. Upon tracing out either of the system, we get back a thermal density matrix

$$
\begin{equation*}
\rho_{L}=\operatorname{tr}_{R}(|\mathrm{TFD}\rangle\langle\mathrm{TFD}|)=\frac{e^{-\beta H}}{Z(\beta)} \tag{1.8}
\end{equation*}
$$

Since the two $L$ and $R$ systems are completely disconnected, we would have expected the bulk state to also consist of disconnected spacetimes. But the particular pattern of quantum entanglement in the $|\mathrm{TFD}\rangle$ state generates a single connected smooth bulk spacetime. At zero temperature, the TFD state reduces to the product state $|0\rangle_{L} \otimes|0\rangle_{R}$ without any entanglement. In the bulk this would correspond to two disconnected pure AdS spacetimes [12]. But for any non-zero temperature, the state consists of an infinite superposition of the $L$ and $R$ energy eigenstates. Thus, entanglement is a necessary condition for the bulk spacetime to be connected and acts as a glue that builds spacetime!


Figure 1.3: Entanglement builds spacetime: As the inverse temperature $\beta$ approaches infinity, $|T F D\rangle$ becomes unentangled product state $|0\rangle_{L} \otimes|0\rangle_{R}$, in turn resulting in two disconnected copies of vacuum AdS spacetime.

An important measure of entanglement between two systems is given by the von Neumann entropy, sometimes also called entanglement entropy in this context. Given any state $|\psi\rangle \in \mathcal{H}_{A} \otimes \mathcal{H}_{B}$, this is defined as

$$
\begin{equation*}
S=-\operatorname{tr}\left(\rho_{A} \log \rho_{A}\right), \quad \rho_{A}=\operatorname{tr}_{\mathcal{H}_{B}}(|\psi\rangle\langle\psi|) \tag{1.9}
\end{equation*}
$$

We can easily calculate the entanglement entropy in the TFD state. The entropy of the total state is of course zero, since its a pure state. If we have access to only one system, then the reduced state is a thermal state. Computing the entanglement entropy for a thermal density matrix gives us back the thermal entropy of that state. But we know that the entropy for the single system is just the black hole
entropy given by the area of its horizon. Thus, we see that the entanglement entropy is geometrised as the area of the black hole horizon.

It was observed by Ryu and Takayanagi [13] that the entanglement entropy of any spatial subregion of the boundary CFT has a natural geometric interpretation in the bulk geometry. To state it precisely, consider a subregion $A$ of the boundary CFT and consider its entanglement entropy with respect to its complement $A^{c}$. This is given by the holographic entanglement entropy as

$$
\begin{equation*}
S\left(\rho_{A}\right)=\frac{\operatorname{Area}\left(\chi_{A}\right)}{4 G_{N}} \tag{1.10}
\end{equation*}
$$

where $\chi_{A}$ is a special codimension- 2 bulk surface (sometimes called the RT surface) which is homologous to $A$ and having minimal area. When we have a black hole in the bulk spacetime, and we take $A$ to be the entire timeslice of the boundary, this surface is just the event horizon of the black hole reproducing the black hole entropy from the von Neumann entropy of the boundary thermal density matrix. This formula as stated above is valid only for static spacetimes, and ignores higher order corrections. There have been several updates of this including a covariant (HRT) formula applying to non-static spacetimes [14], and also a quantum corrected (QES) formula which takes into account the effects of the bulk quantum fields [15]. There are also derivations of these statements using the semi-classical gravitational path integral techniques $[16,17]$.


Figure 1.4: The entanglement entropy of a spatial subregion $A$ (thick black curve) with respect to its complement $A^{c}$ (dashed black curve). In the bulk, this entanglement entropy can be computed geometrically as the area of its corresponding $R T$ surface $\chi_{A}$ (red curve).

Ever since the discovery of this precise connection between entanglement entropy and geometry, there has been a remarkable progress in understanding AdS/CFT through a quantum information-theoretic perspective. There have been numerous non-trivial consequences that directly follow from the RT formula (1.10) and in general the geometrisation of entanglement. Since the RT surfaces directly probe the bulk geometry, by varying the entanglement entropy we can in certain cases rederive the linearised Einstein's equations [18]. Furthermore, we have gained insight into how the bulk information is encoded in the boundary, a phenomenon called as entanglement wedge reconstruction [19]. This has also highlighted the fact that the map from boundary to bulk acts like a quantum error correcting code [20]. Moreover, there are several toy models involving tensor networks which reproduce many of these features [21]. Tensor networks are used to efficiently prepare states in many-body systems, especially those with a particular entanglement structure [22,23]. It was realised that the bulk can be interpreted as a (discrete) tensor network that is preparing the boundary state. Building on much of the above progress, recent developments have also demonstrated the unitarity of black hole evaporation in the bulk, making way to resolving the long-standing black hole information paradox.

### 1.2.1 Entanglement entropy is not enough

In spite of these findings, there is compelling evidence indicating that measures like entanglement entropy alone may not constitute the definitive solution to address all the challenges posed by quantum gravity. For instance, there are bulk regions where the RT surfaces cannot ever penetrate, no matter what boundary subregion we choose. These regions were called as entanglement shadows and are expected to occur generically $[24,25]$. One reason is that the RT surfaces go through phase transitions and due to this they fail to probe the entire bulk geometry, giving rise to these shadows.

Another illustration that entropy-based quantities are not alone sufficient comes from studying black hole interiors. It is known that the interior region of black holes grow indefinitely in classical general relativity. For example, consider again the eternal AdS black hole. The entanglement entropy of boundary subsystems grows linearly for a short time before it quickly saturates [26]. While the length of the wormhole connecting the two sides still keeps growing for an exponential amount of time. Thus, we are in need of a new concept to explain this bulk phenomenon. It turns out that another notion from quantum information called quantum complexity has exactly the right properties to explain these issues.

### 1.3 Quantum Complexity

Intuitively the notion of complexity deals with quantifying the difficulty of a given task, with a certain set of resources at hand. In the context of quantum computation this difficulty is measuring the optimal number of simple steps that are required to form an initial reference state to a given final state. We can think of such a computation using the model of quantum circuits. The simple steps that build quantum circuits are a set of elementary quantum gates using which we can implement any given operator to a given precision. Given this input, we can define the circuit complexity (or gate complexity) of an operator as

$$
\begin{equation*}
C_{\epsilon}(U)=\min \left\{n:\left\|U-\prod_{i=1}^{n} g_{i}\right\| \leq \epsilon\right\} \tag{1.11}
\end{equation*}
$$

The definition above depends on a tolerance parameter $\epsilon$, and a set of gates $g_{i} \in \mathcal{G}$. This is a well defined quantity for finite qubit systems. For a system of $N$ qubits, the maximum complexity of a unitary acting on them is exponential in $N$, and one can show that most unitary operators are near-maximally complex. Similar to complexity of operators there is also a notion of state complexity. The complexity of a target state is defined using a reference state as

$$
\begin{equation*}
C_{\epsilon}\left(\left|\psi_{T}\right\rangle\right)=\min \left\{C_{\epsilon}(U): U\left|\psi_{R}\right\rangle=\left|\psi_{T}\right\rangle\right\} \tag{1.12}
\end{equation*}
$$

The reference state is usually chosen to be simple state such as a product state. Note that these definitions of complexity are highly discontinuous as $\epsilon$ is taken to zero [27]. There is also no known practical method of solving the above minimization problems, or in other words, finding the optimal circuit that generates a given unitary. To address these issues, Nielsen came up with a geometric method of estimating complexity [28]. In this method, we write the unitary as a path order exponential over a time-dependent Hamiltonian

$$
\begin{equation*}
U=\mathcal{P} \exp \left(\int_{0}^{1} H(s) d s\right) \tag{1.13}
\end{equation*}
$$

The Hamiltonian is then expanded over a chosen basis of operators, analogous to an elementary gate set as

$$
\begin{equation*}
H(s)=\sum_{I} Y^{I}(s) \mathcal{O}_{I} \tag{1.14}
\end{equation*}
$$

The functions $Y_{I}$ control the presence of the gates at each step of the circuit. To define the complexity of the unitary, we minimize over a suitably chosen cost functional of these $Y_{I}$ 's as

$$
\begin{equation*}
C_{F}(U)=\min _{\left\{Y_{I}\right\}} \int d s F\left[Y_{I}(s)\right] \tag{1.15}
\end{equation*}
$$

The cost functional $F\left[Y_{I}(s)\right]$ can be thought of as defining a metric on the space of unitaries, which is a smooth manifold. This metric can be chosen such that it penalises some directions corresponding to gates which are difficult to realise. Calculating the complexity of a given unitary is then translated into a problem of finding geodesics on this manifold. This continuous notion of complexity was used to calculate circuit complexity in quantum field theory [29,30]. Though in most cases the computations are not general enough and are performed in certain subspaces like those generated by Gaussian states [31] or involving primary states of the conformal group [32-34].

A method that is inspired from tensor networks and directly suited for conformal field theories was introduced in [35] called path-integral complexity. As the names suggests, this method involves states that are prepared using the Euclidean path integral. Say we are preparing the vacuum state of some Hamiltonian via a tensor network computation. This computation can be optimised by removing unnecessary tensors such that we do not lose much accuracy in the prepared state. This reduces the lattice sites of the tensor network making the computation more efficient. In the continuum limit, this change in the network structure would correspond to changing the metric over which a path integral is being performed.

This is best understood in the context of 2 d CFTs. For example consider the preparation of the vacuum state of a 2 d CFT. In this case we know exactly how the path integral changes under a change in the underlying geometry. Let us compare the wavefunctional on a flat geometry $d s^{2}=\left(d \tau^{2}+d x^{2}\right)$ to the one on a general curved two-dimensional geometry $d s^{2}=e^{2 \phi(\tau, x)}\left(d \tau^{2}+d x^{2}\right)$. We know that both wavefunctionals are proportional to each other

$$
\begin{equation*}
\Psi_{g_{i j}=e^{2 \phi} \delta_{i j}}=e^{S_{L}[\phi]-S_{L}[0]} \Psi_{g_{i j}=\delta_{i j}} \tag{1.16}
\end{equation*}
$$

with the proportionality factor given by the Liouville action

$$
\begin{equation*}
S_{L}[\phi]=\frac{c}{24 \pi} \int d \tau d x\left(\left(\partial_{\tau} \phi\right)^{2}+\left(\partial_{x} \phi\right)^{2}+\mu e^{2 \phi}\right) \tag{1.17}
\end{equation*}
$$

By varying the conformal factor $\phi(\tau, x)$ we change the geometry on which the path integral is performed, and the Liouville action naturally associates a cost for each geometry. It was proposed that minimisation of Liouville action gives the most efficient computation. This means that the complexity is just given by

$$
\begin{equation*}
C(\Psi)=\min _{\phi} S_{L}[\phi(\tau, x)] \tag{1.18}
\end{equation*}
$$

The minimization is done such the the metric at $\tau=0$ is held fixed. This procedure also seems to shed light on the emergence of bulk $A d S_{3}$ spacetime. For example, the conformal factor $\phi$ that minimises the Liouville action for the vacuum state
gives rise to a hyperbolic metric which can be interpreted as a time slice of the dual $A d S_{3}$ vacuum. It was shown that the Liouville action can be derived as a gate counting cost function, relating this procedure to circuit complexity.

As is evident from the various definitions of complexity above, all notions of complexity involve some sort of optimization procedure, thus making it an inherently challenging quantity to calculate exactly for generic states and operator in quantum field theory. However, in most physical scenarios, we are interested in the dynamics of complexity. A central question in this regard is of how complexity evolves in time. We can either study the complexity of a state as it time evolves, or the operator complexity of the time evolution operator $U(t)=e^{i H t}$ itself. The maximal complexity for an $N$ qubit system was exponential in $N$, more generally it is exponential in the degrees of freedom of a system. For chaotic quantum systems the complexity is expected to grow linearly until it reaches the maximal value. Random quantum circuits serve as a proxy to model the quantum chaotic dynamics of black holes, and for unitary operators built out of random circuits this linear growth conjecture was proven in [36], see also [37].


Figure 1.5: Time evolution of complexity in quantum chaotic systems. The growth is linear until a time of order $e^{S}$, after which it fluctuates around $C_{\max }$. There will be recurrences at times doubly exponential in the entropy.

Another universal feature of complexity dynamics is its behaviour under small perturbations. Say we perturb a quantum state at some time in the past by an operator $W$. We can imagine $W$ to be a simple operator acting only a single qubit within the circuit that is preparing the state. How does the complexity of this perturbation grow over time, i.e., what is the complexity of the operator $W(t)=$
$U^{\dagger}(t) W U(t)$ ? If $W$ were absent then the $U$ and $U^{\dagger}$ cancel, giving the identity operator, whose complexity doesn't grow. With $W$ present, such a cancellation happens initially for a small time, but the presence of the perturbation makes the operator $W(t)$ more complex over time, leading again to a linear growth in time. This delay in the onset of linear growth is called the switchback effect, and is the demonstration of the butterfly effect on the dynamics of complexity.

Both the linear growth in time and the switchback effect are essential and universal features of complexity in quantum chaotic systems. Remarkably, these features are reproduced by various geometric quantities in gravity, suggesting the need to study complexity in the context of holography.

### 1.4 Holographic Complexity

The role of quantum complexity in holography was motivated by studying the physics of black hole interiors. There were early hints [38] that that complexitytheoretic arguments are essential to understand black hole thought experiments. More concrete evidence was soon found in AdS/CFT. We already know that AdS black holes are thermal quantum systems with a given temperature and entropy. Moreover, the dynamics governing the evolution of black holes is maximally chaotic. Once a black hole is formed from a collapse, it quickly reaches thermal equilibrium. There are various observables in the boundary CFT that we can use to probe the bulk black hole during its thermalisation. For example, local observables like correlation functions, or non-local observables like Wilson loops expectation values or even entanglement entropies [26]. All such observables rapidly thermalise and reach their stable thermal values.

In spite of this one can see that there the black hole is still undergoing non-trivial dynamics after thermalisation. Most strikingly, the interior "size" of a black hole keeps growing for a very long time past thermalisation. In classical general relativity this growth persists forever, though this (semi)-classical description would breakdown before the black hole completely evaporates. There were also hints from tensor network models [26] in which this growth of the black hole interior is reflected in the growth of the size (or the number of nodes) of the tensor network. In a series of conjectures [39-42], various bulk quantities that capture this growth were related to the complexity of the state in the boundary CFT.

The first of these is the Complexity $=$ Volume $(\mathrm{CV})$ proposal. This proposal relates the complexity of the CFT state $|\psi(t)\rangle$ to the volume of the maximal volume slice
$\Sigma_{t}$ that is anchored to the boundary time $t$ as

$$
\begin{equation*}
C(|\psi(t)\rangle)=\frac{\operatorname{Vol}\left(\Sigma_{t}\right)}{G_{N} l} \tag{1.19}
\end{equation*}
$$

Here $l$ is some length scale. To gain some intuition for how this works, let us again consider the two-sided eternal AdS black hole. In this case the boundary state $|\psi(t)\rangle=e^{-i\left(H_{L}+H_{R}\right) t}|\mathrm{TFD}\rangle$, and we need to find maximal volume slices in the bulk. These surfaces can be computed given the metric of the corresponding AdS-Schwarzschild geometry, which is

$$
\begin{gather*}
d s^{2}=-f(r) d t^{2}+\frac{d r^{2}}{f(r)}+r^{2} d \Omega_{d-1}^{2}  \tag{1.20}\\
f(r)=1+\frac{r^{2}}{l_{A d S}^{2}}-\frac{\mu}{r^{d-2}}
\end{gather*}
$$



Figure 1.6: Left: Two maximal volume surfaces $\Sigma\left(t_{1}\right)$ and $\Sigma\left(t_{2}\right)$ anchored at boundary times $t_{2}>t_{1}$. Right: A depiction of the spatial geometries at various times $t=0, t_{1}, t_{2}$ showing a linear growth in time of the tube-like region inside the horizon.

The mass of the black hole is given as $M=(d-1) \operatorname{vol}\left(S^{d-1}\right) \mu / 16 \pi G_{N}$. We can see from the figure 1.6 that unlike RT surfaces, the maximal volume slices naturally probe the interior. Moreover, the (regularized) volume of these slices linearly increases with respect to the boundary time $t$, and at late time

$$
\begin{equation*}
\operatorname{Vol}\left(\Sigma_{t}\right) \sim \gamma|t|, \quad t \rightarrow \infty \tag{1.21}
\end{equation*}
$$

where $\gamma$ is a constant that is proportional to the black hole's mass. Thus we see that these surfaces indeed probe the linear growth of the interior, with the complexity growth rate determined by the mass of the black hole.

Instead of relating complexity to a single maximal surface, [42] put forward a new proposal called Complexity $=$ Action (CA), with potential improvements. The relevant bulk quantity here is the action of a certain spacetime region called the Wheeler-deWitt (WdW) patch. Let us denote the WdW patch anchored at time $t$ as $W_{t}$. Then, the complexity of the boundary state at time $t$ is given by the action of the region $W_{t}$ as

$$
\begin{equation*}
C(|\psi(t)\rangle)=S\left[W_{t}\right] \tag{1.22}
\end{equation*}
$$

The Wheeler-deWitt patch is defined as the domain of dependence of a bulk Cauchy slice that is anchored at the boundary state, see the figure 1.7 below. By definition this region doesn't pick one preferred bulk slice, rather it is the entire bulk region that can be associated to a given boundary time slice. In black hole spacetimes extends from the asymptotic boundaries and into the black hole horizon till the singularity.


Figure 1.7: The Wheeler-deWitt patch in global AdS. It is the domain of dependence of a given bulk Cauchy slice anchored to the boundary.

We need to evaluate the gravitational action functional in this region

$$
\begin{equation*}
S[M]=\frac{1}{16 \pi G} \int_{M} d^{d+1} x \sqrt{g}(R-2 \Lambda)+\frac{1}{8 \pi G} \int_{\partial M} d^{d} x \sqrt{h} K+\cdots \tag{1.23}
\end{equation*}
$$

The first term is the familiar Einstein Hilbert action. Since the WdW patch is a region that consists codimension- 1 boundaries we also need the second term involving the Gibbons-Hawking-York action. As the region also contains codimension-2


Figure 1.8: $A W d W$ patch $W\left(t_{L}, t_{R}\right)$ in the eternal black hole. This region extends from the asymptotic boundaries to inside the horizons, and intersects the future singularity.
corners which could in general be null, timelike or spacelike surfaces. To account for these, the above action needs to be supplemented with additional terms (as denoted by the ellipsis.. ) to be well-defined. These corner terms have been studied in [43-45]. Due to the boost symmetry outside the horizon, the action is time-independent there. The entire time dependence in the action for late time comes from the region inside the future horizon. The rate of change of the action is surprisingly simple and is given by the mass of the black hole as

$$
\begin{equation*}
\frac{d C}{d t}=\frac{d S\left[W_{t}\right]}{d t}=2 M \tag{1.24}
\end{equation*}
$$

Both the CV and CA proposals of holographic complexity are supported by various arguments justifying them to be bulk duals of complexity. They are probes of the deep black hole interior and grow linearly for very long times, an essential feature of complexity. What is even more remarkable is that they possess exactly the right behaviour under perturbations that is required for a good definition of complexity. We know that complexity exhibits a universal time delay under a perturbation, known as the switchback effect. Such a perturbation in the bulk can be modelled by sending in shockwaves into the black hole. Computing either the volume or the action in the shockwave geometry exhibits the exact universal time delay, showing that these proposals satisfy the switchback effect. This is a strong check of these conjectures, placing them on a firmer footing.

Nevertheless, there are many things about holographic complexity that need to understood more clearly. There are ambiguities in the definitions of holographic complexity. In the CV proposal there is a need of an undetermined length scale $l$ such that the ratio of volume and $G_{N} l$ is dimensionless. The CA proposal at first looks cleaner without the requirements of undetermined parameters, but there are
ambiguities depending on how we choose the boundary terms in the action. It has also been shown that there exists a huge class of geometric bulk quantities that share the same qualitative behaviour as of the CV or CA proposals.

In the next two chapters, using the combination of the ideas of path integral optimization and holographic $T \bar{T}$ we will study costs of bulk spacetime regions. These regions of spacetime can be interpreted as a quantum circuit that map between boundary states at different times and cutoffs. We will perform a thorough analysis of various geometric cost proposals and relate existing holographic complexity proposals to the optimization of such costs.


## Cost Proposals for Holographic Path Integrals

## Contents

2.1 Introduction ..... 19
2.2 Path integrals between geometric states ..... 20
2.2.1 Holography at a finite cutoff and $T \bar{T}$ ..... 20
2.2.2 Bulk path integrals ..... 24
2.2.3 Path integral cost and holographic state complexity ..... 28
2.3 Holographic path integral cost proposals ..... 30
2.3.1 Path integral cost ..... 31
2.3.2 Physical properties of path integral cost ..... 33
2.3.3 The space of all proposals: from boundary path inte- grals to functions on bulk subregions ..... 35
2.3.4 Reducing the space of cost proposals ..... 37

### 2.1 Introduction

How complex are states in semiclassical gravity? What is the least complex way of evolving from one state to another via a path integral? To address these questions we will first discuss how a given semiclassical bulk state in AdS gravity can be the solution to not one but a continuous family of mixed boundary conditions at the asymptotic boundary, and so have representations in many deformed holographic CFTs. Next, since this is a fine-grained description and the costs and complexities of these states are UV divergent, we will describe how precisely to coarse-grain the holographic theories to get UV-finite results. Lastly we explain in what sense we can 'optimise' the path integral between bulk states.

We will momentarily give a more precise description for the case of pure $\mathrm{AdS}_{3}$ gravity, but first give a more heuristic picture of the general situation. Given a semiclassical bulk configuration which is asymptotically $\operatorname{AdS}$, it is a priori not
yet clear in what theory this configuration describes a state, as we have not yet specified the boundary conditions. In other words, we have not yet defined which degrees of freedom fluctuate and which ones are kept fixed (i.e. what are the sources and what are the expectation values of operators), or equivalently, we have only provided one bulk configuration rather than the full phase space of solutions.

The standard choice would be to impose standard asymptotic AdS boundary conditions where the bulk configuration would correspond to a state in the dual CFT. We can however also make other interesting choices. Here, we will be interested in imposing Dirichlet boundary conditions on a timelike hypersurface in the bulk (for Lorentzian spacetimes). Linearized on-shell fluctuations around this background which preserve the Dirichlet boundary conditions will have a mix of nonnormalizable and normalizable modes turned on near the AdS boundary, which should therefore be interpreted as belonging to a deformed CFT with sources for multi-trace operators turned on. At the linearized level, one only finds doubletrace deformations, and if the backreaction of the matter fields on the geometry can be neglected one only finds a double-trace deformation for the stress-tensor which looks like a $T \bar{T}$ deformation with a space-time dependent source. Including the backreaction of matter will generate other double-trace deformations which involve other operators in the CFT. Going beyond the linearized approximation, one will generically encounter higher-trace deformations as well. A more precise analysis would consider the full non-linear phase space of solutions with the relevant Dirichlet boundary conditions, but we do not expect this phase space to have simple asymptotics at infinity, as at the non-linear level the irrelevant higher-trace deformations will generally lead to solutions which are not asymptotically AdS. This full non-linear analysis is in general intractable, but luckily the situation in pure $\mathrm{AdS}_{3}$ is more favorable and we can make some of these statements more precise as we will do next. The reader should keep in mind though that ultimately we are interested in the more general situation sketched here.

### 2.2 Path integrals between geometric states

### 2.2.1 Holography at a finite cutoff and $T \bar{T}$

We consider path integrals bulk theories which are holographically dual to $T \bar{T}$ deformed CFTs. We start with a review of holographic $T \bar{T}$ in order to understand the precise holographic map between bulk and boundary path integrals. First we follow the perspective and presentation of [46], that $T \bar{T}$-deformed holographic CFTs are UV-complete but non-local field theories, and that they are dual to gravity in asymptotically AdS spacetime, i.e. whose bulk slices have infinite volume, with mixed boundary conditions at infinity. In the next subsection we discuss the
coarse-grained descriptions of both sides: gravity with Dirichlet boundary conditions inside a finite cutoff surface, and the $T \bar{T}$ low energy effective theory.

## Fine-grained description

For concreteness and simplicity we consider pure three dimensional semiclassical Einstein gravity in AdS with all bulk matter background fields in their vacuum configurations. In this case the holographic $T \bar{T}$ deformation, i.e. the deformation which brings in the bulk Dirichlet boundary conditions to finite cutoff, is the original $T \bar{T}$ deformation as studied by Zamolodchikov [47,48]. As alluded to above, our discussion generalises to other dimensions and with non-trivial bulk background fields turned on with only a modification to the boundary field theory deformation [49-52]. The most general asymptotically $\mathrm{AdS}_{3}$ metric solving Einstein's equations can be written in Fefferman-Graham gauge as

$$
\begin{array}{r}
d s^{2}=G_{\mu \nu} d x^{\mu} d x^{\nu}=\frac{L^{2}}{4 \rho^{2}} d \rho^{2}+G_{i j}(\rho, x) d x^{i} d x^{j} \\
G_{i j}(\rho, x)=\frac{G_{i j}^{(0)}(x)}{\rho}+G_{i j}^{(2)}(x)+\rho G_{i j}^{(4)}(x) \tag{2.1}
\end{array}
$$

Here, $\rho$ is a radial coordinate with the asymptotic boundary at $\rho=0^{1}$. For $\mathrm{AdS}_{3}$, the Fefferman-Graham expansion ends at $G^{(4)}$, and on-shell $G^{(4)}$ is fully determined by $G^{(2)}$ and $G^{(0)}$ :

$$
\begin{equation*}
G_{i j}^{(4)}=\frac{1}{4} G_{i k}^{(2)} G^{(0) k l} G_{l j}^{(2)} \tag{2.2}
\end{equation*}
$$

In the standard AdS/CFT dictionary, Dirichlet boundary conditions are imposed by holding $G^{(0)}$ fixed on a UV cutoff surface such as $\rho=\epsilon$. The subleading metric falloff $G^{(2)}$ maps to the stress tensor one-point function in the dual field theory and, up to conservation and tracelessness, is unconstrained. Now consider a particular fixed asymptotically $\mathrm{AdS}_{3}$ spacetime, with metric denoted by $G$. This fixed metric $G$ is the solution to Einstein's equations for a set of not one but many different boundary conditions at the asymptotic boundary. A special set of such boundary conditions parametrised by a variable $\lambda$ holds fixed at $\rho=\epsilon$ the combination

$$
\begin{equation*}
G_{i j}^{(0)}-\frac{\lambda}{4 \pi G_{N} L} G_{i j}^{(2)}+\left(\frac{\lambda}{4 \pi G_{N} L}\right)^{2} G_{i j}^{(4)} . \tag{2.3}
\end{equation*}
$$

The boundary conditions (2.3) are mixed; they hold a combination of the metric and its derivatives fixed. As an example of how a given metric can be the solution to many different boundary conditions take $G$ equal to be pure $\mathrm{AdS}_{3}$, for which $G^{(2)}=G^{(4)}=0$ and $G_{i j}^{(0)}=\eta_{i j}$. This metric $G$ satisfies the mixed boundary conditions (2.3) for all $\lambda$, if the combination is fixed to $\eta_{i j}$.

[^1]The mixed boundary conditions (2.3) for the bulk metric at the asymptotic boundary are special [46]. In the bulk this is because they are equivalent to Dirichlet boundary conditions at finite radial cutoff, i.e. holding fixed the induced metric $G_{i j}\left(\rho_{c}, x\right)$ on a finite radial cutoff surface

$$
\begin{equation*}
\rho_{c}=-\frac{\lambda}{4 \pi G_{N} L} \tag{2.4}
\end{equation*}
$$

This can be easily checked from the Fefferman-Graham expansion (2.1). Dirichlet boundary conditions at finite cutoff are interesting because then the physics in the interior of the cutoff surface is an effective description of semiclassical gravity in a spatial volume that is finite, unlike AdS [48]. The bulk gravitational action evaluated in the whole bulk does not depend on $\lambda$, but the set of on-shell metric configurations does. Since the mixed metric boundary conditions at the asymptotic boundary are equivalent to Dirichlet boundary conditions with metric equal to that which $G$ induces on a finite radial cutoff surface, the gravitational theories with these different boundary conditions all by construction include $G$ amongst their on-shell metric configurations, but in general have little other overlap in on-shell metric configurations.

In the dual holographic field theory, it is an old story that replacing Dirichlet with mixed boundary conditions for the bulk metric at infinity maps to deforming the original undeformed field theory with a double trace deformation [53]. The particular double-trace deformation that the mixed boundary conditions (2.3) corresponds to is the $T \bar{T}$ deformation. Specifying a bulk metric $G$ and mixed boundary condition parameter $\lambda$ is sufficient to fix the dual $T \bar{T}$-deformed holographic CFT living on a manifold $M$. The field theory background metric on $M$ is most simply expressed in terms of the metric induced by $G$ on the $\rho=\rho_{c}$ surface denoted $\tilde{M}$ :

$$
\begin{equation*}
\gamma_{i j}(x)=\rho_{c} G_{i j}\left(\rho_{c}, x\right)=\left.\rho_{c} G\right|_{\tilde{M}} \tag{2.5}
\end{equation*}
$$

The metric on $M$ is $\gamma$, and the metric on $\tilde{M}$ is $g$. Note that, from this map between metrics, a subregion of $M$ is empty if and only if the corresponding subregion of $\tilde{M}$ is also empty. In general $\gamma$ has a non-trivial $\lambda$ dependence. The action of the deformed theory is the solution to the flow equation

$$
\begin{equation*}
\frac{d}{d \lambda} S^{(\lambda)}=\int d^{2} x \sqrt{\gamma} \mathcal{O}_{T \bar{T}}^{(\lambda)} \quad \text { with } \quad \mathcal{O}_{T \bar{T}}^{(\lambda)}:=-\frac{1}{2}\left(\gamma_{i k} \gamma_{j l}-\gamma_{i j} \gamma_{k l}\right) T^{i j} T^{k l} \tag{2.6}
\end{equation*}
$$

The seed action is that of the undeformed CFT, $S^{(0)}=S_{C F T}$, which is dual to the gravitational theory with asymptotic Dirichlet boundary conditions. The deforming operator $\mathcal{O}_{T \bar{T}}^{(\lambda)}$ has a $\lambda$ dependence because the stress tensor changes as the action flows. From the bulk gravitational perspective, what is special about the $T \bar{T}$ deformation in the dual holographic CFT is that the corresponding mixed
boundary conditions are equivalent to Dirichlet boundary conditions on a finite radial surface.

## Coarse-grained description

Here we introduce the coarse-grained version of holographic $T \bar{T}$ : the effective description of gravity within a finite box, and the EFT of a $T \bar{T}$ deformed CFT. Consider a gravitational partition function with Dirichlet boundary conditions $Z_{\text {grav }}[g]$, which depends on the value of the induced metric $g$ on its boundary. As a consequence of diffeomorphism invariance, this dependence is constrained by (a radial version of) the Wheeler-DeWitt (WDW) equation [54]

$$
\begin{equation*}
H_{W D W} Z_{\text {grav }}[g]=0, \tag{2.7}
\end{equation*}
$$

where $H_{W D W}$ is the WDW Hamiltonian ${ }^{2}$

$$
\begin{equation*}
H_{W D W}=g_{i j} \frac{\delta}{\delta g_{i j}}+\frac{1}{\sqrt{g}}\left(g_{i k} g_{j l}-\frac{1}{2} g_{i j} g_{k l}\right) \frac{\delta}{\delta g_{i j}} \frac{\delta}{\delta g_{k l}}+\sqrt{g} R . \tag{2.8}
\end{equation*}
$$

Formally this equation can be solved to relate gravitational partition functions with different metric boundary conditions:

$$
\begin{equation*}
Z_{\text {grav }}[g]=\int D \tilde{g} \mathcal{K}[g, \tilde{g}] Z_{\text {grav }}[\tilde{g}] \tag{2.9}
\end{equation*}
$$

The kernel can be calculated for any bulk field content, including with matter $[56,57]$, but we won't need the precise form for our discussion. The initial data we want to use when solving (2.9) is the gravitational partition function with AdS Dirichlet boundary conditions, i.e. the undeformed CFT generating functional. Suppose we take an asymptotically $\operatorname{AdS}$ metric $G$ and set $g$ equal to the metric induced on a radial cutoff surface $\left.\tilde{M}(\lambda)\right|_{G}$ parametrised by $\lambda$. With (2.9) we have an effective description $Z_{\text {grav }}\left[g=\left.G\right|_{\tilde{M}(\lambda)}\right]$ of gravity with Dirichlet boundary conditions that fix the induced metric equal to the metric induced on $\tilde{M}(\lambda)$ by $G$. The effective description 'throws away' everything outside the cutoff surface. $Z_{\text {grav }}\left[g=\left.G\right|_{\tilde{M}(\lambda)}\right]$ has the geometry of $\operatorname{Int}(\tilde{M}) \cap G$ as one allowed onshell metric configuration among many. Other on-shell metric configurations differ at the boundary in their normal derivatives, corresponding to different one-point functions in the dual cut-off theory.

Gravity with a Dirichlet metric boundary condition $g=\left.G\right|_{\tilde{M}(\lambda)}$ has a dual nongravitational description as the effective theory of a $T \bar{T}$-deformed CFT on a manifold $M$ with degrees of freedom above the scale set by $\lambda$ integrated out [48,50].

[^2]The map between EFT generating functional and gravitational partition function is

$$
\begin{equation*}
Z_{E F T}^{(\lambda)}(\gamma)=Z_{\text {grav }}\left[g=\left.G\right|_{\tilde{M}(\lambda)}\right] \tag{2.10}
\end{equation*}
$$

This is a continuous family of EFT's parameterised by $\lambda$ with non-dynamical background metrics $\gamma$.

What is the map from the bulk metric boundary condition $g$ to quantities in the dual non-gravitational EFT? The metric $g$ fixes the conformal class of the field theory background metric:

$$
\begin{equation*}
\gamma_{i j}(x)=\lambda^{2 / d}(x) g_{i j}(x) \tag{2.11}
\end{equation*}
$$

with the deformation parameter related to the radial coordinate in FeffermanGraham gauge by $\lambda(x) \propto-\rho_{c}(x)^{d / 2}$ [56]. In general we can have a non-constant deformation, corresponding to a non-constant radial cutoff. The holographic $T \bar{T}$ deformation in dimensions other than two and with sources turned on is defined by the property that it is dual to a bulk gravitational theory with Dirichlet conditions at finite cutoff. The $T \bar{T}$ deformed non-gravitational effective action is formally the solution to the flow equation

$$
\begin{equation*}
\frac{\delta}{\delta \lambda(x, t)} \log Z_{E F T}^{(\lambda)}(\gamma)=\int_{M} d^{d} x\left\langle X^{(\lambda, \gamma)}\right\rangle \tag{2.12}
\end{equation*}
$$

This flow equation, including the precise form for $X$, is derived by mapping the gravitational WDW equation (2.7) to an exact RG equation in the field theory [48, $50,56]$. Schematically, first we take the semiclassical limit where the gravitational partitional function is given by the on-shell gravitational action. Then, functional variations $\delta_{g} Z_{\text {grav }}[g]$ with respect to $g$ become functions of the Brown-York stress tensor of the on-shell action, and in the boundary effective action this maps to deformations by the field theory stress tensor. In the special case of 2 d , with a constant deformation parameter $\lambda$, and with no sources except a flat background metric turned on, the deformation is the standard $T \bar{T}$ operator, $X=\mathcal{O}_{T \bar{T}}$ defined in (2.6).

### 2.2.2 Bulk path integrals

We have a one-to-many map from a Cauchy slice of fixed bulk spacetime $G$ to a geometric state in the Hilbert spaces of not one but a continuous family of $T \bar{T}$ deformed holographic CFTs. Each mixed boundary condition for which $G$ is an on-shell solution maps to a different $T \bar{T}$ deformed theory. The whole of $G$ maps to the causal development of these geometric states. We thus have many field theoretic descriptions for the causal evolution of a spatial slice of $G$. This leads to
the key question we want to focus on: which of the deformed CFTs describes the evolution of the bulk state in the least 'complex' way?

To be concrete, let us focus on transition amplitudes between geometric states with metric and conjugate momentum induced by a given fixed $G$ on two arbitrary Cauchy slices $\Pi_{1}$ and $\Pi_{2}$, see figure 2.1. By construction $G$ is the dominant spacetime saddlepoint contribution in the semiclassical approximation to the transition amplitude between those two geometric states. This transition amplitude has a formal path integral representation

$$
\begin{equation*}
\left\langle\psi_{2} \mid \psi_{1}\right\rangle=\int D \tilde{G} e^{i S_{g r a v}[\tilde{G}]} \delta\left(\left.\tilde{G}\right|_{\Pi_{1}}-\left.G\right|_{\Pi_{1}}\right) \delta\left(\left.\tilde{G}\right|_{\Pi_{2}}-\left.G\right|_{\Pi_{2}}\right) \tag{2.13}
\end{equation*}
$$

The $\delta$-functions are the wavefunctions of the initial and final states in the basis of geometric states, and they in effect impose spacelike boundary conditions on the path integral. We also implicitly impose the timelike boundary condition (2.3). Finding the classical gravitational solution for these boundary conditions is not an inconsistent overdetermined problem because $G$ is by construction a solution. $S_{\text {grav }}$ is the Einstein-Hilbert action including, if necessary, boundary, corner, and holographic counterterms [58,59].

We would like to interpret the bulk path integral (2.13) and its dominant saddle point $G$ as originating from the continuum limit of a tensor network representation of the transition amplitude, and associate a cost, i.e. a properly understood regularised number of gates, to that tensor network. Keeping $G$ fixed and changing the asymptotic boundary conditions does not affect the gravitational transition amplitude (2.13) in the saddle-point limit ${ }^{3}$, but are the costs of the associated path integrals the same? We propose that the 'fundamental gates' the gravitational theories use are in fact different, and so the cost of their path integrals even those that prepare the same final state from a given initial state - may be different. This is backed up by the fact that each mixed boundary condition maps to a field theory with different $T \bar{T}$-deformation parameter, which at least naively have different operators and gates available to them [46, 60, 61]. This introduces a $\lambda$ dependence to the 'cost' of $G$, which we can minimise over in our pursuit of finding the least complex way of evolving between two geometries.

