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Close Binary Stars Modeled by Two Prolate Ellipsoids in Synchronous Rotation

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Abstract

The presence of tidal deformations in close binary stars has already been confirmed by astronomical observations. The present paper aims to simply address an astronomy problem, studying the relative movement of close binaries disturbed by their mutual deformation through some basic concepts and tools of celestial mechanics. For this purpose, the tidal effect is modeled by considering that each star is an elongated revolution ellipsoid in such a way that axes of revolution are coincident, and their largest axes point toward each other along the motion. The potential for mutual attraction is then obtained, resulting in a perturbed Keplerian system with perturbation proportional to the inverse of the cubic distance between the stars, thus being a one-degree-of-freedom problem and, therefore, integrable. The effective potential, the integrals of energy and angular momentum, and the Laplace vector are used to obtain qualitative information about the dynamics before integrating it. The motion describes a rosette-like orbit with periodic osculating elements, or a circle when the energy is a local minimum. Finally, an analytical solution is presented in terms of elliptic functions by using a regularizing and linearizing function.

Unified Astronomy Thesaurus concepts: Binary stars (154); Celestial mechanics (211); Analytical mathematics (38)

1. Introduction

Nonrigid celestial bodies close to other massive bodies present deformations in their shapes due to tidal effects. Earth attracted by the Moon and the Sun is perhaps the most conspicuous and studied example of this effect. However, there are other cases such as the one of the close binary stars, where those effect must be considered to compute their mutual orbits.

Usually, it is not possible to distinguish such deformations due to the fact they are remote astronomical objects, but thanks to advances in observation techniques it has been possible to detect the effect in some pairs. This is the case, for example, of the well-studied star Algol (β Per, HD 19356)—which is actually a hierarchical system composed of a close inner pair formed by Algol A and Algol B and a third more distant star, Algol C, orbiting the internal pair. Different observation techniques (see Baron et al. 2012) show evidences of an extended corona around Algol B. Also, additional treatments on the observed images show that while Algol A appears to be almost a perfect circular disk, Algol B fills its Roche lobe, being elongated toward Algol A (see Figures 5 and 10 of Baron et al. 2012).

In the scenario in which both stars would fill their respective Roche lobes, the overall image would be two elongated axisymmetric ellipsoids with their greatest axes of symmetry pointing toward each other during their orbital motion; in other words, a 1:1 spin-orbit resonance. We model this arrangement as the movement of two rigid bodies with an elongated ellipsoidal shape in *spoke* configuration using Duboshin's terminology (Duboshin 1974, 1982, 1984).

After the pioneering works of Duboshin, the problem of considering spheroids instead of point masses has been widely considered by many authors in the framework of n bodies, in particular on the circular restricted three-body problem (RTBP; see e.g., Vidyakin 1973; Bhatnagar & Chawla 1977; Cid & Elipe 1985; Elipe & Ferrer 1985; Elipe 1992; among others). In the papers just mentioned, primaries are considered spheroids with their planes of symmetry coincident with the fixed plane of motion of the centers of mass because, as shown in Duboshin (1982), circular motion (*regular solution* in Duboshin's words) is possible in those configurations.

Later on, some authors extend the problem to the elliptic RTBP, i.e., they consider that the primaries (spheroids) move on an elliptic orbit. However, this assumption is not correct, due that the only *regular* motion is the circular (Duboshin 1982), and, as we shall show in this paper, although the motion of the primaries is bounded, it is not elliptic because the orbit is precessing.

As we shall prove in Section 2, the potential of the ellipsoids considered here is

$$U = -\mu\left(\frac{1}{r} + \frac{\varepsilon}{r^3}\right), \qquad \varepsilon \ll 1; \tag{1}$$

that is, we are dealing with a two-body central force problem.

The problem belongs to the type of quasi-Keplerian Hamiltonians

$$\mathcal{H} = \mathcal{H}_K + \sum_{j=2}^{N} \frac{B_j}{r^j}, \tag{2}$$

with $\mathcal{H}_K$ being the Hamiltonian of the Kepler problem. This type of Hamiltonian represents several problems in post-Newtonian physics, like Manev's, Fock's, or Schwarzschild-type potentials, and is used to explain the nodal precession of Mercury, for instance. Besides, in artificial satellite theory, we find the so-called intermediaries that are quasi-Keplerian Hamiltonians (Equation (2)), and that play an important role; indeed, an intermediary is a simplification of the original Hamiltonian obtained by a perturbation theory and still containing most of the characteristics of the original problem. In the literature, we find several intermediaries, but among the radial ones (depending only on r), the two most relevant are: the one given by Cid & Lahulla (1969), which contains a perturbation term of the type $1/r^3$ in the potential, and Deprit's intermediary with a term $1/r^2$ (Deprit 1981).

Since the Hamiltonian in Equation (2) depends on only one variable, the radial distance r, the problem is integrable and one could think that there is no need to dedicate more time to such "a simple" problem. Albeit the fact of its integrability, the list of works dedicated to the problem is long, facing several aspects of motion; in particular, the time evolution of orbital elements (Lara & Gurfil 2012; Belen'kii 1981; Cid et al. 1986; Ferrándiz 1986; Deprit & Ferrer 1987; Abad et al. 2020, 2021).