Both $T \bar{T}$-deformed CFTs and gravitational duals are thought to be UV complete [46, 48, 60, 62], so we expect UV divergences in state complexity and the cost of preparing or evolving between states. This is associated to the infinite volume of all spatial slices of all asymptotically AdS spacetimes. To regulate we need to coarsegrain and consider effective theories with a cutoff. In principle we could choose

[^3]

Figure 2.1: Coarse-grained vs fine-grained bulk path integrals. Left: A path integral representation of the transition amplitude between two compact manifolds $\Sigma_{1}$ and $\Sigma_{2}$ with specified metrics. The gravitational path integral has Dirichlet boundary conditions g. Right: Embedding in a larger UV complete theory of gravity in AdS. Fix a bulk metric $G$, which is a solution to a family of mixed boundary conditions at the asymptotic boundary parameterised by $\lambda$. The mixed boundary conditions are on-shell equivalent to Dirichlet boundary conditions on surface $\tilde{M}(\lambda)$. The light blue surfaces are Cauchy slices $\Pi_{1}$ and $\Pi_{2}$, which $\Sigma_{1}$ and $\Sigma_{2}$ are compact subregions of.
any UV cutoff as our regulator, but $T \bar{T}$-deformed theories and their generalizations come with a natural effective UV scale, which is the scale above which the theory becomes approximately non-local. It is therefore very natural to integrate out all degrees of freedom above this scale, but keep the degrees of freedom below this scale, as the same scale is also expected to set the effective size of the smallest possible tensor in a tensor network description. In the $T \bar{T}$ example discussed above the relevant scale is set by the deformation parameter $\lambda$. In the bulk, there is a natural dual description of this UV cutoff, namely the existence of the Dirichlet surface $\tilde{M}(\lambda)$. In effect $\tilde{M}(\lambda)$ splits the bulk spacetime $G$ into an exterior UV region, and an interior IR region, and we 'integrate out' the exterior. This is an important point so let us emphasize this once more: The bulk Dirichlet surface defines both the boundary conditions for the dual field theory, as well as the UV cutoff in that theory.

We will make this more precise: The path integral depicted by figure 2.1 is the transition amplitude between two spatial geometries on $\Sigma_{1}$ and $\Sigma_{2}$ in an effective gravitational theory $Z_{\text {grav }}[g]$ which holds the boundary metric fixed to $g$. This can be embedded in a UV complete theory of gravity in AdS if there exists a $G$ and a set of asymptotic boundary conditions parametrised by $\lambda$ such that $g=\left.G\right|_{\tilde{M}(\lambda)}$, and $\Sigma_{1}$ and $\Sigma_{2}$ can be embedded in Cauchy slices of $G$, i.e. the metrics on $\Sigma_{1}$ and $\Sigma_{2}$ equal the metrics $G$ induces on $\Sigma_{1} \subset \Pi_{1}$ and $\Sigma_{2} \subset \Pi_{2}$. Then by construction the dominant saddlepoint geometry in the effective theory coincides with $G$ restricted to the interior of $\tilde{M}(\lambda)$. To 'optimise' the path integral we keep $G$ and the compact
submanifolds $\Sigma_{1}$ and $\Sigma_{2}$ fixed, and vary $\lambda$. Pictorially, this amounts to varying the shape of the red surface in the left half of figure 2.1 while keeping on-shell interior geometry equal to the geometry inside $\tilde{M}(\lambda)$ in the right half of the figure. This gives us a continuous family of gravitational path integrals that prepare the same final bulk state $\left|\Sigma_{2}\right\rangle$ from a given reference state $\left|\Sigma_{1}\right\rangle^{4}$. Note that to keep $\Sigma_{1}$ and $\Sigma_{2}$ fixed and $\lambda$-independent we need the intersection of $\tilde{M}(\lambda)$ with $\Pi_{1}$ and $\Pi_{2}$ to be $\lambda$-independent, which in general requires $\tilde{M}(\lambda)$ to have non-constant radius, which in turn requires a non-constant $\lambda$ parameter [56].

We have given the explicit map between the bulk metric $g$ on $\tilde{M}$ and the background field theory metric $\gamma$ on $M$ in (2.11). If we wish we can use the same coordinates $x^{i}$ on both manifolds. By extension, given a bulk path integral between Cauchy slices as depicted in figure 2.1 we know how to map between intersections of those bulk Cauchy slices with $\tilde{M}$ and boundary Cauchy slices of $M$. From this we have a complete and precise holographic duality between a given path integral in an effective gravitational theory and a path integral in a $T \bar{T}$-deformed CFT. In a slight abuse of notation, when discussing path integrals on the subregions of $M$ and $\tilde{M}$ between Cauchy slices, we will also call these subregions $M$ and $\tilde{M}$. While in the remaining part of the chapter we will not make an explicit use of the above discussion about holographic $T \bar{T}$, we believe it can be taken as a starting point in building a bridge between gravitational notions of cost functions and operatorial expressions for circuits living on $\tilde{M}(\lambda)$.

We consider bulk path integrals in both Lorentzian or Euclidean signature. In some ways they are similar: for both signatures the path integrals on manifolds with one boundary prepare states, and when there are two boundaries the path integrals calculate transition amplitudes. Lorentzian and Euclidean bulks differ however in the possible metric signatures of their embedded surfaces. For Euclidean bulks all surfaces $\tilde{M}$ are spacelike. Path integrals on $\tilde{M}$ with two boundaries correspond to unnormalised density matrices,

$$
\begin{equation*}
\rho=P e^{-\int d \tau H(\tau)} \tag{2.14}
\end{equation*}
$$

with infinite Euclidean time evolution preparing the projection operator onto the vacuum state $\rho=|0\rangle\langle 0|$. For Lorentzian bulks the embedded surface $\tilde{M}$ can be spacelike, timelike, or even non-constant signature. Since the signature of $\tilde{M}$ is the same as $M$, as follows from (2.11), in the $T \bar{T}$ deformed boundary theory a timelike $\tilde{M}$ corresponds to a Lorentzian path integral while spacelike gives Euclidean ones. States on time-reflection symmetric slices in Lorentzian AdS spacetimes can be prepared by a Euclidean path integral, while other states cannot and need some Lorentzian time evolution.

[^4]We consider path integral cost proposals for both Euclidean and Lorentzian bulks. A cost proposal that is physically reasonable, e.g. non-negative, in one signature need not be in the other, so part of specifying a cost proposal is saying whether it is applicable to Lorentzian or Euclidean bulks, or both. When we come to giving specific cost proposals we will always specify which metric signature it is applicable to. The question as far as path integral optimisation of state preparation goes is which set of $\tilde{M}$ one is minimising cost over. For Lorentzian bulks if one wishes to find the shortest path in a space of unitary operators, then one should restrict to only timelike $\tilde{M}$. Spacelike components of $\tilde{M}$ in Lorentzian bulks that are achronal with respect to each other and $\Sigma_{1}$ and $\Sigma_{2}$ can be thought of as extensions of the partial Cauchy slices on which the initial and final states are defined. Whether they are part of the past or future slice depends on the time-orientation of their normals. The states on $\Sigma_{1}$ and $\Sigma_{2}$ are then reduced state with respect to the larger slices. We will only consider timelike $\tilde{M}$ in Lorentzian bulks.

One last closing comment is in order. In the present work we start with an asymptotically AdS geometry and identify in it the cut-off surface $\tilde{M}(\lambda)$ and its interior, as in figure 2.1 (right). Starting with the situation depicted in figure 2.1 (left), it is a priori not clear if it can be embedded in an asymptotically (perhaps locally) AdS geometry. There are several reasons for it. One is a possible issue of singularities arising as one tries to extend the geometry towards infinity and another are matter fields that may enforce non-AdS asymptotics. A special case is pure gravity with negative cosmological constant in three bulk dimensions, in which case the geometry is guaranteed to be a portion of the $\mathrm{AdS}_{3}$ manifold.

### 2.2.3 Path integral cost and holographic state complexity

Suppose we have a prescription for associating a cost to the gravitational path integral depicted in the left half of figure 2.1. Heuristically, this cost denoted $\mathscr{C}(\tilde{M}(\lambda))$ 'counts' the number of gates in different spacetime tensor networks parametrised by $\lambda(x, t)$. Each $\lambda(x, t)$ defines a different path integral that maps the same fixed initial state $\left|\Sigma_{1}\right\rangle$ to final state $\left|\Sigma_{2}\right\rangle$, and a given one of these bulk path integrals will not correspond to the least complex circuit between those states, which is why we are discussing cost rather than complexity.

There is however a sense in which the bulk path integrals can be optimised, by minimising path integral cost $\mathscr{C}(\tilde{M}(\lambda))$ over $\lambda$. Recall that we keep $G, \Sigma_{1}$ and $\Sigma_{2}$ fixed, so the set of $\lambda$ to minimise over are those for which

$$
\begin{equation*}
\partial \tilde{M}(\lambda)=\partial \Sigma_{1} \cup \partial \Sigma_{2} \tag{2.15}
\end{equation*}
$$

In some cases the minimal path integral cost can be interpreted as state complexity, and this allows us to connect holographic path integral cost to holographic state
complexity. The subtlety is in which set of path integrals it is meaningful to compare.

Stepping back for a moment, when does it make sense to optimise path integrals? Path integrals in a fixed seed field theory but allowing for different background geometries, field theory sources and field theory deformations define different operators, and it is not meaningful to compare the costs of the path integrals. Minimising cost over all such path integrals cannot meaningfully be interpreted as 'optimisation' of state preparation, if indeed they even act on the same Hilbert spaces. The key idea is that there are a set of bulk path integrals it is meaningful to optimise: those that take a given initial state to the same final state.

To connect path integral cost to state complexity, we also need to choose a suitable initial state, such that the final state is prepared from 'nothing'. In the complexity literature this is often taken to be a spatially unentangled product state, but this does not have an approximate semiclassical description, so we cannot use it. Our initial state takes $\Sigma_{1}$ to be a bulk point. It is helpful to think of this as limit of a small spatial region placed deep in the IR of the asymptotically AdS spacetime, which we shrink to zero volume, see figure 2.2 . As $\tilde{M}(\lambda)$ moves outwards from the deep IR towards $\partial \Sigma_{2}$, the Hilbert space of the effective gravitational theory grows to non-trivial size. In the dual boundary theory this initial state lives in a theory where we have taken the $T \bar{T}$ deformation parameter $\lambda \rightarrow \infty$. This sets the effective RG scale to zero, and the effective theory integrates out everything above that scale which leaves a trivial theory.

The bridge from path integral cost to state complexity is still not complete. Even the optimum path integral may not correspond to the shortest path in the space of states, because not all unitary operators are generated by the set of path integrals we have considered. This means that we only expect the cost of the optimum path integral to still only upper bound the complexity of state $\left|\Sigma_{2}\right\rangle$ :

$$
\begin{equation*}
\min _{\lambda} \mathscr{C}(\tilde{M}(\lambda)) \geq C\left(\left|\Sigma_{2}\right\rangle\right) \tag{2.16}
\end{equation*}
$$

It is not clear when if ever this inequality is saturated, i.e. when if ever the least complex unitary operator taking $\left|\Sigma_{1}\right\rangle$ to $\left|\Sigma_{2}\right\rangle$ has a path integral representation.

The semiclassical bulk path integrals and their associated costs are defined on a specified and fixed bulk geometry, but the set of path integrals we want to optimise over do not necessarily have the same background geometry. We can choose to keep the bulk fixed and optimise over the set of path integrals on subregions of that single fixed bulk defined by their boundary $\Sigma_{1} \cup \tilde{M}(\lambda) \cup \Sigma_{2}$. A larger and more natural superset allows for different bulk geometries $G$, as well as different $\tilde{M}(\lambda)$, that also keep the initial and final states $\left|\Sigma_{1}\right\rangle$ and $\left|\Sigma_{2}\right\rangle$ fixed. When we


Figure 2.2: A fine-grained description of the preparation of a bulk state from 'nothing'. The initial state lives on a surface $\Sigma_{1}$ embedded deep in the $I R$ of an asymptotically AdS spacetime. This corresponds to a state in a $T \bar{T}$-deformed CFT with very large deformation parameter. We take this to be the reference state when defining the state complexity of the final state on $\Sigma_{2}$.
come to connecting to holographic state complexity proposals we will choose, for the sake of simplicity, path integral cost proposals for which we can show that the minimal cost is the same for either set.

This concludes our discussion of transition amplitudes between bulk states. We have discussed what Hilbert spaces the bulk states live in, what the exact map is from bulk theory and state to $T \bar{T}$-deformed CFT and state, constructed a set of path integrals we can 'optimise' over that prepare the same state from an initial state, how to coarse-grain the description on both sides, and how to prepare a state from nothing. Next we will consider proposals for the cost $\mathscr{C}(\tilde{M}(\lambda))$ of the coarse-grained bulk path integrals, which gives us a quantity to minimise and so optimise.

### 2.3 Holographic path integral cost proposals

In AdS/CFT the boundary path integral defines a (Lorentzian/Euclidean) timeevolution operator. The goal of this section is to consider holographic proposals for the cost of the path integral, i.e. the length of the path through the space of
operators.

### 2.3.1 Path integral cost

Let us start by defining what we mean by path integral cost. This subsection is mostly field theoretic and logically separate from gravity and holography. We may associate an operator $U_{M}$ to the path integral on any manifold $M$ with two boundaries and unspecified boundary conditions, see figure 2.3. Matrix elements of the operator are defined by the path integral with specified boundary conditions for the fields of interest $\phi=\phi_{1,2}$ on the two boundaries, which computes a transition amplitude:

$$
\begin{equation*}
\left\langle\phi_{2}\right| U_{M}\left|\phi_{1}\right\rangle:=\int_{\phi=\phi_{1}}^{\phi=\phi_{2}} D \phi e^{-S[\phi]} . \tag{2.17}
\end{equation*}
$$

We are interested in both Euclidean and Lorentzian path integrals. For the latter there is an insertion of $i$ in the path integral representation of the transition amplitude. We denote the operator defined in (2.17) by $U_{M}$ whether or not the operator is unitary, i.e. even if the path integral is Euclidean. $U_{M}$ depends not only on the geometry of $M$, but on the field theory itself. We take the field theory to be holographic and consider not only changes to the background geometry but also allow the addition of sources and deformations to the theory. When we come to embedding $M$ in a bulk spacetime, these sources and deformations will correspond to adding bulk excitations and bringing in the boundary to finite cutoff with $T \bar{T}$ deformations. From the context we hope it is clear whether $M$ refers only to the manifold, or to all the data required to define the path integral including the manifold, seed theory, sources, and theory deformations.

We define path integral cost $\mathscr{C}$ as the length of the path generated by time evolution from the identity operator to $U_{M}$ in a metric space of operators. For the path integral cost to be well-defined the metric space must be specified: both which set of operators to include and which metric to impose on it. Expressed in the terminology of Nielsen's geometric formulation [28], cost functions determine the metric on the set of operators, and control functions specify paths in that metric space. The path integral generates a path from $\mathbb{1}$ to $U_{M}$ which is not generally geodesic, and so the path integral cost upper bounds the operator complexity of $U_{M}$, because that is defined as the length of the shortest path between $\mathbb{1}$ and $U_{M}$. An example illustrating the difference between path integral cost and operator complexity is given in figure 2.4. The operator complexity of $U_{M}$ itself upper bounds the state complexity of the state $|f\rangle=U_{M}|i\rangle$, because state complexity can be defined as operator complexity minimised over those unitary operators that map a fixed reference state $|i\rangle$ to the target state $|f\rangle$. The path integral operator $U_{M}$ does not in general correspond to the shortest path between $|i\rangle$ and $|f\rangle$,


Figure 2.3: Left: $M$ is a Euclidean or Lorentzian manifold with two boundaries from which operators can be defined. Right: The space of all such operators, which are unitary when $M$ is Lorentzian. $U_{M}$ is the operator whose matrix elements are calculated by the path integral on $M$.
which is why its complexity is only an upper bound to the state complexity. These statements can be summarised as

$$
\begin{equation*}
\mathscr{C}(M) \geq C\left(U_{M}\right) \geq C(|f\rangle) \tag{2.18}
\end{equation*}
$$

where $\mathscr{C}(M)$ is the path integral cost of $M, C\left(U_{M}\right)$ the operator complexity of $U_{M}$ and $C(|f\rangle)$ the state complexity of $|f\rangle=U_{M}|i\rangle$. Note the calligraphic font that distinguishes cost $\mathscr{C}$ from complexity $C$.

Note that the same operator $U_{M}$ can be represented in an infinite number of ways as a circuit in physical time upon picking a time foliation, see figure 2.5. From the circuit perspective, constant time slices can be thought of as layers of the circuit, and these different time foliations as different ways of assigning gates amongst the layers. The circuit as a whole is independent of its time foliation, and this is a physical reason for why the cost should be foliation independent. This we impose on the bulk side of the holographic proposal through a covariance requirement. On the boundary side one implication is that the lengths of the red, blue and other paths in figure 2.5 from $\mathbb{1}$ to $U_{M}$ are the same. Hence, while in general two randomly selected paths connecting $\mathbb{1}$ and $U_{M}$ will have different lengths (due to describing physically different circuits), each such path will come with an equivalence class of paths of the same length that can be generated by changes in


Figure 2.4: An example illustrating the difference between path integral cost and operator complexity. Suppose we have a manifold $M$ such that Hamiltonian evolution from the initial to final boundary traces out a closed path through the space of unitary operators, i.e. a Poincaré recurrence with $e^{i H t_{f}} \approx \mathbb{1}$. The path integral cost is the length of the closed path, which is non-zero, while the complexity of the time evolution operator is trivially zero. This is an example where $\mathscr{C}(M) \neq C\left(U_{M}\right)$.
time-foliation. This is a symmetry of the metric space that is a consequence of the physical equivalence of different time foliations.

### 2.3.2 Physical properties of path integral cost

Necessary conditions that holographic proposals must satisfy in order to be reasonably interpreted as path integral cost, i.e. the length of a path in a metric space of operators, are the following:

1. The trivial path integral has zero cost.

When $M=\varnothing$ the path integral is trivial with no time evolution, the operator associated to it is the identity, and holographic proposals should evaluate to zero cost: $\mathscr{C}=0$. We can, and will, even strengthen this requirement by demanding that the trivial path integral is the only one that has zero cost.
2. Additivity.

Concatenating path integrals joins paths in the space of operators, and cost is the total length of the path, so cost is additive. This means that if we have $M$ and $M^{\prime}$ which share a boundary, then $\mathscr{C}\left(M \cup M^{\prime}\right)=\mathscr{C}(M)+\mathscr{C}\left(M^{\prime}\right)$. This distinguishes cost from complexity which is subadditive: $C\left(U_{M} \cdot U_{M^{\prime}}\right) \leq$


Figure 2.5: Left: $M$ is taken to be a Lorentzian cylinder, and the blue and red ellipses represent constant time slices of two different time foliations of the cylinder. Right: $U_{M}$ is the unitary operator whose matrix elements are calculated by the path integral on $M$. Different time foliations define different paths to $U_{M}$.

$$
C\left(U_{M}\right)+C\left(U_{M^{\prime}}\right) .
$$

## 3. Symmetry.

The length of a path traced through a metric space of operators from $A$ to $B$ is the same as from $B$ to $A$. This means that holographic proposals for path integral cost cannot depend on which way around the two connected components of $\partial M$ are labeled the 'initial' and 'final' boundaries.
4. Covariance.

The cost of a path integral on a manifold should be independent of the coordinates used to describe the manifold.
5. Non-negativity.

A discretised path integral is a circuit, and path integral cost is a measure of the number of gates in that circuit. This number cannot be negative so path integral cost must be non-negative.

The above are essential points in order to sensibly associate path integral cost to spacetime regions in holography. In relation with the properties of the existing holographic complexity proposals, one may also want to impose that for TFD states and their gravitational representation in terms of eternal black holes, optimal paths in our proposals give rise to late-time linear growth and switchback effect [63]. These effects are only conditions in Lorentzian setups and on TFD states, not on every state like the other requirements, so for us they are not as fundamental as items 1-5 above.

### 2.3.3 The space of all proposals: from boundary path integrals to functions on bulk subregions

We are looking for proposals for the gravitational dual of the cost of a holographic field theory path integral. Naturally it is equally valid to think of the path integral in the gravitational or non-gravitational description, and we have given the explicit map between the two, but quantities such as cost need not manifest the same way on the two sides. The set of cost proposals we consider take inspiration from existing holographic state complexity proposals and have two aspects in common: (1) a geometric map from the subregion of $M$ or $\tilde{M}$ on which the path integral is defined to bulk subregion $X_{M}$, and (2) a function $f\left(X_{M}\right)$ on that bulk subregion. These two shared aspects take inspiration from existing holographic state complexity proposals, which are also functions on bulk subregions.

We want to consider all such pairs of maps which together define a tentative holographic cost proposal:

$$
\begin{equation*}
\mathscr{C}(M)=f\left(X_{M}\right) . \tag{2.19}
\end{equation*}
$$

The set of cost proposals we start from contains an infinite number of ways of specifying $X_{M}$ given $M$, and an infinite number of functions $f$. Cost, the length of path in a metric space, obeys certain mathematical and physical properties, and we will see the extent to which the space of possible gravitational duals can be reduced by imposing these properties.

## Specifying the bulk subregion: $M \rightarrow X_{M}$

We want to work within a single fixed bulk spacetime. As discussed in section 2.1, fixing which mixed boundary conditions to use in the bulk theory fixes the deformation parameter $\lambda$ in the boundary theory, and fixing the bulk geometry $G$ fixes the actual deformation, the background field theory sources including the metric, and the boundary state including its causal evolution. Functions on bulk subregions of a fixed $G$ can only be dual to the cost of the boundary path integral that corresponds to Hamiltonian evolution of the boundary state dual to that bulk geometric state, rather than an arbitrary path integral in the boundary theory.

The two boundaries of $\tilde{M}$ have to be attached to the hypersurfaces $\Sigma_{1}$ and $\Sigma_{2}$ as in figure 2.1, which should be parts of bulk Cauchy slices and hence achronal. In section 2.1 these hypersurfaces were specified and fixed, but in this section where we start from the boundary theory with a specified fixed $M$ they are not unambiguously defined. For the bulk path integral the choice of $\Sigma_{1}$ and $\Sigma_{2}$ is as fundamental as the choice of $\tilde{M}$, but we may still wonder whether in a given prescription these can be defined according to some unambiguous and covariant rule. Some examples of ways to define hypersurfaces $\Sigma_{1}$ and $\Sigma_{2}$ that we attach to the boundaries of $\tilde{M}$ are, in increasing degree of generality,

1. Future/past directed null surfaces, as in the CA and CV2.0 proposals.
2. The extrema of some functional defined on the hypersurface, as in the CV proposal.
3. The solution to $\xi=0$, where $\xi$ is some function of the local intrinsic and extrinsic geometry. This need not be the extremum of any given functional.
4. The same as 3 ., except now allowing for non-local and global data in the definition. An example would be to define the hypersurfaces as the solution to $K_{\Sigma}=\operatorname{Vol}(\tilde{M})$.

This list includes the codimension-one surfaces appearing in holographic complexity proposals, and a large family of generalisations, though the list is not exhaustive.

Consider the bulk spacetime subregions $X_{M}$ that can be covariantly defined with respect to a codimension- 1 surface $\tilde{M}$. There are of course an infinite number of such prescriptions; let us first consider those that are similar in nature to the holographic complexity proposals. One natural candidate is the interior of $\Sigma_{1} \cup$ $\tilde{M} \cup \Sigma_{2}$ which is codimension-0 and which we label $N$, see figure 2.6. Codimension0 bulk regions are used in the CA and CV2.0 state complexity proposals. We can also take $X_{M}$ to be codimension- 1 with respect to the bulk. Natural candidates include $\tilde{M}, \Sigma_{1}$ or $\Sigma_{2}$, or any union of these. We will show later how taking $X_{M}=\tilde{M}$ and evaluating its volume gives a cost proposal that when 'optimised' reduces to the CV state complexity proposal.

These candidates for $X_{M}$ only scratch the surface of possibilities. At this stage there is nothing to favour one candidate over another; it is only when we impose the physical properties of cost that we can rule out possibilities.

## Functions on bulk subregions

With a specified bulk region $X_{M}$, we may propose the cost $\mathscr{C}(M)$ of the path integral on $M$ to be a function evaluated on that region: $f\left(X_{M}\right)$. To complete the cost proposal the function $f$ needs to be specified. Some examples in order of decreasing simplicity include the region's:

1. Volume,
2. Gravitational action, including the Einstein-Hilbert term on codimensionzero regions, the Gibbons-Hawking-York (GHY) boundary term on non-null codimension-one regions, Hayward-type corner terms on codimension-two regions, or appropriate terms on null-boundaries, see e.g. [45],
3. Local functionals of curvature invariants: $f=\int \sqrt{|g|} \xi\left(R_{\mu \nu \rho \sigma}, K_{m n}\right)$,


Figure 2.6: We are looking for holographic proposals for the cost of the path integral on $M . \tilde{M}, \Sigma_{1}, \Sigma_{2}$ and $N$ are representive of bulk subregions of various codimension that can be covariantly defined and on which functions can be evaluated as part of a holographic proposal.
4. More general local functionals: $f=\int \sqrt{|g|} \xi\left(R, R^{2}, R_{\mu \nu} R^{\mu \nu}, \ldots, K, K^{2}, \ldots, \phi, \ldots\right)$ where, for example, $\phi(x)$ could be some non-physical auxiliary scalar field that appears in the complexity proposal but not in the bulk Lagrangian,
5. Non-local functionals, e.g. $f=\int d^{d} x d^{d} y R(x) R(y)$,

Again, these candidates for functions on bulk subregions only scratch the surface of possibilities, and except for appeals to simplicity there is nothing to favour one candidate over another until we impose the physical requirements of computational cost.

### 2.3.4 Reducing the space of cost proposals

We are considering the set of holographic proposals for the cost of a boundary path integral, which consists of a map from the surface on which the path integral is defined to a bulk subregion, and a function on that subregion. Let us apply the physical requirements of section 2.3.2 to identify which functions on bulk subregions can be interpreted as cost.

1. Only the trivial path integral has zero cost.

This requires $f\left(X_{M}\right)=0$ when $M=\tilde{M}=\varnothing$. This rules out for example $\operatorname{Vol}(\tilde{M})$ for Lorentzian bulks, because we can have an $\tilde{M}$ that is non-empty but has zero volume because it is a null surface. The same cost proposal applied to Euclidean bulks is allowed.

This requirement fixes additive constants. It does not rule out any proposal of the form (2.20) if $X_{M}=N$ as long as the integrand is non-singular and $N \rightarrow 0$ in that limit. This rules out some $X_{M}$, such as $\Sigma_{1}$ being past-directed null and $\Sigma_{2}$ future-directed null.

## 2. Additivity.

Additivity does not rule out any proposal that is the integral of a local density, such as volume or gravitational action, as long as the contribution from $X_{M_{1}} \cap X_{M_{2}}$ vanishes. A non-trivial example of how this can occur is when $f$ includes GHY boundary terms, because if $X_{M_{1}}$ and $X_{M_{2}}$ share a boundary then the outward normal on one is the inward normal on the other, so the GHY terms cancel.

The gravitational action including GHY and corner terms is additive, if the joints are spacelike though generally not generally for timelike joints [43]. A codimension-2 joint is spacelike if its metric is Euclidean, and timelike if it is Lorentzian; it is not determined by the metric signatures of the codimension1 segments, i.e. $\tilde{M}$ and the $\Sigma$ 's, whose shared boundary is the joint. Since additivity is only an issue for timelike joints, we only have to worry about Lorentzian bulks. There are joints between $\tilde{M}$ and the partial Cauchy slices $\Sigma_{1}$ and $\Sigma_{2}$, but since the boundaries of partial Cauchy slices are spacelike the gravitational action is additive.

The requirement does rule out all choices of $X_{M}$ bulk subregions for which $X_{M_{1} \cup M_{2}} \neq X_{M_{1}} \cup X_{M_{2}}$, such as $X_{M}=\Sigma_{1}$, if $f$ is extensive. It also rules out some functions $f$, such as non-local ones like $\int d x d y R(x) R(y)$, which will not generally give additive cost proposals.
3. Symmetry.
$f\left(X_{M}\right)$ should be invariant under relabelling $\Sigma_{1}$ and $\Sigma_{2}$. An example which satisfies this requirement would be if both $\Sigma$ 's are defined the same way, such as minimum volume surfaces in a Euclidean bulk. An example which does not satisfy symmetry would be if $\Sigma_{1}$ satisfies $K=a$ while $\Sigma_{2}$ satisfies $K=b$ with $a$ and $b$ different constant trace extrinsic curvatures, as then the embedding of the partial Cauchy slices will change under relabelling. It is
sufficient that $X_{M}$ is invariant under the relabelling, though not necessary as in the trivial example $f\left(X_{M}\right)=0$.
4. Covariance.

Requiring the proposal to be covariantly defined leaves a large space of proposals. All the bulk subregions $X_{M}$ defined in section 2.3 .3 as well as the proposals $f\left(X_{M}\right)$ for assigning numbers to those regions given in 2.3.3 are defined in a coordinate-independent way.
5. Non-negativity.

We need $f\left(X_{M}\right) \geq 0$ for all bulks and subregions thereof to which the proposal is applicable. On the one hand it is trivial to define manifestly nonnegative functions $f$. With a non-negative scalar density $\mathcal{F}$, the following is manifestly non-negative:

$$
\begin{equation*}
f\left(X_{M}\right)=\int_{X_{M}} \mathcal{F} \tag{2.20}
\end{equation*}
$$

Volume-type cost proposals use a constant $\mathcal{F}$. Taking the absolute value or an even power of any real-valued function makes it non-negative, so the space of non-negative $f$ is not small.

On the other hand it can be difficult to know whether a given proposal is non-negative for its whole domain. Suppose one has a proposal for which $X_{M}$ is a codimension-zero region of the bulk, and $f$ is the Einstein-Hilbert action of that region, with or without boundary and corner terms. The problem is that there are on-shell asymptotically AdS spacetimes with arbitrarily negative Einstein-Hilbert action, which gives an unphysical negative cost. The action is unbounded from below and not merely negative, so nonnegativity cannot be restored simply by adding a constant. Examples that demonstrate this unboundedness of the Einstein-Hilbert term can be constructed in two ways: either by making the action arbitrarily negative over a finite spacetime volume, or by making it negative (but finite) over an arbitrarily large spacetime volume. For the first kind of example, we consider a Weyl transformation of the bulk metric $G_{\mu \nu} \rightarrow e^{2 \omega} G_{\mu \nu}$ with $\omega$ supported strictly inside $X_{M}$, so that the spacetime is still asymptotically $\operatorname{AdS}$ and so boundary and corner terms of the gravitational action are unaffected. For Lorentzian or Euclidean gravity the contribution to the gravitational action from the Einstein-Hilbert term does change under the Weyl transformation:

$$
\begin{equation*}
\sqrt{G} \mathcal{R} \rightarrow e^{(d-2) \omega} \sqrt{G}\left(\mathcal{R}-2(d-1) \nabla^{2} \omega-(d-2)(d-1)(\partial \omega)^{2}\right) \tag{2.21}
\end{equation*}
$$

The Einstein-Hilbert action can thus be made arbitrarily negative with a rapidly oscillating Weyl factor. To be an on-shell solution to Einstein's
equation requires ${ }^{5}-\left(\frac{d}{2}-1\right) \mathcal{R} \approx T_{\mu}^{\mu}$. This means that the matter-fields involved would have to arbitrarily strongly violate the trace energy condition (TEC) $T_{\mu}^{\mu} \leq 0$. While the TEC is satisfied for simple matter models such as pressure-less dust it does not hold in all physical situations [64,65]. An example are neutron stars which are believed to be accurately described as perfect fluids with equation of state $p=\rho$, which violates the TEC [66]. However, our construction would require $T_{\mu}^{\mu}$ to become unbounded, and it is unclear to us whether this can be accomplished by any form of reasonable matter. See also [67] for a discussion of stability issues of spacetimes in which the TEC is violated. The second kind of example, where a finite negative term is integrated over an arbitrarily large volume, was essentially already constructed in [68], where it was shown that the complexity of an $\mathrm{AdS}_{3}$ black hole with generic topology behind the horizon can be made arbitrarily negative by adding handles to the Einstein-Rosen bridge. This led the authors of [68] to propose a bound on the genus of bulk spacetimes.

From our point of view, these are arguments that seem to rule out holographic proposals for cost (or complexity) that include only the EinsteinHilbert action. This is on the grounds that within the space of all asymptotically AdS spacetimes there are those on which the proposal evaluates to a negative value, and so cannot be interpreted as cost or complexity. As a corollary of this argument, the domain of validity of the CA proposal cannot be all asymptotically AdS spacetimes; there are those on which the action of the WDW patch will be negative. This is not to say that the CA proposal does not give reasonable results for spacetimes such as the eternal black hole to which it was originally applied, nor that simple modifications of the proposal say by adding the matter action cannot remedy the unboundedness of the total action.

Since path integral cost upper bounds holographic state complexity, there will be additional checks coming from late-time linear growth and the switchback effect. These are specific to TFD states and hence are only secondary to the above primary requirements. In the case of observables defined using codimension-one surfaces, [69] showed that there are infinite classes of proposals which satisfy both linear growth and the switchback effect. This could mean that these conditions on complexity are not too restrictive. However, there are valid covariant proposals which violate linear growth and/or the switchback effect. In the case of linear growth, consider when $X_{M}$ is a codimension-one constant curvature slice (with $R=-2$ ) in a BTZ black hole background. The volume of these slices satu-

[^5]rates quickly, and hence this $f\left(X_{M}\right)$ can be ruled out. Similar restrictions apply when $X_{M}$ is codimension-zero. In the next section, we give an example of a new codimension-zero complexity proposal that exhibits late time linear growth. Furthermore, since the complexity of a perturbed TFD state is expected to exhibit switchback effect, this will constrain how we choose $X_{M}$ and $f\left(X_{M}\right)$ in shockwave geometries.

This concludes our preliminary discussion of the space of holographic cost proposals. We found that non-negativity in particular is a subtle and difficult to verify requirement, and that new proposals, unless manifestly non-negative, need to be carefully checked with a skeptical eye. A natural direction to take from here is to consider proposals of increasing intricacy, and check their non-negativity case by case. Proposals where $f$ is the volume of $X_{M}$ are in some sense the simplest, and their non-negativity is manifest at least in Euclidean setups. In section 3.1 we give an argument for why the area of $\tilde{M}$ is a physically well-motivated proposal for the complexity of $U_{M}$, from the perspective of $T \bar{T}$ deformations. In the next chapter we will look in more detail at various costs, including gravitational actiontype proposals, and in particular if and when they run afoul of the non-negativity requirement.


## Holographic Complexity from Optimising Costs

## Contents

3.1 CV from optimising boundary volume ..... 44
3.2 Towards CV2.0 from optimising bulk volume ..... 46
3.3 CV from optimising Euclidean gravitational action ..... 48
3.4 Bulk action and $T \bar{T}$ ..... 59
3.5 Relation to kinematic space ..... 63
3.6 Obstacles to obtain the CA proposal from a cost ..... 66
3.7 Linear growth at late times for BTZ black hole ..... 68
3.8 General methods for gravitational action proposals ..... 70
3.8.1 Equations of motion ..... 71
3.8.2 Solution method, totally geodesic foliations ..... 72
3.8.3 Examples ..... 75
3.8.4 Implications of the Gauss-Bonnet theorem ..... 77
3.8.5 Lemons in Lorentzian $\mathrm{AdS}_{3}$ ..... 79
3.9 Discussion ..... 85

In the previous chapter we discussed path integral costs and their connection to state complexity. The path-integral cost $\mathscr{C}(M)$ in general only provides an upper bound for the operator complexity $C\left(U_{M}\right)$, which in turn bounds the state complexity of the final state $|f\rangle=U_{M}|i\rangle$,

$$
\begin{equation*}
\mathscr{C}(M) \geq C\left(U_{M}\right) \geq C(|f\rangle) \tag{3.1}
\end{equation*}
$$

In certain special cases we might expect that an optimal path integral cost gives a reasonable state complexity.

In this chapter we will give some illustrative examples of path integral cost proposals that reduce to existing holographic state complexity proposals. In each case we fix a proposal for cost of the bulk path integral on a bulk subregion, minimise this cost over an appropriate set of $\tilde{M}$, and show that the resulting minimal cost
matches a state complexity proposal. When optimising over path integral state preparation what we hold fixed is the final state, and we should allow for different bulk geometries as well as different subregions of each geometry. We will minimise cost with respect to $\tilde{M}$ within a fixed bulk geometry, and then argue that the cost cannot be lowered by varying the geometry, but this should be considered a simplifying feature of the particular cost proposals we are dealing with rather than a general feature. Note also that more than one path integral cost proposal can reduce to a given state complexity proposal; we give two that reduce to the CV conjecture.

### 3.1 CV from optimising boundary volume

We now give a path integral cost proposal that when optimised reduces to the CV state complexity proposal. Consider any asymptotically Euclidean AdS spacetime of any dimension. We are looking to connect to state complexity, so as per the discussion from section 2.2.3 the appropriate set of codimension- 1 surfaces $\tilde{M}$ over which we will minimise path integral cost all have fixed boundaries in common, see figure 2.2. We take $\Sigma_{1}$ to be a bulk point, so the path integrals really are preparing the state from nothing. Each $\tilde{M}$ in the set we are 'optimising' over then has one boundary, which is fixed. The cost proposal we will use is

$$
\begin{equation*}
\mathscr{C}=\operatorname{Vol}[\tilde{M}] \tag{3.2}
\end{equation*}
$$

We just wish to show that the $\tilde{M}$ that minimises or 'optimises' this cost is the maximal volume slice in the Lorentzian continuation of the space, so we suppress constants of proportionality. The $\tilde{M}$ that minimises the path integral cost (3.2) is the minimal volume surface in the set with fixed boundary. We label the minimal volume surface $\tilde{M}^{*}$.

Naively we could leave $\Sigma_{2}$ unspecified because it does not play a role in the cost proposal. Can we lower the volume of $\tilde{M}^{*}$ and so the path integral cost by allowing the background geometry to vary? The answer is generally yes, but suppose $\Sigma_{2}$ is a subregion of a minimal volume slice of a given Euclidean geometry. Since $\Sigma_{2}$ lies on a minimal volume slice, and by definition $\partial \tilde{M}=\Sigma_{2}$, we have that $\tilde{M}^{*}=\Sigma_{2}$. This means that we cannot lower the volume of $\tilde{M}^{*}$ without changing the geometry on $\Sigma_{2}$, which is forbidden by the requirement of keeping the final state fixed in this optimisation procedure.

Gaussian normal coordinates adapted to $\tilde{M}^{*}$ are

$$
\begin{equation*}
d s^{2}=d \tau^{2}+g_{i j}(\tau, x) d x^{i} d x^{j} \tag{3.3}
\end{equation*}
$$

with $\tilde{M}^{*}: \tau\left(x^{i}\right)=0$. Suppose this minimal volume surface lies on a time reflection symmetric slice $\Sigma$ of the Euclidean space, which implies that the extrinsic curvature
$K_{i j}^{(\tau)}$ vanishes. This won't generally be the case since being a minimal volume surface only guarantees that the trace $K^{(\tau)}=0$ vanishes but let us assume it. We may then analytically continue to Lorentzian signature $\tau \rightarrow i t$, and second order shape variations in the direction normal to $\tilde{M}$ flip sign:

$$
\begin{equation*}
\frac{\delta^{2}}{\delta \tau(y) \delta \tau\left(y^{\prime}\right)} \operatorname{Vol}[\tilde{M}]=-\frac{\delta^{2}}{\delta t(y) \delta t\left(y^{\prime}\right)} \operatorname{Vol}[\tilde{M}] \tag{3.4}
\end{equation*}
$$

and so $\tilde{M}^{*}$, which is the global minimal volume in the Euclidean space, is a local maximum in the analytic continuation to Lorentzian spacetime.

We have shown how to reduce to the CV state complexity proposal after finding the $\tilde{M}$ which minimises or 'optimises' the path integral cost (3.2). Our assumptions are that $\tilde{M}^{*}$ lies on a time reflection slice, and that the surface is the global, not just a local, maximum in volume in the Lorentzian spacetime. Note that we have only shown how to match CV conjecture at the point of time reflection symmetry.

It would have been preferable to find a cost proposal that applies to the same Lorentzian bulk as the CV conjecture, rather than the Euclidean continuation. The basic obstacle is that our optimisation procedure involves minimising a cost, while the CV conjecture maximises a volume. We do not rule out the possibility of a Lorentzian cost proposal that reduces to the CV conjecture, but we were not able to find one that satisfies all the physical requirements.

## Heuristic justification for cost proposal from $T \bar{T}$

We are taking a phenomenological approach to cost proposals, rather than try to justify them from a bottom-up gate-counting physical picture. We do however have a heuristic justification for this subsection's cost equals boundary volume proposal which we find appealing and will describe. Similar arguments have previously been made in $[61,70]$.

Consider a Euclidean path integral of a $T \bar{T}$-deformed theory defined on some twodimensional manifold $M$. The deformation parameter $\lambda$ in a $T \bar{T}$-deformed theory is related to the scale of non-locality $L_{n l}$ by

$$
\begin{equation*}
L_{n l}^{2} \sim \lambda \tag{3.5}
\end{equation*}
$$

One way of arguing for this relation is from the fact that the $T \bar{T}$ deformation of the free boson action is the Nambu-Goto string action, with string length $l_{s}^{2} \sim \lambda$ [62].

We may discretize the path integral with a tensor network if we assume that each region of proper area $L_{n l}^{2}$ represents one tensor. Then the total number of tensors is

$$
\begin{equation*}
\mathscr{C}(M) \sim \int_{M} d x d \tau \frac{\sqrt{\gamma}}{L_{n l}^{2}} \sim \int_{M} d x d \tau \frac{\sqrt{\gamma}}{\lambda(x, \tau)} \tag{3.6}
\end{equation*}
$$

The state that the path integral prepares depends on the manifold, boundary conditions, and field theory action, especially through the deformation parameter $\lambda(x, \tau)$. Following section 2.2.1, we can now take the CFT to be holographic, with metric inherited from the induced metric on a finite cutoff surface $z=\rho(\tau)$ in Poincaré $\mathrm{AdS}_{3}$ :

$$
\begin{equation*}
\frac{1}{\rho^{2}} \gamma_{i j}=g_{i j} \tag{3.7}
\end{equation*}
$$

with

$$
\begin{equation*}
\sqrt{g}=\frac{1}{\rho^{2}} \sqrt{1+\dot{\rho}(\tau)^{2}} \tag{3.8}
\end{equation*}
$$

and the $T \bar{T}$ relation between cutoff and deformation parameter

$$
\begin{equation*}
\lambda \sim G_{N} \rho^{2} \tag{3.9}
\end{equation*}
$$

Substituting in (3.6) we get an heuristic estimate for the effective number of gates in the path integral on $M$ :

$$
\begin{equation*}
\mathscr{C}(M) \sim \frac{1}{G_{N}} \int_{\tilde{M}} d x d \tau \frac{\sqrt{1+\dot{\rho}^{2}}}{\rho^{2}}=\frac{\operatorname{Vol}(\tilde{M})}{G_{N}} \tag{3.10}
\end{equation*}
$$

This completes a heuristic derivation of the cost equals boundary volume proposal. Just like in the complexity=volume proposal [40], the proportionality factor in this equation will have to depend on the choice of an additional length scale, as $\mathscr{C}(M)$ has to be dimensionless. Also, notice that although the state preparation we are considering is similar to the one in [1], the holographic path integral cost derived from a $T \bar{T}$ gate counting procedure (3.10) is quite different from the one based on the on-shell gravitational action that was provided there,

$$
\begin{equation*}
I[\rho] \sim \frac{1}{G_{N}} \int d x d \tau \frac{1+\dot{\rho} \arctan \dot{\rho}}{\rho^{2}} \tag{3.11}
\end{equation*}
$$

### 3.2 Towards CV2.0 from optimising bulk volume

The CV2.0 proposal asserts that the spacetime volume of the boundary-anchored WdW patch represents a holographic notion of complexity in dual quantum field theories. It seems obvious to try to obtain this from the optimization of a cost functional which is simply given by the volume of $X_{M}$. There are however some subtleties when trying to make this precise and as we will see the CV2.0 proposal does not quite follow. Instead, we obtain something which is more like the volume of half the WdW patch.

The first issue we need to address is the choice of the initial and final slices $\Sigma_{1}$ and $\Sigma_{2}$. It is tempting to choose these to be the future and past null cones emanating from the boundaries of $\tilde{M}$, but this would lead to a problem with the additivitiy criterion for the cost function: if we combine $\tilde{M}_{1}$ and $\tilde{M}_{2}$, the future null cone attached to the future boundary of $\tilde{M}_{1}$ obviously does not agree with the past null cone attached to the past boundary of $\tilde{M}_{2}$ (which equals the future boundary of $\tilde{M}_{1}$ ). Hence $X_{M_{1}} \cap X_{M_{2}} \neq \emptyset$ and because of this additivity will generically fail. One can also choose both $\Sigma_{1}$ and $\Sigma_{2}$ to be simultaneously future or past directed null cones, but this would manifestly lead to violations of the time-reversal criterion. Moreover, it would lead to situations where $X_{M}$ could become the empty set in the limit where $\tilde{M}$ becomes null. In what remains we will choose $\Sigma_{1}$ and $\Sigma_{2}$ by the property that they have vanishing scalar extrinsic curvature, but the conclusions will not be substantially different for other choices of $\Sigma_{1}$ and $\Sigma_{2}$. Given this choice for $\Sigma_{1}$ and $\Sigma_{2}$, consider the candidate cost functional

$$
\begin{equation*}
\mathscr{C}(M) \sim \int_{N=\operatorname{Int}\left(\tilde{M} \cup \Sigma_{1} \cup \Sigma_{2}\right)} \sqrt{|G|}+\alpha \int_{\tilde{M}} \sqrt{|g|} \tag{3.12}
\end{equation*}
$$

where $\alpha$ is a non-negative dimensionful constant. Such simple cost functions satisfy all the properties listed in section 2.3.2 that we require from a good notion of a gravitational cost. A precise expression for this cost functional also requires an overall dimensionful prefactor which we did not include in (3.12) and which can be chosen arbitrarily.

Let us consider this cost function in the context of the situation depicted in figure 3.1 in the Lorentzian context, where we want the path integral to remain defined on a timelike surface. We restrict $\tilde{M}$ to be timelike in the set to be optimised over, in order to find the unitary time evolution operator with the lowest cost. If one then performs optimization of (3.12) for timelike separated initial and final state, the second term gets arbitrarily small for an almost null boundary and the latter also leads to the minimal enclosed bulk spacetime volume. If one optimizes also over time duration at fixed initial and final state, then one gets a portion of the WdW patch. Shrinking one state to a bulk point gives rise to a 'past half' of the WdW patch bounded by $\Sigma_{2}$. The volume of this half of the WdW patch equals to the optimum of the cost (3.12), which is as close as we can get to the CV2.0 proposal. We cannot change the geometry in the past domain of dependence of $\Sigma_{2}$ without changing the geometry on $\Sigma_{2}$, so having minimised the cost while working with a fixed bulk geometry to the volume of this 'half' WdW patch we cannot lower it further by varying the interior geometry without changing the final state. Rather than creating the state from a single bulk point, we could also have asked the question what the minimum cost is of the reverse process where we use a circuit to map an initial state to a single bulk point. One could perhaps think of this as a circuit which maps the state to a completely unentangled and therefore


Figure 3.1: Optimisation of the Lorentzian cost functional given by a sum of bulk and boundary volume. The minimal cost is obtained as $\tilde{M}$ approaches an almost null surface. As $\Sigma_{1}$ is shrunk to a point, one obtains the past half of the WdW patch anchored to $\Sigma_{2}$.
non-geometric state represented by a single bulk point. The optimal cost for this state demolition process is then given by the volume of the future half of the WdW patch where the WdW patch is cut in two pieces by $\Sigma_{1}$. Overall, the conclusion of this analysis could be that the CV2.0 proposal is not just computing the cost of creating the state but rather the sum of the creation and demolition cost. It would be interesting to explore this interpretation further.

### 3.3 CV from optimising Euclidean gravitational action

In section 2.3.4 we argued that the gravitational action of codimension-0 bulk subregions is not a reasonable cost proposal because there are asymptotically AdS spacetimes, both Lorentzian and Euclidean, for which the action is negative. These actions are negative due to the conformal mode of the Ricci scalar, and are on-shell for matter configurations that violate the trace energy condition. In this section we will nonetheless use gravitational action as a cost proposal. What we have in mind is a corrected proposal that is non-negative: either one which excludes problematic negative action fringe cases in an ad hoc fashion, or a modification such as adding the matter action which makes the total action positive even for trace energy condition violating configurations, though we have not proven that this works. In any case we will only apply our proposal here to subregions of pure global AdS, with the matter in its vacuum configuration, so we are far from the problematic fringe cases where we would have to specify precisely how we correct our proposal to ensure positivity. We only stipulate that the presumptive
correction is negligible when evaluated on pure AdS.
Cost proposals which use the gravitational action can also in some cases reduce to existing complexity proposals when optimised. In previous work we considered Euclidean Poincaré $\mathrm{AdS}_{3}$ and gave a gravitational action cost proposal that reduces to the volume of the constant time slice when optimised [1]. Our cost proposal was the on-shell gravitational action of the codimension-0 bulk region bounded by two constant Euclidean Poincaré time slices and a finite cutoff radial boundary. This we claimed is dual to the cost of the Euclidean path integral in the $T \bar{T}$-deformed boundary CFT on that radial cutoff boundary. The basic idea is that we have a set of path integrals on different radial cutoff surfaces that prepare the same state, and when the gravitational action cost proposal is minimised over this set we found that it matches the CV state complexity proposal. Minimising path integral cost with respect to background geometry is inspired by the work of [71], and for Poincaré $\mathrm{AdS}_{3}$ our proposal reduces to the Liouville action in agreement with their work, in the limit of a slowly varying cutoff surface. The optimum path integral maps between ground states of theories with different UV cutoffs by building up or coarse-graining away the UV structure with as little Euclidean time evolution as possible.

In this subsection we will show that when our gravitational action proposal is applied to global AdS we again match with the CV state complexity proposal. The purpose is two-fold: (1) to show that there is more than one cost proposal, in this case cost equals boundary volume and cost equals gravitational action, that can reduce to a given complexity proposal when minimised over a suitable set of path integrals, and (2) to give further evidence that the cost proposal based on gravitational action that we gave in [1] is a reasonable one. For our gravitational action proposal we do not know whether (or necessarily expect that) it reduces to the CV conjecture in other asymptotically AdS spacetimes than the Euclidean Poincaré and global AdS examples that we have explicitly checked.

## Gravitational action in Poincare AdS

For simplicity and concreteness, we are going to consider the preparation of the ground state of a 2d CFT on a line using the Euclidean path integral. To this end, we take the standard Euclidean AdS solution, with the curvature scale $l_{A d S}=1$,

$$
\begin{equation*}
d s^{2}=\frac{d z^{2}+d t^{2}+d x^{2}}{z^{2}} \tag{3.13}
\end{equation*}
$$

and the partition function of the CFT equals the exponent of minus the on-shell bulk action

$$
\begin{equation*}
I=\frac{1}{\kappa} \int_{M} d^{3} x \sqrt{G}(R+2)+\frac{2}{\kappa} \int_{\partial M} d^{2} x \sqrt{g} K+I_{c} . \tag{3.14}
\end{equation*}
$$

$M$ is the bulk region bounded by $\rho(t) \leq z \leq \infty$ and $t_{i} \leq t \leq t_{f}$, as shown in figure 3.2. The a priori finite function $\rho(z)$ interpolates between the values $z=z_{i}$ at $t=t_{i}$ and $z=z_{f}$ at $t=t_{f}$, with $t_{i} \leq t_{f}$ and $z_{f}<z_{i}$. For simplicity we also take the setup to be independent of the transverse direction $x$. Furthermore, we write $\kappa=16 \pi G_{N}, G$ for the 3 d metric on $M, g$ for the induced 2 d metric on $\partial M$, and $K$ is the trace of the extrinsic curvature. $\partial M$ is only piecewise smooth and has a kink or joint at $t=t_{f}$ and $t=t_{i}$ as shown in figure 3.2. Each joint contributes a term

$$
\begin{equation*}
I_{c}=\frac{2}{\kappa} \int d x \sqrt{j} \alpha \tag{3.15}
\end{equation*}
$$

to the gravitational action. Herein, $\sqrt{j}$ is the length element along the joint and $\alpha$ is simply the angle between the two normal vectors of the two surfaces coming together at the joint (which may have either sign). Joint-terms of this type were studied by Hayward in [43, 44], but in the Euclidean setting, which is of interest here, this was already done earlier in [72], see also the discussion in [45].


Figure 3.2: We consider a subregion $M$ of Euclidean Poincaré $A d S_{3}$. We introduce two time-slices $t=t_{i}$ and $t=t_{f}$ corresponding to the field theory ground states $|0\rangle_{z_{i}}$ and $|0\rangle_{z_{f}}$, which are prepared for different values of the radial cutoff. The radial boundary is at finite cutoff, $z=\rho(t)$. Our proposal is that the complexity of the circuit that maps between these ground states with different finite Wilsonian cutoffs is given by the gravitational action on $M$.

As discussed above, we are going to interpret the on-shell value of the bulk effective
action of the region $M$ as the complexity of the circuit defined by the surface $z=\rho(t)$ which maps the vacuum state $|0\rangle_{z_{i}}$ with cutoff $z_{i}$ to the vacuum state $|0\rangle_{z_{f}}$ with cutoff $z_{f}$. If we use the relation between a finite radial cutoff and the coefficient $\mu$ of the $T \bar{T}$ deformation via [48],

$$
\begin{equation*}
\mu(t)=\kappa \rho(t)^{2} \tag{3.16}
\end{equation*}
$$

we can reinterpret the states $|0\rangle_{\rho(t)}$ as ground states of the $T \bar{T}$ deformed CFT with a time-dependent coefficient $\mu(t)$.

Concretely, the induced line element on the boundary surface is

$$
\begin{equation*}
d s^{2}=\frac{\left(1+\dot{\rho}^{2}\right) d t^{2}+d x^{2}}{\rho^{2}} \tag{3.17}
\end{equation*}
$$

its Ricci scalar reads

$$
\begin{equation*}
R^{(d-1)}=\frac{2\left(\rho \ddot{\rho}-\dot{\rho}^{2}\left(1+\dot{\rho}^{2}\right)\right)}{\left(1+\dot{\rho}^{2}\right)^{2}} \tag{3.18}
\end{equation*}
$$

the trace of the extrinsic curvature reads

$$
\begin{equation*}
K=\frac{\rho \ddot{\rho}+2\left(1+\dot{\rho}^{2}\right)}{\left(1+\dot{\rho}^{2}\right)^{3 / 2}} \tag{3.19}
\end{equation*}
$$

and from (3.14) we obtain

$$
\begin{align*}
I & =\frac{-4}{\kappa} \int_{M} d^{2} x \int_{z=\rho}^{\infty} \frac{d z}{z^{3}}+\frac{2}{\kappa} \int_{\partial M} d^{2} x \frac{\rho \ddot{\rho}+2\left(1+\dot{\rho}^{2}\right)}{\rho^{2}\left(1+\dot{\rho}^{2}\right)}+I_{c} \\
& =\frac{2 V_{x}}{\kappa} \int_{t_{i}}^{t_{f}} d t \frac{\rho \ddot{\rho}+\left(1+\dot{\rho}^{2}\right)}{\rho^{2}\left(1+\dot{\rho}^{2}\right)}+I_{c} \tag{3.20}
\end{align*}
$$

for the on-shell bulk action, where we have introduced $V_{x}=\int d x$. For the corner term, we also find

$$
\begin{equation*}
I_{c}=\frac{2 V_{x}}{\kappa}\left(\frac{\pi / 2-\arctan \dot{\rho}\left(t_{f}\right)}{z_{f}}+\frac{\pi / 2+\arctan \dot{\rho}\left(t_{i}\right)}{z_{i}}\right) \tag{3.21}
\end{equation*}
$$

Integrating by parts, this action can be written only using first derivatives of $\rho$, yielding

$$
\begin{equation*}
I=\frac{2 V_{x}}{\kappa} \int_{t_{i}}^{t_{f}} d t\left(\frac{1}{\rho^{2}}+\frac{\dot{\rho} \arctan \dot{\rho}}{\rho^{2}}\right)+\frac{\pi V_{x}}{\kappa}\left(\frac{1}{z_{f}}+\frac{1}{z_{i}}\right) . \tag{3.22}
\end{equation*}
$$

The terms which are independent of $\rho$ do not affect the equations of motion, and can always be removed by a suitable counter term, which we will assume to be done from now on. We believe this is justified, as it is known [43, 45] that the joint
term can spoil the additivity of the action under combining bulk regions, which besides the formulation of a well defined variational principle is usually the second main reason for adding boundary terms to the action (3.14). ${ }^{1}$

The equations of motion obtained by extremizing (3.22) read

$$
\begin{equation*}
\frac{\rho \ddot{\rho}+\left(1+\dot{\rho}^{2}\right)}{\rho^{3}\left(1+\dot{\rho}^{2}\right)^{2}}=0 \tag{3.23}
\end{equation*}
$$

The most immediately visible solution to this equation is the one where we formally take the limit $\dot{\rho} \rightarrow \infty$. This corresponds to the boundary surface turning into an equal-time slice, which is in fact where, based on the intuition surrounding holographic complexity and tensor networks, we expect the most optimised circuit preparing the state $|0\rangle_{z_{f}}$ to live, see e.g. [73]. The generic solution to (3.23) reads

$$
\begin{equation*}
\rho(t)=\sqrt{\mathcal{R}^{2}-\left(t-t_{0}\right)^{2}} \tag{3.24}
\end{equation*}
$$

and describes circular arcs of radius $\mathcal{R}$ centered on the boundary point at $t=t_{0}$. The formal solution $\dot{\rho} \rightarrow \infty$ corresponds to the limit of infinite radius.

Our proposal is that the Euclidean action (3.22) (excluding the $\rho$-independent remnants of the joint terms) is a measure of the complexity of preparing the state $|0\rangle_{z_{f}}$ from the state $|0\rangle_{z_{i}}$ using the circuit described by $\rho(t)$. The optimal circuit, with fixed Euclidean time distance $\Delta t=\left|t_{f}-t_{i}\right|$, is then of the form (3.24), and the complexity of this circuit is given by evaluating the Euclidean action on this solution. With the explicit boundary conditions being $\rho\left(t_{f}\right)=z_{f}$ and $\rho\left(t_{i}\right)=z_{i}$, the value of the Euclidean action in the first term of (3.22) is

$$
\begin{equation*}
I=\frac{2 V_{x}}{\kappa}\left(\frac{1}{z_{f}} \arctan \frac{z_{i}^{2}-z_{f}^{2}+\Delta t^{2}}{2 z_{f} \Delta t}-\frac{1}{z_{i}} \arctan \frac{z_{i}^{2}-z_{f}^{2}-\Delta t^{2}}{2 z_{i} \Delta t}\right) . \tag{3.25}
\end{equation*}
$$

Note that this result comes entirely from the corner terms, as the first term in (3.20) exactly vanishes on-shell. Interpreting it as a function of the variable $t_{i} \leq t_{f}$ while keeping $z_{i} \neq z_{f}$ fixed, we can verify that the above expression is minimized by $t_{i}=t_{f}$. This corresponds to the limit $\mathcal{R} \rightarrow \infty$ or $\dot{\rho} \rightarrow \infty$ and hence the equal time slice that is intuitively expected to play a special role in describing the complexity of the state $|0\rangle_{z_{f}}$. Using $1 / \kappa=c / 24[6]$, the minimum value is given by

$$
\begin{equation*}
I_{\min }=\frac{c \pi V_{x}}{24}\left(\frac{1}{z_{f}}-\frac{1}{z_{i}}\right) \tag{3.26}
\end{equation*}
$$

[^6]which is proportional to the spatial volume of the strip $z_{f} \leq z \leq z_{i}$ on the equal time slice at $t=t_{f}$. Of course, if we send $z_{f} \rightarrow \epsilon \ll 1$ and $z_{i} \rightarrow \infty$, this reproduces the standard result of the volume proposal for the complexity of the CFT ground state. Clearly, this result also vanishes if $z_{i}=z_{f}$, which we take as a non-trivial consistency check and further justification for excluding the remnants of the joint terms in (3.22). ${ }^{2}$

To close this section, let us compare our results to the ones that can be obtained from the Liouville action. For $\dot{\rho} \ll 1$, equation (3.22) can be approximated as

$$
\begin{equation*}
I=\frac{2 V_{x}}{\kappa} \int d t\left(\frac{1}{\rho^{2}}+\frac{\dot{\rho}^{2}}{\rho^{2}}\right) \tag{3.27}
\end{equation*}
$$

which, assuming no $x$-dependence, is equivalent to the Liouville Lagrangian

$$
\begin{equation*}
S_{L}=\frac{c}{24 \pi} \int d t \int d x\left(\eta e^{2 \omega}+\left(\partial_{t} \omega\right)^{2}+\left(\partial_{x} \omega\right)^{2}\right) \tag{3.28}
\end{equation*}
$$

after a change of variables $\rho(t) \rightarrow(1 / \sqrt{\eta}) e^{-\omega(t)}$. Note that the physically interesting solution $\dot{\rho} \rightarrow \infty$ falls outside of the range of applicability of the approximation necessary to obtain the Liouville action from (3.22). The equations of motion derived from (3.27) take the form

$$
\begin{equation*}
\frac{\rho \ddot{\rho}+\left(1-\dot{\rho}^{2}\right)}{\rho^{3}}=0 . \tag{3.29}
\end{equation*}
$$

As we will see below, these field equations also arise if we introduce a new time coordinate in order to bring the induced metric on the boundary into conformal gauge.

## Conformal time and extremizing the action

There is a subtle but crucial difference between our setup discussed in the previous subsection and the calculations of [73], which we will discuss in this subsection in order to avoid confusion.

In order to do so, we note that [73] investigates a setup similar to the one depicted in figure 3.2, and up to notation (3.20) also appears in the appendix of that paper. Following [73], we can now introduce a conformal time $u$, with

$$
\begin{equation*}
d u=\sqrt{1+\dot{\rho}(t)^{2}} d t \tag{3.30}
\end{equation*}
$$

[^7]such that the line element (3.17) is transformed into the conformal gauge form
\[

$$
\begin{equation*}
d s^{2}=\frac{d u^{2}+d x^{2}}{\varrho(u)^{2}} \tag{3.31}
\end{equation*}
$$

\]

Here, we have introduced a new variable such that $\varrho(u(t))=\rho(t)$. Under (3.30), the action (3.22) changes to [73]

$$
\begin{equation*}
I=\frac{2 V_{x}}{\kappa} \int_{u_{i}[\varrho]}^{u_{f}[\varrho]} d u\left(\frac{\sqrt{1-\varrho^{\prime 2}}+\varrho^{\prime} \arcsin \varrho^{\prime}}{\varrho^{2}}\right) . \tag{3.32}
\end{equation*}
$$

If we were to just identify the integrand in (3.32) as a Lagrangian and compute naively the Euler equations, we arrive at

$$
\begin{equation*}
\frac{\varrho \varrho^{\prime \prime}+2\left(1-\varrho^{\prime 2}\right)}{\varrho^{3}\left(1-\varrho^{\prime 2}\right)^{2}}=0 \tag{3.33}
\end{equation*}
$$

which, up to notation and the addition of a nonzero tension term, are the equations which where studied in [73].

The subtlety announced at the beginning of the subsection is that (3.30) is a reparametrization of time which is dependent on the variable with respect to which we want to vary the action, hence formally in going from (3.22) to (3.32) the integration bounds $u_{i}$ and $u_{f}$ become themselves functionals of $\varrho$, and will lead to a nontrivial contribution according to Leibniz's rule when varying the action. In fact it can be checked that introducing (3.30) and $\varrho(u)$ in the equation of motion (3.23) gives a result

$$
\begin{equation*}
\frac{\varrho \varrho^{\prime \prime}+\left(1-\varrho^{\prime 2}\right)}{\varrho^{3}}=0 \tag{3.34}
\end{equation*}
$$

that is inequivalent to (3.33). Interestingly, (3.34) has the form of the Liouville equation (3.29), just for $\varrho(u)$ instead of $\rho(t)$.

The most commonly known example where a field-dependent reparametrization can be useful is the Lagrangian for geodesic motion, which becomes a constant when introducing affine parametrisation. Of course, this does not mean that the equations of motion degenerate, as the full information about the value of the action - i.e. the length of the curve - is now entirely encoded in the integration domain. Unfortunately, the expression (3.32) rather inelegantly falls into a middle ground between the two possible extremes, as both the integrand and the integration bounds are functionals of the variable $\varrho$, and for this reason we found it intractable to work with.

This does not mean that either our work or [73] are wrong, just that we are studying a different variational problem. We work with the action (3.22) where explicitly
we assume Dirichlet boundary conditions for $\rho(t)$ at the fixed values $t=t_{f}$ and $t=t_{i}$, while [73] works with the action (3.32) with the implicit assumption of Dirichlet boundary conditions for the field $\varrho$ at fixed values of $u_{i}, u_{f}$, which is an inequivalent mathematical exercise.

## Comparison to AdS/BCFT models

We can investigate this issue a bit further. So far, we have essentially considered what amounts to minisuperspace models, by plugging in an ansatz into the action and deriving equations of motion for the function parametrizing that ansatz, instead of first deriving general equations of motion and then simplifying them with a given ansatz. How can we write our equations of motion in a form that is more suggestive for their general meaning and potential origin? We will do this in the next section, but as an aside, we will now demonstrate that the semicircle solutions that we found can also be obtained if we interpret the boundary of the bulk domain as an "end of the world brane" with an energy-momentum tensor describing matter with a very specific equation of state. The covariant equations of motion of this end of the world brane will imply the general equation that we will derive in the next section. The derivation in the next section does not rely on an end of the world brane interpretation, and it remains to be seen whether this agreement is more than a technical coincidence.

We should also point out that the work of [73] was strongly influenced by the type of AdS/boundary CFT (BCFT) models introduced in [74,75]. In such models the boundary of the space on which the BCFT lives is also extended into the bulk spacetime in the form of an end of the world brane, on which Neumann boundary conditions are imposed. Besides the bulk Einstein equations, this leads to an equation of motion of the form

$$
\begin{equation*}
K_{\mu \nu}-K g_{\mu \nu}=\frac{\kappa}{2} T_{\mu \nu} \tag{3.35}
\end{equation*}
$$

which determines the embedding of the end of the world brane into the ambient space. These models allow for considerable bottom-up toy-model building freedom, and $T_{\mu \nu}$ is the energy-momentum tensor of any matter that lives in the brane worldvolume. In practice, it is often set to be a constant tension term

$$
\begin{equation*}
T_{\mu \nu}=\lambda g_{\mu \nu} \tag{3.36}
\end{equation*}
$$

with tension $\lambda$. As reported in [73], their equation of motion is consistent with (3.35). As we ignore tension terms, we would set the right hand side of (3.35) to zero, and apart from the equal time slice obtained by $\dot{\rho} \rightarrow \infty$, our semicircular embeddings do not satisfy this equation.

Interestingly, in a Lorentzian AdS/BCFT context, semicircular embeddings into Poincaré AdS were derived in [76] for a simple model of $T_{\mu \nu}$ given by a perfect
fluid with equation of state $p=a \sigma$ ( $p=$ pressure, $\sigma=$ energy density) in the limit $a \rightarrow \infty$. So we see that semicircular embeddings into a Poincaré AdS do satisfy an equation of the form (3.35), just with a specific non-trivial right hand side. Due to the peculiar limit in the parameter $a, T_{\mu \nu}$ satisfies the condition

$$
\begin{equation*}
\operatorname{det}\left[T_{\mu \nu}\right]=0 \tag{3.37}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
T_{\mu \nu} T^{\mu \nu}-T^{2}=0 \tag{3.38}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\operatorname{det}\left[K_{\mu \nu}-K g_{\mu \nu}\right]=0 \tag{3.39}
\end{equation*}
$$

respectively

$$
\begin{equation*}
\left(K_{\mu \nu}-K g_{\mu \nu}\right)\left(K^{\mu \nu}-K g^{\mu \nu}\right)-\operatorname{Tr}\left[K_{\mu \nu}-K g_{\mu \nu}\right]^{2}=K_{\mu \nu} K^{\mu \nu}-K^{2}=0 \tag{3.40}
\end{equation*}
$$

for our semicircular embeddings (3.24), even though they were not derived from an AdS/BCFT ansatz. We will give a direct derivation of equation (3.40) as a flow equation for our complexity proposal in the following section.

## Gravitational action in global AdS

Let us calculate the on-shell gravitational action between constant time slices in global AdS, with a variable finite cutoff boundary, see figure 3.3. Consider Euclidean $\mathrm{AdS}_{d+1}$ with unit AdS length in global coordinates:

$$
\begin{equation*}
d s^{2}=\left(1+r^{2}\right) d \tau^{2}+\frac{d r^{2}}{1+r^{2}}+r^{2} d \Omega_{d-1}^{2} \tag{3.41}
\end{equation*}
$$

We assume the cutoff surface $\tilde{M}$ has spherical symmetry, $r=\rho(\tau)$. We define $\Sigma_{1}$ and $\Sigma_{2}$ to be $K=0$ surfaces, which in this case will be just constant time $\tau=\tau_{1,2}$ slices. We will be minimising the on-shell gravitational action over cutoff boundary surfaces $r=\rho(\tau)$ with fixed initial and final cutoff $r_{1}=\rho\left(\tau_{1}\right)$ and $r_{2}=$ $\rho\left(\tau_{2}\right)$. Different boundary surfaces define different path integrals which evaluate the transition amplitude between the ground states of a holographic CFT with Euclidean time dependent $T \bar{T}$ deformation. Fixing $r_{1}$ and $r_{2}$ fixes the initial and final $T \bar{T}$ deformation.

Let us calculate the on-shell gravitational action of the region depicted by figure 3.3. We assume the cutoff surface has spherical symmetry. Consider Euclidean $\mathrm{AdS}_{3}$ with unit AdS length in global coordinates:

$$
\begin{equation*}
d s^{2}=\frac{1}{\cos \theta^{2}}\left(d \tau^{2}+d \theta^{2}+\sin \theta^{2} d \phi^{2}\right) \tag{3.42}
\end{equation*}
$$

The asymptotic boundary is at $\theta=\pi / 2$. We want the on-shell gravitational action of the region $N$, which is bounded by $\tau_{1}=0, \tau_{2}=T$, and the radial cutoff surface $\theta=\theta(\tau)$. The full gravitational action including corner terms is

$$
\begin{equation*}
I=\frac{1}{\kappa} \int_{N} d^{3} x \sqrt{G}(\mathcal{R}+2)+\frac{2}{\kappa} \int_{\tilde{M}} d^{2} x \sqrt{g} K+I_{c} . \tag{3.43}
\end{equation*}
$$



Figure 3.3: The subregion of Euclidean global AdS whose gravitational action we propose to be the cost of the path integral on the finite cutoff boundary.

The extrinsic curvature of the surface $\theta=\theta(\tau)$ is

$$
\begin{equation*}
K=\frac{-\ddot{\theta} \tan \theta+\left(1+2 \tan \theta^{2}\right)\left(1+\dot{\theta}^{2}\right)}{\sec \theta \tan \theta\left(1+\dot{\theta}^{2}\right)^{\frac{3}{2}}} \tag{3.44}
\end{equation*}
$$

Using this, the action takes the simple form

$$
\begin{equation*}
I=\frac{2}{\kappa} \int d \phi d \tau\left(\frac{1}{\cos \theta^{2}}-\frac{\ddot{\theta} \tan \theta}{\left(1+\dot{\theta}^{2}\right)}\right)+I_{c} \tag{3.45}
\end{equation*}
$$

This can further be simplified by partially integrating over $\tau$ as

$$
\begin{equation*}
I=\frac{2}{\kappa} \int d \phi d \tau\left(\frac{1}{\cos \theta^{2}}+\frac{\dot{\theta} \arctan \dot{\theta}}{\cos \theta^{2}}\right)-\frac{2}{\kappa} \int d \phi \tan \theta \arctan \dot{\theta}+I_{c} \tag{3.46}
\end{equation*}
$$

If the boundary is not smooth at a corner situated at $\tau=\tau_{c}$, then the action receives an additional contribution given by the Hayward term

$$
\begin{equation*}
I_{c}=\frac{2}{\kappa} \int d \phi \tan \theta\left(\tau_{c}\right)\left(\arctan \dot{\theta}\left(\tau_{c}\right)+\pi / 2\right) \tag{3.47}
\end{equation*}
$$

Including this term from a single corner, the total action now is

$$
\begin{equation*}
I=\frac{2}{\kappa} \int d \phi d \tau\left(\frac{1+\dot{\theta} \arctan \dot{\theta}}{\cos \theta^{2}}\right)+\frac{2 \pi^{2} \tan \theta\left(\tau_{c}\right)}{\kappa} \tag{3.48}
\end{equation*}
$$

This result is closely related to our previous result for Euclidean Poincaré $\mathrm{AdS}_{3}$ [1]. Varying this action allows us to find surfaces that extremise the gravitational action. The equations of motion are

$$
\begin{equation*}
\frac{\ddot{\theta}-\tan \theta\left(1+\dot{\theta}^{2}\right)}{\cos \theta^{2}\left(1+\dot{\theta}^{2}\right)^{2}}=0 \tag{3.49}
\end{equation*}
$$

Before solving the above equation, we see that there will always be a solution when $|\dot{\theta}| \rightarrow \infty$. In this limit the surface turns in to a equal-time slice. The most general solution for $\theta(\tau)$ is given by

$$
\begin{equation*}
\theta(\tau)=\arcsin (\alpha \sinh (\tau)+\beta \cosh (\tau)) \tag{3.50}
\end{equation*}
$$

As expected from the discussion in [1], these $\theta(\tau)$ describe surfaces of constant scalar curvature $R=-2$. The circuits whose boundary surface is given by the above $\theta(\tau)$ or in terms of $r(\tau)=\tan \theta(\tau)$ extremise the action (3.48) and hence the cost of the circuit. More specifically, consider the circuit preparing the ground state $|0\rangle_{\theta_{f}}$ at some cut-off $\theta_{f}$ starting from a trivial initial state. Such a circuit $\theta(\tau)$, running from $\tau=0$ to $\tau=T>0$ is given by (3.50) with $\theta(0)=0$ and $\theta(T)=\theta_{f}$. The cost can now be calculated from the value of the on-shell action, and is given by

$$
\begin{equation*}
I=\frac{4 \pi}{\kappa}\left(T+\arctan \left(\frac{\tan \theta_{f}}{\tanh T}\right) \tan \theta_{f}\right)+\frac{2 \pi^{2} \tan \theta_{f}}{\kappa} \tag{3.51}
\end{equation*}
$$

Minimum value of this optimised cost for preparing $|0\rangle_{\theta_{f}}$ is achieved for $T=0$, when the surface is a constant time slice. The minimum value is

$$
\begin{equation*}
I_{\min }=\frac{4 \pi^{2} \tan \theta_{f}}{\kappa} \tag{3.52}
\end{equation*}
$$

The volume of the constant time slice with a radial cut-off at $\theta_{f}, \operatorname{Vol}\left(\theta_{f}\right)$ is equal to $2 \pi \tan \theta_{f} \tan \frac{\theta_{f}}{2}$. Using this, and $\frac{1}{\kappa}=\frac{c}{24 \pi}$ we can rewrite the above value as

$$
\begin{equation*}
I_{\min }=\frac{c}{12} \frac{\operatorname{Vol}\left(\theta_{f}\right)}{\tan \frac{\theta_{f}}{2}} \tag{3.53}
\end{equation*}
$$

The minimum cost is indeed proportional to the volume of the constant time-slice. As the radial cut-off is taken to infinity, $\theta_{f} \rightarrow \pi / 2$, we see that the proportionality constant is exactly $\frac{c}{12}$.

We should again ask whether the bulk path integral cost can be lowered by allowing the background geometry to vary. In a similar resolution to the previous subsection we simply specify $\Sigma_{2}$ to lie on the constant time slice. Then the cost-minimising $\tilde{M}$ and $\Sigma_{2}$ coincide and it is not possible to lower the cost without changing the final state.

### 3.4 Bulk action and $T \bar{T}$

We have considered the on-shell action of a cutout region of Poincaré $\mathrm{AdS}_{3}$, and interpreted it as a complexity functional of states in $T \bar{T}$-deformed holographic CFTs. The relation (3.16) between the coefficient of the $T \bar{T}$ deformation and the radial location has been derived for constant radial cutoff [48-50], but not for time-dependent $\rho(t)$. In this section we consider the flow equations which describe movement of the cutoff surface in a fixed background. By integrating these flow equations we should be able to derive a more precise relation between the coefficient of the $T \bar{T}$ deformation and the location of the bulk surface. In addition, these flow equations will tell us how complexity changes as we change the surface locations, and for which surfaces complexity is optimized while keeping the initial and final state fixed.