The two-body system with a central force is introduced in Mechanics textbooks (see, for instance, Goldstein 1980 and Scheck 2003), and it is not difficult to plot the orbit, which for bounded motions is a *rosette* orbit, and based on this fact it is frequent to read that the motion consists of an ellipse that is precessing on its plane. But, is this true? Is the osculating eccentricity constant along the motion? what happens to others osculating elements? How can we obtain the orbital elements at some specific instant of time? This paper aims to answer these questions by using different techniques and also present an analytical solution to the equations of motion.

With this goal, in Section 3, we write the planetary Lagrange equations for the orbital elements on the plane of motion. With this, even without integrating them, we may observe that, indeed, the argument of the pericenter has a secular component, and that eccentricity is periodic. Both facts are also determined by analyzing the time derivative of the Laplace vector.

The second part of the paper is related with the analytical integration of the equations of motion, which is performed by means of a regularizing and linearizing function (Cid et al. 1983) that converts the original problem into the motion of an elliptic oscillator; thus, the solution is simple and, at the same time, gets rid of singularities that may be present in case (1) of Section 3. However, the main difficulty arises when we want to relate the solution of the harmonic oscillator to the original problem because the relation between both involves elliptic functions and integrals, as shown in Section 4.

2. Potential Function and Equations of Motion

Let us consider an inertial reference frame $Oxyz$ and two stars treated as ellipsoids S_i ($i = 1, 2$) of masses m_i and centers of mass O_i, both of revolution about the y-axes, of semi-axes a_i, b_i, $c_i = a_i$, and $a_i < b_i$, in such a way that their major-axes continuously point to each other. This configuration is what Duboshin (1974) dubbed "spoke" configuration, resembling the spokes of a wheel.

Let the origin of the reference frame O be located at O_1. We are interested in finding the motion of O_2 assuming that the spoke configuration remains along the time; that is to say, both ellipsoids rotate synchronously about their z-axes with the same angular velocity as the orbital angular velocity of the motion of O_2 around O_1 (see Figure 1).

It is known that the mutual gravitational potential of two rigid bodies may be expanded in harmonic series, and its first

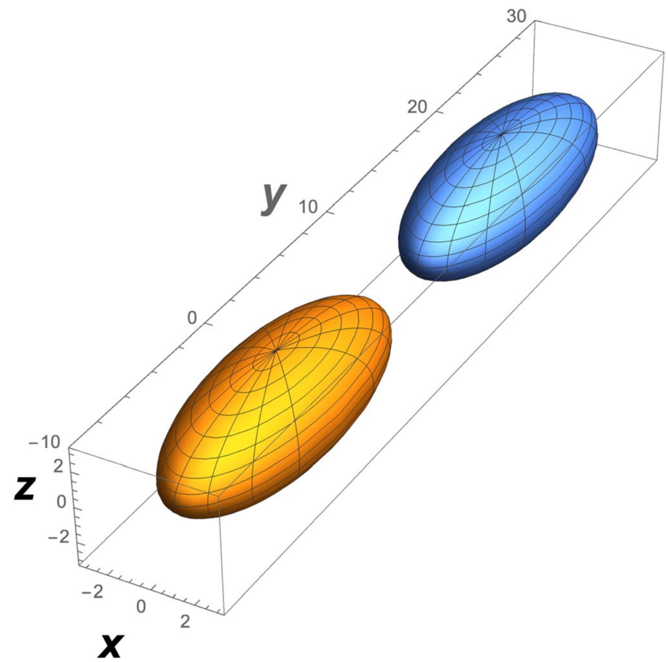


Figure 1. Two axisymmetric ellipsoids in the spoke configuration. Their major semi-axes are always on a straight line.

terms are (see, e.g., Leimanis 1965; Elipe & Ferrer 1985)

$$U = -\mathcal{G}\frac{m_1 m_2}{r} - \mathcal{G}m_1\frac{A_2 + B_2 + C_2 - 3I_{21}}{2r^3} - \mathcal{G}m_2\frac{A_1 + B_1 + C_1 - 3I_{12}}{2r^3}, \tag{3}$$

where $\mathcal{G}$ is the gravitational constant; I_{ij} is the moment of inertia of the body S_i with respect to the straight line $(\overrightarrow{O_i O_j})$ joining the mass centers O_i and O_j; (A_i, B_i, C_i) are the principal moments of inertia of body S_i; and $r = \|\overrightarrow{O_1 O_2}\|$ is the distance between the centers of mass of the rigid bodies.

It is well known from vectorial calculus (Marsden & Tromba 1988) that for a triaxial ellipsoid of semi-axes a, b, c, and mass m, its principal moments of inertia are

$$A = \frac{m}{5}(b^2 + c^2), \qquad B = \frac{m}{5}(a^2 + c^2), \qquad C = \frac{m}{5}(a^2 + b^2), \tag{4}$$

and if $a = c \neq b$, there results that $A = C \neq B$.