## Excluding counterterms

The relevant flow equation can most easily be derived using the ADM formalism [7]. We will keep the number of spacetime dimensions free in what follows, and write the metric as

$$
\begin{equation*}
d s^{2}=N^{2} d r^{2}+g_{\mu \nu}(x, r)\left(d x^{\mu}+N^{\mu} d r\right)\left(d x^{\nu}+N^{\nu} d r\right) \tag{3.54}
\end{equation*}
$$

This contains the usual lapse and shift functions, for which one can locally choose a convenient gauge $N=1$ and $N^{\mu}=0$. Following ADM and choosing units so that $\kappa=1$, we now write the Lagrangian in terms of canonical variables

$$
\begin{equation*}
\mathcal{L}=\sqrt{g}\left(\pi^{\mu \nu} \partial_{r} g_{\mu \nu}-N H-N^{\mu} H_{\mu}\right), \tag{3.55}
\end{equation*}
$$

where the lapse and shift functions appear as Lagrange multipliers enforcing the Hamiltonian and momentum constraints

$$
\begin{equation*}
H=H^{\mu}=0 . \tag{3.56}
\end{equation*}
$$

The canonical momenta are given by [77]

$$
\begin{align*}
\pi_{\mu \nu} & =\frac{1}{\sqrt{g}} \frac{\partial S}{\partial g^{\mu \nu}} \\
& =-\left(K_{\mu \nu}-K g_{\mu \nu}\right)  \tag{3.57}\\
& =-\frac{1}{2}\left(\partial_{r} g_{\mu \nu}-g_{\mu \nu} g^{\rho \sigma} \partial_{r} g_{\rho \sigma}\right)
\end{align*}
$$

where in the second step we used the fact that metric variations are given by the Brown-York tensor, and in the last step we used the explicit form of the extrinsic curvature for the metric (3.54) in the gauge $N=1, N^{\mu}=0$. Of course, the same
result can also be obtained by explicitly rewriting the action as in (3.55). Using (3.57), we find for the radial derivative

$$
\begin{equation*}
\partial_{r} g_{\mu \nu}=-2 \pi_{\mu \nu}+\frac{2}{d-2} g_{\mu \nu} \pi_{\rho}^{\rho} \tag{3.58}
\end{equation*}
$$

where $d$ is the total number of bulk spacetime dimensions (for now we are predominantly interested in $d=3$ ). The Hamiltonian constraint can be computed from (3.55) and, for unit AdS radius, one finds

$$
\begin{align*}
H & =R^{(d-1)}-2 \Lambda-\left(K^{2}-K^{\mu \nu} K_{\mu \nu}\right) \\
& =R^{(d-1)}+(d-1)(d-2)+\pi^{\mu \nu} \pi_{\mu \nu}-\frac{1}{d-2}\left(\pi_{\rho}^{\rho}\right)^{2} \tag{3.59}
\end{align*}
$$

It is fairly straightforward to include matter fields in the discussion; the Hamiltonian constraint will then also contain the Hamiltonian of the matter sector, but we will for simplicity restrict to the purely gravitational case. To describe the flow we imagine starting with a surface at constant $r$ and moving the cutoff slightly so that $r \rightarrow r+\epsilon(x)$. For any surface, we can always locally find coordinates such that the surface is located at fixed value of $r$ and the metric is in the ADM gauge, so there is no loss of generality in this assumption. Then

$$
\begin{align*}
\delta_{\epsilon} S & =\int \epsilon(x) \partial_{r} g^{\mu \nu} \frac{\partial S}{\partial g^{\mu \nu}} \\
& =\int \sqrt{g} \epsilon(x) \partial_{r} g^{\mu \nu} \pi_{\mu \nu}  \tag{3.60}\\
& =2 \int \sqrt{g} \epsilon(x)\left(\pi^{\mu \nu} \pi_{\mu \nu}-\frac{1}{d-2}\left(\pi_{\rho}^{\rho}\right)^{2}\right)
\end{align*}
$$

where we used equation (3.58) for the radial dependence of the metric in terms of momenta. Interestingly, this is precisely of $T \bar{T}$ form, but with $T$ and $\bar{T}$ defined with respect to the metric variations of the finite surface, not the boundary at infinity. A more coordinate independent way of stating the result is that as we move a surface in a given AdS background, we turn on a local $T \bar{T}$-deformation with a coefficient given by the orthogonal distance between the original and deformed surface. If we could relate the local $T$ and $\bar{T}$ on a given surface to the $T$ and $\bar{T}$ as defined at infinity, we could integrate these flow equations and write the final result in terms of a finite $T \bar{T}$ deformation of the theory at infinity. We leave a further exploration of this interesting question to future work but thinking of finite $T \bar{T}$ deformations in terms of a change in the boundary conditions for the metric we expect it to involve the linearized Einstein equations around the background [46].

Clearly, using (3.57) for $d=3$ the variation of the action vanishes if equation (3.40) is satisfied. As is clear from the Hamiltonian constraint, this condition can
also be phrased as $R^{(d-1)}+(d-1)(d-2)=0$, i.e. the boundary surface has constant scalar curvature. Therefore, to optimize the complexity of the process we should use constant scalar curvature surfaces; the metric on a Euclidean $\operatorname{AdS}_{d-1}$ manifold precisely has the required scalar curvature. This is consistent with the observation that complexity is minimized if we take $t_{i}=t_{f}$ and consider a purely radial surface at the $t=t_{f}$ constant timeslice in section 3.3.

## Including counterterms

So far the discussion has used the standard bulk AdS action without the inclusion of additional counterterms, which would render the on-shell value of the action finite as one takes the surface to the asymptotic boundary. As alluded to in the beginning, in the original appearance of Liouville theory as defining path integral complexity, the absence of the volume counterterm was important. Here we briefly discuss what happens if we add a volume term for the boundary surface with an arbitrary coefficient. In our discussion of the on-shell value of the action, it would add an extra term

$$
\begin{equation*}
S_{c . t}=-2 \lambda \int d^{2} x \sqrt{g}=-2 \lambda \int d t d x \frac{\sqrt{1+\dot{\rho}^{2}}}{\rho^{2}} \tag{3.61}
\end{equation*}
$$

Adding the counterterm modifies the field equations to

$$
\begin{equation*}
\frac{1}{\rho^{3}\left(1+\dot{\rho}^{2}\right)}\left(\left(\rho \ddot{\rho}+1+\rho^{2}\right)-\lambda \sqrt{1+\dot{\rho}^{2}}\left(\frac{1}{2} \rho \ddot{\rho}+1+\dot{\rho}^{2}\right)\right)=0 \tag{3.62}
\end{equation*}
$$

We can also reconsider the flow equations in the presence of the volume counterterm. Denoting the volume counterterm as

$$
\begin{equation*}
S_{\mathrm{vol}}=-2 \lambda(d-2) \int_{\partial M} \sqrt{g} \tag{3.63}
\end{equation*}
$$

so that $\lambda=1$ is precisely the counterterm which would cancel the volume divergence near the AdS boundary, we now introduce $\tilde{\pi}_{\mu \nu}=\pi_{\mu \nu}-\lambda(d-2) g_{\mu \nu}$ so that these are precisely the canonical momenta in the presence of the extra boundary volume term. The Hamiltonian constraint can be rewritten as

$$
\begin{equation*}
H=R^{(d-1)}+\left(1-\lambda^{2}\right)(d-1)(d-2)+\tilde{\pi}^{\mu \nu} \tilde{\pi}_{\mu \nu}-\frac{1}{d-2}\left(\tilde{\pi}_{\rho}^{\rho}\right)^{2}-2 \lambda \tilde{\pi}_{\rho}^{\rho}=0 \tag{3.64}
\end{equation*}
$$

We can now consider two types of flows. We can consider the variation of the action as we change the radial surface in a given background, but we can also consider the variation of the action as we perform a conformal rescaling of the metric on the radial surface. In $d=3$, the latter does not require an adjustment of the
bulk geometry, but in higher dimensions this is no longer true. It is therefore not clear whether conformal rescalings of the induced metric on the boundary surface are in general compatible with keeping the initial and final states fixed in $d>3$. Regardless, the change of the action under the first type of flow now reads

$$
\begin{align*}
\delta_{\epsilon} S & =\int \epsilon(x) \partial_{r} g^{\mu \nu} \frac{\partial \tilde{S}}{\partial g^{\mu \nu}} \\
& =\int \sqrt{g} \epsilon(x) \partial_{r} g^{\mu \nu} \tilde{\pi}_{\mu \nu}  \tag{3.65}\\
& =2 \int \sqrt{g} \epsilon(x)\left(\tilde{\pi}^{\mu \nu} \tilde{\pi}_{\mu \nu}-\frac{1}{d-2}\left(\tilde{\pi}_{\rho}^{\rho}\right)^{2}-\lambda \tilde{\pi}_{\rho}^{\rho}\right)
\end{align*}
$$

and for the second type of flow with $\delta g^{\mu \nu}=\epsilon(x) g^{\mu \nu}$

$$
\begin{align*}
\delta_{\epsilon} \tilde{S} & =\int \epsilon(x) g^{\mu \nu} \frac{\partial \tilde{S}}{\partial g^{\mu \nu}} \\
& =\int \sqrt{g} \epsilon(x) \tilde{\pi}_{\rho}^{\rho} \\
& =\frac{1}{2 \lambda} \int \sqrt{g} \epsilon(x)\left(R^{(d-1)}+\left(1-\lambda^{2}\right)(d-1)(d-2)+\tilde{\pi}^{\mu \nu} \tilde{\pi}_{\mu \nu}-\frac{1}{d-2}\left(\tilde{\pi}_{\rho}^{\rho}\right)^{2}\right) \tag{3.66}
\end{align*}
$$

We see that both flows take the form of $T \bar{T}$ deformations, with various extra terms such as the scalar curvature and the trace of the stress tensor. Just as in the case without counterterm $(\lambda=0)$ it would be interesting to integrate these flows to finite flows starting at the AdS boundary.

The first flow is extremized when the surface obeys

$$
\begin{equation*}
R^{(d-1)}+(d-1)(d-2)-\lambda(d-2) K=0 \tag{3.67}
\end{equation*}
$$

which still holds for an $\operatorname{AdS}_{d-1}$ equal time slice in $\mathrm{AdS}_{d}$. As expected, for our setup (3.67) is equivalent to (3.62). The second flow, on the other hand, is extremized when $K=\lambda(d-1)$. This does not have an extremum for an $\mathrm{AdS}_{d-1}$ equal time slice in $\operatorname{AdS}_{d}$ unless $\lambda=0$. Moreover, as we indicated above, it is not clear whether the initial state and final state are kept fixed along the flow, and therefore the precise interpretation of this flow is somewhat unclear. In any case, it would be interesting to explore whether surfaces obeying (3.67) or $K=\lambda(d-1)$ have the potential to define a new notion of complexity.

Finally, we notice that it is also possible to add higher order counterterms, but for those the connection to $T \bar{T}$ deformations becomes more complicated.

### 3.5 Relation to kinematic space

In the above, we have often tacitly assumed that the information about the bulk surface $z=\rho(t)$ is encoded locally in the boundary theory. However, as our discussion of flows shows, it is highly questionable whether this is a reasonable assumption. A better way to encode the information of the surface $z=\rho(t)$ in the boundary theory is through pairs of points $\left(t_{1}(t), t_{2}(t)\right.$ ) (with $x=0$ ) on the boundary, such that the geodesic that starts at $t_{1}(t)$ and ends at $t_{2}(t)$ is tangent to the bulk surface at the point $(z=\rho(t), t, 0)$, see figure 3.4.


Figure 3.4: We can parameterize a generic bulk curve $\rho(t)$ by the pairs of boundary points $\left(t_{1}(t), t_{2}(t)\right)$, such that a bulk geodesic connecting these two points is tangent to the bulk curve at $z=\rho(t)$. This way, the profile $\rho(t)$ is encoded as a path in kinematic space, the space of bulk geodesics.

This construction has the benefit of being covariant, and viewing Euclidean time as another spatial coordinate, these geodesics encode precisely the entanglement wedges which touch the surface but do not cross it. In other words, they precisely encode the information about those regions of spacetime we try to omit in our bulk path integral construction. One can ask whether there is a natural geometry associated to the pairs of points of this type, and the answer is yes. Conformal invariance produces a natural metric on the space of pairs of points, also known as kinematic space [78]. For the case at hand it is given up to an undetermined constant prefactor by the 2 d de Sitter metric

$$
\begin{equation*}
d s_{k s}^{2}=\frac{-d t_{1} d t_{2}}{\left(t_{1}-t_{2}\right)^{2}} \tag{3.68}
\end{equation*}
$$

In the spirit of defining complexity by assigning a metric to a group of transformations [28-30], we can now ask what the length of the path in this geometry associated with $\rho(t)$ is. To compute it explicitly, we need the explicit form of $t_{1}(t)$ and $t_{2}(t)$. These are given by

$$
\begin{equation*}
t_{1,2}(t)=t+\rho \dot{\rho} \pm \rho \sqrt{\dot{\rho}^{2}+1} \tag{3.69}
\end{equation*}
$$

Consider now the action

$$
\begin{equation*}
S_{k s} \sim \int \frac{d x}{\rho} d s_{k s}(t) \tag{3.70}
\end{equation*}
$$

where we included the coordinate $x$ in units of the cutoff $\rho$, and the distance $d s$ obtained from (3.68) upon inserting (3.69). This results in

$$
\begin{equation*}
S_{k s} \sim \int d t d x\left|\frac{\rho \ddot{\rho}+\left(1+\dot{\rho}^{2}\right)}{\rho^{2}\left(1+\dot{\rho}^{2}\right)}\right|, \tag{3.71}
\end{equation*}
$$

which agrees precisely with the bulk action in the form (3.20) as long as $\ddot{\rho} \geq$ $-\rho^{-1}\left(1+\dot{\rho}^{2}\right)$. This is related to the fact that the kinematic space is a Lorentzian manifold and the condition in question is the one that one moves there along a timelike path.

This strongly suggests that the relevant circuit geometry for these types of finite bulk surface computations is a version of kinematic space ${ }^{3}$. Note that on-shell, (3.71) vanishes exactly for the semi-circular arcs that solve (3.23), as they are also geodesics in AdS-space. In other words, for these solutions the path traversed in kinematic space shrinks to a point.

Note that alternatively one may use the standard kinematic space prescription built around entanglement entropy of intervals on constant $t$ time slices. The metric (3.68) is the same but now with $t_{1}$ and $t_{2}$ replaced simply by $x_{1}$ and $x_{2}$ with

$$
\begin{equation*}
x_{1,2}(t)= \pm \rho(t) \tag{3.72}
\end{equation*}
$$

Using again (3.70) gives this time

$$
\begin{equation*}
S_{k s^{\prime}} \sim \int d t d x \frac{|\dot{\rho}|}{\rho^{2}} \tag{3.73}
\end{equation*}
$$

This is clearly a different expression than (3.71), which however bears a striking similarity with the gate counting approach of [28-30] when the latter uses a Manhattan norm.

[^8]Let us mention that generalizing the kinematic space consideration leading to (3.71) to more complicated geometries is not obvious as minimal geodesics do not necessarily penetrate the whole spacetime. In the case of geodesics computing the entanglement entropy, these are entanglement shadows [24] and they appear, for example, in the case of double-sided black holes.

Finally, let us mention that the relation between kinematic space and complexity was explored earlier in two different instances in [79] and [80], however, these proposals are distinct from ours and use a standard entanglement-based kinematic space.

## Kinematic space analysis in global AdS

The on-shell gravitational action for global $\mathrm{AdS}_{3}$ with a time-dependent boundary surface $\theta(\tau)$, given in (3.45), can be rewritten as

$$
\begin{align*}
I & =\frac{2}{\kappa} \int d \phi d \tau\left(\frac{1}{\cos \theta^{2}}-\frac{\ddot{\theta} \tan \theta}{\left(1+\dot{\theta}^{2}\right)}\right)+I_{c} \\
& =\frac{2}{\kappa} \int d \phi d \tau\left(\tan \theta\left(\frac{\tan \theta\left(1+\dot{\theta}^{2}\right)-\ddot{\theta}}{\left(1+\dot{\theta}^{2}\right)}\right)+1\right)+I_{c} \tag{3.74}
\end{align*}
$$

Ignoring an overall additive factor and the corner term, the remaining action reads

$$
\begin{equation*}
I=\frac{2}{\kappa} \int d \phi d \tau \tan \theta\left(\frac{\tan \theta\left(1+\dot{\theta}^{2}\right)-\ddot{\theta}}{\left(1+\dot{\theta}^{2}\right)}\right) . \tag{3.75}
\end{equation*}
$$

This action can in fact be reproduced by considering the kinematic space of bulk curves. The data of time-dependent bulk surfaces $\theta(\tau)$ can be encoded in the boundary. This is done by giving a pair of boundary points $\left(\tau_{1}(\tau), \tau_{2}(\tau)\right)$ such that the bulk geodesic starting at $\tau_{1}(\tau)$ and ending at $\tau_{2}(\tau)$ is tangent to the bulk surface at the point $(\tau, \theta(\tau), 0)$. Such pairs of points form the kinematic space, with a metric fixed by conformal invariance

$$
\begin{equation*}
d s_{k s}^{2}=-\frac{4 d \tau_{1} d \tau_{2}}{\sinh ^{2}\left(\tau_{1}-\tau_{2}\right)} \tag{3.76}
\end{equation*}
$$

The explicit dependence of the boundary points $\tau_{1,2}(\tau)$ on the bulk time $\tau$ is

$$
\begin{equation*}
\tau_{1,2}(\tau)=\tau+\log \left(\frac{1 \pm \cos \theta \sqrt{1+\dot{\theta}^{2}}}{\sin \theta+\dot{\theta} \cos \theta}\right) \tag{3.77}
\end{equation*}
$$

Now consider an action built using kinematic space, as

$$
\begin{equation*}
S_{k s}=\frac{2}{\kappa} \int(\tan \theta d \phi) d s_{k s} \tag{3.78}
\end{equation*}
$$



Figure 3.5: A subregion of Lorentzian Poincaré AdS between two constant time slices with a time dependent boundary. The path integral of the $T \bar{T}$-deformed theory on the $z=$ $\rho(t)$ boundary defines a path through the space of unitaries, and the gravitational action of the shaded region, including boundary and corner terms, is the natural Lorentzian extension of our previous proposal [1] for the length or cost of that path.

Using (3.77) and the distance in kinematic space $d s_{k i n}$ the above action equals

$$
\begin{equation*}
S_{k s}=\frac{2}{\kappa} \int(\tan \theta d \phi) d s_{k s}=\frac{2}{\kappa} \int d \phi d \tau \tan \theta\left(\frac{\tan \theta\left(1+\dot{\theta}^{2}\right)-\ddot{\theta}}{\left(1+\dot{\theta}^{2}\right)}\right) . \tag{3.79}
\end{equation*}
$$

This matches exactly to the on-shell gravitational action obtained above after ignoring an overall additive factor and the corner term, and is the global AdS equivalent to the result for Poincaré $\operatorname{AdS}$ that we had obtained in section 4.2 of [1].

### 3.6 Obstacles to obtain the CA proposal from a cost

Here we will see that the naive extension of our Euclidean cost proposal from [1] to Lorentzian Poincaré $\mathrm{AdS}_{3}$ gives unphysical results. The proposal is to extremise the action of the subregion of Lorentzian Poincaré $\mathrm{AdS}_{3}$,

$$
\begin{equation*}
d s^{2}=\frac{-d t^{2}+d z^{2}+d x^{2}}{z^{2}} \tag{3.80}
\end{equation*}
$$

shown in figure 3.5 to see whether these extrema can be sensibly interpreted as circuit cost or complexity of Lorentzian time evolution between the initial and
final time slice. In close analogy to the Euclidean result given in equation (2.10) of [1], the gravitational action of this subregion is (see appendix A for details of the calculation)

$$
\begin{equation*}
I=\frac{1}{8 \pi G_{N}} \int d x \int_{t_{i}}^{t_{f}} d t\left(\frac{1-\dot{\rho} \operatorname{arctanh} \dot{\rho}}{\rho^{2}}\right) \tag{3.81}
\end{equation*}
$$

This action is unbounded from below; it can be seen that $I \rightarrow-\infty$ in the limit of the cutoff surface becoming null, $|\dot{\rho}| \rightarrow 1$. Whether the action is bounded from above depends on the boundary conditions. The solution to the Euler-Lagrange equations for general boundary conditions $\rho\left(t_{i}\right)=\rho_{i}, \rho\left(t_{f}\right)=\rho_{f}$ is

$$
\begin{equation*}
\rho(t)=\sqrt{t^{2}+A t+B} \tag{3.82}
\end{equation*}
$$

where

$$
\begin{equation*}
A=-\frac{t_{f}^{2}-t_{i}^{2}+\rho_{i}^{2}-\rho_{f}^{2}}{t_{f}-t_{i}}, \quad B=\frac{\left(t_{f}-t_{i}\right) t_{i} t_{f}+t_{f} \rho_{i}^{2}-t_{i} \rho_{f}^{2}}{t_{f}-t_{i}} \tag{3.83}
\end{equation*}
$$

This solution is a local maximum of (3.81). In contrast to what we observed in the Euclidean case in [1], this timelike cutoff surface bends outwards from $\rho_{i / f}$ towards the asymptotic boundary, as it tries to maximise the $\rho^{-2}$ factor. The solution fails to be real when the time interval becomes too large,

$$
\begin{equation*}
t_{f}-t_{i}>\rho_{i}+\rho_{f} \tag{3.84}
\end{equation*}
$$

Roughly speaking this is because when the time interval is large compared to the spatial initial and final cutoffs, the cutoff surface can get to the asymptotic boundary and back without $\dot{\rho}$ becoming large enough to flip the sign of numerator in (3.81). Once at the asymptotic boundary, the denominator diverges, leading to an action that is unbounded from above, which is why we find no (real) solution that extremises the action. In fact, the action can be arbitrarily negative for other cutoff surfaces as well. For a more generic cutoff surface $\rho(t, x)=\sqrt{r(x)^{2}+t^{2}}$ (this is derived as a solution to equation (3.96) which we discuss in section 3.8), the action is given by

$$
\begin{equation*}
I=\frac{2}{\kappa} \int d t d x \frac{r(x) r^{\prime \prime}(x)}{\left(t^{2}+r(x)^{2}\right)\left(1+r^{\prime}(x)^{2}\right)} \tag{3.85}
\end{equation*}
$$

We see that the above integral can turn out to be negative depending on the choice of $r(x)$. For example, with $r(x)=\sin (\omega x)$ the action computed in the region $t, x \in[0,1]$ is negative and proportional to $\omega^{2}$. Thus, by making the cutoff surface more wavy the action can be made arbitrarily negative.

### 3.7 Linear growth at late times for BTZ black hole

The notion of holographic complexity was initially used for describing the growth of black hole interiors for long times. Any measure of complexity must exhibit a late time linear growth in black hole backgrounds. We saw that the cost function given by Euclidean gravitational action in the region bounded by two $K=0$ slices $\Sigma_{1,2}$ and $\tilde{M}$ was well-defined and gave sensible results in global $A d S$. Moreover, choosing a trivial $\Sigma_{1}$ and optimising over $\tilde{M}$ lead to the action between a constant scalar curvature slice and $\Sigma_{2}$. In this subsection, we will assume we can evaluate the Lorentzian action between these surfaces in the BTZ black hole and verify that this exhibits linear growth at late times. Therefore, it is a candidate new holographic complexity proposal.

Consider a BTZ black hole with horizon radius $r_{h}=1$ and $\operatorname{AdS}$ radius $L=1$. Then using Kruskal coordinates, the metric takes the simple form [81]

$$
\begin{equation*}
d s^{2}=-\frac{4 d U d V}{(1+U V)^{2}}+\frac{(1-U V)^{2}}{(1+U V)^{2}} d \phi^{2} \tag{3.86}
\end{equation*}
$$

The mass of the black hole is $M=\frac{r_{n}^{2}}{8 G L^{2}}=\frac{1}{8 G}$. We have the asymptotic AdS boundaries located at $U V=-1$ and the horizons at $U V=0$. Maximal volume slices have vanishing trace of extrinsic curvature

$$
\begin{equation*}
K=\frac{\left(U^{2} V^{2}-1\right) U^{\prime \prime}+2(U V-3) U^{\prime}\left(U-U^{\prime} V\right)}{4(U V-1)\left|U^{\prime}\right|^{3 / 2}}=0 \tag{3.87}
\end{equation*}
$$

Here, ' denotes a $V$ derivative. In fact, the maximal surfaces are best described in Eddington-Finkelstein coordinates

$$
\begin{equation*}
d s^{2}=-f(r) d v^{2}+2 d v d r+r^{2} d \phi^{2} \tag{3.88}
\end{equation*}
$$

with $f(r)=r^{2}-1$. Then, the shape $v(r)$ of the maximal surfaces is given as $[82,83]$

$$
\begin{equation*}
\frac{d v}{d r}=\frac{\sqrt{f(r) r^{2}+c^{2}}-c}{f(r) \sqrt{f(r) r^{2}+c^{2}}} \tag{3.89}
\end{equation*}
$$

The constant $c$ determines the boundary time at which the maximal surface is anchored. It goes from $c=0$ for the surface anchored at $t_{L}=t_{R}=0$ to $c=\frac{1}{2}$ for the final slice. The final maximal slice for the BTZ black hole is at $r=\frac{1}{\sqrt{2}}$ and at late times maximal surfaces pile up very close to this surface. Constant scalar curvature slices satisfying $R+2=0$ are much easier to find, and are given by

$$
\begin{equation*}
U V+\lambda U+\mu V-1=0 \tag{3.90}
\end{equation*}
$$



Figure 3.6: Growth of action in the BTZ black hole between constant curvature surfaces (blue) and maximal volume surfaces (green).

Here $\lambda$ and $\mu$ again determine where these surfaces are anchored at the boundary.
Now consider the gravitational action within the region bounded by the maximal surface and a constant curvature surface both fixed at the same boundary time $t_{L}=t_{R}=t$. To compute the growth rate, consider two such regions separated by a small boundary time $\delta t$. Let us compute the difference in the actions of these nearby regions. The total action outside the horizon is time-independent and can be ignored. This leaves us with two regions inside the horizon: action in the blue region between constant curvature surfaces minus action in the red region between maximal surfaces in figure 3.6. For the blue region inside the horizon, the bulk action is given by

$$
\begin{equation*}
\delta I_{b u l k}=-\frac{1}{2 G_{N}}\left(\tanh t+\frac{t}{\cosh t^{2}}\right) \delta t \tag{3.91}
\end{equation*}
$$

To this, we also need to add boundary and corner contributions. For the boundary terms, we have the usual GHY surface terms along $A B$ and $C D$ (see figure 3.6), and also null boundaries along the horizon segments $A C$ and $B D$. We choose the normals to the null surface to be affinely parameterised, hence the latter terms can be set to zero. As the expansion parameter along these (Killing) horizon segments vanishes, we can likewise ignore the counter terms proposed for null surfaces [45]. The boundary terms $\delta I_{b d y}$ from the segments $A B$ and $C D$ exactly cancel the above bulk term, leaving us with four corner terms. Since these corners arise from the intersection of a spacelike surface and the null horizon, the appropriate action [84] is

$$
\begin{equation*}
I_{\text {corner }}=\frac{1}{8 \pi G_{N}} \int_{\partial \Sigma} d \phi \sqrt{\sigma} a \tag{3.92}
\end{equation*}
$$

where $\sigma$ is the metric on the corner $\partial \Sigma$ and $a= \pm \log (\mathbf{k} \cdot \mathbf{n})$, with $\mathbf{k}$ and $\mathbf{n}$ being the normals to both the surfaces at the corner. Calculating this for corners on both sides gives

$$
\begin{equation*}
\delta I_{\text {corner }}=\frac{1}{2 G_{N}} \tanh t \delta t \tag{3.93}
\end{equation*}
$$

For the red region, we only have the bulk and corner terms, since $K=0$ along the boundaries. We shall argue that both these terms can be safely ignored in the late time limit. The bulk action can be computed numerically from the shape of surface in (3.89). At late boundary time, since these surfaces pile up close to the final surface, this action is negligible. Computing the corner term from (3.92) gives

$$
\begin{equation*}
\delta I_{\text {corner }}=\frac{1}{2 G_{N} c(t)} \frac{d c(t)}{d t} \delta t \tag{3.94}
\end{equation*}
$$

where $c(t)$ is implicitly given by (3.89). Again, at late boundary times $t \gg 1$, we have $c \approx \frac{1}{2}$, hence this term doesn't contribute as well. Now, adding up all the contributions we have

$$
\begin{equation*}
\frac{d I_{\text {total }}}{d t_{b d y}}=\frac{\tanh t}{4 G_{N}}=2 M \tanh \frac{t_{b d y}}{2} \tag{3.95}
\end{equation*}
$$

where $t_{b d y}=t_{L}+t_{R}=2 t$ is the total boundary time. Since at late boundary times $\tanh \frac{t_{b d y}}{2} \approx 1$, the action grows linearly.

### 3.8 General methods for gravitational action proposals

Let us quickly summarize some of the achievements of the preceding section. We showed in section 3.3 how a cost-proposal based on the bulk gravitational action can reproduce the complexity=volume proposal on a Euclidean global AdS background and we discussed how this result can be connected to the geometry of kinematic space, i.e. the space of spacelike bulk geodesics, in section 3.5. While there are problems with the generalisation of this ansatz to Lorentzian cases as discussed in section 3.6 , it is interesting to note the prominent role that surfaces of a constant intrinsic curvature (such as (3.50) and (3.82)) play in all these attempts as solutions to the equations of motion derived by extremising the action. This motivated us in section 3.7 to propose a new complexity proposal based partially on constant intrinsic curvature surfaces that was shown to pass at least one important plausibility check, namely late time linear growth in a black hole background.

For this reason, in this section we will now give a more general analysis of the general equations derived in [1] (of which sections 3.3 and 3.6 only provide special
examples). As we are about to explain, a quite generic solution method can be formulated based on foliating surfaces by geodesic curves, which in turn might suggest a deeper and more general connection to the physics and geometry of the kinematic space than what we discussed in [1] and section 3.5. However, a more detailed study of such a possible deeper connection will be left for future research.

### 3.8.1 Equations of motion

In [1] we essentially analysed a problem where co-dimension one hypersurfaces $\tilde{M}$ were embedded into $\mathrm{AdS}_{3}$ according to the equation ${ }^{4}$

$$
\begin{equation*}
K_{m}^{n} K_{n}^{m}-K^{2}=0 \tag{3.96}
\end{equation*}
$$

Herein $K_{m n}$ is the extrinsic curvature tensor of the surface and $K=K_{n}^{n}$ is its trace. Latin indices are raised and lowered with the induced metric $g_{m n}$. The potential physical interpretations of this equation are manifold. Our main interpretation in [1] was that when deriving a notion of state complexity by extremising the action of a bulk region bounded by initial and final time slices as well as a variable boundary surface, (3.96) arises as the equation of motion of that surface. Additionally, in section 3 of [1] we pointed out how surfaces satisfying (3.96) arise from flow equations which describe movement of the cutoff surface in a fixed background, while in section 4 of [1] we pointed out a connection with kinematic space. In the following, we will continue to explore these possible interpretations in more generality than what was possible in [1].

To do so, we should first point out that the derivation given in section 3.1 of [1] is independent of the number of dimensions and equally applicable to the Lorentzian case, hence from now on we take equation (3.96) to be the equation of interest even in the general case. ${ }^{5}$ Also, due to the Hamiltonian constraint ${ }^{6}$

$$
\begin{equation*}
0 \equiv H=R-2 \Lambda-\left(K_{m}^{n} K_{n}^{m}-K^{2}\right) \tag{3.97}
\end{equation*}
$$

equation (3.96) corresponds to demanding that the Ricci curvature $R$ of the induced metric of the surface is constant. Specifically, if we focus on three bulk

[^9]dimensions and set the AdS-radius to $L=1 \leftrightarrow \Lambda=-1$, then $\mathcal{R}=-6$ and $R=-2$. Of course, the problem of constant curvature surfaces embedded into maximally symmetric ambient spaces is well studied in the mathematical literature, see e.g. [86-93] and references therein for interesting results. However due to differences in notation and nomenclature in the mathematical literature, in the following sections we will spell out the most relevant facts for our case in our own language and try to give them a physical interpretation from the perspective of holography.

The first observation we can make about (3.96) is that it can be written solely in terms of the object $K_{m}^{n}$. This is reminiscent of the paper [94], where the authors studied (one-dimensional) curves with more complicated equations of motion than merely geodesic equations. The authors there found that in some cases, it was possible to phrase these equations in terms of extrinsic curvature as a function of an affine parameter. Then, a solution can be obtained in a two-step procedure: first by solving the equation for the extrinsic curvature, and then finding an embedding for a curve that actually has this extrinsic curvature as a function of the affine parameter. Similarly, we could try to solve (3.96) by firstly finding any tensor $K_{m}^{n}$ (dependent on generic induced coordinates $y^{a}$ ) that satisfies this equation ${ }^{7}$, and then solving for the embedding of a hypersurface in the ambient spacetime that, for the correct choice of induced coordinate system, has the extrinsic curvature found in the first step of the solution procedure. Unfortunately we have not been able to carry out this procedure in general, hence in the next subsection we will study a particular ansatz to solve (3.96).

### 3.8.2 Solution method, totally geodesic foliations

It is trivial to see that an ansatz of the form

$$
\begin{equation*}
K_{m n}=m_{m} m_{n} k \tag{3.98}
\end{equation*}
$$

with some vector $m$ and some function $k$ will automatically satisfy (3.96). We can demand $m$ to be normalized, or alternatively we could allow $m$ to be unnormalised and absorb $k$ into its norm up to an overall sign. Which convention is more useful depends on the problem at hand. Firstly, let us discuss how general this ansatz is. For two dimensional surfaces, (3.96) is equivalent to det $K_{m n}=0$ and hence to (3.98), i.e. this ansatz is generic in this case. For higher dimensions however, (3.98) only covers a small subset of the solutions of (3.96).

For a hypersurface embedded into an ambient spacetime, we can utilize the Codazzi

[^10]equations. Besides (3.96), a consistent embedding into an ambient space with given $K_{m n}$ needs to satisfy the following equations [85]:
\[

$$
\begin{align*}
\mathcal{R}_{\alpha \beta \gamma \delta} e_{a}^{\alpha} e_{b}^{\beta} e_{c}^{\gamma} e_{d}^{\delta} & =R_{a b c d} \pm\left(K_{a d} K_{b c}-K_{a c} K_{b d}\right)  \tag{3.99}\\
\mathcal{R}_{\mu \beta \gamma \delta} n^{\mu} e_{b}^{\beta} e_{c}^{\gamma} e_{d}^{\delta} & =K_{b c \mid d}-K_{b d \mid c}  \tag{3.100}\\
\left(\mathcal{R}_{\alpha \beta}-\frac{1}{2} \mathcal{R} G_{\alpha \beta}\right) n^{\beta} e_{a}^{\alpha} & =K_{a \mid b}^{b}-K_{, a} \tag{3.101}
\end{align*}
$$
\]

Note that the bracket in (3.99) automatically vanishes with our ansatz, hence if the ambient space has a Riemann-tensor of the form of a maximally symmetric spacetime

$$
\begin{equation*}
\mathcal{R}_{\alpha \beta \gamma \delta}=\frac{\mathcal{R}}{d(d-1)}\left(G_{\alpha \gamma} G_{\beta \delta}-G_{\alpha \delta} G_{\beta \gamma}\right), \tag{3.102}
\end{equation*}
$$

then due to the projections in (3.99) the Riemann tensor of the induced metric will have a similar maximally symmetric form in terms of the induced metric and its Ricci scalar. Hence our ansatz (3.98) necessarily describes a hypersurface whose induced metric is (locally ${ }^{8}$ ) maximally symmetric, and because in an $\operatorname{AdS}_{d}$ background (3.96) implies a negative induced curvature, the induced metric has to be locally $\mathrm{AdS}_{d-1}$. Furthermore, under the assumption of embedding into a locally AdS space, the left-hand sides of (3.100) and (3.101) will vanish because $G_{\alpha \beta} n^{\beta} e_{a}^{\alpha}=0$, giving us an interesting set of differential equations for $m$ and $k$. Assuming we can set $k= \pm 1$ at least in certain regions of the hypersurface, (3.100) gives:

$$
\begin{equation*}
0=m_{c} \nabla_{d} m_{b}+m_{b} \nabla_{d} m_{c}-m_{d} \nabla_{c} m_{b}-m_{b} \nabla_{c} m_{d} \tag{3.103}
\end{equation*}
$$

So in general, we would have to find a vector field $m$ in a locally AdS space that satisfies (3.103), and then see whether there actually is a surface embedded into AdS that has the corresponding extrinsic curvature and induced metric in the induced coordinate system of our choice.

Let us now focus on $d=3$ dimensional ambient spaces, i.e. two dimensional hypersurfaces for the moment. Clearly, $m^{a}$ is a vector in the tangent space to the hypersurface, so there is one perpendicular direction in the tangent space, and we introduce the tangent vector field $l^{a}$, such that $l^{a} m_{a}=0, l^{a} l_{a}=$ const. What can we learn about this vector field? Take equation (3.103), and contract it with $l^{c} l^{b}$ :

$$
\begin{equation*}
0=m_{d} l^{c} l^{b} \nabla_{c} m_{b} \Rightarrow 0=l^{c} l^{b} \nabla_{c} m_{b} \tag{3.104}
\end{equation*}
$$

[^11]We hence know

$$
\begin{equation*}
l^{b} m_{b}=0 \Rightarrow 0=l^{c} \nabla_{c}\left(l^{b} m_{b}\right)=\left(l^{c} \nabla_{c} l^{b}\right) m_{b}+\underbrace{l^{c} l^{b} \nabla_{c} m_{b}}_{=0} . \tag{3.105}
\end{equation*}
$$

That means the projection of the vector $l^{c} \nabla_{c} l^{b}$ on the $m_{b}$ direction has to vanish. As we assume the hypersurface worldvolume to be 2-dimensional, the only other direction is $l_{b}$. We find:

$$
\begin{equation*}
\left(l^{c} \nabla_{c} l^{b}\right) l_{b} \propto l^{c} \nabla_{c} l^{b} l_{b}=0 \tag{3.106}
\end{equation*}
$$

It follows that (3.105) and (3.106) together imply the geodesic equation

$$
\begin{equation*}
l^{c} \nabla_{c} l^{b}=0 \tag{3.107}
\end{equation*}
$$

in the induced metric of the hypersurface. Thus our ansatz (3.98) implies that the integral lines of the normalised vector-field perpendicular to the direction $m_{a}$ have to be geodesics which foliate the hypersurface. So far, we are explicitly talking about the geodesic equation with respect to the induced metric, but as (3.98) implies $K_{a b} l^{a} l^{b}=0$ these curves also have vanishing extrinsic curvature in the normal direction to the hypersurface. Hence these curves foliating the hypersurface will also be geodesics with respect to the ambient metric. We can show this explicitly. The relation between the covariant derivative in the ambient space $X_{; \beta}$ and the covariant derivative in the induced metric $X_{\mid b}$ gives [85]

$$
\begin{equation*}
l_{; \beta}^{\alpha} e_{b}^{\beta}=l_{\mid b}^{a} e_{a}^{\alpha} \pm l^{a} K_{a b} n^{\alpha} . \tag{3.108}
\end{equation*}
$$

Contracting (3.108) with $l^{b}$, we find

$$
\begin{equation*}
\underbrace{l^{\beta} l_{; \beta}^{\alpha}}_{\text {ambient space geodesic eq. }}=\underbrace{l^{b} l_{l b}^{a}}_{\text {induced metric geodesic eq. }} e_{a}^{\alpha} \pm l^{b} l^{a} K_{a b} n^{\alpha} . \tag{3.109}
\end{equation*}
$$

Herein, $l^{\beta}$ is the ambient space form of the vector field $l^{b}$ in the hypersurface. Thus, if the 2 d hypersurface is foliated by curves (with tangent vector $l$ ) that are both geodesics of the ambient space and the induced metric (i.e. totally geodesic), then necessarily $K_{a b} l^{a} l^{b}=0$. On the other hand, if $K_{a b} b^{a} l^{b}=0$ is given and as derived above the geodesic equation with respect to the induced metric is satisfied, then so will be the geodesic equation with respec to the ambient metric.

To summarise, we have shown that in $\mathrm{AdS}_{3}$, the constant curvature surfaces that we are trying to find as solutions of (3.96) (which implies (3.98) in three bulk dimensions) are foliated by curves that are geodesics both with respect to the ambient space and the induced metric. This is just the AdS equivalent of the well known statement in $\mathbb{R}^{3}$ that all developable surfaces (i.e. $R=0$ ) are ruled surfaces
(i.e. foliated by straight lines in $\mathbb{R}^{3}$ ) [93], however both in this and in our case, the converse is not true. We can use this realisation to construct hypersurfaces that will solve (3.96) subject to quite generic boundary conditions, as we demonstrate in section 3.8.3. Interestingly, the result of this section hence implies a relation between solutions of (3.96) in three bulk dimensions and the abstract space of geodesics of the bulk spacetime. This space of geodesics generalises the well known kinematic space [78] which we use for example in section 3.5 by including geodesics not restricted to an equal time slice as well as timelike geodesics [34, 95, 96]. In three bulk dimensions, this space will be four dimensional, and as we have shown in this section, a surface solving (3.96) will correspond to a curve in this space of geodesics, each point along this curve corresponding to one geodesic which constitutes a slice of the codimension-one surface $\tilde{M}$ in the bulk. While there is a considerable freedom of how such curves in the space of geodesics can look like, corresponding in part to our freedom of choosing arbitrary boundary conditions for the surface $\tilde{M}$ in the bulk, not every such curve generates a bulk surface that solves (3.96). It would hence be interesting to try and rephrase equation (3.96) as a constraint on curves in the space of bulk geodesics, but we leave this for future work. Furthermore, it was discussed in [34] that the space of timelike geodesics in $\mathrm{AdS}_{3}$ can be mapped to the space of coherent states of the CFT. Under this identification, the Lorentzian solutions which we will later construct in section 3.8.5 would receive the interpretation of corresponding to (closed) paths in this space of states, however we will also leave it to future research to investigate the possible significance of this observation.

### 3.8.3 Examples

In [1], we derived solutions to (3.96) in Euclidean Poincaré $\mathrm{AdS}_{3}$ anchored to two constant time slices at different times on the boundary. The solution was a translation invariant hypersurface with semi-circular cross sections, and we remarked that these semicircular cross-sections are geodesics of the ambient space. In light of the results discussed in the previous subsection, this observation is now not surprising anymore. In fact, we can quite easily generalise our solution to the case where translation invariance is broken, assuming only a mirror symmetry between initial and final time slice. This is done by the ansatz

$$
\begin{equation*}
z(t, x)=\sqrt{r(t)^{2}-x^{2}} \tag{3.110}
\end{equation*}
$$

where $r(t)$ is an arbitrarily varying (half) width along the $t$-axis, and we have used the usual coordinate system on (Euclidean) Poincaré $\mathrm{AdS}_{3}$ that gives us the line element

$$
\begin{equation*}
d s^{2}=\frac{1}{z^{2}}\left(d t^{2}+d x^{2}+d z^{2}\right) \tag{3.111}
\end{equation*}
$$

where from now on we set the AdS-scale to $L=1$ for simplicity. The case in [1] was simply $r(t)=$ const. This embedding indeed satisfies (3.96). Likewise, in Lorentzian global $\mathrm{AdS}_{3}$ with line element

$$
\begin{equation*}
d s^{2}=\frac{1}{\cos (\theta)^{2}}\left(-d t^{2}+d \theta^{2}+\sin (\theta)^{2} d \phi^{2}\right) \tag{3.112}
\end{equation*}
$$

we can now easily construct the surface

$$
\begin{equation*}
t(\phi, \theta)=t_{b d y}\left[\arctan \left(\sqrt{\csc ^{2}(\theta) \sec ^{2}(\phi)-1}\right)\right] \tag{3.113}
\end{equation*}
$$

which can be verified to satisfy (3.96), and where $t_{b d y}[\phi]$ is the boundary condition at the asymptotic boundary $\theta=\pi / 2$ which we assume to be symmetric under $\phi \rightarrow-\phi$. See figure 3.7 for examples.


Figure 3.7: Left: Example of (3.110) for $r(t)=3+\frac{1}{2} \sin (t)-\cos \left(t^{2} / 5\right)$. This satisfies (3.96). For this solution the sign of $K$ switches in between points on the surface, hence $k$ in (3.98) can not be globally absorbed into the normalisation of $m$. Note we are working in the Euclidean case, so the norm of $m$ has to be positive. Right: Example of (3.113) for $t_{b d y}[\phi]=\frac{1}{8} \cos (4 \phi)-\frac{\cos (\phi)}{4}$. The asymptotic boundary of global AdS is depicted as a grey cylinder.

The ease with which we can now construct solutions to (3.96) allows us to directly settle certain interesting physical questions, such as those concerning uniqueness of solutions. Consider again Euclidean Poincaré $\mathrm{AdS}_{3}$, and on the boundary we want our hypersurface to be anchored on an ellipse in the $t-x$-plane with semi minor axis $=1$ along the $t$-axis and semi major axis $=2$ along the $x$-axis. Interestingly, we can construct hypersurface embeddings similar to (3.110) in two ways: with a foliation in terms of semi-circular arcs parallel to the $t$-axis, or with a foliation in terms of semi-circular arcs parallel to the $x$-axis, see figure 3.8. Both these embeddings satisfy (3.96), but one reaches farther into the bulk than the other.

Hence for given boundary conditions, solutions to (3.96) will generally not be unique.


Figure 3.8: Two hypersurfaces satisfying (3.96) with the same boundary condition at $z=0$. For the solution on the left, we find $K<0$ everywhere, while for the one on the right we find $K>0$ everywhere.

We will make more use of this solution generating method in subsection 3.8.5, but before that we will comment on the importance of the Gauss-Bonnet theorem in our context.

### 3.8.4 Implications of the Gauss-Bonnet theorem

As we are searching for surfaces of constant scalar curvature, in the case of twodimensional surfaces $\tilde{M}$ it is quite natural to consider the implications of the Gauss-Bonnet theorem

$$
\begin{equation*}
\int_{\tilde{M}} \frac{R}{2} d V+\int_{\partial \tilde{M}} k_{g} d s+\sum_{\text {corners } c} \alpha_{c}+\sum_{\text {conical sing. } s} \beta_{s}=2 \pi \chi \tag{3.114}
\end{equation*}
$$

see e.g. [97]. Herein, the first term is an integral of the Gaussian curvature over the volume of the surface. The second term is an integral over the geodesic curvature along the boundary lines of the manifold. The third term takes into account contributions from corners in these boundaries. Here, $\alpha_{c}$ is the external angle at every corner by which the boundary changes direction, i.e. $\pi$ minus the interior angle at the corner. This angle has to be defined with a positive sign at convex corners and a negative sign at concave corners. Lastly, the fourth term (see [98]) takes into account contributions from conical singularities in the manifold $\tilde{M}$, where $\beta_{s}$ is the conical deficit angle. These specific terms are rarely mentioned in descriptions of the Gauss-Bonnet theorem, but they will be especially important in our context, and so we will explain them in more detail in appendix B. On the
right hand side of the equation, $\chi$ is the Euler characteristic. Importantly, the theorem is valid both in the Euclidean and Lorentzian case, however, in the latter angles have to be replaced by Lorentzian analogues and $\chi \equiv 0$, see [99-102] and the discussion in appendix B.

As we are concerned with constant curvature surfaces, the first term in (3.114) can be simplified to the product $R V / 2$ where $V$ is the total volume of the surface. Let us for the moment assume smooth surfaces, without corners or conical singularities. We can hence write:

$$
\begin{equation*}
\frac{R V}{2}=2 \pi \chi-\int_{\partial \tilde{M}} k_{g} d s \Rightarrow V=\frac{2 \int_{\partial \tilde{M}} k_{g} d s-4 \pi \chi}{-R} \tag{3.115}
\end{equation*}
$$

Hence, because we fix the value of $R$, there is (for fixed topology) a direct relation between volume $V$ and geodesic curvature of the edge of the surface $\int_{\partial \tilde{M}} k_{g} d s$. The later in turn is related to the boundary conditions that we impose on the surface, i.e. when prescribing a curve at a cutoff-surface near the boundary where the surface $\tilde{M}$ is supposed to be anchored.

Let us assume $\chi \geq 0$, which covers the cases of Lorentzian surfaces $(\chi \equiv 0)$, diskshaped Euclidean ones $\chi=1$ and spherical Euclidean ones $\chi=2$. As we also assume $R<0$, that yields a bound

$$
\begin{equation*}
V \leq \frac{2 \int_{\partial \tilde{M}} k_{g} d s}{-R} \tag{3.116}
\end{equation*}
$$

As $k_{g}$ is the curvature of the edge within the surface $\tilde{M}$, we cannot compute it before having found the surface. However, if that surface is embedded into a larger space with non-vanishing extrinsic curvature, we assume the curvature $k$ of the geodesic within that ambient space to obey $|k| \geq\left|k_{g}\right|$ (a curve in a certain submanifold may be a geodesic with respect to the induced metric $\left(k_{g}=0\right)$ but not the ambient metric $(k \neq 0)$ ). Equation (3.116) clearly implies that the average over $k_{g}$ along the boundary is positive. Assuming now that both $k_{g}$ and $k$ are positive everywhere, we obtain ${ }^{9}$

$$
\begin{equation*}
V \leq \frac{2 \int_{\partial \tilde{M}} k d s}{-R} \tag{3.117}
\end{equation*}
$$

This bound can be computed solely from the boundary conditions, i.e. the curve on a cutoff slice near the asymptotic boundary where we demand the surface $\tilde{M}$ to be anchored. Thus, even though the surfaces we are looking for are not extremal area surfaces, their total volume is bounded from above. Hence, we expect them

[^12]not to be too "wild" in the bulk, and especially for $\chi \geq 0$ there can be no smooth constant negative curvature submanifolds embedded into AdS that don't reach out to the asymptotic boundary. However, in section 3.8 .5 we will study surfaces in AdS that include conical singularities, and they can be contained entirely within the bulk.

We will now quickly discuss a potential application of these results, whose full exploitation we however leave to future research. In holography, for example when dealing with the complexity=volume proposal, we are often tasked with finding extremal volume slices in a bulk spacetime. For simplicity, let us consider the case of a Euclidean bulk, where these extremal area slices actually minimise the area. Then, clearly $V_{e x t} \leq V$. However, for generic non-translation invariant boundary conditions, the extremal volume slices are not easy to find, as seen e.g. in [103] where it was possible to solve the relevant partial differential equation only perturbatively. Hence, our results may be useful in occasions where only a bound on the volume is needed ${ }^{10}$. Not only could one then employ the bound (3.117) but, as shown in sections 3.8 .2 and 3.8.3, the constant curvature surfaces can be directly constructed given quite generic boundary conditions (only subject to a symmetry condition) without the need to solve additional differential equations. Some information might then already be gleaned from these surfaces, or they might be used as well motivated initial guess in numerical relaxation schemes.

### 3.8.5 Lemons in Lorentzian AdS $_{3}$

In this subsection, we will now put the methods explained in section 3.8.2 to use in order to construct generic timelike hypersurfaces solving (3.96) in global Lorentzian $\mathrm{AdS}_{3}$. As we had realised in section 3.8.4, timelike surfaces embedded into AdS with constant negative curvature can only be fully contained inside the bulk (without boundary) if they have conical singularities, which will of course be the case here. See also appendix B.0.3 for further details.

It is well known that in Lorentzian global $\mathrm{AdS}_{3}$, there are timelike geodesics that oscillate, i.e. pass through the center of AdS regularly, turning around at finite radial coordinate without ever reaching the boundary. We can now construct co-dimension one hypersurfaces which are foliated by such geodesics, obtaining structures such as the one shown in the top left of figure 3.9. Specifically, using the global AdS metric (3.112) (with boundary at $\theta=\pi / 2$ ), the embedding of (the

[^13]branch valid for $-\pi / 2<t<\pi / 2$ of) a radial timelike geodesic is given by
\[

$$
\begin{equation*}
t(\theta)=\arctan \left(\frac{E \sin (\theta)}{\sqrt{-1+E^{2} \cos (\theta)^{2}}}\right), \quad \phi=\text { const. } \tag{3.118}
\end{equation*}
$$

\]

where the "energy" $E>1$ of the geodesic is related to its turning point $\theta_{\max }$ by $\theta_{\max }=\arccos 1 / E$. Following the methods of section 3.8.2, we can construct surfaces of the form

$$
\begin{equation*}
t(\theta, \phi)=\arctan \left(\frac{E(\phi) \sin (\theta)}{\sqrt{-1+E(\phi)^{2} \cos (\theta)^{2}}}\right) \tag{3.119}
\end{equation*}
$$

where we have promoted $E$ to a $\phi$-dependent parameter. The relevant equations can become a bit cumbersome, but in the special case where $E(\phi)=E=$ const., the induced metric (in $\theta-\phi$ coordinates) reads

$$
g_{m n}=\left(\begin{array}{cc}
-\frac{\sec ^{2}(\theta)}{-1+E^{2} \cos ^{2}(\theta)} & 0  \tag{3.120}\\
0 & \tan ^{2}(\theta)
\end{array}\right)
$$

while the extrinsic curvature takes the form

$$
K_{m n}=\left(\begin{array}{cc}
0 & 0  \tag{3.121}\\
0 & E \tan (\theta)
\end{array}\right), \quad K=E \cot (\theta)
$$

Curiously, $K$ diverges at $\theta=0$ where the surfaces will have a conical singularity.
Due to the presence of these conical singularities at time coordinates $t=0$ and $t=\pi$ (a consequence of the periodicity of the timelike geodesics), we have adopted the term "lemons" for these shapes ${ }^{11}$. See figure 3.9 for a number of examples. It is easy to verify that surfaces of the form (3.119) will automatically satisfy (3.96), even if $E(\phi)$ is an arbitrary function. Note that $E>1$, and in the limit $E \rightarrow \infty \Leftrightarrow \theta_{\max }=\pi / 2$, i.e. the surface touches the AdS boundary with its equator in this limit. In fact, in this limit $t(\theta, \phi)=\theta$, hence the surface becomes the null boundary (the past part of it for this branch) of the WdW patch of the $t=\pi / 2$ time slice on the boundary. This is interesting because the WdW patch, which plays a central role in the complexity=action proposal [41] ${ }^{12}$, thus emerges very naturally from our construction. We would like to contrast this with the situation in [106] (see also $[73,107,108]$ for more recent works in this direction), where the

[^14]

Figure 3.9: Gallery of generalised lemons. In each plot, we use the coordinate system of (3.112), where the $A d S$-boundary is mapped to the grey cylinder at $\theta=\pi / 2$. The time axis is shown explicitly. Each yellow surface is an embedding described by (3.119), with individual timelike geodesics shown as grey lines. In the $t=0$ plane, the red line indicates the shape of the cut through the surface at its equator. What all of these surfaces have in common is the existence of tips with conical singularities, as demanded by consistency with the Gauss-Bonnet theorem. Apart from this, these surfaces can have not only many different shapes (top left and middle), they can also have self intersections (top right), reach out to touch the boundary (bottom left and middle), or even reach out to intersect the boundary (bottom right). The latter case may actually seem somewhat confusing at first: As discussed earlier in section 3.8.2, the induced metric has to be maximally symmetric, and hence homogeneous. But evidently, the signature of the metric switches from timelike to spacelike as we travel along the surface, and hence it can not be really homogeneous. This can happen because along the transition line, the induced metric is sufficiently continuous, but not analytic. From top-left to bottom-right, these surfaces are given by $E(\phi)=\sqrt{2}, E(\phi)=2 \sin (2 \phi)+\cos (4 \phi)+5, E(\phi)=5 \sin ^{2}\left(\frac{\phi}{4}\right)+\sqrt{2}$, $E(\phi)=\tan ^{4}\left(\frac{\phi}{2}\right)+\sqrt{2}, E \rightarrow \infty$, and $E(\phi)=\left(\frac{\cos (\phi)}{\sin (\phi)}\right)^{2}+2$ for $0<\phi<\pi, E(\phi)=$ $-i\left(\left(\frac{\cos (\phi)}{\sin (\phi)}\right)^{2}+2\right)$ for $\pi<\phi<2 \pi$, respectively. The bottom middle figure shows how the WdW patch arises naturally in this construction.
authors introduce a tension term $T$ to their equations as the simplest possible term (however, this tension is then given an a posteriori interpretation as an emergent holographic time). The null-boundaries of the WdW patch for the given boundary time slice are obtained as solution only in the rather unphysical seeming limit $T \rightarrow-\infty$. So the fact that the WdW patch arises naturally in our construction is rather encouraging, but as discussed in section 3.6 the null limit for the surface $\tilde{M}$ is related to a divergence in the value of the action. Also, as explained in [109,110], WdW patches in non-translation invariant settings can get quite complicated. So it would be interesting to see whether our method can be adapted to this and help analyse the features of such non-trivial WdW patches by first constructing a foliation of the interiour of the WdW patch in terms of lemon surfaces, and then taking the appropriate limit. Going further, we can even allow imaginary values of $E(\phi)$ in (3.119) which leads to spacelike surfaces that reach out towards the asymptotic boundary, as also shown in figure 3.9.

In our calculations motivated by complexity so far, we have always assumed the presence of an initial and final time slices like in figure 3.5, respectively section 3.6. But as the lemon surfaces start and end on conical singularities, we can as well calculate the action of their interior, without any additional boundary surfaces. There are no joints in this case, hence this only requires the bulk term and the Gibbons-Hawking-York boundary term. We assume the conical singularities to make no contribution to the action, which can be checked by a limiting argument similar to appendix B of [111] where it was shown that caustic points do not contribute to the action.

As (3.119) describes one half of a lemon, from the conical singularity at $t=0$ to the equator at $t=\pi / 2$ (i.e. from $\theta=0$ to $\theta=\theta_{\max }$ along one branch), the bulk and boundary terms read

$$
\begin{align*}
I_{E H} & =\int_{0}^{\theta_{\max }} d \theta \int_{t(\theta, \phi)}^{\pi / 2} d t \int_{0}^{2 \pi} d \phi \sqrt{-G}(\mathcal{R}-2 \Lambda),  \tag{3.122}\\
I_{G H Y} & =2 \times 2 \int_{0}^{\theta_{\max }} d \theta \int_{0}^{2 \pi} d \phi \sqrt{-g} K . \tag{3.123}
\end{align*}
$$

where we ignore the common prefactor involving the Newton constant. Together, we find

$$
\begin{align*}
& I_{E H}+I_{G H Y}  \tag{3.124}\\
& =8 \pi \int_{0}^{\theta_{\max }} d \theta \frac{E \sec (\theta)}{\sqrt{E^{2} \cos ^{2}(\theta)-1}}-\frac{\tan (\theta)}{\cos ^{2}(\theta)}\left(\pi-2 \arctan \left(\frac{E \sin (\theta)}{\sqrt{E^{2} \cos ^{2}(\theta)-1}}\right)\right) \tag{3.125}
\end{align*}
$$

$$
\begin{equation*}
=4 \pi^{2} \tag{3.126}
\end{equation*}
$$

Hence, for all lemons that do not reach the asymptotic boundary, we obtain the same value for the action. This is not surprising, because in [1] we explicitly derived equation (3.96) as a flow-equation from the bulk-action. The idea was that such a flow might be triggered by turning on a $T \bar{T}$ deformation, moving the boundary into the bulk [48] (see also [112-114]), and the surfaces that satisfy (3.96) would receive the physical interpretation of being those surfaces on which such a flow can come to rest. As an alternative description, these surfaces bound regions of the bulk whose action does not change under infinitesimal deformations of their boundary. But as the interior of the WdW patch can be foliated by such surfaces, and all are valid solutions to the equations of motion, it follows that the action evaluated inside all of these lemons has to have the same value. To provide an analogy, suppose you are looking for the extrema of a potential $V(x)$, where a particle might potentially be at rest, even if unstable. The equation of motion for this is $V^{\prime}(x)=0$. If all points $x \in I$ inside an interval satisfy this equation, it follows that the potential is constant in that interval. Concerning the action of the lemon surfaces, this argument is valid not only in the case of constant $E$ as assumed above. It can be checked tediously but explicitly that even for functions ${ }^{13}$ $E(\phi)$ the above action calculation yields the same result, as we should now expect.

The solutions of (3.96) hence have the physical interpretation of defining a foliation of a part of the bulk spacetime in terms of timelike surfaces such that the action inside of each such surface has the same constant value. This also implies that the action evaluated in the region between any two lemons vanishes identically. As said above, based on [1] we hope to interpret these surfaces as potential endpoints of a flow of the asymptotic boundary into the bulk triggered by turning on a $T \bar{T}$ deformation in the boundary theory. Given the time-periodic nature of the lemons, this would clearly have to be done in a time-dependent manner, and it would be interesting to construct such a $T \bar{T}$-deformation explicitly and analyse it from a field theory point of view. Apparently the field theory in question, if it exists, naturally is described by Dirichlet boundary conditions on a bulk submanifold resembling a cyclic universe, starting from an initial (conical) singularity, expanding, contracting, and ending in a final (conical) singularity with a period that has to be exactly $\Delta t=\pi$ before the cycle starts all over again. We leave an investigation of this for future work.

One additional thing that we want to quickly comment on is the action for lemons which do reach the asymptotic boundary of AdS. The WdW patch is obtained in our construction by taking the limit $E \rightarrow \infty$ of the lemon surfaces, and as the action inside every lemon for finite $E$ is constant, we might be tempted to

[^15]assign this value also to the action inside of the entire WdW patch. However, the WdW patch is bounded by null-surfaces which have to be treated in their own special way in action calculations as explained for example in [45], and generally the correct value of the action can not be obtained by a continuous limit from regions bounded by timelike or spacelike surfaces. Nonetheless, some of the terms proposed in [45] for null-boundaries, the so-called counter terms, are not unique (see also [115]) and it has been shown that, for example when translation invariance is slightly broken, they can cause problematic results in the complexity=action proposal $[110,116]$. Consequently, there is some interest in alternative methods of treating null-boundaries in the calculation of the bulk action, see e.g. [117]. It might thus be interesting to explore whether there is a well defined alternative prescription for calculating the action inside of WdW patches that would yield the same value that our limiting procedure suggests. Alternatively, one might propose to use timelike lemon surfaces with $\theta_{\max }=\pi / 2-\epsilon, \epsilon \ll 1$, as a regularisation of the WdW patch and the associated UV divergences that does not need null boundary surfaces, as opposed to using a WdW patch intersected by a cutoff surface at $\theta_{\text {cutoff }}=\pi / 2-\epsilon$ which is usually done. Such prescriptions for a modified CA proposal would however yield finite values for complexity, without any $\epsilon$-dependent divergent terms, defying physical expectations for how complexity should behave in a quantum field theory. Turning back to the analysis of the action associated to general lemon surfaces, when $E$ is given an imaginary value, we obtain spacelike surfaces that intersect the asymptotic boundary. To calculate the action inside such a "peeled lemon" (like the last example in figure 3.9) by standard methods, we would have to introduce a cutoff-surface, and the resulting value of the action would be divergent in the limit of vanishing UV regulator $\epsilon$. However, when allowing imaginary values for $E$, we obtain values for the turning radius of the form $\theta_{\max }=\pi / 2+i \operatorname{arccsch}(\operatorname{Im}(E))$. Curiously, this could be taken to suggest that at least on a formal level, the embeddings for the spacelike solutions of (3.96) can be extended beyond the AdS boundary $(\theta=\pi / 2)$ by using a complex $\theta$-coordinate, and one might speculate whether such embeddings have an interpretation in terms of "wrong sign" $T \bar{T}$-deformations. For example, by careful analytic continuation the integral (3.125) can thusly be lifted to a contour integral from $\theta=0$ to $\theta=\theta_{\max }$ in the complex plane, and be shown to still yield the same result even for imaginary $E$. However, it should be pointed out that this complexification approach fails when directly applied to earlier steps in the calculation such as (3.122) and (3.123), especially because the volume elements $\sqrt{-G}, \sqrt{-g}$ introduce branch-cuts due to the square-roots.

Given the importance of geodesics in our solution method explained in sections 3.8.2 and 3.8.5, there appears to be an interesting parallel to the recent work on Lorentzian bit-threads in [118,119], where likewise geodesic flows were employed.

Especially, some of the figures in [119] appear familiar from the construction of lemons in section 3.8.5. We leave an in depth exploration of possible connections between our work and $[118,119]$ for future research, however for now we want to caution the reader that the apparent connection explained above may only be superficial, for the following reasons: In [119], Lorentzian flows are defined as timelike, divergenceless, future directed vector fields $v$ with a bound on the norm. Such flows are not unique, and using congruences of timelike geodesics is just one convenient way to construct such flows explored in [119], but not the only one. In contrast, in $2+1$ bulk dimensions, solutions of (3.96) have to be foliated by geodesics as we have shown. These geodesics, however, can be both timelike or spacelike. Because of this, there is a difference between how this work and [119] construct foliations of the bulk spacetime outside of the WdW patch, even though the foliations given for the inside of the WdW patch might agree. Furthermore, the similarities end when going to higher dimensions. While the construction of [119] using timelike geodesics still works in higher bulk dimensions, we do not think that such geodesics will have a particular role to play for obtaining solutions of (3.96) in more than $2+1$ bulk dimensions. See also appendix C, where we will study spherically symmetric lemons in higher dimensional global AdS, and quickly comment on their qualitative differences to their lower dimensional counterparts. Of course, the explicit equations given in $[118,119]$ would still have practical uses in our kind of investigations, e.g. when constructing lemons in the BTZ black hole background.

### 3.9 Discussion

In this chapter we explored proposals for the cost of path integrals that prepare and transition between states in gravitational theories. We described these path integrals in gravitational theories with Dirichlet boundary conditions on a finite radial surface, which are holographically dual to $T \bar{T}$ deformed CFTs, and gave the precise map between path integrals in the bulk and the boundary. We have given bulk proposals for the cost of such path integrals that satisfy a set of physical requirements, and shown explicitly how such path integrals can be optimised: by minimising their cost over a suitable set of bulk subregions to reduce to existing holographic state complexity proposals. Lastly we developed general methods for gravitational action-type proposals.

Our work was partly inspired by the idea that holographic complexity proposals and their possible generalizations originate from coarse-graining circuits represented by moving the asymptotic boundary inwards [120]. See figure 2.1. Such an approach to holographic complexity of bringing in the asymptotic AdS boundary to finite cutoff can be made precise through the language of $T \bar{T}$-deformed holo-
graphic CFTs $[1,48,61,70]$. The aim of our work was to generalise this approach to consider general bulk subregions within finite cutoffs, functions on which we propose to be the cost of the path integral on the subregion.

Our approach is complementary to and generalises existing work on holographic state complexity. Maximum volume slices and WdW patches from our perspective are bulk subregions that minimise path integral cost for a suitably chosen proposal. Holographic complexity arises from the optimisation of path integral preparation of states. Note that once the function on bulk subregions is fixed, the 'optimal' subregion that minimises the path integral cost is dynamically determined; we do not independently specify the optimal bulk subregion and the function on it. This is in contrast to the two-functional holographic complexity proposals pursued in [69], and it would be worthwhile to combine their approach, complexity=anything, with ours, cost=anything, and see what subset of their proposals arise from the minimisation of carefully chosen path integral cost proposals over suitable bulk subregions.

We were able to find path integral cost proposals that reduce to some of the existing holographic state complexity proposals. Cost = boundary volume in a Euclidean bulk reduces to complexity $=$ volume at the time reflection symmetric slice. We gave an physical justification for this proposal in terms of a $T \bar{T}$-motivated notion of discretisation of the boundary path integral. Cost $=$ bulk volume in Lorentzian signature reduced to complexity $=$ volume 2.0. We also showed that, in the special case of pure global Euclidean AdS, the cost=gravitational action proposal from our previous paper [1] reduces to complexity=volume, though again only on a slice that is time reflection symmetric. Lastly, and much in the spirit of [69], we applied our cost=anything philosophy to conjecture novel complexity proposals. Our new codimension-0 candidate holographic complexity proposal satisfies at least persistent linear growth in thermofield double states, though the proposal was not derived by minimising a cost, but rather through providing new covariantly defined boundary anchored bulk regions.

We were not able to find a path integral cost proposal that reduces to the complexity=action conjecture, or the complexity=volume conjecture except on time symmetric slice where we are free to analytically continue between Euclidean and Lorentzian signature. The key issue is the existence of Lorentzian bulk subregions for which the gravitational action is unbounded in both directions, which prevents the use of Lorentzian gravitational action as a cost proposal that reduces to complexity=action. Our analysis does not rule out the complexity=action proposal, since failure to find a suitable cost proposal does not prove its non-existence. Furthermore, the shortest path in the Hilbert space may involve non-geometric states, which would be outside the scope of our work.

There a several avenues for future research. Our discussion of bulk cost functions has been entirely phenomenological. In the enormous set of cost proposals that satisfy our physical requirements, we have given no reason to favour one over another, besides perhaps simplicity. Moreover, except for cost=boundary volume we have given no physical justification for any of our proposals. That said, cost and complexity are inherently ambiguously defined, so even if one could find a gate set and metric on the space of operators that gives one of our cost proposals, that would not favour that proposal over others as there is no reason to favour that definition of cost over others. We view the size of our set of cost proposals as directly related to the inherent ambiguity of definition. As further justification, note that the set of complexity proposals that satisfy reasonable requirements is similarly enormous [69].

## Part II

## Euclidean Wormholes

## Recent Developments

### 4.1 Introduction

In quantum mechanics, we are accustomed to the idea of summing over all classical trajectories with specific boundary conditions within the path integral framework. Similarly, in quantum field theory, we perform path integrals by varying over all configurations of the fields over a fixed spacetime background. It is then natural in quantum gravity to consider the gravitational path integral, where we integrate over all possible spacetime metrics

$$
\begin{equation*}
Z_{\text {grav }}=\int D g D \phi e^{-S[g, \phi]} \tag{4.1}
\end{equation*}
$$

The above gravitational path integral is only formal and has many issues in general, and can be computed exactly only in very rare cases. We usually approximate it using a saddle-point analysis. A longstanding question has been if the above path integral should sum over all the possible spacetimes with given boundary conditions including those with a non-trivial topology. If we take the gravitational path integral seriously, we are lead to studying spacetime geometries with a nontrivial topology which are called as wormholes. A change in the topology of a bulk spacetime can occur in several ways. We will consider wormholes which are smooth bulk geometries connecting two or more distinct boundaries. These are topologically distinct from the manifolds in which the multiple boundaries are disconnected. Often, these are refereed to as spacetime wormholes or as Euclidean wormholes ${ }^{1}$. This distinguishes them from spatial wormholes which only connect region in space instead of regions of spacetime.

[^16]In the last few years, a lot of progress has been achieved by understanding the contributions of spacetime wormholes in several contexts. We will briefly review these in this chapter.

### 4.2 Black hole evaporation and replica wormholes

One of the main puzzles a consistent theory of quantum gravity has to resolve is the black hole information paradox. By doing a computation in semi-classical gravity, Hawking showed that black holes have a temperature and give rise to thermal radiation. This famously gives rise to the black hole information paradox. Consider a black hole that has formed from a collapse of a pure state. After its evaporation we are left with thermal Hawking radiation. This process takes a pure quantum state to a mixed state, destroying information and thus violating unitarity of quantum mechanics.

A quantitative way of measuring this is the Page curve, which plots the von Neumann entropy of a subsystem as a function of time. In all unitary systems this curve is non-monotonic, initially rising and falling back at later times. The entropy of Hawking radiation though, in Hawking's calculation, keeps monotonically increasing till the end of the black hole evaporation process. A natural expectation


Figure 4.1: The von Neumann entropy of Hawking radiation $S(t)$ as a function of time. The dotted line depicts Hawking's semi-classical calculation which violates unitarity. The solid blue curve arises due to contributions from replica wormholes, and is consistent with unitarity.
would be that semi-classical gravity alone is insufficient to reproduce a unitary Page curve and we need a microscopic theory of quantum gravity for this. Recent results instead showed that calculations in semi-classical gravity can indeed lead to a unitary Page curve for the Hawking radiation. This happens crucially due to contributions from spacetime wormholes that arise in the calculations of the entropy of the radiation. The entropy is computed via path integral tech-
niques using multiple replicas of the original system. In the usual saddle point of such a path integral, the replicas are disconnected. It was realised $[121,122]$ that there are other saddle points in which different replicas are connected through a smooth geometry, hence these are called as replica wormholes. These wormholes lead $[121,123,124]$ to a new and simple formula for the von Neumann entropy $S(t)$ of systems that are coupled to gravity. Using this formula does not require us to explicitly construct the replica wormholes, and it also leads to a Page curve compatible with unitarity as demonstrated in the figure 4.1.

### 4.3 Spectral statistics and wormholes

The fact that black holes are quantum systems with a discrete spectrum is quite mysterious to understand from the smooth gravitational physics in the bulk spacetime. It has long been known that we can estimate the coarse-grained density of states from the Euclidean path integral of a black hole. Moreover, we also expect [125] that the physics of black holes is governed by random matrices, a characteristic feature of quantum chaotic systems. In light of this we could wonder if Euclidean gravity knows not only about the coarse grained density, but also about the coarse-grained level statistics of the black hole microstates.

An essential diagnostic of a discrete spectrum is the spectral form factor (SFF), which is defined as

$$
\begin{equation*}
Z(\beta+i T) Z(\beta-i T)=\sum_{n, m} e^{-(\beta+i T) E_{n}} e^{-(\beta-i T) E_{m}} \tag{4.2}
\end{equation*}
$$

The large $T$ behaviour of the SFF directly probes the discreteness of the spectrum. For large $T$, the phases in the above expression are wildly oscillatory. By performing an average over $T$ we can see that this function is essentially bounded above by $Z(2 \beta)$. On the other hand, if we compute the SFF holographically in the bulk, we see that it keeps decaying forever ${ }^{2}$.

The SFF has a very characteristic shape for quantum chaotic systems. See figure 4.2 below for the log-log plot of a (normalised) SFF involving a Gaussian random matrix ensemble. This initially decays in a downward slope due to the cancellations among the phases. But, after a characteristic time scale, it begins to rise erratically in a region called the ramp, and finally oscillates erratically around its late time value in a region called the plateau. We can see from the figure that the spectral form factor is not a self-averaging quantity in the ramp and plateau regions. The disorder averaged SFF cleanly displays a linear growth in the ramp region and saturation at the plateau. The linear rise is due to the long range

[^17]level repulsion, and the saturation is due to the short range level repulsion in the spectrum. Overall, the SFF of a quantum chaotic system displays spectral rigidity of random matrix theory, as the eigenvalue repulsion has to be precisely balanced to lead to a linear ramp.

Random matrix universality suggests that we should be able to see similar features for the black hole spectrum in quantum gravity. Then, an important puzzle is to explain the behaviour of the SFF from a bulk gravitational perspective. We will see that Euclidean wormholes encode the spectral statistics of the black hole microstates and also give a gravitational explanation for the ramp.


Figure 4.2: The normalised spectral form factor for the Gaussian Unitary Ensemble (GUE) as a log-log plot. We have set $\beta=5$, and the size of the GUE matrices is $500 \times 500$, with a Gaussian width of $\frac{1}{500}$. The yellow curve is an average over 100 different samples, and the blue curve is that of a single sample.