The moments of inertia I_{ij} with respect to the line $\overrightarrow{O_i O_j}$ are

$$I_{12} = A_1\alpha_{12}^2 + B_1\beta_{12}^2 + C_1\gamma_{12}^2, \qquad I_{21} = A_2\alpha_{21}^2 + B_2\beta_{21}^2 + C_2\gamma_{21}^2, \tag{5}$$

where α_{ij}, β_{ij}, and γ_{ij} are the cosines of the angles made by $\overrightarrow{O_i O_j}$ with the principal inertia axes (O_i; ξ_i, η_i, ζ_i) of the body S_i (Leimanis 1965). In as much as in the previous configuration the principal inertia axes of the body S_1 are parallel to the ones of the body S_2; $O_1\eta_1 = \overrightarrow{O_1 O_2}/\|\overrightarrow{O_1 O_2}\|$; hence, there results that $\alpha_{ij} = \gamma_{ij} = 0$ and $\beta_{12} = -\beta_{21} = 1$. With all that, we obtain that

$$A_i + B_i + C_i - 3I_{ij} = 2(A_i - B_i), \tag{6}$$

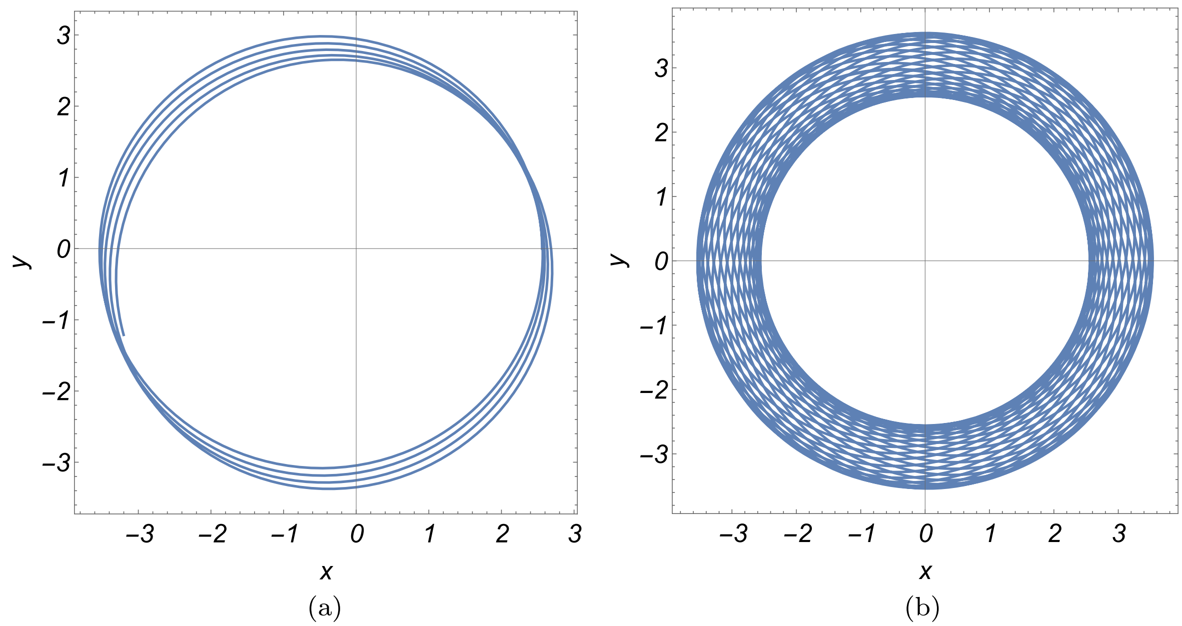


Figure 2. Rosette orbit for $\tilde{\varepsilon} = 0.1$ along a few revolutions (a) and for a wider span of time (b).

and then

$$A_i + B_i + C_i - 3I_{ij} = 2(A_i - B_i) = \frac{2}{5}m_i(b_i^2 - a_i^2). \quad (7)$$

Thus, the potential given by Equation (3) becomes

$$U = -\mathcal{G}\frac{m_1 m_2}{r} - \mathcal{G}m_1 m_2 \frac{1}{5r^3}(b_2^2 - a_2^2) - \mathcal{G}m_1 m_2 \frac{1}{5r^3}(b_1^2 - a_1^2) \quad (8)$$

or

$$U = U(r) = -\mathcal{G}\frac{m_1 m_2}{r} - \tilde{\varepsilon}\, \mathcal{G}m_1 m_2 \frac{1}{r^3}, \quad \text{with } \tilde{\varepsilon} = [(b_1^2 - a_1^2) + (b_2^2 - a_2^2)]/5. \quad (9)$$