### 4.3.1 Wormholes in JT gravity

The puzzle stated above has been understood in great detail for a two-dimensional model of gravity called Jackiw-Teitelboim (JT) gravity with a negative cosmological constant. The chaotic behaviour of the underlying microscopic system is very apparent here, since this JT gravity is dual to a double-scaled random matrix integral, as shown in [126]. The bulk theory consists of two fields, the metric $g$ and
a scalar $\phi$ called the dilaton. Its action on a two-dimensional surface $M$ is

$$
\begin{equation*}
S_{\mathrm{JT}}=-S_{0} \chi(M)-\frac{1}{2} \int_{M} d x^{2} \sqrt{g} \phi(R+2)-\int_{\partial M} d u \sqrt{h} \phi(K-1) \tag{4.3}
\end{equation*}
$$

Here $\chi(M)=2-2 g-n$ is the Euler characteristic of the surface $M$ of genus $g$ and $n$ boundaries. The equation of motion for the dilaton ensures that the surface $M$ is hyperbolic. The Euclidean gravitational path integral for this theory can be computed exactly, even when we have multiple boundaries as in the case of wormhole geometries. In fact, denoting the partition function with $n$ asymptotic boundaries, each with a given inverse temperature $\beta_{i}$ as $Z_{\text {grav }}\left(\beta_{1}, \cdots, \beta_{n}\right)$ is given as a sum over distinct topologies as

$$
\begin{equation*}
Z_{\text {grav }}\left(\beta_{1}, \cdots, \beta_{n}\right)=\sum_{g=0}^{\infty} e^{S_{0}(2-2 g-n)} Z_{g, n}\left(\beta_{1}, \cdots, \beta_{n}\right) \tag{4.4}
\end{equation*}
$$

where $e^{S_{0}}$ is an expansion parameter and $Z_{g, n}$ is a known contribution from surfaces of fixed genus $g$. The simplest case is $Z_{0,1}(\beta)$ which is the partition function of JT gravity on the hyperbolic disk geometry. All the other $Z_{g, n}$ can be computed once we have two ingredients. The first is the partition function $Z_{\mathrm{T}}(\beta, b)$ on a hyperbolic geometry called the trumpet which has a holographic boundary with inverse temperature $\beta$ and a geodesic boundary of length $b$. The second ingredient consists of the Weil-Petersson volumes of hyperbolic surfaces, which satisfy a recursion relation. With these in hand, all the $Z_{g, n}$ can be calculated.
The holographic dictionary between JT gravity and the random matrices then says that the partition function is equivalently given in the matrix ensemble as

$$
\begin{equation*}
Z_{\text {grav }}\left(\beta_{1}, \cdots, \beta_{n}\right)=\left\langle\operatorname{Tr}\left(e^{-\beta_{1} H}\right) \cdots \operatorname{Tr}\left(e^{-\beta_{n} H}\right)\right\rangle_{\text {ensemble }} \tag{4.5}
\end{equation*}
$$

In the above equation on the right hand side, $H$ denotes a random Hamiltonian acting on the black hole Hilbert space, and the angular brackets perform an average over the ensemble. Now we can compute the spectral form factor in JT gravity. From the boundary matrix ensemble, the disorder average of the spectral form factor is simply given by

$$
\begin{align*}
\langle Z(\beta+i T) Z(\beta-i T)\rangle & =\left\langle\operatorname{Tr}\left(e^{-(\beta+i T) H}\right) \operatorname{Tr}\left(e^{-(\beta-i T) H}\right)\right\rangle \\
& =\int d E_{1} d E_{2}\left\langle\rho\left(E_{1}\right) \rho\left(E_{2}\right)\right\rangle e^{-\beta\left(E_{1}+E_{2}\right)} e^{-i T\left(E_{1}-E_{2}\right)} \tag{4.6}
\end{align*}
$$

The spectral density correlator $\left\langle\rho\left(E_{1}\right) \rho\left(E_{2}\right)\right\rangle$ is a well known function $[127,128]$ in random matrix theory, and for an ensemble in the unitary symmetry class, it is

$$
\begin{equation*}
\left\langle\rho\left(E_{1}\right) \rho\left(E_{2}\right)\right\rangle \approx\left\langle\rho\left(E_{1}\right)\right\rangle\left\langle\rho\left(E_{2}\right)\right\rangle-\frac{\sin ^{2}\left(\pi\langle\rho(E)\rangle\left(E_{1}-E_{2}\right)\right)}{\pi^{2}\left(E_{1}-E_{2}\right)^{2}}+\langle\rho(E)\rangle \delta\left(E_{1}-E_{2}\right) \tag{4.7}
\end{equation*}
$$

Here we have used $E=\left(E_{1}+E_{2}\right) / 2$. Each of the three terms in the above expression have a distinctive behaviour, and roughly correspond to the slope, ramp and plateau respectively in the SFF. We can now give a gravitational explanation for the behaviour of SFF in JT gravity as follows. To do this, let us look at the bulk partition function with two asymptotic boundaries. Schematically, it it looks like


This is given by genus expansion as we saw above in (4.4). It contains a leading disconnected term of order $O\left(e^{2 S_{0}}\right)$, a connected wormhole term of order $O(1)$, and several terms which are subleading of order $O\left(e^{-2 S_{0}}\right)$. The first term comes from two disconnected geometries, each of which is topologically a disk. We can see that this is similar to the first term in the spectral correlator in (4.7). Analytically continuing $\beta_{1}, \beta_{2}$ to $\beta \pm i T$, this product of two disconnected geometries gives us the factorised product $\left\langle Z_{0,1}(\beta+i T)\right\rangle\left\langle Z_{0,1}(\beta-i T)\right\rangle$. This is easily computed, since we know the partition function of the disk geometry [129,130], and is given as

$$
\begin{equation*}
\left|\left\langle Z_{0,1}(\beta+i T)\right\rangle\right|^{2} \sim \frac{e^{2 S_{0}}}{\left(\beta^{2}+T^{2}\right)^{\frac{3}{2}}} e^{\frac{2 \pi^{2} \beta}{\beta^{2}+T^{2}}} \tag{4.9}
\end{equation*}
$$

This decays forever and goes zero for large $T$, and is the explanation of the slope in the SFF. Note that each of these disk geometries is a saddle point of the gravitational path integral.

The more interesting term in $Z_{\text {grav }}\left(\beta_{1}, \beta_{2}\right)$ above is the connected geometry equal to $Z_{0,2}\left(\beta_{1}, \beta_{2}\right)$. This geometry is topologically a cylinder, with its two boundaries having regularized lengths $\beta_{1}$ and $\beta_{2}$. This is computed using the trumpet partition function as

$$
\begin{align*}
Z_{0,2}\left(\beta_{1}, \beta_{2}\right) & =\int_{0}^{\infty} b d b Z_{\mathrm{T}}\left(\beta_{1}, b\right) Z_{\mathrm{T}}\left(\beta_{2}, b\right)  \tag{4.10}\\
& =\frac{1}{2 \pi} \frac{\sqrt{\beta_{1} \beta_{2}}}{\beta_{1}+\beta_{2}}
\end{align*}
$$

We can see that this gives rise to a linear in $T$ ramp, once we continue $\beta_{1}, \beta_{2}$ to $\beta \pm i T$ and $T \gg \beta$

$$
\begin{equation*}
Z_{0,2}(\beta+i T, \beta-i T)=\frac{\sqrt{\beta^{2}+T^{2}}}{4 \pi \beta} \rightarrow \frac{|T|}{4 \pi \beta} \tag{4.11}
\end{equation*}
$$

This linear ramp with the exact coefficient is in fact produced by the average value of the sine kernel in (4.7)

$$
\begin{equation*}
\int d E_{1} d E_{2}\left(-\frac{1}{2 \pi^{2}} \frac{1}{\left(E_{1}-E_{2}\right)^{2}}\right) e^{-\beta\left(E_{1}+E_{2}\right)} e^{-i T\left(E_{1}-E_{2}\right)}=\frac{|T|}{4 \pi \beta} \tag{4.12}
\end{equation*}
$$

Thus we see that Euclidean wormholes contain really important information and lead to a linearly growing ramp in the SFF. And in general, the two boundary partition function encodes the spectral statistics of the black hole microstates. A crucial fact about these wormholes is that they are not a saddle point in the gravitational path integral. In fact, the computation above in (4.10) is a completely off-shell calculation.

Finding a spacetime explanation of the plateau in the SFF requires more work. We can see that from on the matrix integral side, the last term in (4.7) sets the value $Z(2 \beta)$ at which the SFF saturates. There is evidence $[131,132]$ that a geometrical explanation for transition from the ramp to the plateau arises from appropriately resumming subleading terms in the topological expansion.

### 4.3.2 Wormholes in AdS/CFT

We expect black holes to display chaotic dynamics in all dimensions [125]. Then, it is very reasonable [133] to expect that black hole microstates exhibit a quantum chaotic (i.e., random matrix) spectral statistics in gravity above two dimensions, and not just in toy models such as JT gravity. What would then be the role of connected two boundary Euclidean wormholes in higher dimensions?

Unlike the case for JT gravity, the standard holographic dictionary for AdS/CFT in higher dimensions asserts that the boundary dual is a single quantum theory and not an ensemble average. The presence of Euclidean wormholes then raises an immediate concern: it contributes to the connected correlators such as $\left\langle Z\left(\beta_{1}\right) Z\left(\beta_{2}\right)\right\rangle-\left\langle Z\left(\beta_{1}\right)\right\rangle\left\langle Z\left(\beta_{2}\right)\right\rangle$, but this should automatically vanish, since in a single well-defined theory objects such as $Z$ are fixed without any statistical interpretation.

This is known as the factorization puzzle in AdS/CFT, where quantities such as the gravitational partition functions computed with multiple boundaries fail to factorize: $Z_{\text {grav }}\left(\beta_{1}, \beta_{2}\right) \neq Z_{\mathrm{CFT}}\left(\beta_{1}\right) Z_{\mathrm{CFT}}\left(\beta_{2}\right)$. A possible resolution of this puzzle is that theories of gravity in which Euclidean wormholes contribute are dual to an ensemble of CFTs on the boundary. Unless we discover a principle that forbids all generic Euclidean wormholes, this viewpoint would likely require a major revision of the standard AdS/CFT duality in which the boundary is a single well-defined CFT. A different approach to resolving the factorization puzzle is to realise that semi-classical Euclidean gravity is a low energy effective field theory which can only
probe coarse-grained quantities and does not have access to the full microscopic theory. Stated in a slightly different language, the Hamiltonian in the case of JT gravity was a true random matrix from an ensemble, while in more realistic systems the Hamiltonian would rather only be pseudo-random. Then, the situation is very similar to other complex chaotic systems where we use a coarse-grained description of the precise microscopic description.

With this in mind, it would be worthwhile to understand the contribution of Euclidean wormholes in more than two spacetime dimensions and see how they encode the quantum chaotic level statistics of the black hole microstates. Euclidean wormholes having a non-negative boundary scalar curvature do not occur as solutions to the vacuum Einstein's equation [134].

Thus, in the absence of such saddle points, it is a difficult task to compute Euclidean wormholes amplitudes. In a manner similar to JT gravity, a fully off-shell calculation of the $\mathrm{AdS}_{3}$ gravity path integral with two torus asymptotic boundaries was achieved in $[135,136]$, thanks to the relative simplicity of $\mathrm{AdS}_{3}$ gravity. The resulting two-boundary torus partition function is one that is consistent with Virasoro symmetry, modular invariance and also displays random matrix statistics. In fact, this $\mathrm{AdS}_{3}$ partition function $Z_{\mathrm{T}^{2} \times \mathrm{I}}$ in appropriate limits behaves very similarly to $Z_{0,2}$ we saw in JT gravity. For Virasoro primary states of fixed spins $s_{1}, s_{2}$ on the boundary, and at low temperatures we have

$$
\begin{equation*}
Z_{\mathrm{T}^{2} \times \mathrm{I}}\left(\beta_{1}, \beta_{2}\right)=e^{-\beta_{1} E_{1}-\beta_{2} E_{2}}\left(\frac{1}{2 \pi} \frac{\sqrt{\beta_{1} \beta_{2}}}{\beta_{1}+\beta_{2}} \delta_{s_{1}, s_{2}}+O\left(\frac{1}{\beta}\right)\right) \tag{4.13}
\end{equation*}
$$

where $E_{1}, E_{2}$ are the threshold energies above which black holes of spins $s_{1}, s_{2}$ exist. From this expression we can see that the off-shell wormhole encodes random matrix level statistics. This implies level repulsion between the energy eigenvalues, and thus a linear ramp in each spin sector. Recently, by developing a trace formula for chaotic 2d CFTs in [137], this torus wormhole partition function was shown to be a coarse-grained object from a single microscopic theory without the need for an ensemble interpretation.

In spacetime dimensions greater than three, computing a fully off-shell wormhole partition function is out of reach. Since there are no wormhole saddles in pure gravity, a useful approach is to find geometries that become saddles after adding a constraint. For example, by either constraining the length between the two boundaries of the wormhole or by fixing the boundary energies by hand, several two-boundary classical Euclidean wormhole solutions of vacuum Einstein equations were found in [138-140].

A slightly different, yet related class of wormholes are the double cone geometries introduced in [141]. These are constructed as a quotient of any stationary two-


Figure 4.3: An off-shell Euclidean wormhole in $A d S_{3}$ with two torus boundaries.
sided black hole, see figure 4.4. Since the Killing time runs in opposite direction on the two sides of the maximal extension of such a black hole, the quotient produces a geometry that has Lorentzian periods $\pm T$ on the two boundaries. Thus, it is a candidate bulk saddle that contributes to the SFF at infinite temperature, $Z(i T) Z(-i T)$. They are different from the wormholes we have been discussing as they are constructed in a Lorentzian setting. The quotient is singular at the bifurcation surface and is problematic especially when there are propagating fields. It was suggested [141] that this can be easily resolved by making the metric slightly complex in a way that avoids the singularity. The resulting complex spacetime is still satisfies the relevant boundary conditions (i.e., has periods $\pm T$ on the opposite sides) and thus is a saddle-point that contributes to the spectral form factor $Z(i T) Z(-i T)$. These geometries have a compact zero mode, which physically corresponds to the shift in the relative time coordinates of the two boundaries. This shift also has a period $T$, corresponding to the volume of the compact zero mode. This exactly gives rise to the linear ramp in the SFF.


Figure 4.4: The double cone geometry is constructed by periodically identifying the Killing time $t \sim t+T$ for a general two-sided black hole. This results in a two-boundary spacetime wormhole.

### 4.4 Operator statistics and wormholes

In the previous section we learned that Euclidean wormholes contain statistical information about the quantum chaotic aspects of the spectrum. The signatures of quantum chaos are not only present in the spectrum of a chaotic theory, but also in the matrix elements of its operators. An often useful statement about this is the eigenstate thermalization hypothesis (ETH) [142-144], where we can write the matrix elements of simple operators as

$$
\begin{equation*}
\left\langle E_{i}\right| O_{\alpha}\left|E_{j}\right\rangle=f_{\alpha}\left(E_{i}\right) \delta_{i j}+e^{-\frac{S}{2}} g_{\alpha}\left(E_{i}, E_{j}\right) R_{i j} \tag{4.14}
\end{equation*}
$$

expressed in terms of smooth functions. The matrix elements $\left\langle E_{i}\right| O_{\alpha}\left|E_{j}\right\rangle$ are determined by the micro-canonical expectation values, up to exponentially suppressed corrections and $R_{i j}$ are approximately Gaussian random variables with mean zero and unit variance. ETH is a form of coarse-graining, giving a statistical description of the precise and complicated data of a single theory.

Such a statistical description of operators allows us to compute the averages and moments of correlation functions and partition functions that depend on these matrix elements. Assuming such random statistics for theories with a holographic dual leads to non-factorizing quantities which are captured by classical Euclidean wormhole solutions in the bulk [145].

In chaotic 2d CFTs a stronger form of ETH holds for the OPE coefficients $C_{i j k}$ of primary operators. Due to constraints on the theory coming from crossing symmetry and modular invariance, the average values of OPE coefficients have universal asymptotic formulae, whenever at least one of the operators in $C_{i j k}$ is heavy (i.e., in the chaotic regime)

$$
\begin{equation*}
\overline{\left|C_{i j k}\right|^{2}} \sim C_{0}\left(h_{i}, h_{j}, h_{k}\right) C_{0}\left(\bar{h}_{i}, \bar{h}_{j}, \bar{h}_{k}\right) \tag{4.15}
\end{equation*}
$$

where the $C_{0}$ is a function of the weights of the primary operators, related to the DOZZ formula for the structure constants of Liouville theory. Using this, [146] studied an large- $c$ ensemble of CFT data. The spectrum in this ensemble determined by the Cardy formula above the black hole threshold up to a few discrete set of states below the threshold representing massive particles. The OPE coefficients, in the spirit of ETH (4.14), are given by

$$
\begin{equation*}
C_{i j k}=\sqrt{C_{0}\left(h_{i}, h_{j}, h_{k}\right) C_{0}\left(\bar{h}_{i}, \bar{h}_{j}, \bar{h}_{k}\right)} R_{i j k} \tag{4.16}
\end{equation*}
$$

where $R_{i j k}$ is an approximately Gaussian random variable with $O(1)$ variance. Note that this is just an ensemble of CFT data, and not an actual ensemble of microscopic CFTs.


Figure 4.5: An example of an on-shell wormhole in 3d gravity with three primary operator insertions $O_{1}, O_{2}, O_{3}$ corresponding to massive particles in the bulk. The onshell action of this wormhole computes the average of the OPE coefficient $\overline{\left|C_{123}\right|^{2}}$.

Several classical wormhole solutions with disconnected boundaries exist in three dimensional gravity once we allow the bulk metric to be given in terms of a hyperbolic slicing. This occurs whenever the genus of the boundary Riemann surface is greater than two, or when the boundary is at least once punctured torus or at least a thrice punctured sphere. The on-shell gravitational action of these wormholes exactly reproduces the averages of correlation functions and partition functions defined using the above ensemble, see figure 4.5 for a depiction. Finally, let us conclude this section by mentioning that there are some limited examples of wormholes capturing operator statistics in higher dimensions as well, $[147,148]$.

## Wormholes with Matter

## Contents

5.1 Brief Introduction ..... 103
5.2 General strategy ..... 104
5.3 Explicit examples ..... 106
5.3.1 Free field ..... 106
5.3.2 Interacting field: Cubic superpotential ..... 110
5.4 More general wormholes ..... 111
5.4.1 Backreacted double cone ..... 113
5.5 Constrained saddles ..... 115
5.6 Conclusions ..... 120

### 5.1 Brief Introduction

The recent developments involving spacetime wormholes connecting disconnected boundaries in two and three-dimensional quantum gravity have revived an interest in understanding the role of similar geometries in higher dimensions as well. In lower dimensions, we have simple models of gravity without bulk propagating degrees of freedom. This has allowed us to calculate wormhole amplitudes by exactly performing the gravitational path integral in many instances $[126,135]$. In higher dimensions we do not have this convenience, and as usual we need to rely on saddle-points of the gravitational path integral.

Quite intriguingly, Euclidean AdS spacetimes that are solutions of pure Einstein gravity are disallowed to have multiple boundaries when the boundary curvature is positive everywhere, as proved in a theorem in [134]. This was also extended to the case when the boundary has vanishing scalar curvature in [149]. Thus, these theorems forbid wormholes to occur as classical saddles in pure Einstein gravity for the relevant cases of non-negative boundary curvature. The Euclidean bulk
manifold is permitted to have multiple disconnected boundaries once we allow for negatively curved boundary manifolds. Though this route leads to interesting wormholes in 3d gravity [150, 151] which encode the OPE statistics for 2d CFTs [145, 146], it is not the most relevant case as CFTs in $d>2$ defined on a negatively curved space are usually not well-defined [152].

There have been many interesting constructions of Euclidean wormhole solutions which are supported by certain types matter, see [151,153-157]. A stability analysis of several generic constructions of wormholes was done in [157]. In this chapter, we will only consider the simplest cases of Euclidean wormholes and attempt to understand their properties and their implications. To avoid the no-go theorem stated above, we will study these wormholes in the presence of a scalar field.

### 5.2 General strategy

In the following, we will be interested in two-sided $\operatorname{AdS}_{d+1}$ Euclidean wormholes with boundaries that have zero scalar curvature. To be precise, we will assume the boundary topology to be that of a $d$-dimensional torus. One-sided Euclidean black hole solutions with a similar boundary topology also exist, where one of the periods of the torus shrinks to zero size in the bulk. Since such wormhole solutions are absent in pure gravity by the above theorem, we will look for two-sided solutions for Einstein gravity minimally coupled to a scalar field. Let us write the bulk action including a potential for the scalar field as

$$
\begin{equation*}
S=-\frac{1}{16 \pi G} \int d^{d+1} x \sqrt{g}\left(R-\partial_{\mu} \phi \partial^{\mu} \phi+V(\phi)\right) \tag{5.1}
\end{equation*}
$$

Since we are in AdS, the action also has a negative cosmological term which we absorb into the definition of the potential $V(\phi)$. We are interested in metrics for which the line elements take form

$$
\begin{equation*}
d s^{2}=d r^{2}+e^{2 A(r)} d x_{d}^{2} \tag{5.2}
\end{equation*}
$$

Here $d x_{d}^{2}=\sum_{i=1}^{d} d x_{i}^{2}$ represents a flat $d$-dimensional torus of unit volume. Let us assume that the function $A(r)$ and the scalar $\phi(r)$ depend only on the radial direction. The two boundaries of the spacetime are reached as the radial coordinate is taken to plus or minus infinity. The equation of motion for the metric and the scalar field can be summarised as the following coupled system of differential equations

$$
\begin{align*}
(d-1) A^{\prime \prime}(r)+\phi^{\prime}(r)^{2} & =0 \\
d(d-1) A^{\prime}(r)^{2}+(d-1) A^{\prime \prime}(r) & =V(\phi)  \tag{5.3}\\
\phi^{\prime \prime}(r)+d A^{\prime}(r) \phi^{\prime}(r)+\frac{1}{2} \frac{\partial V}{\partial \phi}(r) & =0
\end{align*}
$$

The analysis of the above equations can be hugely simplified by rewriting the potential $V(\phi)$ in terms of a function $W(\phi)$, as is familiar in the literature [158-161]

$$
\begin{equation*}
V(\phi)=d(d-1) W^{2}-(d-1)^{2}\left(\frac{\partial W}{\partial \phi}\right)^{2} \tag{5.4}
\end{equation*}
$$

The function $W(\phi)$ is often called as the superpotential, which usually arises in the truncation of supergravity theories but can be used in much more general settings. Substituting (5.4) into (5.3), the field equations for both $A(r)$ and $\phi(r)$ reduce to first order equations which are as follows

$$
\begin{align*}
A^{\prime} & =W \\
\phi^{\prime} & =-(d-1) \frac{\partial W}{\partial \phi} \tag{5.5}
\end{align*}
$$

Another way of realising this is to rewrite the radial part of action in (5.1) using the scalar curvature for the metric in (5.2) and using (5.4)

$$
\begin{align*}
S= & -\frac{1}{16 \pi G} \int d r e^{d A}\left(d(d-1)\left(A^{\prime}-W\right)^{2}-\left(\phi^{\prime}+(d-1) \frac{\partial W}{\partial \phi}\right)^{2}\right)  \tag{5.6}\\
& -\frac{1}{8 \pi G} \int d r \frac{d}{d r}\left(d e^{d A} A^{\prime}\right)+\frac{d-1}{8 \pi G} \int d r \frac{d}{d r}\left(e^{d A} W\right)
\end{align*}
$$

From the first line of the action above, we see that (5.5) are indeed extrema of the action. The terms in the second line are total derivatives, out of which the former is cancelled by the Gibbons-Hawking-York boundary term.

In the absence of a scalar field, the potential term only contains the cosmological constant, $V=d(d-1)$ and hence we can set $W= \pm 1$. This gives us $A=r, \phi=0$ as we can see from (5.5), which is the familiar one-sided Euclidean Poincare AdS spacetime. To get two-sided configurations, we need the behaviour of $A(r)$ such that $A^{\prime}(r) \rightarrow \pm 1$ as $r \rightarrow \pm \infty$. This requires $A^{\prime \prime}>0$ as we traverse from one boundary to another, but this is in odds with the first equation in (5.3) for a $\phi^{\prime}$ that is real everywhere. To resolve this, we will allow imaginary sources for the scalar field.

Let us now understand some properties of the superpotential $W(\phi)$, and how to choose one so as to obtain a wormhole solution. For a normalizable deformations of AdS such as a massive field in the bulk, the superpotential has the form $W=$ $1+h \phi^{2}+\cdots$, where $h$ determines the mass of the field (in $d=2$ it is just the conformal dimension of the dual operator). For sake of simplicity let us find wormhole solutions which are symmetric under $r \rightarrow-r$. The $\mathbb{Z}_{2}$ symmetry implies that $W$ is an odd function, and the potential $V$ is an even function of $\phi$.

This symmetry guarantees us that $A^{\prime}(r)=-A^{\prime}(-r)$, but we have to pay the price by working with complex solutions. This can be directly seen from the defining equation for the potential in (5.4). At $\phi=0$, the potential is equal to the cosmological constant $V=d(d-1)$, so this requires the coefficient of the linear term around $\phi=0$ in $W$ to be purely imaginary and equal to $\pm i \sqrt{d /(d-1)}$.

How should the scalar field behave at both the boundaries? A little bit of analysis shows that $\phi$ should approach a constant at the boundaries. If the field were to go to zero, like in the case of a normalizable operator, we are lead to a contradiction. Since $A^{\prime}(r) \rightarrow \pm 1$ at infinity, $W$ should take two different values at the boundaries. But if the scalar field goes to zero and $W$ is odd, this cannot occur. The scalar field cannot also grow near the boundaries, as this would imply that $A^{\prime}(r)$ itself would also be growing there.

Therefore, at the boundaries the scalar must approach a constant. And due to the $\mathbb{Z}_{2}$ symmetry, $\phi \rightarrow \pm \phi_{0}$ as $r \rightarrow \pm \infty$. Let us look at some explicit examples to see how this works.

### 5.3 Explicit examples

### 5.3.1 Free field

Let us start with the simplest case of a free massless minimally coupled scalar field. In this case, the action is

$$
\begin{equation*}
S=-\frac{1}{16 \pi G} \int d^{d+1} x \sqrt{g}\left(R+d(d-1)-\partial_{\mu} \phi \partial^{\mu} \phi\right) \tag{5.7}
\end{equation*}
$$

Since the potential $V=d(d-1)$ is just the cosmological constant, a superpotential that satisfies (5.4) and that has the required properties we discussed above is easy to find. It is given by

$$
\begin{equation*}
W(\phi)=-i \sinh \sqrt{\frac{d}{d-1}} \phi \tag{5.8}
\end{equation*}
$$

The general solutions for the metric and the scalar field are

$$
\begin{align*}
& d s^{2}=d r^{2}+\left(\frac{\lambda_{1} e^{d r}+\lambda_{2} e^{-d r}}{2}\right)^{\frac{2}{d}} d x_{d}^{2}  \tag{5.9}\\
& \phi(r)=2 i \sqrt{\frac{d-1}{d}} \arctan \left(\tanh \frac{d\left(r+r_{0}\right)}{2}\right)
\end{align*}
$$

The scalar field $\phi$ runs between $\pm i \frac{\pi}{2} \sqrt{\frac{d-1}{d}}$ on the two boundaries. Since we solved two first order equations, we should only have two constants of integration. The
parameters $\lambda_{1}, \lambda_{2}$ which set the relative sizes of the boundary tori are actually not independent and are related by $\lambda_{1}=e^{2 d r_{0}} \lambda_{2}$. By expanding the scalar field near the two boundaries and using the standard AdS/CFT dictionary we see that the operator dual to the scalar picks up non-zero expectation values on the two sides, equal to $\mp 2 i \sqrt{\frac{d-1}{d}} e^{-2 r_{0}}$.

Looking at the boundary values of the scalar field, we can tell that this is an unusual solution because the sources of the dual operator are set to very specific values, and not allowed to change. To understand this better, we can try to construct a wormhole solution with an arbitrary sources.

Instead of finding exact solutions, let us consider solutions to (5.3) when we turn on a small source for the scalar field. Near a boundary, we know that in general the field has the behaviour

$$
\begin{equation*}
\phi \sim C_{1} \phi_{1}(r)+C_{2} \phi_{2}(r) \tag{5.10}
\end{equation*}
$$

where the ratio $C_{1} / C_{2}$ is determined by boundary conditions. Say we are in empty AdS, and we turn on an infinitesimal source $\epsilon$ for the scalar field. At leading order the metric is $A=r+\mathcal{O}\left(\epsilon^{2}\right)$. The scalar field only has a radial dependence, and the solution of the field of mass $m^{2}=\Delta(\Delta-d)$ is simply

$$
\begin{equation*}
\phi(r)=\epsilon e^{-(d-\Delta) r}+\alpha \epsilon e^{-\Delta r} \tag{5.11}
\end{equation*}
$$

Using this solution we can now find the $\mathcal{O}\left(\epsilon^{2}\right)$ corrections to the metric:

$$
\begin{equation*}
A(r)=r-\frac{\epsilon^{2}}{4(d-1)}\left(e^{-2 r(d-\Delta)}+\alpha^{2} e^{-2 r \Delta}-\frac{8 \alpha m^{2}}{d^{2}} e^{-d r}\right)+\mathcal{O}\left(\epsilon^{3}\right) \tag{5.12}
\end{equation*}
$$

Since a wormhole metric has to turn around, we need $A^{\prime}\left(r_{*}\right)=0$. From the above expression we see that in general no real $r_{*}$ exists, which is in line with expectations for real sources. So as we did above, we shall consider imaginary sources for the scalar field

$$
\begin{equation*}
\phi(r)=i\left(\epsilon e^{-(d-\Delta) r}+\alpha \epsilon e^{-\Delta r}\right) \quad \epsilon, \alpha \in \mathbb{R} \tag{5.13}
\end{equation*}
$$

To compare with an exact solution let us first look at the case of a free massless field with $d=2, \Delta=2$ for the moment. In this case we see that

$$
\begin{equation*}
A(r)=r+\frac{\epsilon^{2}}{4}\left(\alpha^{2} e^{-4 r}\right)+\mathcal{O}\left(\epsilon^{3}\right) \tag{5.14}
\end{equation*}
$$

where we have ignored an additive constant. To this order $r_{*}=1 / 4 \ln \left(\alpha^{2} \epsilon^{2}\right)$. We can attempt to take this one-sided solution and glue it to a copy of itself around $r=r_{*}$ such that the resulting solution has the required asymptotics as $r \rightarrow \pm \infty$

$$
A_{\mathrm{wh}}(r)=\left\{\begin{array}{lr}
A(r), & r_{*} \leq r<\infty  \tag{5.15}\\
A\left(2 r_{*}-r\right), & -\infty<r \leq r_{*}
\end{array}\right.
$$

Similarly we also have to glue the solutions for the scalar field. Though $A_{\mathrm{wh}}(r)$ is continuous and $A_{\mathrm{wh}}^{\prime}(r) \rightarrow \pm 1$ at both the boundaries, this function is not smooth at $r=r_{*}$, and hence is not a valid two-sided solution. Let us compute the metric $A(r)$ to higher orders in $\epsilon$. It can be checked that the non-trivial corrections for the scalar come only from the odd orders, and even orders for the metric. So we can write an ansatz as

$$
\begin{equation*}
\phi(r)=i \sum_{n=0}^{\infty} \epsilon^{2 n+1} \phi_{2 n+1}(r) \quad A(r)=\sum_{n=0}^{\infty} \epsilon^{2 n} A_{2 n}(r) \tag{5.16}
\end{equation*}
$$

Using the field equations (5.3), we can solve for $\phi_{2 n+1}(r), A_{2 n}(r)$ perturbatively. For now, we will ignore additive constants in $\phi$ and $A$ as both of them possess a shift symmetry in this example. We then get the following expressions

$$
\begin{align*}
& \phi(r)=i\left(\epsilon+\alpha \epsilon e^{-2 r}-\frac{\alpha^{3} \epsilon^{3}}{6} e^{-6 r}+\frac{\alpha^{5} \epsilon^{5}}{20} e^{-10 r}-\frac{\alpha^{7} \epsilon^{7}}{56} e^{-14 r}+\mathcal{O}\left(\epsilon^{9}\right)\right)  \tag{5.17}\\
& A(r)=r+\frac{1}{4}\left(\alpha^{2} \epsilon^{2} e^{-4 r}-\frac{1}{4} \alpha^{4} \epsilon^{4} e^{-8 r}+\frac{1}{12} \alpha^{6} \epsilon^{6} e^{-12 r}+\mathcal{O}\left(\epsilon^{8}\right)\right)
\end{align*}
$$

If we had halted at a specific order of $\epsilon$ to create a two-sided solution, one of two outcomes would have occurred. Either $A(r)$ would have no turning point whatsoever, or it would fail to constitute a continuous solution. But we can solve to all orders and the above series can be resummed. Doing this, we obtain

$$
\begin{align*}
& \phi(r)=i \sqrt{2} \arctan \left(\frac{\sqrt{2}-\alpha \epsilon e^{-2 r}}{\sqrt{2}+\alpha \epsilon e^{-2 r}}\right) \\
& A(r)=r+\frac{1}{2} \ln \left(1+\frac{\alpha^{2} \epsilon^{2}}{2} e^{-4 r}\right) \tag{5.18}
\end{align*}
$$

These solutions are now valid to all orders in $\epsilon$, and $A(r)$ has a turning point at $r_{*}=\frac{1}{4} \ln \frac{\alpha^{2} \epsilon^{2}}{2}$. But notice that by expanding the field near infinity, the source term is $\epsilon$ independent, and no longer infinitesimal. In fact, gluing a reflected solution about the turning point, we get back the exact solution that we obtained from the superpotential in (5.9), with the identification $r_{*}=\frac{1}{4} \ln \frac{\alpha^{2} \epsilon^{2}}{2}=-r_{0}$, and $\lambda_{1}=e^{4 r_{0}} \lambda_{2}$

$$
\begin{align*}
d s^{2} & =d r^{2}+\left(\frac{\lambda_{1} e^{2 r}+\lambda_{2} e^{-2 r}}{2}\right) d x_{2}^{2}  \tag{5.19}\\
\phi(r) & =\sqrt{2} i \arctan \left(\tanh \left(r+r_{0}\right)\right)
\end{align*}
$$

This exercise teaches us that these wormholes in (5.9) cannot be analysed with infinitesimally small sources. Thus, these are completely backreacted geometries which depend on the specific finitely tuned sources, which are necessary for their existence. Before proceeding further, let us calculate the on-shell action of these
wormholes with a free scalar. Let's first fix Dirichlet boundary conditions at finite cutoffs as $\phi\left(r_{1,2}\right)=i \phi_{1,2}$ and $A\left(r_{1,2}\right)=\rho_{0}$, and later send these cutoffs to infinity. We can easily write $\phi_{1,2}$ in terms of $r_{1}, r_{2}$ and $\rho_{0}$. A useful relation is

$$
\begin{equation*}
\cos \left(\frac{\phi_{2}-\phi_{1}}{\sqrt{2}}\right)=\sqrt{\lambda_{1} \lambda_{2}} e^{-2 \rho_{0}} \tag{5.20}
\end{equation*}
$$

By taking $r_{2} \rightarrow \infty$ and $r_{1} \rightarrow-\infty$, we also send $\rho_{0} \rightarrow \infty$ and we recover the relation we saw before $\phi_{2}-\phi_{1}=\frac{\pi}{\sqrt{2}}$. The on-shell action of this wormhole at finite cutoff can be split into the bulk and boundary terms. The bulk action with cutoff surfaces placed at $r=r_{1}, r_{2}$ is

$$
\begin{align*}
S_{b u l k} & =-\int_{r_{1}}^{r_{2}} d r \sqrt{g}\left(R+2-(\partial \phi)^{2}\right)  \tag{5.21}\\
& =\left(\lambda_{1} e^{2 r_{2}}-\lambda_{2} e^{-2 r_{2}}\right)-\left(\lambda_{1} e^{2 r_{1}}-\lambda_{2} e^{-2 r_{1}}\right) \tag{5.22}
\end{align*}
$$

and the GHY terms at both the boundaries give

$$
\begin{align*}
S_{G H Y} & =-2\left(\int_{r=r_{2}} d^{2} x \sqrt{h} K-\int_{r=r_{1}} d^{2} x \sqrt{h} K\right)  \tag{5.23}\\
& =2\left(\lambda_{1} e^{2 r_{1}}-\lambda_{2} e^{-2 r_{1}}\right)-2\left(\lambda_{1} e^{2 r_{2}}-\lambda_{2} e^{-2 r_{2}}\right)
\end{align*}
$$

Thus, the total on-shell action for the finite cutoffs at $r=r_{1}, r_{2}$ is

$$
\begin{align*}
S & =S_{b u l k}+S_{G H Y} \\
& =\left(\lambda_{1} e^{2 r_{1}}-\lambda_{2} e^{-2 r_{1}}\right)-\left(\lambda_{1} e^{2 r_{2}}-\lambda_{2} e^{-2 r_{2}}\right) \\
& =-4 \sqrt{e^{4 \rho_{0}}-\lambda_{1} \lambda_{2}}  \tag{5.24}\\
& =-4\left|\sin \left(\frac{\phi_{1}-\phi_{2}}{\sqrt{2}}\right)\right| e^{2 \rho_{0}}
\end{align*}
$$

In the last line above, we traded the variables $r_{1,2}$ for $\rho_{0}$ since we know that $A\left(r_{1,2}\right)=\rho_{0}$. The renormalised action, as we take the cutoffs $r_{2} \rightarrow \infty$ and $r_{1} \rightarrow-\infty$, includes counterterms proportional to the volume of the cutoff surfaces

$$
\begin{align*}
& S_{c t}=2\left(\int_{r=r_{1}} d^{2} x \sqrt{h}+\int_{r=r_{2}} d^{2} x \sqrt{h}\right)  \tag{5.25}\\
& \quad=\left(\lambda_{1} e^{2 r_{1}}+\lambda_{2} e^{-2 r_{1}}\right)+\left(\lambda_{1} e^{2 r_{2}}+\lambda_{2} e^{-2 r_{2}}\right)=4 e^{2 \rho_{0}}
\end{align*}
$$

Thus, the total action including counterterms at finite cutoffs is

$$
\begin{equation*}
S_{r e n}=2\left(\lambda_{1} e^{2 r_{1}}+\lambda_{2} e^{-2 r_{2}}\right)=4 e^{2 \rho_{0}}\left(1-\left|\sin \left(\frac{\phi_{1}-\phi_{2}}{\sqrt{2}}\right)\right|\right) \tag{5.26}
\end{equation*}
$$

which vanishes as we send the cutoffs $r_{2}, r_{1}$ to $\pm \infty$ respectively.

### 5.3.2 Interacting field: Cubic superpotential

As described in the general strategy, the field equations are drastically simplified into two first order equations using the superpotential $W(\phi)$. This method is most useful when we have a non-trivial interacting fields. Let us illustrate this by constructing a wormhole solution that is sourced by an interacting scalar field. We saw that the scalar in the case of symmetric wormholes should go to $\pm \phi_{0}$ at the boundaries. This implies that the shifted field ( $\phi \mp \phi_{0}$ ) decays at the boundaries, and the superpotential must behave like $W=a+b\left(\phi-\phi_{0}\right)^{2}+O\left(\phi-\phi_{0}\right)^{4}$ at one boundary and $W=-a-b\left(\phi+\phi_{0}\right)^{2}+O\left(\phi-\phi_{0}\right)^{4}$ at the other. The simplest such function with two critical points at $\pm i \mu$ is

$$
\begin{equation*}
W(\phi)=\frac{i}{3 \mu^{2}} \sqrt{\frac{d}{d-1}}\left(3 \mu^{2} \phi+\phi^{3}\right) \tag{5.27}
\end{equation*}
$$

The bulk action now includes a potential and is given by

$$
\begin{align*}
S & =-\frac{1}{16 \pi G} \int d^{d+1} x \sqrt{g}\left(R-\partial_{\mu} \phi \partial^{\mu} \phi+V(\phi)\right) \\
V(\phi) & =d(d-1)\left(1+\frac{\phi^{2}}{\mu^{2}}\right)^{2}-d^{2}\left(\phi+\frac{\phi^{3}}{3 \mu^{2}}\right)^{2} \tag{5.28}
\end{align*}
$$

Using the above superpotential, we can solve the first order equations in (5.5). We find the solutions to be

$$
\begin{align*}
& d s^{2}=d r^{2}+e^{\frac{d \tanh ^{2}\left(\alpha\left(r+r_{0}\right)\right)}{3 \alpha^{2}}}\left(\frac{\lambda_{1} e^{\alpha r}+\lambda_{2} e^{-\alpha r}}{2}\right)^{\frac{4 d}{3 \alpha^{2}}} d x_{d}^{2}  \tag{5.29}\\
& \phi(r)=i \mu \tanh \left(\alpha\left(r+r_{0}\right)\right)
\end{align*}
$$

where $\alpha=\sqrt{d(d-1)} / \mu$, and $\lambda_{1}=e^{2 \alpha r_{0}} \lambda_{2}$. Unlike in the case of the free field, the boundary values of the field are $\pm i \mu$ are now allowed to vary as we change $\mu$ or $\alpha$. This parameter also sets the mass of the field. Choosing an arbitrary $\alpha$ destroys AdS asymptotics at the boundaries. To retain them, we need to fine tune $\alpha$ to be exactly $2 d / 3$.