Let us remark that the semimajor axes b_1 and b_2 of the stars are quantities much smaller than the mutual distance r between their centers, then $\tilde{\varepsilon}$ can be considered a small parameter ($0 < \tilde{\varepsilon} \ll 1$). Note also that the potential only depends on r; therefore, the problem is integrable. As a matter of fact there are 10 first integrals of motion as Duboshin (1972) proved for the motion of N rigid bodies. Namely, the center of mass of the system formed by the two ellipsoids moves with constant velocity on a straight line; the angular momentum vector is constant, thus the motion is planar; and finally the energy is also constant because time t does not appear explicitly in the potential function. We may express the relative motion of the point O_2 with respect to the inertial frame centered at O_1 with the help of the first integrals of the center of mass, resulting in the following potential

$$U = -\frac{\mu}{r} - \tilde{\varepsilon}\,\frac{\mu}{r^3} \quad (10)$$

and, consequently, in the following equations of motion

$$\ddot{\boldsymbol{r}} = -\nabla_{\boldsymbol{r}} U(r) = -\frac{\mu}{r^3}\boldsymbol{r} - \tilde{\varepsilon}\,\frac{3\mu}{r^5}\boldsymbol{r}, \quad (11)$$

where $\mu = \mathcal{G}(m_1 + m_2)$.

If we integrate numerically the above equations, the solution is a rosette orbit that looks like an ellipse that is precessing on its plane (Figure 2). Notwithstanding, neither the osculating eccentricity nor the semimajor axis nor even the pericenter angle of the orbit are constant, as we will see in next section.

In what follows, for the sake of simplifying the notation, we shall put $\varepsilon = \mu\,\tilde{\varepsilon}$.

3. Qualitative Aspects of the Motion

In our problem, as has been said before, the angular momentum vector, $\boldsymbol{G} = \boldsymbol{r} \times \dot{\boldsymbol{r}}$, is constant; it follows from Equation (11) that

$$\dot{\boldsymbol{G}} = \dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} + \boldsymbol{r} \times \ddot{\boldsymbol{r}} = 0 \quad \Longrightarrow \quad \boldsymbol{G} = \text{constant}. \quad (12)$$

Consequently, the motion is planar, so we can take the plane of motion as the reference xy plane, then the orbital inclination is zero ($i = 0$). Thus, on the plane of motion, we can describe the motion by means of polar coordinates $r > 0$ and $\theta \in [0, 2\pi)$

$$\begin{aligned} \boldsymbol{r} &= r(\cos\theta, \sin\theta), \\ \dot{\boldsymbol{r}} &= (\dot{r}\cos\theta - r\dot{\theta}\sin\theta, \\ &\quad \dot{r}\sin\theta + r\dot{\theta}\cos\theta), \end{aligned} \quad (13)$$

and, therefore, $\|\dot{\boldsymbol{r}}\|^2 = \dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = \dot{r}^2 + (r\,\dot{\theta})^2$, and

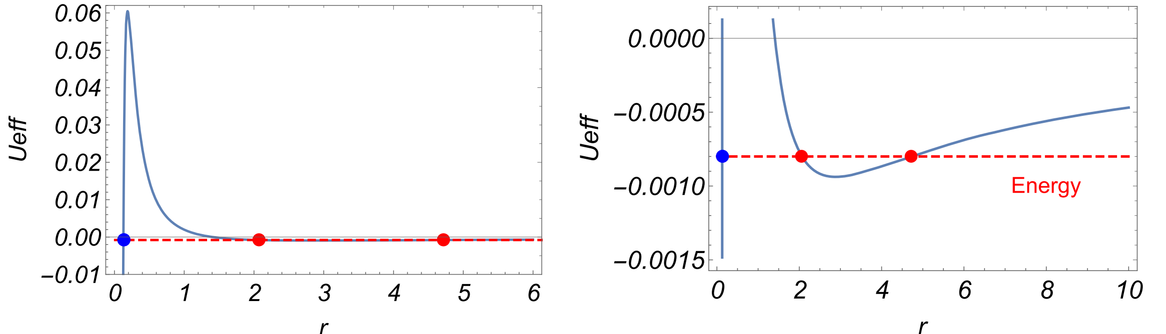
$$G = \|\boldsymbol{G}\| = \Theta = r^2\dot{\theta}. \quad (14)$$

In polar-nodal variables (R, Θ, r, θ), the Hamiltonian of the problem may be written as

$$\mathcal{H} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} - \frac{\varepsilon}{r^3}. \quad (15)$$

With this, the energy, which is constant along the motion because it does not explicitly depend on the time t, takes the form

$$h = \frac{1}{2}\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} + U(r) = \frac{1}{2}\dot{r}^2 + \frac{\Theta^2}{2r^2} - \frac{\mu}{r} - \frac{\varepsilon}{r^3}, \quad (16)$$



(a) The effective potential $U_{\text{eff}}(r)$ and the constant energy h. (b) Plot (a) zoomed in order to appreciate the relative minimum.

Figure 3. (a) Plot of the effective potential $U_{\text{eff}}(r)$ and the constant energy h, and (b) a magnification of the previous plot in order to appreciate details close to the relative minimum. There is movement only when $U_{\text{eff}}(r) \leqslant h$, hence the three possible motions are: (1) $r \in (0, r_0]$ (the blue dot); (2) when r oscillates between the two red points ($r \in [r_p, r_a]$); (3) unbounded motion when $h \geqslant 0$, and $r \geqslant r_{\max}$. In the plots, $h = -0.0008$.

and the effective potential becomes

$$U_{\text{eff}} = \frac{\Theta^2}{2r^2} - \frac{\mu}{r} - \frac{\varepsilon}{r^3} = U_{\text{Keff}} - \frac{\varepsilon}{r^3}, \tag{17}$$

where the term U_{Keff} is the effective potential for the Kepler problem.

Note that the case considered here presents a quite a difference with respect to the pure Keplerian problem due to the $-\varepsilon/r^3$ term in the effective potential (Equation (17)). In truth,

$$\lim_{r\to 0} U_{\text{Keff}} = +\infty \neq \lim_{r\to 0} U_{\text{eff}} = -\infty,$$
$$\lim_{r\to+\infty} U_{\text{Keff}} = \lim_{r\to+\infty} U_{\text{eff}} = 0. \tag{18}$$

The function $U_{\text{Keff}}(r)$ has a minimum at $r_{\text{Kmin}} = \Theta^2/\mu$ whereas the function $U_{\text{eff}}(r)$ has a relative maximum at $r_{\max} = (\Theta^2 + \sqrt{\Theta^2 - 12\mu\varepsilon})/(2\mu)$ and a relative minimum at $r_{\min} = (\Theta^2 + \sqrt{\Theta^2 + 12\mu\varepsilon})/(2\mu)$, with $r_{\max} < r_{\min}$ (see Figure 3). Observe that those critical points correspond to two circular orbits; one unstable ($r = r_{\max}$) and the other stable ($r = r_{\min}$).

Furthermore, from the graph of the effective potential function given by Equation (17) for a certain level of energy, h, arbitrarily fixed (constant by the fact aforementioned that the Hamiltonian is conservative)

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + U_{\text{eff}} = h, \tag{19}$$

and the motion is only possible when $U_{\text{eff}}(r) \leqslant h$ (see Figure 3).

There are three types of possible motions. Let us denote by $r_0 \leqslant r_p \leqslant r_a$, the three possible roots of the equation $U_{\text{eff}}(r) = h$ (see Figure 3); hence, we have:

1. Bounded motion when $0 < r \leqslant r_0$.
2. Oscillatory motion with lower and upper bounds; that is, $r_p \leqslant r \leqslant r_a$.
3. Unbounded motion for $h \geqslant 0$.

In what follows, we restrict our analysis to motion of type 2 because it is the only motion of interest in studying close binary stars.

The motion is planar as it was proven at the beginning of this section, so there is no need to compute the planetary Lagrange equations for all orbital elements, just for those on the plane of motion. Besides, we will make use of the fact that the force is pure radial which will result in simpler equations.

The Laplace vector is defined as

$$\boldsymbol{A} = \dot{\boldsymbol{r}} \times \boldsymbol{G} - \frac{\mu}{r}\boldsymbol{r}. \tag{20}$$

In the Keplerian problem, it is a constant vector that always points to the pericenter and is related with the eccentricity ($e = \|\boldsymbol{e}\|$) in such a way that $\boldsymbol{e} = \boldsymbol{A}/\mu$. Let us verify whether in our problem the Lagrange vector is constant or not. After some algebra, its time derivative may be put as

$$\frac{d\boldsymbol{A}}{dt} = -\varepsilon\,\frac{1}{r^5}(\boldsymbol{r} \times \boldsymbol{G}) \neq 0. \tag{21}$$

In consequence, the Laplace vector is not constant, and then neither the eccentricity e nor the pericenter angle ω are constant. From Equation (21), Laplace's vector suffers a tangential push and its norm is

$$\|\dot{\boldsymbol{A}}\| = \varepsilon\frac{G}{r^4}, \tag{22}$$

but the radial distance r oscillates in the interval $[r_p, r_a]$ (see Figure 3 (b)); thus, $\|\dot{\boldsymbol{A}}\|$ has an oscillatory behavior, and hence e does.

Even more information can be extracted about the dynamics of the problem before integrating it. To do this, we will write the Lagrange's planetary equations to analyze the time variation of the orbital elements. Because the motion is planar and takes place on the xy plane, there is no need to consider neither the inclination i nor the angle of the node Ω. Thus, we will use the version of the planetary equations for purely radial perturbations (F_r) presented in Pollard's textbook, *vide* Pollard (1966) - pp. 33–37, which, for our case, are given by:

$$\dot{e} = G\mu^{-1}\sin(f)\,F_r, \tag{23}$$

$$\dot{\omega} = -G\mu^{-1}e^{-1}\cos(f)\,F_r, \tag{24}$$

$$\dot{a} = 2a^2eG^{-1}\sin(f)\,F_r. \tag{25}$$

From Equation (11), the radial force is $F_r = -3\,\varepsilon\,r^{-4} < 0$. Thus, we may conclude from Equations (23)–(25) that