Hence, we observe that the behavior of this wormhole closely resembles that of the free field example. In order to maintain well-defined AdS asymptotics at both boundaries, and that we have the same central charge on either sides i.e., the cosmological constant remains unchanged, it is essential to carefully select precisely adjusted sources. This implies that these are fully backreacted geometries, and they cease to exist if we alter the sources at the boundary.

### 5.4 More general wormholes

Up to this point, we have been examining spacetime wormholes with the boundary topology of a $d$-dimensional torus, where the metric is determined by a single scale factor $e^{2 A(r)}$. However, these are not the most general class of spacetimes where the metric only depends on the radial coordinate. We will now consider a more general metric ansatz with the same boundary topology as before which allows for a wider class of wormhole solutions. Let us consider the metric ansatz in the following manner

$$
d s^{2}=d r^{2}+e^{2 A(r)} d \tau^{2}+e^{2 B(r)} d x_{d-1}^{2}
$$

In the above, we separated the Euclidean time $S^{1}$ factor from the other $(d-1)$ spatial dimensions. This is the generic form of the metric for black hole spacetimes with a time translation symmetry, in Euclidean coordinates and compatible with a torus boundary topology. In general, searching for wormhole solutions means we have to solve a complicated system of differential equations with the appropriate boundary conditions. Instead, we want to see if such metrics can be solutions of first-order flow equations like we encountered before. To do so, it will be useful to make a change of coordinates into

$$
\begin{equation*}
d s^{2}=e^{d A(\rho)} d \rho^{2}+e^{A(\rho)+(d-1) B(\rho)} d \tau^{2}+e^{A(\rho)-B(\rho)} d x_{d-1}^{2} \tag{5.30}
\end{equation*}
$$

We are slightly abusing notation by reusing the same notation for the coordinates on the torus and also for the ansatz functions $A(\rho), B(\rho)$. The reasoning behind new ansatz is that, in these new coordinates the square root of the metric determinant is just $e^{d A(\rho)}$, which cancels a factor of $e^{-d A(\rho)}$ present in the scalar curvature. Explicitly, the radial configuration of the Einstein-scalar action simplifies in terms of the $A, B, \phi$ variables as

$$
\begin{align*}
S & =-\frac{1}{16 \pi G} \int d^{d+1} x \sqrt{g}\left(R-\partial_{\mu} \phi \partial^{\mu} \phi+V(\phi)\right) \\
& =-\frac{1}{16 \pi G} \int d \rho\left(\frac{d(d-1)}{4}\left(A^{\prime 2}-B^{\prime 2}\right)-d A^{\prime \prime}-\phi^{\prime 2}+e^{d A} V(\phi)\right) \tag{5.31}
\end{align*}
$$

where we have suppressed the coordinates on the torus, since the variables $A, B, \phi$ are functions of $\rho$ alone. All the derivatives are taken with respect to $\rho$. The Einstein field equations and the scalar equations of motion can now be rearranged into the following system of coupled ODEs

$$
\begin{align*}
B^{\prime \prime} & =0  \tag{5.32}\\
(d-1) A^{\prime \prime}-2 e^{d A} V(\phi) & =0  \tag{5.33}\\
2(d-1) A^{\prime \prime}-d(d-1)\left(A^{\prime 2}-B^{2}\right)+4 \phi^{\prime 2} & =0  \tag{5.34}\\
2 \phi^{\prime \prime}+e^{d A} \frac{\partial V}{\partial \phi} & =0 \tag{5.35}
\end{align*}
$$

We could in principle write them in a similar fashion to (5.5) as first order flow equations, but to find an analogue of a superpotential in this needs more work. Nevertheless, let us rearrange them in a more convenient manner as follows

$$
\begin{align*}
A^{\prime} & = \pm\left(b^{2}+4 \frac{\phi^{\prime 2}+e^{d A} V(\phi)}{d(d-1)}\right)^{\frac{1}{2}} \\
B^{\prime} & =b  \tag{5.36}\\
\phi^{\prime} & =c-\frac{1}{2} \int d \rho e^{d A} \frac{\partial V}{\partial \phi}
\end{align*}
$$

A new feature of the above system of equations possess non-trivial solutions even in the absence of the scalar field. If we set $\phi=0$, it is easy to solve (5.36) for $A(\rho), B(\rho)$ as

$$
\begin{equation*}
e^{d A}=\frac{1}{4 \sinh ^{2} \frac{d \rho}{2}}, \quad B=\rho \tag{5.37}
\end{equation*}
$$

We can change back into the more familiar $r$ coordinate system using

$$
\begin{equation*}
\left(\frac{d r}{d \rho}\right)^{2}=e^{d A(\rho)} \tag{5.38}
\end{equation*}
$$

we see that the metric now is

$$
\begin{equation*}
d s^{2}=d r^{2}+\cosh ^{\frac{4}{d}}\left(\frac{d r}{2}\right)\left(\tanh ^{2}\left(\frac{d r}{2}\right) d \tau^{2}+d x_{d-1}^{2}\right) \tag{5.39}
\end{equation*}
$$

This metric is closely related to the double cone class of wormholes $[140,141,162$, 163] we discussed in the last chapter. In $d=2$, this is just a quotient of the BTZ metric written with a Euclidean time coordinate. The double cone metric is usually written in Lorentzian signature, so let us consider $t=i \tau T$, rescaled such that $t \sim t+T$. Now, instead of a Euclidean time circle, we have a Lorentzian circle with periods $\pm T$ at both the boundaries. This is depicted in figure 4.4.

This metric has a singularity at $r=0$, which we can check by computing the Kretschmann scalar in these coordinates. Normally, this singularity is avoided in Euclidean signature by appropriately choosing a real Euclidean period, but we will not do that. Then we have an actual conical singularity at the bifurcation surface, which needs to be resolved. To do so, we will consider the complexification of the spacetime, and choose a particular complex section of this geometry that is non-singular. For example, we can define a contour for $r$ that instead of running from $-\infty$ to $\infty$, is deformed as

$$
\begin{equation*}
r=\tilde{r}-i \epsilon, \quad \epsilon>0 \tag{5.40}
\end{equation*}
$$

with real $\tilde{r}$, [141]. This procedure of choosing a complex contour can be seen to be topologically equivalent to smoothing out the conical singularity. We now have a smooth spacetime wormhole that has a (slightly) complex metric. There are several examples of physically meaningful spacetimes where we need to allow for complex spacetime metrics, such as rotating black holes. A criterion for the allowability of complex metrics was proposed in $[164,165]$, and the complex double cone satisfies it. The on-shell action of this classical solution vanishes.

### 5.4.1 Backreacted double cone

By utilizing the coupled system of equations presented in Equation (5.36), we can investigate solutions when the scalar field is active. To begin, let's look at the case of a free, massless field. Typically, when dealing with backreacting fields, we are obliged to tackle the equations perturbatively, considering small amplitudes of a probe field. However, owing to the straightforward nature of these equations, we are able to get a fully backreacted metric in the presence of a free field. In the $\rho$ coordinate system, the configuration is very similar and as follows

$$
\begin{equation*}
e^{d A(\rho)}=\frac{a^{2}}{4 \sinh ^{2}\left(\frac{a d \rho}{2}\right)}, \quad B=b \rho, \quad \phi^{\prime}=c \tag{5.41}
\end{equation*}
$$

The constant $c$ plays an important role and determines the gradient of $\phi$. The equations of motion relate the constants $a, b, c$ as

$$
\begin{equation*}
c=\frac{\sqrt{d(d-1)\left(a^{2}-b^{2}\right)}}{2} \tag{5.42}
\end{equation*}
$$

Using (5.38), let us rewrite the above metric in the more familiar $r$ coordinate system. Denoting $\xi=b(d-1) / a$, we obtain

$$
\begin{equation*}
d s^{2}=d r^{2}+a^{\frac{2}{d}} \cosh ^{\frac{4}{d}}\left(\frac{d r}{2}\right)\left(\tanh ^{\frac{2(\xi+1)}{d}}\left(\frac{d r}{2}\right) d \tau^{2}+\tanh ^{\frac{2(d-1-\xi)}{d(d-1)}}\left(\frac{d r}{2}\right) d x_{d-1}^{2}\right) \tag{5.43}
\end{equation*}
$$

Turning off the scalar field, sets $a=b=1$, and $\xi \rightarrow \xi_{0}=(d-1)$ in the equations above. As we can easily verify, this results in the empty double cone spacetime. The total on-shell action, now also including the free scalar, is zero.

Just like the empty double cone, we can regulate this spacetime by introducing a deformation as shown in Equation (5.40). In terms of the coordinate $r$, the scalar field can be expressed as follows

$$
\begin{equation*}
\phi(r)=\phi_{0}+\frac{2}{d} \frac{c}{a} \log \left(\tanh \frac{d r}{2}\right) \tag{5.44}
\end{equation*}
$$

For this to be well-defined and smooth on the entire wormhole, we also need to use the complex contour we choose for $r$, as in (5.40). By tuning the free parameters in the equation above we can impose the boundary conditions for the field as

$$
\begin{equation*}
\phi( \pm \infty)= \pm \phi_{0} \tag{5.45}
\end{equation*}
$$

This requires tuning $\xi$ above $\xi_{0}$ by a small amount. Choosing $\xi^{2}=(d-1)^{2}+$ $4 d(d-1) \phi_{0}^{2} / \pi^{2}$, we obtain

$$
\begin{equation*}
\phi=\phi_{0}\left(1-\frac{2 i}{\pi} \log \left(\tanh \frac{d r}{2}\right)\right) \tag{5.46}
\end{equation*}
$$

For an infinitesimal source $\phi_{0}$, the backreaction in the metric (5.43) can be ignored, doing so we get the empty double cone. Recently, it was suggested in [163] that the complex double cone in the Lorentzian time $t=i \tau T$ can be thought of as computing a trace of an "evolution" operator as $\operatorname{Tr}\left(e^{-i \tilde{K} T}\right)$. Here $\tilde{K}$ is the boost generator in the complexified geometry. This also generates time translations in the complex spacetime, in opposite directions on both the boundaries. It was shown that computing this on the empty double cone leads to a linear in $|T|$ ramp. When we have matter on the double cone, we also need to calculate $\operatorname{Tr}\left(e^{-i \tilde{K} T}\right)$ for the matter Hilbert space. The classical value of $\tilde{K}$ is

$$
\begin{equation*}
\tilde{K}=\int d^{d-1} x d r \sqrt{|g|}\left(\partial_{r} \phi\right)^{2}=-\frac{2 a d^{2} \phi_{0}^{2}}{\pi^{2}} \int \frac{d r}{\sinh d r}=-i \frac{2 a d \phi_{0}^{2}}{\pi} \tag{5.47}
\end{equation*}
$$

To evaluate the $r$ integral, we need to use an appropriate contour passing through the lower half complex plane. Thus, we see that including the linear ramp $T$ factor, the effect of the scalar with slightly different couplings $\pm \phi_{0}$ is

$$
\begin{equation*}
\tilde{Z} \sim T e^{-2 a d \phi_{0}^{2} T / \pi} \tag{5.48}
\end{equation*}
$$

where we used $\tilde{Z}$ for the wormhole partition function that also includes the matter contribution.

On the boundary, turning on such sources for a massless operator would correspond to changing the couplings of the Hamiltonians on the two boundaries as

$$
\begin{equation*}
H_{L}=H-\phi_{0} O \quad \text { and } \quad H_{R}=H+\phi_{0} O \tag{5.49}
\end{equation*}
$$

Then, this double cone with a scalar on it, would be dual to a generalised version of the spectral form factor, where the two Hamiltonians are slightly different as shown above. The random matrix theory prediction [140, 166-168] for such a quantity exactly matches with (5.48) above, and is

$$
\begin{equation*}
\tilde{Z}:=\left\langle\operatorname{Tr}\left(e^{i H_{L} T}\right) \operatorname{Tr}\left(e^{-i H_{R} T}\right)\right\rangle \sim \frac{T}{2 \pi} e^{-C \phi_{0}^{2} T} \tag{5.50}
\end{equation*}
$$

Instead of the usual linearly rising ramp in the spectral form factor, we have an exponential decay in the parameter $T$. The constant $C>0$ sets the decay rate and is related to the two-point function of the operator $O$, see [140].

We saw in the previous chapter in section 4.3 , that the linear ramp in the spectral form factor arises due to the connected pair correlator of the eigenvalue density

$$
\begin{equation*}
\left\langle\rho\left(E_{1}\right) \rho\left(E_{2}\right)\right\rangle_{c}=-\frac{1}{2 \pi^{2}} \frac{1}{\left(E_{1}-E_{2}\right)^{2}} \tag{5.51}
\end{equation*}
$$

This function has a pole in the energy difference, at $\Delta E=\left(E_{1}-E_{2}\right)=0$. If we were instead computing the eigenvalue density correlator for two slightly different Hamiltonians, we would have have a completely different behaviour. In the regime where $\Delta E$ is small, we get

$$
\begin{equation*}
\left\langle\rho_{L}\left(E_{1}\right) \rho_{R}\left(E_{2}\right)\right\rangle_{c}=-\frac{1}{2 \pi^{2}} \frac{\left(E_{1}-E_{2}\right)^{2}-\lambda^{2}}{\left(\left(E_{1}-E_{2}\right)^{2}+\lambda^{2}\right)^{2}} \tag{5.52}
\end{equation*}
$$

This behaviour can be checked by computing this pair correlator in a matrix integral involving two different Gaussian matrices in the large $N$ limit. There also exists a calculation using a supersymmetric sigma model in $[166,169]$. From the form of the pair correlator above, we can see that the pole in $\Delta E$ is not at zero, but along the imaginary axis at $\Delta E= \pm i \lambda$. This shift of the poles to the imaginary axis is responsible for the exponential decay, which can be explicitly checked by Fourier transforming (5.52) to obtain (5.50).

### 5.5 Constrained saddles

In the preceding section we analysed the simplest wormhole solutions with the general metric ansatz as in (5.4). We saw that even in the absence of matter, there exists a two-sided classical wormhole solution, the double cone. We also studied the backreacted double cone geometry in the presence of a free massless scalar. These wormholes as we first encountered them, are singular solutions. This required us to complexify the geometry in order to resolve this singularity, which can be seen as a drawback. Another disadvantage of the double cone is that they can only describe wormholes that have identical tori on both the boundaries. It would be useful to have examples of wormhole configurations that posses similar properties of the double cone, but which are real, non-singular and which can have boundaries possessing distinct conformal structures.

This is overly demanding, especially if we want these wormholes to occur as saddle-points of the path integral in pure Einstein gravity. But, it was shown in $[135,138,139]$ that a wide class of wormhole configurations with the above requisite properties do exist, if we instead look for constrained saddles. These are
classical geometries that are not solutions to the field equations, but they do extremise the action subject to an additional constraint. The precise form of the constraint will not be essential for our discussion, this could either be a noncovariant constraint such as fixing the length between the two boundaries of the wormhole to be a constant, or a more covariant constraint such as holding the ADM energies perceived on the two boundaries to be a fixed constant.

For the torus boundary topology that we have been considering in this chapter, the most generic family of solutions is

$$
\begin{equation*}
d s^{2}=d r^{2}+b^{2}\left(2 \cosh \frac{d r}{2}\right)^{\frac{4}{d}}\left(\left(\frac{\beta_{1} e^{\frac{d r}{2}}+\beta_{2} e^{-\frac{d r}{2}}}{2 \cosh \frac{d r}{2}}\right)^{2} d \tau^{2}+d x_{d-1}^{2}\right) \tag{5.53}
\end{equation*}
$$

These wormholes configurations can have independent inverse temperatures $\beta_{1}, \beta_{2}$ at each of the boundaries, and they are all labelled by a single parameter $b \geq 0$. This important parameter determines the bottleneck size of the wormhole, the length of the wormhole, the ADM energy $E(b)$ at the boundaries and also the (renormalized) gravitational action of the wormhole. The action $S$, and the energy can be written in terms of $b$ as

$$
\begin{equation*}
S=\left(\beta_{1}+\beta_{2}\right) E \quad E(b)=\frac{(d-1) b^{d}}{4 \pi G} \tag{5.54}
\end{equation*}
$$

These wormholes are not saddles in the usual unconstrained path integral, since the modulus $b$ is unstable. The path integral receives dominant contributions from configurations near $b=0$, where these wormholes degenerate. We are stabilizing this by adding a constraint that fixes $b$, by either fixing the length between boundaries or by fixing the energy. As in the case of the spectral form factor, we can analytically continue $\beta_{1}$ and $\beta_{2}$ to $\beta \pm i T$ with $\operatorname{Re}\left(\beta_{1,2}\right)=\beta \geq 0$. The action vanishes for $\beta=0$, for which these wormholes are on-shell solutions, identical to the double cone with periodicities $\pm i T$ at the boundaries.

Let us now understand these constrained saddles when we couple them to scalar field. Consider again the case of a free massless scalar. The general solution for such a field with boundary conditions as $\phi( \pm \infty)= \pm \phi_{0}$ is

$$
\begin{equation*}
\phi(r)=\frac{\phi_{0}}{\log \left(\frac{\beta_{1}}{\beta_{2}}\right)} \log \left(\frac{1}{\beta_{1} \beta_{2}}\left(\frac{\beta_{1} e^{\frac{d r}{2}}+\beta_{2} e^{-\frac{d r}{2}}}{2 \cosh \frac{d r}{2}}\right)^{2}\right) \tag{5.55}
\end{equation*}
$$

The total action also including the scalar field is now

$$
\begin{equation*}
S=\frac{d-1}{4 \pi G}\left(\left(\beta_{1}+\beta_{2}\right)+\frac{\beta_{1}-\beta_{2}}{\log \left(\frac{\beta_{1}}{\beta_{2}}\right)} \frac{d \phi_{0}^{2}}{d-1}+O\left(\phi_{0}^{4}\right)\right) b^{d} \tag{5.56}
\end{equation*}
$$

In the above, the first term is the gravitational action from (5.54), and the second term is from the action of the scalar field. We are in the probe limit with small $\phi_{0}$, and the there will be a backreaction in the metric of order $O\left(\phi_{0}^{2}\right)$ which contribute to $O\left(\phi_{0}^{4}\right)$ terms in the action above.

To get a meaningful prediction out of the above action and relate it to some putative boundary quantity, we still need to stabilize the wormhole. As we did for the empty wormhole configuration, we can try choosing analytically continued $\beta_{1}, \beta_{2}$ to $\pm i T$ such that the solution is on-shell. Doing this in the presence of the scalar leads to corrections as

$$
\begin{equation*}
\beta_{1}=-\frac{T}{\pi} \frac{d \phi_{0}^{2}}{d-1}+i T+O\left(\phi_{0}^{4}\right) \quad \beta_{2}=-\frac{T}{\pi} \frac{d \phi_{0}^{2}}{d-1}-i T+O\left(\phi_{0}^{4}\right) \tag{5.57}
\end{equation*}
$$

Unlike in the case of the empty wormhole, the above corrections in $\phi_{0}$ to the inverse temperatures have $\operatorname{Re}\left(\beta_{1,2}\right)<0$. Such negative real parts of the inverse temperature give rise to ill-defined quantities when computing partition functions or thermal correlation functions. This also violates the criterion for allowable complex metrics in $[164,165]$.

A different route that bypasses choosing complex values for $\beta_{1}, \beta_{2}$ was taken in [140]. To stabilize the wormhole in the presence of the scalar, they suggested that we can fix the average of both the boundary energies, which in turn fixes the modulus $b$ in terms of the free parameters $\beta_{1}, \beta_{2}$ and $\phi_{0}$. The ADM energies of the two boundaries in the empty wormhole were equal and given in (5.54). In the presence of matter, we have unequal boundary conditions for the scalar on both sides. Then, the two energies are not necessarily equal to each other. They are given as

$$
\begin{align*}
& E_{1}=\frac{(d-1) b^{d}}{4 \pi G}\left(1+\frac{d \phi_{0}^{2}}{d-1}\left(\frac{1}{\ln \left(\frac{\beta_{1}}{\beta_{2}}\right)}-\frac{\beta_{1}-\beta_{2}}{\beta_{1} \ln ^{2}\left(\frac{\beta_{1}}{\beta_{2}}\right)}\right)\right)  \tag{5.58}\\
& E_{2}=\frac{(d-1) b^{d}}{4 \pi G}\left(1-\frac{d \phi_{0}^{2}}{d-1}\left(\frac{1}{\ln \left(\frac{\beta_{1}}{\beta_{2}}\right)}-\frac{\beta_{1}-\beta_{2}}{\beta_{2} \ln ^{2}\left(\frac{\beta_{1}}{\beta_{2}}\right)}\right)\right) \tag{5.59}
\end{align*}
$$

up to higher-order corrections. Now, we stabilize the modulus $b$ by fixing it in terms of the average energy

$$
\begin{equation*}
E_{\mathrm{avg}}=\frac{E_{1}+E_{2}}{2}=\frac{(d-1) b^{d}}{4 \pi G}\left(1+\frac{d \phi_{0}^{2}}{2(d-1)} \frac{\left(\beta_{1}-\beta_{2}\right)^{2}}{\beta_{1} \beta_{2} \ln ^{2}\left(\frac{\beta_{1}}{\beta_{2}}\right)}\right) \tag{5.60}
\end{equation*}
$$

Inverting the above equation, we can determine $b$ in terms of a fixed $E_{\text {avg }}$. The
stabilized action then is given as

$$
\begin{equation*}
S=\left(\beta_{1}+\beta_{2}\right) E_{\mathrm{avg}}-\frac{\beta_{1}^{2}-\beta_{2}^{2}-2 \beta_{1} \beta_{2} \ln \left(\frac{\beta_{1}}{\beta_{2}}\right)}{\beta_{1} \beta_{2} \ln ^{2}\left(\frac{\beta_{1}}{\beta_{2}}\right)} \frac{d\left(\beta_{1}-\beta_{2}\right)}{2(d-1)} E_{\mathrm{avg}} \phi_{0}^{2} \tag{5.61}
\end{equation*}
$$

The path integral for the two-sided boundary conditions involving a constraint on the energy can now be estimated using the above action. The path integral for generic $\beta_{1}$ and $\beta_{2}$ includes contributions from twist zero modes which relate the coordinate translations of one boundary to another. For $\beta_{1,2}=\beta \pm i T$ such that $0<\beta \ll T$, the zero mode volume should come just from the relative shift of the time coordinates. Since both the boundary times are periodic with periods $\pm T$, the relative shift between these coordinates also shares the same property. Then the zero mode volume is just $T$. The action above in (5.61) in the regime $0<\beta \ll T$ simplifies to

$$
\begin{equation*}
S=2 \beta E_{\mathrm{avg}}+\frac{2 d E_{\mathrm{avg}} T}{\pi(d-1)} \phi_{0}^{2}+O\left(\frac{\beta^{2}}{T}, \phi_{0}^{4}\right) \tag{5.62}
\end{equation*}
$$

Thus, the partition function for the wormhole geometry, stabilized by constraining the average energy, in the regime $\beta \ll T$ is

$$
\begin{equation*}
\tilde{Z}_{E_{\text {avg }}}(\beta+i T, \beta-i T) \sim T e^{-2 \beta E_{\text {avg }}-C \phi_{0}^{2} T} \tag{5.63}
\end{equation*}
$$

where the $T$ factor in front is from the zero mode volume, and the rest is $e^{-S}$, with $C=2 d E_{\text {avg }} / \pi(d-1)$.

Let us now compare the above action of the constrained saddle in the presence of a scalar field with the corresponding prediction coming from the boundary. Such a boundary quantity would be

$$
\begin{align*}
\tilde{Z}_{E_{\text {avg }}}\left(\beta_{1}, \beta_{2}\right) & =\left\langle\operatorname{Tr}\left(e^{-\beta_{1} H_{R}}\right) \operatorname{Tr}\left(e^{-\beta_{2} H_{L}}\right)\right\rangle_{E_{\text {avg }}} \\
& =\int d E_{1} d E_{2}\left\langle\rho_{R}\left(E_{1}\right) \rho_{L}\left(E_{2}\right)\right\rangle e^{-\beta_{1} E_{1}} e^{-\beta_{2} E_{2}} \delta\left(E_{1}+E_{2}-2 E_{\text {avg }}\right) \tag{5.64}
\end{align*}
$$

As before, the presence of a bulk massless scalar field with unequal boundary conditions $\pm \phi_{0}$ implies that we are turning on two slightly opposite couplings for the Hamiltonian on both sides: $H_{R, L}=H \pm \phi_{0} O$. Additionally, as indicated by the subscript $E_{\text {avg }}$ on the left, we have constrained the average energy. This is implemented via a delta function on the right. Since we cannot calculate this directly in any given boundary theory, we need to rely on universal predictions coming from random matrix theory.

Before we do that let us consider a simpler exercise where we can perform an exact computation involving a constraint on the average energy. Consider the usual spectral form factor in random matrix theory:

$$
\begin{align*}
Z\left(\beta_{1}, \beta_{2}\right) & =\left\langle\operatorname{Tr}\left(e^{-\beta_{1} H}\right) \operatorname{Tr}\left(e^{-\beta_{2} H}\right)\right\rangle \\
& =\int d E_{1} d E_{2} e^{-\beta_{1} E_{1}-\beta_{2} E_{2}}\left\langle\rho\left(E_{1}\right) \rho\left(E_{2}\right)\right\rangle \tag{5.65}
\end{align*}
$$

For double-scaled random matrix models, we know that [126]

$$
\begin{equation*}
\left\langle\rho\left(E_{1}\right) \rho\left(E_{2}\right)\right\rangle=-\frac{1}{2 \pi^{2}} \frac{E_{1}+E_{2}}{\sqrt{E_{1}} \sqrt{E_{2}}\left(E_{1}-E_{2}\right)^{2}} \tag{5.66}
\end{equation*}
$$

Using the above spectral two-point function, and changing variables to $\beta_{ \pm}=$ $\beta_{1} \pm \beta_{2}$ and $E^{ \pm}=E_{1} \pm E_{2}$ we get

$$
\begin{align*}
Z\left(\beta_{1}, \beta_{2}\right) & =-\frac{1}{2 \pi^{2}} \int_{0}^{\infty} d E^{+} \int_{-\infty}^{\infty} d E^{-} e^{-\left(\beta_{+} E^{+}+\beta_{-} E^{-}\right) / 2} \frac{2 E^{+}}{\sqrt{\left(E^{+}\right)^{2}-\left(E^{-}\right)^{2}}\left(E^{-}\right)^{2}} \\
& =-\frac{1}{\pi^{2}} \int_{0}^{\infty} \frac{d t}{t} e^{-s t} \int_{-1}^{1} d x \frac{e^{-t x}}{\sqrt{1-x^{2}}} \frac{1}{x^{2}} \tag{5.67}
\end{align*}
$$

where we have made a few more change of variables, $x=E^{-} / E^{+}, t=\beta_{-} E^{+} / 2$ and $s=\beta_{+} / \beta_{-}$. Though the $x$ integral in the above is formally undefined, we can proceed by computing its principle value which turns out to be given by a hypergeometric function

$$
\begin{align*}
Z\left(\beta_{1}, \beta_{2}\right) & =-\frac{1}{\pi^{2}} \int_{0}^{\infty} \frac{d t}{t} e^{-s t}\left(2+\frac{\pi t^{2}}{2}{ }_{1} F_{2}\left(\frac{1}{2} ; \frac{3}{2}, 2 ; \frac{t^{2}}{4}\right)\right) \\
& =-\frac{1}{2 \pi}\left(1-\sqrt{1-\frac{1}{s^{2}}}\right)-\frac{2}{\pi^{2}} \int_{0}^{\infty} \frac{d t}{t} e^{-s t} \tag{5.68}
\end{align*}
$$

Notice that the integral in the second term is actually independent of $s$. Substituting $s=\beta_{+} / \beta_{-}$back into the integral, we see that up to a $\beta_{1}, \beta_{2}$ additive constant

$$
\begin{equation*}
Z\left(\beta_{1}, \beta_{2}\right)=\frac{1}{2 \pi} \frac{\sqrt{\beta_{1} \beta_{2}}}{\beta_{1}+\beta_{2}} \tag{5.69}
\end{equation*}
$$

Now let us insert a delta function in $E^{+}$thereby constraining the average energy. We can then write

$$
\begin{align*}
Z_{E_{\mathrm{avg}}}\left(\beta_{1}, \beta_{2}\right) & =-\frac{1}{\pi^{2}} \int_{0}^{\infty} d E^{+} e^{-\beta_{+} E^{+} / 2} f\left(\beta_{-}, E^{+}\right) \delta\left(E^{+}-2 E_{\mathrm{avg}}\right)  \tag{5.70}\\
& =-\frac{e^{-\beta_{+} E_{\mathrm{avg}}}}{\pi^{2}} f\left(\beta_{-}, 2 E_{\mathrm{avg}}\right)
\end{align*}
$$

where we have used the result in (5.68) to denote

$$
\begin{equation*}
f\left(\beta_{-}, E^{+}\right)=\frac{2}{E^{+}}+\frac{\pi \beta_{-}^{2} E^{+}}{8}{ }_{1} F_{2}\left(\frac{1}{2} ; \frac{3}{2}, 2 ; \frac{\left(\beta_{-} E^{+}\right)^{2}}{16}\right) \tag{5.71}
\end{equation*}
$$

Now continuing $\beta_{1,2}=\beta \pm i T$, for $T \gg 1$ we can write

$$
\begin{equation*}
Z_{E_{\text {avg }}}(\beta+i T, \beta-i T)=\frac{1}{\pi} T e^{-2 \beta E_{\text {avg }}}+O\left(T^{0}\right) \tag{5.72}
\end{equation*}
$$

To summarise this exercise, we see that the spectral form factor with an average energy constraint also displays a linear ramp in $T$ for large enough $T$, although with a different coefficient.

Let us briefly return to the main case that we were interested in, involving the modified spectral form factor. We are not aware of a universal result for $\left\langle\rho_{R}\left(E_{1}\right) \rho_{L}\left(E_{2}\right)\right\rangle$ that is valid for all ranges of energies, similar to (5.66). But we know the that behaviour for small energy differences should be

$$
\begin{equation*}
\left\langle\rho_{L}\left(E_{1}\right) \rho_{R}\left(E_{2}\right)\right\rangle_{c}=-\frac{1}{2 \pi^{2}} \frac{\left(E_{1}-E_{2}\right)^{2}-\lambda^{2}}{\left(\left(E_{1}-E_{2}\right)^{2}+\lambda^{2}\right)^{2}} \tag{5.73}
\end{equation*}
$$

This is not sufficient to predict the modified spectral form factor for general $\beta_{1,2}$, but it is exactly what we need to estimate it in the regime $\beta_{1,2}=\beta \pm i T$ for $T \gg \beta$. Inserting (5.73) into (5.64) in this regime, we obtain

$$
\begin{equation*}
\tilde{Z}_{E_{\text {avg }}}(\beta+i T, \beta-i T) \sim T e^{-2 \beta E_{\mathrm{avg}}-\lambda T} \tag{5.74}
\end{equation*}
$$

which of course matches with the bulk calculation obtained in (5.63), for $\lambda=C \phi_{0}^{2}$.

### 5.6 Conclusions

In this chapter we considered the simplest examples of Euclidean wormholes in $\operatorname{AdS}_{d+1}$ with two asymptotic boundaries. For simplicity, we assumed that the boundaries were flat, with the topology to be that of $d$-dimensional torus. This could easily be extended to the case of positively curved boundaries. We first studied examples of a minimally coupled scalar field to gravity where an ansatz for the metric and the potential of the field allowed us to solve the field equations using first order flow equations. These had the peculiar property that the boundary sources for the scalar were fixed to specific constants.

Subsequently, we explored additional instances of wormholes by permitting a broader class of spacetime metrics. By choosing a judicious ansatz for the metric, we again expressed the field equations as first order equations. In this case, the
two-sided solutions we obtained were singular, analogous to the double cone wormholes of [141]. We saw that these solutions exist even in the absence of matter. In the presence of a free field, we studied an example of a backreacted double cone geometry. The proper treatment here required us to consider complex spacetime metrics.

Finally we considered an even more general class of wormholes configurations which are constrained saddles. These have similar properties to the double cone geometries, without unnecessary complexities. We again studied these constrained saddles in $d$-dimensions, in the presence of a scalar field. To stabilize these wormholes, we used a constraint that fixes the average of the two boundary energies, as introduced in [140]. We looked at an example involving random matrices with this fixed average energy constraint, and compared the results of the constrained saddles with some boundary predictions.

## Summary \& Outlook

The research presented in this dissertation explores two topics: complexity and wormholes, both of which have proven to be essential ingredients in deepening our understanding of holography. In Part I, we discuss holographic complexity, and in Part II, we explore Euclidean wormholes. At the beginning of each part, we have provided background information and motivations for studying these topics. In this brief concluding chapter, we will gather our final thoughts, reflect on some of the motivations, and offer an outlook for the future.

## Complexity

In chapters 2 and 3 we explored various ways of assigning a cost to holographic path integrals. The path integrals we considered were in bulk gravitational theories, which have Dirichlet boundary conditions on a varying radial cutoff surface. On the boundary, these would be path integrals in $T \bar{T}$ deformed CFTs. These path integrals can be interpreted as continuous quantum circuits which map an initial state at a given bulk cutoff to some final state at another bulk cutoff. We took a phenomenological approach towards quantifying the costs of such path integrals and imposed a set of physical conditions that any reasonable cost functional should satisfy. Then, in chapter 3, we explicitly showed that by optimising costs over suitably chosen bulk subregions we can re-obtain some familiar complexity proposals present in the literature.

The general approach towards complexity in holography has been that, despite encapsulating a single and very useful concept, there exist various definitions or methods for measuring it. We have seen this in the huge space of possibilities for defining cost functions that satisfy all of our physical requirements. Similarly, there are also an infinite class of diffeomorphism invariant bulk quantities all of
which have the requisite properties of holographic complexity [69]. This is in stark contrast to something like entanglement entropy, which has a precisely defined bulk dual. Boundary definitions of complexity such as circuit complexity similarly have a lot of freedom in choosing a gate set and penalty factors for costly gates. Recently, another notion of complexity, called as Krylov complexity has been the subject of significant study [170-173]. This can be defined for all quantum systems and naturally avoids many ambiguities present in other definitions. For the case of a specific one-dimensional quantum system this was shown $[174,175]$ to be dual to a bulk volume in JT gravity, and there is a more general argument that this holds in higher dimensions in [176]. Even with definitions like Krylov complexity, which appear to isolate the simplest bulk dual for complexity (the maximal volume), it remains mysterious how we should interpret all the other diffeomorphism-invariant quantities, all of which exhibit a behavior expected from complexity.

The principle reason for studying complexity in holography was to understand the growth of black hole interiors. The growth and late time saturation in particular has been well understood [177, 178] again for the volume of black hole interior in JT gravity. It is currently unclear how such a saturation is achieved for holographic complexity proposals other than maximal volume. It would be interesting to understand these issues in the near future.

## Wormholes

The chapters 4 and 5 were devoted to a different topic, Euclidean wormholes. We began by reviewing some important recent developments which arose as a direct consequence of including contributions from Euclidean wormholes with disconnected boundaries. We learned in chapter 4 that the Euclidean wormholes encoded statistical information of some underlying microscopic data. Replica wormholes capture statistical information about the entanglement spectrum of the black hole radiation which leads to a unitary Page curve. Similarly, Euclidean wormholes with two disconnected boundaries capture the spectral statistics of the black hole microstates. These also gave a gravitational explanation of the ramp in the spectral form factor. We also saw examples where Euclidean wormholes capture the statistics of OPE coefficients in 3d gravity. In JT gravity, which is dual to an ensemble of random matrices, the wormhole with disconnected boundaries compute moments in the ensemble. But such wormholes in higher dimensions, where we do not expect an ensemble in the boundary, lead to a factorization puzzle. To overcome this, we choose the viewpoint that calculations in semi-classical gravity only have access to a coarse-grained version of the underlying microscopic data.

In chapter 5, inspired by the recent developments, our aim was to explore worm-
holes in general dimensions. Wormholes with multiple disconnected (flat) boundaries do not arise as solutions to vacuum Einstein equations [134]. To overcome this, we first studied wormholes that are supported by a scalar field with imaginary sources at the boundaries. We developed a general strategy using which many such generic wormhole solutions can be constructed. All of them have the property that the sources for the scalar have to be precisely fine-tuned at the boundaries. Later, we expanded our search by considering a more general ansatz, and found wormholes that were singular solutions, akin to the double cone wormholes studied in [141]. We regulated these solutions and also constructed a backreacted double cone geometry in the presence of a massless scalar. When the boundary conditions of the field are set to be slightly different on both the boundaries, we observed that this geometry, instead of having a linear ramp, has an exponential decay. We also considered a similar analysis for constrained saddles [138, 140] with matter, where we had to stabilise the wormholes by adding a constraint on the energy. Furthermore, we showed how such a constraint works in a simple model involving random matrices.

The wormholes similar to double cones and the constrained wormhole saddles both reaffirm the statistical interpretation of semi-classical gravity. In fact, as we showed, in the presence of a massless scalar with different boundary conditions, the computations of these wormholes lead to a modified (smooth) spectral form factor which can be reproduced from general arguments in random matrix theory. It would be interesting to see if this persists for massive scalars and other types of matter.

On the other hand, the initial instances we encountered of wormholes with matter, which included scalar fields sourced by specific imaginary sources, appear to be somewhat challenging to reconcile with this statistical interpretation. These wormholes are related to axionic wormholes [156], and also are very similar to holographic RG flows, with the scalar acting like a running coupling. One outcome could be that these wormholes are not stable and such saddle points never really local minima in the path integral, and are thus unreliable. On a slightly unrelated note, there exist several stable and generic wormhole solutions with matter as constructed in [157]. Do these wormholes also have a statistical interpretation? We leave these questions for future work.

## Cost as Lorentzian gravitational action

Let us calculate the gravitational action of the subregion of Lorentzian Poincaré $\mathrm{AdS}_{3}$ depicted in figure 3.5. The gravitational action is

$$
\begin{equation*}
I=I_{E H}+I_{G H Y}+I_{\text {Hayward }} \tag{A.1}
\end{equation*}
$$

The Einstein-Hilbert term is

$$
\begin{equation*}
I_{E H}=\frac{1}{16 \pi G_{N}} \int \sqrt{-G}(\mathcal{R}-2 \Lambda) \tag{A.2}
\end{equation*}
$$

where $\mathcal{R}=-6$ and $\Lambda=-1$ in $\mathrm{AdS}_{3}$ with $L=1$. Then $I_{E H}$ is proportional to the spacetime volume,

$$
\begin{equation*}
I_{E H}=-\frac{1}{8 \pi G_{N}} \int d x \int_{t_{i}}^{t_{f}} d t \frac{1}{\rho^{2}} \tag{A.3}
\end{equation*}
$$

Our region has two corners, both of which are spacelike surfaces meeting a timelike surface, both of which contribute to the gravitational action ${ }^{1}$

$$
\begin{equation*}
I_{\text {Hayward }}=\frac{1}{8 \pi G_{N}} \int \sqrt{\sigma} \eta \tag{A.4}
\end{equation*}
$$

where $\sigma$ is the induced metric on the joint, and

$$
\begin{equation*}
\sinh \eta=-t_{1} \cdot n_{2} \tag{A.5}
\end{equation*}
$$

where $t_{1}$ is the (timelike) normal to the $t=\left\{t_{i}, t_{f}\right\}$ slices, and $n_{2}$ is the (spacelike) normal to $z=\rho(t)$. For our setup it's easy to show that

$$
\begin{equation*}
\eta= \pm \operatorname{arctanh} \dot{\rho} \tag{A.6}
\end{equation*}
$$

with $+(-)$ at the $t_{f}\left(t_{i}\right)$ joint. This gives

$$
\begin{equation*}
I_{\text {Hayward }}=\frac{1}{8 \pi G_{N}} \int d x\left[\frac{\operatorname{arctanh} \dot{\rho}\left(t_{f}\right)}{\rho\left(t_{f}\right)}-\left(t_{f} \leftrightarrow t_{i}\right)\right] . \tag{A.7}
\end{equation*}
$$

[^18]Let us calculate the contribution of the finite cutoff time-like boundary to the gravitational action through the Gibbons-Hawking-York (GHY) term. The boundary is the hypersurface

$$
\begin{equation*}
z=\rho(t) \tag{A.8}
\end{equation*}
$$

in Poincaré $\mathrm{AdS}_{3}$ (3.80). We want to calculate the extrinsic curvature of this surface

$$
\begin{equation*}
K=\nabla_{\mu} n^{\mu}=z^{3} \partial_{\mu}\left(z^{-3} n^{\mu}\right) \tag{A.9}
\end{equation*}
$$

where the unit normal to the hypersurface is given by

$$
\begin{equation*}
n^{\mu}:=\frac{G^{\mu \nu} \zeta_{\nu}}{|\zeta|} \tag{A.10}
\end{equation*}
$$

with un-normalised normal

$$
\begin{equation*}
\zeta_{\mu}=\partial_{\mu}(\rho(t)-z) ; \quad \zeta_{x}=0, \zeta_{t}=\dot{\rho}, \zeta_{z}=-1 \tag{A.11}
\end{equation*}
$$

and

$$
\begin{equation*}
|\zeta|=z \sqrt{1-\dot{\rho}^{2}} \tag{A.12}
\end{equation*}
$$

Plugging these in to the formula for $K$ gives

$$
\begin{equation*}
\left.K\right|_{z=\rho}=\frac{-\rho \ddot{\rho}+2\left(1-\dot{\rho}^{2}\right)}{\left(1-\dot{\rho}^{2}\right)^{3 / 2}} \tag{A.13}
\end{equation*}
$$

which is exactly what you get if you Wick rotate the Euclidean answer, so that $\dot{\rho}^{2} \rightarrow-\dot{\rho}^{2}$, and $\ddot{\rho} \rightarrow-\ddot{\rho}$. The GHY term is

$$
\begin{align*}
I_{G H Y} & =\frac{1}{8 \pi G_{N}} \int \sqrt{-g} K \\
& =\frac{1}{8 \pi G_{N}} \int d x \int_{t_{i}}^{t_{f}} d t \frac{-\rho \ddot{\rho}+2\left(1-\dot{\rho}^{2}\right)}{\rho^{2}\left(1-\dot{\rho}^{2}\right)} \tag{A.14}
\end{align*}
$$

Integrating the double derivative term by parts gives a boundary contribution that cancels the Hayward terms in our case (however, this cancellation would not have happened with general spacelike boundaries rather than the constant time slices we have considered). Combining everything gives

$$
\begin{equation*}
I=\frac{1}{8 \pi G_{N}} \int d x \int_{t_{i}}^{t_{f}} d t\left(\frac{1-\dot{\rho} \operatorname{arctanh} \dot{\rho}}{\rho^{2}}\right) \tag{A.15}
\end{equation*}
$$

which agrees with the Wick rotation of the Euclidean result of [1]. We note that while the GHY term (A.14) by itself may remain finite when a null-limit of the surface $\rho(t)$ is taken (see [179]), the full action (A.15) diverges in this limit. This can be shown to be a consequence of the Hayward-type corner terms (A.4). In general, the procedure of obtaining gravitational action of a region with null boundaries as a null-limit of timelike or spacelike regions is ambiguous, see [45].

# B <br> <br> Conical singularities in the <br> <br> Conical singularities in the Gauss-Bonnet formula 

 Gauss-Bonnet formula}

## B.0.1 Euclidean case

While most sources don't state the Gauss-Bonnet theorem explicitly including terms necessary for conical singularities, it is not hard to find the appropriate terms. While a formal proof was given in [98], a less rigorous but quite simple approach would be the one of [180] where it was simply postulated that the GaussBonnet theorem should continue to hold in the presence of conical singularities, and then the necessary correction term was derived by looking at one simple example ${ }^{1}$. In this section, we will give our own argument which easily generalises to the Lorentzian case in the next subsection.

To do so, we consider a body with a conical singularity like the one sketched on the left side of figure B.1. Of course we know that a conical singularity can be resolved, in a sense, by introducing a cut and spreading the cone on a flat plane as indicated in the figure. Hence, let us now assume that, as indicated by the dashed red lines in the figure, we introduce a cut that goes from the exact location of the conical singularity to a point elsewhere in the surface where it is locally smooth. Before introducing this cut, we assume the Gauss-Bonnet theorem holds in a form

$$
\begin{equation*}
\int_{\tilde{M}} \frac{R}{2} d V+\int_{\partial \tilde{M}} k_{g} d s+\sum_{\text {(old) corners } c} \alpha_{c}+X_{\text {conical sing. }}=2 \pi \chi \tag{B.1}
\end{equation*}
$$

Herein, $X_{\text {conical sing. }}$ is the as of now unkown contribution from the conical singularity which we want to derive.

How does introducing the cut change both sides of this equation? The first term stays the same because we don't assume to spread the cut open by deforming

[^19]

Figure B.1: Left: Illustration for the derivation of the Gauss-Bonnet-theorem in the presence of conical singularities. We consider a two dimensional surface, either Euclidean or Lorentzian, with a conical singularity, and introduce a cut (red dashed lines) from the conical singularity to a point elsewhere in the surface, where it is locally smooth. The conical deficit of the singularity (or its Lorentzian analogue) is $\delta$. Right: Construction of the lemon's induced metric by taking a region between two timelike geodesics in $A d S_{2}$ and identifying the boundary. At the intersection of the two geodesics that form the boundary of the region, their two tangent vectors cross with a relative boost factor $\delta$, which is the Lorentzian analogue of the deficit angle for the conical singularity that is formed at this location due to the identification. The two vertical lines indicate the $A d S_{2}$ boundaries at $\theta_{2}= \pm \frac{\pi}{2}$.
the surface, both edges of the cut remain at the same location. We have merely indicated a slight spread of the cut as a visual aid in the figure. Due to the cut, new contributions to the second term could in principle appear, however we argue this won't matter for multiple reasons. Firstly, the contributions from the two sides of the cut should cancel exactly. Secondly, we could choose the cut to be geodesic, setting $k_{g}=0$. Thirdly, we can take a limit where the cut is infinitesimally short. The fourth term will not be present anymore due to the resolution of the conical singularity, hence on the left hand side all changes come down to the additional terms due to the two corners at both ends of the cut. Of course, introducing the cut also causes a change of the topology like removing a disk, and due to the behaviour of the Euler characteristic under connected sums this means $\chi$ is reduced by 1. Hence, we find

$$
\begin{equation*}
\int_{\tilde{M}} \frac{R}{2} d V+\int_{\partial \tilde{M}} k_{g} d s+\sum_{\text {(old) corners } c} \alpha_{c}+\sum_{\text {new corners } c} \alpha_{c}=2 \pi(\chi-1) \tag{B.2}
\end{equation*}
$$

Comparing (B.1) and (B.2), we find that the correct contribution for conical singularities is hence determined by the contributions for corners along boundaries via

$$
\begin{equation*}
X_{\text {conical sing. }}=\sum_{\text {new corners } c} \alpha_{c}+2 \pi . \tag{B.3}
\end{equation*}
$$

So what are now the contributions from the two corners at which the cut starts and ends? Firstly, at the point in a locally smooth neighbourhood of the surface, essentially the new boundary introduced by the cut makes a 180-degree turn there, i.e. $\alpha_{c 1}=-\pi$. Note the negative sign because this corner is a concave one from the point of view of the surface. The corner located at the position of the conical singularity is also concave from the point of view of the surface, but there, with respect to the local geometry, the angle by which the boundary changes its direction is reduced by the deficit angle $\delta$ of the conical singularity as evident from the figure B.1. Hence, the contribution is $\alpha_{c 2}=-(\pi-\delta)$, and we find

$$
\begin{equation*}
X_{\text {conical sing. }}=\alpha_{c 1}+\alpha_{c 2}+2 \pi=\delta . \tag{B.4}
\end{equation*}
$$

This means that the contribution of a conical singularity in the Gauss-Bonnet theorem should be simply its deficit angle $\delta$, as also realised in $[98,180]$. Our derivation makes it obvious why the terms coming from conical singularities and terms coming from corners of the boundary are so similar (just sums over angles), and can readily be generalised to the Lorentzian case as we show in the next subsection.

## B.0.2 Lorentzian case

Generalisations of the Euclidean Gauss-Bonnet theorem to the Lorentzian case were worked out in [99-102] ${ }^{2}$, and while none of these papers explicitly discusses conical singularities, the generalisation of the concept of an angle to the Lorentzian case lies at the heart of all of these works. As the appropriate terms for conical singularities are just sums over deficit-angles which can be derived from the terms needed for boundaries with corners, as shown in the previous subsection, it is hence easy to generalise this also to the Lorentzian case. The papers [99-102] differ in some of the details of exactly how to define Lorentzian angles, e.g. some use complex quantities, so for concreteness we follow [99] and define the (always real valued) oriented Lorentzian angle or boost parameter $\delta$ between two future pointing normalised timelike vectors $X$ and $Y$ to satisfy ${ }^{3}$

$$
\begin{equation*}
\cosh (\delta)=-X \cdot Y \tag{B.5}
\end{equation*}
$$

The Lorentzian Gauss-Bonnet-theorem then takes the form [99]

$$
\begin{equation*}
\int_{\tilde{M}} \frac{R}{2} d V+\int_{\partial \tilde{M}} k_{g} d s+\sum_{\text {corners } c} \alpha_{c}=0 \tag{B.6}
\end{equation*}
$$

where some care has to be taken concerning the signs of the generalised angles $\alpha_{c}$. In fact, the difference to the Euclidean case is two-fold: Firstly, the right hand side automatically vanishes ( $\chi \equiv 0$ ), secondly, traversing a closed timelike geodesic polygon in flat space yields the total Lorentzian angle

$$
\begin{equation*}
\alpha_{12}+\alpha_{23}+\ldots+\alpha_{n 1}=0 \tag{B.7}
\end{equation*}
$$

whereas in the Euclidean case the exteriour angles of a polygon sum to $2 \pi$. This means that we can quite easily generalise our derivation from the previous subsection to the Lorentzian case, however while there on both the left- and the right-hand side an additional term $2 \pi$ appeared, this will not be the case in the Lorentzian setting, and we find that the appropriate contribution to (B.6) to account for Lorentzian conical singularities will be a term $X_{\text {conical sing. }}=\delta$ where $\delta$ is the Lorentzian analogue of the deficit angle at the conical singularity.

## B.0.3 Application to Lemons

Let us now demonstrate how the Lorentzian version of the Gauss-Bonnet theorem (including terms for conical singularities) can be applied to the example of a lemon

[^20]surface from section 3.8.5, where for simplicity we will assume a $\phi$-independent parameter $E$. To do this, we view the lemon as a boundary-less closed surface which however has two conical singularities, as e.g. the example on the top-left of figure 3.9. Hence as $\chi \equiv 0$ in the Lorentzian case, we need to verify
\[

$$
\begin{equation*}
\frac{R V}{2}+\delta_{\text {past conical sing. }}+\delta_{\text {future conical sing. }}=0 \tag{B.8}
\end{equation*}
$$

\]

To correctly calculate the Lorentzian analogue of the deficit angle, the easiest way in this case (but not necessarily the only or most general one) is to resolve the conical singularity by introducing a cut. For this, consult the right side of figure B.1. As we showed in section 3.8.2, the induced metric on the surface should be locally $\mathrm{AdS}_{2}$. But of course, when thinking about AdS space we usually envision a static spacetime with an asymptotic boundary as opposed to something resembling a periodic cosmology that starts from an initial (conical) singularity, expands, contracts, and ends in a final (conical) singularity, like the surfaces shown in figure 3.9. The resolution of this issue is of course that the induced metric of the lemons is only locally AdS, and we know that global identifications can yield very non-trivial geometries, like for example the BTZ black hole. The right side of figure B. 1 shows how an identification applied to (global) $\mathrm{AdS}_{2}$ with line element

$$
\begin{equation*}
d s^{2}=\frac{1}{\cos \left(\theta_{2}\right)^{2}}\left(-d t_{2}^{2}+d \theta_{2}^{2}\right) \tag{B.9}
\end{equation*}
$$

can yield the induced geometry of a lemon surface. Note that we have introduced coordinates $t_{2}, \theta_{2}$ on $\mathrm{AdS}_{2}$ to distinguish them from the coordinates of global $\mathrm{AdS}_{3}$ (3.112), which in this section we explicitly replace by $t \rightarrow t_{3}, \theta \rightarrow \theta_{3}$. Also note that (3.112) and (B.9) have the same AdS-radius (which we set to one), hence the three dimensional Ricci scalar is $\mathcal{R}=-6$ and the two dimensional one is $R=-2$, as required by our construction (e.g. (3.97)).

To create the lemon, we have to take the region between two intersecting timelike geodesics in $\mathrm{AdS}_{2}$ and then identify these two geodesics. For concreteness, we assume both boundary geodesics in figure B .1 to turn around at a maximal radial coordinate $\left|\theta_{\max , 2}\right|=\arccos \left(1 / E_{2}\right)$ according to the coordinate system (B.9). Concerning the lemon surface embedded into the $\mathrm{AdS}_{3}$ ambient space with coordinate system (3.112) as shown in figure 3.9, we introduce the turnaround radius $\theta_{\max , 3}=\arccos \left(1 / E_{3}\right)$. These two sets of parameters are related because the diameter of the $\mathrm{AdS}_{2}$ region (the shaded region in figure B.1) has to be equal to the circumference of the surface when embedded into $\mathrm{AdS}_{3}$ (the surfaces in 3.9). This yields the relation

$$
\begin{equation*}
4 \operatorname{arctanh}\left(\tan \left(\frac{\theta_{\max , 2}}{2}\right)\right)=2 \pi \tan \left(\theta_{\max , 3}\right) \tag{B.10}
\end{equation*}
$$

The first term in (B.8) is easy to compute from the induced metric (3.120), and we find

$$
\begin{equation*}
\frac{R V}{2}=-4 \pi \sqrt{E_{3}^{2}-1}=-4 \pi \tan \left(\theta_{\max , 3}\right) \tag{B.11}
\end{equation*}
$$

 we note that at the point $\left(\theta_{2}=0\right)$ where the two boundary geodesics intersect, their future pointing normalised tangent vectors (drawn read in figure B.1) read (this can be shown from (3.118))

$$
\begin{equation*}
X_{ \pm}^{m}=\binom{X_{ \pm}^{t_{2}}}{X_{ \pm}^{\theta_{2}}}=\binom{E_{2}}{ \pm \sqrt{E_{2}^{2}-1}} \tag{B.12}
\end{equation*}
$$

and hence are boosted with respect to each other by a boost parameter/Lorentzian angle

$$
\begin{equation*}
\delta_{\text {future conical sing. }}=\operatorname{arccosh}\left(-X_{+} \cdot X_{-}\right)=\operatorname{arccosh}\left(2 E_{2}^{2}-1\right)=2 \pi \tan \left(\theta_{\max , 3}\right) \tag{B.13}
\end{equation*}
$$

even though by the identification of the two boundary geodesics also these two vectors are formally identified. With (B.11) and (B.13), we verify that (B.8) is satisfied as required.

## Lemons in higher dimensions

Let us try to find analogues of the lemon surfaces studied in section 3.8.5 in global Lorentzian $\mathrm{AdS}_{4}$,

$$
\begin{equation*}
d s^{2}=\frac{1}{\cos (\theta)^{2}}\left(-d t^{2}+d \theta^{2}+\sin (\theta)^{2} d \psi^{2}+\sin (\theta)^{2} \sin (\psi)^{2} d \phi^{2}\right), \tag{C.1}
\end{equation*}
$$

with boundary at $\theta=\pi / 2$. Assuming rotational symmetry, we just have to propose an embedding parametrized as

$$
\begin{equation*}
t(\theta, \psi, \phi)=f(\theta) \tag{C.2}
\end{equation*}
$$

After some computations, equation (3.96) then yields the ODE

$$
\begin{equation*}
4 \cot (\theta) f^{\prime}(\theta) f^{\prime \prime}(\theta)-2\left(\csc ^{2}(\theta)+2\right) f^{\prime}(\theta)^{2}\left(f^{\prime}(\theta)^{2}-1\right)=0 \tag{C.3}
\end{equation*}
$$

which effectively is a first order ODE for $f^{\prime}$, as $f$ does not show up in the equation. We find the solution:

$$
\begin{equation*}
f^{\prime}(\theta)=\frac{\sqrt{\sin (2 \theta)} \cot (\theta)}{\sqrt{\mathrm{C}+\sin (2 \theta) \cot ^{2}(\theta)}} \tag{C.4}
\end{equation*}
$$

Unfortunately this is hard to integrate to get an analytic expression for $f$. Nevertheless, we can distinguish three cases, see also figure C.1. For $C>0, f^{\prime} \leq 1$ and the curve $f(\theta)$ we obtain is spacelike and reaches all the way to the boundary at $\theta=\pi / 2$. For $C<0, f^{\prime} \geq 1$ and the curve $t=f(\theta)$ we obtain is timelike and $f^{\prime}$ diverges at some finite $\theta_{\max } \leq \pi / 2$, the turning point of the surface. For $C=0$ we get $f^{\prime}=1$, i.e. we obtain the null boundary of the WDW patch in this limit. This means that these spherically symmetric lemons in $\mathrm{AdS}_{4}$ (and also higher dimensions as we have verified) will share many qualitative features with their $\mathrm{AdS}_{3}$ counterparts, but there are also some interesting qualitative differences that make the $\mathrm{AdS}_{3}$ lemons special.


Figure C.1: Equation (C.4) for $C$ varying in equal steps between -1 (blue) and 1 (red). As before, the $A d S_{4}$ boundary is at $\theta=\pi / 2$.

First of all, while ansatz (3.98) would of course also work in higher dimensions, it is not generic in these cases, and in fact the extrinsic curvature tensor of the solutions discussed here will not have this form. Consequently, the higher dimensional lemons are not foliated by timelike geodesics of the ambient AdS space. Furthermore, note that $f^{\prime}(0)=1$ for any $C$ in (C.4), so at the center the embeddings will always approach the local lightcone. We have discussed the appearance of a conical singularity already in section 3.8.5, but there for finite turning point radius the embedding at the conical singularity did not approach the lightcone. What this means is that unlike the $\mathrm{AdS}_{3}$ case, for $\mathrm{AdS}_{d \geq 4}$ the metric will degenerate close to the conical singularity and this is accompanied by the appearance of a curvature singularity of the induced metric there. While $R$ is of course constant by construction, this happens for example for the Kretschmann scalar. Another qualitative difference between $\mathrm{AdS}_{3}$ lemons and the higher dimensional case is that the former all neatly fit into a time interval of size $\Delta t=\pi$ as shown in figure 3.9. This is because the periodicity of the timelike AdS geodesics is independent of the parameter $E$ introduced in section 3.8.5. In contrast, integrating (C.4) numerically shows that the $\operatorname{AdS}_{d \geq 4}$ lemons will have different sizes, depending on how close to the boundary they reach.

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## Samenvatting

Alle wetten van de natuurkunde zoals wij die kennen berusten grofweg op twee fundamentele pijlers: de kwantummechanica en de algemene relativiteitstheorie. De kwantummechanica beschrijft ons universum op zijn kleinste schaal. Het is een paradigma van de natuurkunde dat de klassieke mechanica vervangt en dat een raamwerk biedt voor het begrijpen van de vele contra-intuïtieve effecten die je tegenkomt op het microscopische niveau van elementaire deeltjes. Het heeft talloze toepassingen in de echte wereld, variërend van elektronenmicroscopen tot de halfgeleiders die aanwezig zijn in alle moderne elektronica. De algemene relativiteitstheorie daarentegen is Einsteins theorie van ruimte en tijd. Het beschrijft de zwaartekracht als gevolg van de kromming veroorzaakt door materie in de ruimtetijd. De algemene relativiteitstheorie is belangrijk op de grootste schaal in het universum. Het geeft aanleiding tot voorspellingen zoals het bestaan van zwarte gaten en de uitdijing van ons heelal.

De meeste natuurkundige scenario's kunnen worden verklaard met behulp van alleen de kwantummechanica of alleen de algemene relativiteitstheorie. Maar er zijn situaties waarin we ze allebei tegelijkertijd moeten begrijpen. Een belangrijk voorbeeld van een dergelijke situatie vormen zwarte gaten. Dit zijn zeer massieve objecten die de ruimtetijd zo sterk krommen dat zelfs licht niet kan ontsnappen zodra het een oppervlak binnengaat dat de gebeurtenishorizon wordt genoemd. Om ze goed te kunnen begrijpen, moeten we de effecten van de kwantummechanica en de algemene relativiteitstheorie combineren. Daarmee liet Hawking zelfs zien dat zwarte gaten niet echt zwart zijn. Ze gedragen zich als thermische objecten en hebben daardoor een temperatuur en entropie en zenden ook Hawking-straling uit.

De entropie van zwarte gaten, die in het algemeen een telling is van het aantal microscopische configuraties dat mogelijk is voor een gegeven macroscopische configuratie, blijkt een belangrijk mysterie te herbergen. Het is evenredig met het gebied van de horizon. Naïef zou de entropie evenredig moeten zijn met het vo-
lume van een bepaald object. Dit suggereert dat de fundamentele bouwstenen van zwarte gaten zich in een lagere dimensie bevinden. Dit feit leidde tot het holografische principe, dat stelt dat elke theorie van kwantumzwaartekracht microscopische vrijheidsgraden heeft in één lagere dimensie. Een precieze realisatie hiervan werd gevonden in de snaartheorie, de zogenaamde AdS/CFT-correspondentie. Dit is een verklaring die ons vertelt dat de kwantumzwaartekracht in een bepaald universum genaamd Anti-de Sitter-ruimtetijd precies tweeledig is met een specifieke kwantumtheorie die een conforme veldtheorie wordt genoemd in een lagere dimensie. Er kan worden aangenomen dat deze CFT op de grens van de AdS-ruimtetijd leeft. Omdat deze verklaring een dualiteit is, kunnen we objecten van de CFT naar de AdS vertalen en omgekeerd. In feite is het begrijpen van zo'n woordenboek een hoofdthema van onderzoek op dit gebied geweest.

De binnenkant van zwarte gaten is verre van saai. Ze blijven heel lang groeien. Dit roept een puzzel op in AdS/CFT, omdat het impliceert dat er een overeenkomstige grootheid moet bestaan in de grenskwantumtheorie die op equivalente wijze lange tijd blijft groeien. Veel gebruikelijke grootheden, zoals correlatiefuncties of entropieën, verzadigen in korte tijd tot hun thermische waarden. Er werd aangenomen dat een grootheid die kwantumcomplexiteit wordt genoemd precies de juiste eigenschappen had om zo'n langdurige groei te beschrijven. Het begrip complexiteit heeft hier betrekking op het kwantificeren van de moeilijkheidsgraad van een bepaalde taak, wanneer we een bepaalde reeks hulpmiddelen ter beschikking krijgen. Dit is van groot belang op het gebied van de informatica, waarbij de moeilijkheidsgraad van het oplossen van algoritmen wordt bestudeerd en deze dienovereenkomstig worden geclassificeerd. In de AdS-ruimtetijd werd voorgesteld dat de kwantumcomplexiteit van de dubbele toestand duaal is ten opzichte van een geometrisch object zoals het volume van een maximaal oppervlak. In het eerste deel van het proefschrift hebben we laten zien dat we subregio's van de bulkruimtetijd kunnen interpreteren als kwantumcircuits, en hebben we verschillende kosten voor deze subregio's voorgesteld. We hebben ook laten zien hoe dergelijke kosten kunnen worden geoptimaliseerd, waardoor we de complexiteit krijgen van het meest efficiënte circuit dat een eindtoestand vanuit een begintoestand voorbereidt.

Het tweede deel van het proefschrift was gewijd aan het begrijpen van wormgaten. Dit zijn geometrieën die verschillende regio's in een universum of zelfs twee of meer losgekoppelde universums met elkaar verbinden. We bestuderen ruimtetijdwormgaten, ook bekend als Euclidische wormgaten, dit zijn verbonden ruimtetijdgebieden met twee of meer niet-verbonden grenzen. Een paar decennia geleden werden dergelijke geometrieën als pathologisch beschouwd in de kwantumzwaartekracht, en zelfs in het bijzonder in AdS/CFT. Maar recente vooruitgang, vooral dankzij het begrijpen van een tweedimensionaal model genaamd JT-zwaartekracht,
heeft aangetoond dat Euclidische wormgaten ons iets vertellen over statistische informatie over het microscopische systeem. Het begrijpen van de bijdragen van dergelijke wormgaten leidt ook tot enorme vooruitgang bij het oplossen van de informatieparadox over zwarte gaten. We bestuderen de constructie van enkele eenvoudige voorbeelden van Euclidische wormgaten in algemene afmetingen in de aanwezigheid van een scalair veld. We hebben ook geprobeerd de implicaties van het bestaan van dergelijke wormgaten te begrijpen.

## Nun@M,

All the laws of physics as we know them broadly stand on two foundational pillars: Quantum Mechanics and General Relativity. Quantum mechanics describes our universe at its smallest scales. It is a paradigm of physics replacing classical mechanics, that provides a framework for understanding the many counter-intuitive effects one encounters at the microscopic level of elementary particles. It has numerous applications in the real world, ranging from electron microscopes to the semi-conductors present in all modern electronics. General relativity on the other hand is Einstein's theory of space and time. It describes gravity as a result of the curvature caused by matter on spacetime. General relativity is important at the largest scales in the universe. It gives rise to predictions such as the existence of black holes and the expansion of our universe.

Most physical scenarios can be explained either using only quantum mechanics or only general relativity. But there are situations when we need to understand both of them simultaneously. An important example of such a situation is given by black holes. These are very massive objects that curve spacetime so much that not even light can escape once it enters a surface called the event horizon. To properly understand them we need to combine effects of quantum mechanics and general relativity. In fact, by doing so Hawking showed that black holes are not really black. They behave like thermal objects, and hence have a temperature, entropy and also emit Hawking radiation.

The entropy of black holes, which in general is a count the number of microscopic configurations possible for a given macroscopic configuration, turns out to hold a key mystery. It is proportional to the area of the horizon. Naively, the entropy should be proportional to the volume of a given object. Then, this suggests that the fundamental building blocks of black holes live in one lower dimension. This fact lead to the holographic principle, which states that any theory of quantum gravity has microscopic degrees of freedom in one lower dimension. A precise realisation of this was found in string theory, called the AdS/CFT correspondence. This
is a statement which tells us that quantum gravity in a certain universe called Anti-de Sitter spacetime is exactly dual to a specific quantum theory called as a conformal field theory in one lower dimension. This CFT can be thought to live on the boundary of the AdS spacetime. This statement being a duality, allows us to translate objects from the CFT to the AdS and vice versa. In fact, understanding such a dictionary has been a main theme of research in this area.

The interiors of black holes are far from dull. They keep on growing for a very long time. This raises a puzzle in AdS/CFT, as it implies that there must exist a corresponding quantity in the boundary quantum theory that equivalently keeps on growing for long times. Many usual quantities such as correlation functions or entropies saturate to their thermal values in a short period of time. It was conjectured that a quantity called as quantum complexity had exactly the right properties to describe such a long time growth. The notion of complexity here deals with quantifying the difficulty of a given task, when we are given a certain set of resources at hand. This is a very important in the field of computer science, which involves studying the difficulty of solving algorithms, and classifying them accordingly. In the AdS spacetime, the quantum complexity of the dual state was proposed to be dual to a geometric object such as the volume of a maximal surface. In the first part of the thesis we showed that we can interpret subregions of the bulk spacetime as quantum circuits, and proposed various costs for these subregions. We also showed how such costs can be optimised, giving us the complexity of most efficient circuit that prepares a final state from an initial state.

The second part of the thesis was devoted to understand wormholes. These are geometries that connect different regions in a universe or even two or more disconnected universes. We study spacetime wormholes, also known as Euclidean wormholes, which are connected spacetime regions having two or more disconnected boundaries. A few decades ago, such geometries were considered pathological in quantum gravity, and even in AdS/CFT specifically. But recent progress, especially coming from understanding a two-dimensional model called JT gravity showed that Euclidean wormholes tell us about some statistical information of the microscopic system. Understanding the contributions of such wormholes also lead to tremendous progress towards solving the black hole information paradox. We study the construction of some simple instances of Euclidean wormholes in general dimensions in the presence of a scalar field. We also attempted to understand the implications of the existence of such wormholes.

## Acknowledgements

I would like to sincerely thank each and every one who has been a part of the journey in this acknowledgements section, which is often the only part of any thesis that gets read from start to finish. Foremost of all, I am profoundly grateful to my advisor, Jan, for his guidance along the past four years. Jan, you have been nothing short of inspiring in so many ways, in your approach towards research and also how you manage things in general. I consider myself very fortunate to have learnt a lot of physics from you. Thank you for your patience and kindness, even in the most trying times.

I also want to extend my gratitude to all my collaborators, whose invaluable contributions have shaped various portions of this thesis. To Andrew, I am particularly thankful for the enriching discussions and support during many challenging moments. Michal, thank you for being such an enthusiastic collaborator. You have been a source of inspiration and learning and your numerous and valuable pieces of career advice is deeply appreciated.

Despite working from home and the office often being eerily quiet for extensive periods of my PhD , the times I spent seeing and working alongside my fellow PhD students were always a highlight, bringing much-needed intellectual stimulation. In particular, I'm grateful to Bahman, Edward and Jeremy who started this academic journey alongside me in the same year. We had countless moments of fun and learning together, in the Solvay school and various other conferences, and in general over the last few years. I would also like to thank all the members of the string theory group. The PhDs and postdocs that I've met along the way: Akash, Alexander, Ana-Maria, Andrei, Ankit, Antonio, Beatrix, Boris, Carlos, Chiara, Davide, Diego, Dominik, Dora, Eleni, Evita, Facundo, Gabriele, Gian-Piero, Giorgos, Greg, Igal, Ignacio, Ioana, Jackson, Jildou, Karthik, Luis, Masataka, Mert, Peng, Ricardo, Shira, Soumangsu, Soner, Stathis, Suzanne, Tarek, Victor. Thank you all, I have had fun discussions and also something to learn from all of you. I would also like to extend my thanks to the secretaries Anna-Marieke, Astrid,

Jirina, Klaartje, Rose, for their support. I have to thank Bahman and Jeremy for also being my paranymphs. Sharing the office over this year with you two has been really rewarding.

I am grateful to all the teachers and educators I had over the years. In particular, a special thanks goes to Sunil, who not only introduced me to theoretical physics but also ignited my passion for it. Thank you for still being a source of inspiration and support. I also extend my warm thanks to all my friends, both back in India and here. Your presence has enriched my life, warding off the monotony and bringing joy to everyday moments. To Chaithanya and Prashanth, we met serendipitously in this country, thank you for being there and for never making me miss home too much.

Above all, my everlasting gratitude is reserved for my family. To my parents and my grandmother, I am deeply grateful for their unconditional love and unwavering support. Without them, none of this would have been possible. Finally, I dedicate this thesis, along with all of its imperfections, to Sri Ramakrishna.


[^0]:    ${ }^{1}$ There were also other hints of the holographic nature of gravity, e.g., $[6,7]$

[^1]:    ${ }^{1}$ The coordinate $\rho$ is related to the Poincaré coordinate $z$ by $\rho=z^{2}$.

[^2]:    ${ }^{2}$ The quantised WDW Hamiltonian has contact terms that can be removed by normal ordering but which leave operator ordering ambiguities in its definition [55]. These are similar to the operator ordering ambiguities arising from the point-splitting regularisation of the $T \bar{T}$ operator. Since our primary focus is on cost proposals rather than $T \bar{T}$ we will not address these subtleties here.

[^3]:    ${ }^{3}$ There are subleading in $1 / N$ corrections to this statement from the metric and matter field fluctuations which we neglect. Our proposals for path integral cost are only accurate to leading order in $1 / N$.

[^4]:    ${ }^{4}$ Herein, we assume both of these states to live effectively in the IR-sector of the Hilbert space of the UV-complete theory that is created by coarse graining.

[^5]:    ${ }^{5}$ We neglect the cosmological constant in this discussion because when considering unboundedness of the action we are interested in the limit $|\mathcal{R}| \gg|\Lambda|$.

[^6]:    ${ }^{1}$ Note that in our Euclidean setting, where spacelike surfaces have spacelike normal vectors, the joints under consideration are more similar to the timelike joints discussed in [43, 45] than spacelike ones in a Lorentzian setting.

[^7]:    ${ }^{2}$ As an illustrative example, imagine a Euclidean axisymmetric spacetime, with a spacetime region in the shape of a regular prism that breaks rotational symmetry around the axis to a discrete subgroup. In the limit where the radius of the prism goes to zero, the action on that region may not go to zero, as while bulk and surface terms vanish in this limit due to the vanishing of bulk volume and surface area, the joint terms will lead to a contribution proportional to an integral along the axis of symmetry. This remnant term is the analogue of the last bracket in (3.22).

[^8]:    ${ }^{3}$ Note that this analysis did not include corner terms contributions to the action. However, in the case of open bulk curves, in the kinematic space framework there are additional contributions that we have not included. It would be certainly interesting to see if they reproduce the corner terms. We would like to thank Bartek Czech for bringing this up.

[^9]:    ${ }^{4}$ We follow here the notation of section 3 of [85], where latin indices refer to the induced geometry of the hypersurface $\tilde{M}$ with coordinates $y^{a}$, greek indices refer to the ambient (bulk) spacetime $N$ with coordinates $x^{\alpha}$, and we can define the projector $e_{a}^{\alpha}=\partial x^{\alpha} / \partial y^{a}$. To avoid confusion concerning e.g. the Ricci scalar, we use $\mathcal{R}$ for curvature tensors of the bulk, and $R$ for curvature tensors of the induced metric. The bulk metric as throughout this section is $G_{\mu \nu}$ while the induced metric is $g_{m n}$.
    ${ }^{5}$ Specifically, in equation (3.7) of [1] (where $d$ stands for the dimension of the entire bulk spacetime), plugging in the relation $\pi_{m n}=-\left(K_{m n}-K g_{m n}\right)$ shows that (3.96) arises as the flow eqation.
    ${ }^{6}$ Here and for the rest of the section, we only consider vacuum spacetimes in the bulk.

[^10]:    ${ }^{7}$ Because (3.96) does not contain derivatives, this reduces to a pointwise matrix equation. Any section dependent on parameters $y^{a}$ in the space of matrices that satisfy the constraint (3.96) would then be a valid solution of the first step of this procedure.

[^11]:    ${ }^{8}$ BTZ black holes [81] are examples for spaces which are locally AdS, but have interesting global properties.

[^12]:    ${ }^{9}$ This bound can be sharpened again by reinstating the term proportional to $\chi$, assuming $\chi>0$. Of course, we also have to keep in mind that such constant curvature surfaces may not exist for arbitrary choice of $R$, see e.g. [87].

[^13]:    ${ }^{10}$ See [104] for a positivity bound on vacuum subtracted volumes with applications to holography.

[^14]:    ${ }^{11}$ Even though similar geometric shapes with a different mathematical definition have been described by the same name [93].
    ${ }^{12}$ Even more, by causality arguments similar to [105] the WdW patch is actually the largest region in the bulk on which any complexity proposal for one given boundary time slice can depend. For example, the extremal volume slice defining complexity in the complexity=volume proposal is always contained inside of the WdW patch by definition.

[^15]:    ${ }^{13}$ We continue to assume, however, that $E(\phi)$ is real and bounded from above, and free from self-intersections, i.e. $E(\phi)$ is a periodic function with period $2 \pi$, unlike the third example in figure 3.9.

[^16]:    ${ }^{1}$ Though we will be working mostly in Euclidean signature, spacetime wormholes also include metrics with Lorentzian signature, or even complex metrics.

[^17]:    ${ }^{2}$ This decay is related to a version of information loss in the bulk [11]

[^18]:    ${ }^{1}$ See appendix A of [84] for Hayward corner terms for every kind of corner.

[^19]:    ${ }^{1}$ This can be justified by observing that at least for the symmetric conical singularities on surfaces of revolution, all conical singularities are locally equivalent up to their deficit angle, and hence the correction term should only depend on the deficit angle.

[^20]:    ${ }^{2}$ See also [181] for an analysis of the Gauss-Bonnet theorem for surfaces of varying signature.
    ${ }^{3}$ For our specific case of future pointing timelike vectors, this follows from the more general equations given in [99] by using the relation $\cosh (\log (x))=\frac{1+x^{2}}{2 x}$, the normalisation of the vecors, and some algebra.