$$\dot{e}\begin{cases}\leqslant 0 \text{ for } f \in [0, \pi], \\ > 0 \text{ for } f \in (\pi, 2\pi),\end{cases}$$
$$\Longrightarrow e\begin{cases}\text{decreases for } f \in [0, \pi], \\ \text{increases for } f \in (\pi, 2\pi),\end{cases}$$

and similarly,

$$\dot{\omega}\begin{cases}\geqslant 0 \text{ for } f\in\left[\dfrac{3\pi}{2},\dfrac{5\pi}{2}\right),\\ <0 \text{ for } f\in\left[\dfrac{\pi}{2},\dfrac{3\pi}{2}\right),\end{cases}$$

$$\Longrightarrow \omega\ \begin{cases}\text{increases for } f\in\left[\dfrac{3\pi}{2},\dfrac{5\pi}{2}\right),\\ \text{decreases for } f\in\left[\dfrac{\pi}{2},\dfrac{3\pi}{2}\right),\end{cases}$$

and

$$\dot{a}\ \begin{cases}\leqslant 0 \text{ for } f\in[0,\pi],\\ >0 \text{ for } f\in(\pi,2\pi),\end{cases}$$

$$\Longrightarrow a\ \begin{cases}\text{decreases for } f\in[0,\pi],\\ \text{increases for } f\in(\pi,2\pi),\end{cases}$$

which results in the oscillatory character of these three orbital elements.

Remark 1. Let us consider the circular stable orbit that corresponds to the value $r = r_{\min}$ of the effective potential. From the radial acceleration $(\ddot{r} - r\dot{\theta}^2)$, we have that

$$\ddot{r} - r\dot{\theta}^2 = -\frac{dU}{dr}, \tag{26}$$

and because the radial distance in a circle is constant, $r = a =$ constant, there results that

$$\dot{\theta}^2 = \frac{\mu}{a^3} + 3\varepsilon\frac{\mu}{a^5} = n_K^2 + 3\varepsilon\frac{\mu}{a^5}, \tag{27}$$

where n_K is the mean motion of the Kepler circular orbit, and, thus, the mean motion of the circular orbit here considered and the Keplerian one are related by

$$n^2 = n_K^2\left(1 + \frac{3\varepsilon}{a^2}\right). \tag{28}$$

This fact must be taken into account when considering problems of the type of circular restricted three-body problem when the primaries are spheroids, and, thus, the angular velocity of the synodic reference frame must be n and not n_K. Incidentally, let us mention that the orbit of Algol B around Algol A is circular (Baron et al. 2012), hence the orbital period of Algol's inner orbit should be computed from n and not from n_K.

Remark 2. In the last years there are several papers dealing with the elliptic restricted three-body problem when the primaries are spheroids; that is, they consider that the motion of the primaries (spheroids) move on a Keplerian elliptic orbit, but as we just shown, this elliptical motion is not possible under these assumptions, and therefore the original hypothesis (elliptic motion of the primaries) is not true.

4. Analytical Solution

The equation of motion by using the integral of energy given by Equation (19), despite its simple appearance, is not easy to integrate analytically, and its solution involves elliptic functions. At this point, let us note that, even in the simplest case, i.e., the Kepler problem, the solution is usually obtained through the true or eccentric anomalies whose relationship with the independent variable is via the Kepler equation. There are several procedures for solving Equation (19). Most of them consist in converting the equations of motion into the ones of a harmonic oscillator by means of a of time and length transformation and then introduce a generalized Kepler equation (Abad et al. 2001), or use the Krylov–Bogoliubov averaging method (Krylov & Bogoliubov 1950) that provides error bounds for the solution in time intervals of size the inverse of the small parameter, which is quite convenient for studies requiring long time intervals and its complexity is similar to a pure Kepler's problem (Abad et al. 2021). Other alternatives use a linearization function to reach the harmonic oscillator problem (Belen'kii 1981; Cid et al. 1986; Ferrándiz 1986) and then obtain the relation between the solution of the harmonic oscillator and the original problem by means of elliptic functions (Ferrándiz 1986). The last one is the approaching chosen to be followed from now on to obtain an exact solution for the problem considered.

Let $g(r)$ be a smooth real function defined just ahead. Let us also introduce a pseudo-time $\tau = \tau(t)$ as

$$d\tau = g^{-1}(r)dt. \tag{29}$$

Thus, the Equation (19) becomes

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = g^2(r)[h - U_{\mathrm{eff}}(r)], \tag{30}$$

and differentiating it with respect to τ, we get

$$\frac{d^2r}{d\tau^2} = \frac{d}{dr}[g^2(r)(h - U_{\mathrm{eff}}(r))]. \tag{31}$$

The linearizing function $g(r)$ is chosen in such a way that Equation (31) represents a harmonic oscillator, such that

$$\frac{d^2r}{d\tau^2} = \frac{d}{dr}[g^2(r)(h - U_{\mathrm{eff}}(r))] = 2c_1 r + c_2, \tag{32}$$

then

$$g^2(r)(h - U_{\mathrm{eff}}(r)) = c_1 r^2 + c_2 r + c_3, \tag{33}$$

where c_1, c_2, c_3 are constants of integration.

There are several possibilities for the linearizing function $g(r)$. Among them, we choose the one of the class given by Cid et al. (1983), namely

$$g(r) = r^{3/2}(r+\beta)^{-1/2}, \quad \text{with } \beta \text{ constant and} (r+\beta) > 0. \tag{34}$$

Replacing Equation (34) into Equation (33) results

$$r^3(h - U_{\mathrm{eff}}(r)) = (r+\beta)(c_1 r^2 + c_2 r + c_3). \tag{35}$$

Substituting the effective potential U_{eff} given by Equation (17) into Equation (35), it follows that

$$r^3\left(h - \frac{\Theta^2}{2r^2} + \frac{\mu}{r} + \frac{\varepsilon}{r^3}\right) = (r+\beta)(c_1 r^2 + c_2 r + c_3). \tag{36}$$

By identifying coefficients of this polynomial equation, Equation (36), there results that

$$c_1 = h, \tag{37}$$

$$c_2 = \mu - \beta h, \tag{38}$$

$$c_3 = -\frac{1}{2}\Theta^2 - \beta h + \mu + h\beta^2, \tag{39}$$

and β is obtained as the positive root of the cubic polynomial in β,

$$P(\beta) = h\beta^3 - \mu\beta^2 - \frac{1}{2}\Theta^2\beta - \varepsilon, \tag{40}$$

which can be obtained numerically because the coefficients are constant. With this, the equation of the harmonic oscillator is

$$r = A_0 \cos(\sqrt{-2h}\ \tau + B_0) + \frac{\mu - \beta h}{-2h}; \tag{41}$$

thus, the motion is τ-periodic with frequency $w = \sqrt{-2h}$. Let us recall that for motions in case 2, $h < 0$ and $r_p \leqslant r \leqslant r_a$). This harmonic oscillator (41) may be put in the form

$$r = a(1 - \cos(E)) \tag{42}$$

with

$$a = \frac{\mu - \beta h}{-2h}, \quad e = A_0\frac{2h}{\mu - \beta h}, \quad \text{and } E = w\tau + B_0. \tag{43}$$

The quantities A_0 and B_0 are constants of integration, determined by the initial conditions. For instance, by assuming that at $\tau = 0$, the particle is at its pericenter, in other words, $r(0) = r_p$ and $r'(0) = 0$, or by the boundary conditions $r(\tau = 0) = r_p$ and $r(\tau = \pi/w) = r_a$. Thus, A_0 and B_0, or equivalently a and e, are obtained by solving the system

$$\begin{cases} a(1 - e) = r_p, \\ a(1 + e) = r_a, \end{cases} \quad \text{or} \quad \begin{cases} 2a = r_p + r_a, \\ a^2(1 - e^2) = r_p r_a. \end{cases} \tag{44}$$

Equation (42) is exactly the same as in the Keplerian problem and gives the radial distance as a function of the eccentric anomaly (E) or, equivalently, in terms of the pseudo-time τ. Concerning the other variable, the angle θ, we proceed from the angular moment integral, $\|\boldsymbol{r} \times \dot{\boldsymbol{r}}\| = r^2\dot{\theta} = \Theta$, an expression that can also be obtained from Hamilton equations of the Hamiltonian given by Equation (15). Taking into account the relation between time (t) and pseudo-time (τ), there results that

$$\frac{d\theta}{d\tau} = g(r)\frac{\partial\mathcal{H}}{\partial\Theta} = \frac{r^{3/2}}{\sqrt{r + \beta}}\,\frac{\Theta}{r^2}, \tag{45}$$

then taking into account that $E = \sqrt{-2h}\,\tau + B_0$,

$$\begin{aligned} \theta &= \frac{\Theta}{a\sqrt{-2h}} \int_0^\tau \frac{dE}{\sqrt{1 - e\cos(E)}\,\sqrt{1 + B - e\cos(E)}}, \\ B &= \beta/a > 0, \end{aligned} \tag{46}$$

and by making the change $\cos(E) = x$,

$$\begin{aligned} \theta &= \frac{\Theta}{a\sqrt{-2h}} \int_{\cos(E)}^1 \\ &\times \frac{dx}{\sqrt{(1 + B - ex)(1 - ex)(1 - x)(x + 1)}}, \end{aligned} \tag{47}$$

or equivalently,

$$\begin{aligned} \theta &= \frac{\Theta}{ae\sqrt{-2h}} \int_{\cos(E)}^1 \frac{dx}{\sqrt{(x_1 - x)(x_2 - x)(1 - x)(x + 1)}} \\ &= \frac{\Theta}{ae\sqrt{-2h}} I. \end{aligned} \tag{48}$$

The four roots of the quartic polynomial inside the square root are

$$x_1 = \frac{1 + B}{e} \quad > \quad x_2 = \frac{1}{e} \quad > \quad x_3 = 1 \geqslant x \quad > \quad x_4 = -1. \tag{49}$$

Using expression (252.00) from the classical book Byrd & Friedman (1971) of elliptic integrals, Equation (48) above becomes

$$I = \frac{2}{\sqrt{(x_1 - x_3)(x_2 - x_4)}} F(\varphi, k), \tag{50}$$

where $F(\varphi, k)$ is the normal elliptic integral of the first kind with modulus k and amplitude φ

$$\begin{aligned} k^2 &= \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_2 - x_4)}, \\ \varphi &= \mathrm{am} u_1 = \sin^{-1}\left(\sqrt{\frac{(x_1 - x_3)(x - x_4)}{(x_3 - x_4)(x_1 - x)}}\right). \end{aligned} \tag{51}$$

At this point it is convenient to define two quantities $\mathcal{G}$ and α^2, which is going to be useful on the next step,

$$\mathcal{G} = \frac{2}{\sqrt{(x_1 - x_3)(x_2 - x_4)}}, \qquad \alpha^2 = \frac{x_4 - x_3}{x_1 - x_3}. \tag{52}$$

In a similar way, we can obtain the relation between time t and pseudo-time τ. Indeed, from Equation (29), there results that

$$\begin{aligned} dt &= \frac{a}{\sqrt{-2h}} \frac{[1 - e\cos(E)]^{\frac{3}{2}}}{\sqrt{1 + B - e\cos(E)}} \\ dE &= \frac{ae}{\sqrt{-2h}} \frac{(x_2 - x)^{\frac{3}{2}}}{\sqrt{(x_1 - x)(x_3 - x)(x - x_4)}}\ dx. \end{aligned} \tag{53}$$

Resorting to the results for elliptic integrals from Byrd & Friedman (1971), once more, we obtain the following expression using the relation (252.23)

$$\begin{aligned} &\int_{x_4}^x \frac{(x_2 - x)^{\frac{3}{2}}}{\sqrt{(x_1 - x)(x_3 - x)(x - x_4)}}\ dx = (x_2 - x_4)^2\ \mathcal{G} \\ &\times \int_0^{u_1} \frac{(1 - k^2\sin^2 u)^2}{(1 - \alpha^2\sin^2 u)^2}\ du, \end{aligned} \tag{54}$$

and from a sequence of formulas in Byrd & Friedman (1971) and after some cumbersome computations, eventually, we arrive at

$$t = \frac{ae}{\sqrt{-2h}} \frac{1}{\alpha^4} [k^4 V_0 + 2k^2(\alpha^2 - k^2)V_1 + (\alpha^2 - k^2)^2 V_2], \tag{55}$$

where

$$
\begin{aligned}
V_0 &= F(\varphi, k),\\
V_1 &= \Pi(\varphi, \alpha^2, k),\\
V_2 &= \frac{1}{2(\alpha^2 - 1)(k^2 - \alpha^2)}\cdot\\
&\quad [\alpha^2 E(u) + (k^2 - \alpha^2)u\\
&\quad + (2\alpha^2 k^2 + 2\alpha^2 - \alpha^4 - 3k^2)\Pi(\varphi, \alpha^2, k)\\
&\quad - \left.\frac{\alpha^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1 - \alpha^2 \operatorname{sn}^2 u}\right],
\end{aligned}
\tag{56}
$$

where $F(\varphi, k)$, $E(\varphi, k) = E(\varphi, k)$, and $\Pi(\varphi, \alpha^2, k)$ are the normal elliptic integral of the first, second, and third kind, respectively, with modulus k and amplitude φ, also known as Legendre's canonical integrals of first, second, and third kind, respectively.

5. Conclusion

We model the tidal effect of two close binary stars by two elongated ellipsoids of revolution, such that their revolution axes are on the same straight lying on a fixed plane; thus the major-axes of the ellipsoids point to each other during the orbital motion of the stars. For this model, we obtain that the mutual potential belongs to the class of quasi-Keplerian motions; that is, a perturbed Kepler problem with a radial perturbation $F_r = -3\varepsilon/r^3$.

A large amount of information is obtained with no need of integrating the problem by checking that the problem is conservative and that the angular momentum vector is constant; hence, the energy is constant and the motion is planar. Laplace vector also provides information because as its time the derivative is not null, neither are eccentricity and argument of the pericenter, and they also show a periodic behavior. These facts are also determined by means of Lagrange equations on the plane for radial forces.

The orbit is a rosette-like orbit, and its osculating elements periodically vary along the orbit. Circular motion is also possible, but in this case the mean motion is different from the mean motion of the Kepler motion.

Lastly, we make an integration of the problem by using a regularizating and linearizating function. The solution is quite cumbersome because it contains intricate relations of elliptic functions.

We also may conclude that the hypothesis that some authors use in an extension of the elliptic restricted three-body problem (by considering the primaries spheroids moving along an ellipse) has no physical meaning because the orbit of the primaries is no longer elliptic, but it is a rosette-like orbit.

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