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An optimal control L_2-gain disturbance rejection design for networked control systems

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***Abstract*— This paper proposes an optimal control solution for the L_2-gain disturbance rejection problem for networked control systems. The problem, usually referred as mixed H_2/H_∞, aims at designing a linear stabilizing controller minimizing a given cost function subject to a L_2-gain disturbance rejection constraint. The problem is formulated assuming network induced time-varying delays and/or packet losses. An LMI-based minimization problem is derived based on a Lyapunov-Krasovskii approach, considering a polytopic covering of the time-delay range. Simulation results are provided to verify the performance of the methodology.**

I. INTRODUCTION

This work addresses the design of an optimal L_2-gain disturbance rejection control for linear systems controlled through a communication network. These control schemes, usually termed as NCS (Networked Control Systems), have captured the attention of many researches in last years. There are a huge range of potential applications of these technologies and, also, challenging control problems have arisen from the presence of unreliable communication channels in the control loops, see [1].

Many authors have studied the conditions which must be verified to stabilize the whole system using different controllers and under specific conditions of the network. To name some of them, we can find works which study the stability of NCSs with delays, [2]; packet losses, [3]; or with limited bandwidth, [4].

In this framework, some remarkable results have been obtained based on the Lyapunov-Krasovskii approach, [5]. Using an appropriate Lyapunov-Krasovskii functional, it is possible to take into account delays and packet dropouts which affect the communication. This technique has been widely used to study the stability of time-delay systems, see for instance [6], [7] and references therein. It is from the work [8], and the input delay approach to sampled-data system, when the contributions of time-delay systems were widely applied to networked control system framework, see [9].

Several works are available in the Literature in the context of optimal control for NCS. In [10], some modifications of Kalman filter are introduced to design an optimal controller dealing with packet dropouts. However, few works resort to the powerful Lyapunov-Krasovskii approach, see [11], [12]. Both works consider time delays in their formulations finding a guaranteed cost control. The guaranteed cost is a function of the initial value of the functional ($V(x_0,t_0)$), so it depends on the initial conditions. To remove this dependence, they propose some conservative bounds of the guaranteed cost, formulating an optimization problem to minimize the cost.

The method proposed in this work deals with linear NCSs subject to L_2 bounded disturbances, where signals are affected by time-varying delays and packet dropouts. No a priori knowledge of the statistical distribution of the delay is assumed, though upper and lower bounds of the delay interval are required. Additionally, it is assumed that the maximum number of consecutive packet losses is bounded.

Thus, given a cost function J and a controlled output $z(t)$, the proposed problem can be informally stated as designing a linear controller such that:

- Stabilizes the unperturbed system ($\omega(t) \equiv 0$), as the cost function J is minimized.
- The controlled output satisfies $\|z(t)\|_2 \leq \gamma \|\omega(t)\|_2$ for any nonzero disturbance $\omega(t)$.

This problem is usually referred in the literature as the mixed H_2/H_∞ control problem (see, for instance, [13] for time-varying delay systems), as the H_2 part accounts for the optimization of a performance index with an L_2-gain disturbance rejection constraint in the H_∞ component.

To find a suitable controller, the Lyapunov-Krasovskii approach is used, yielding an optimization problem involving Nonlinear Matrix Inequalities (NLMI). Then, a well-known procedure is employed to reformulate the optimization problem in terms of Linear Matrix Inequalities (LMIs). In order to reduce the conservatism of the stability criterion, a polytopic covering of the time-delay range is employed. This allows us to retain the maximum and minimum value of the delay to very last formulation of the NLMI problem, thus avoiding unnecessary sources of conservatism.

Finally, we present simulation results for a vehicle tracking problem in which the above mentioned characteristics are featured. We will show that the presented method offers a satisfactory solution for the proposed problem.

The paper is organized as follows. Section II is devoted to the description of the NCS model to be considered. Section III is concerned with the main results. In Section III-A the optimal L_2-gain control problem is formulated. A particular solution for this problem is proposed in Section III-B. Finally, Section IV applies the previous result to an scenario of vehicle tracking. Conclusions are summarized in Section V.

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II. NCS MODEL AND PRELIMINARIES

Consider the following LTI system given by:

$$\begin{aligned} \dot{x}(t) &= Ax(t)+Bu(t)+B_\omega \omega(t), & (1)\\ z(t) &= Cx(t)+Du(t), & (2)\\ x(t) &= \phi(t), \quad t\in[t_0-\tau_M,t_0], & (3)\end{aligned}$$

where $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^m$ and $z(t)\in\mathbb{R}^q$ are the state vector, control input vector and controlled output, respectively; $\omega(t)\in L_2[t_0,\infty)$ denotes the external disturbances; A, B, B_ω, C and D are some constant matrices of appropriate dimensions and $\phi(t)$ denotes the initial conditions.

Consider system (1)-(3) being controlled through a network. The inclusion of such a network in the control loop induces time-varying delays, and possibly packet dropouts. Assume that the sensor samples measurements from the plant in a clock-driven manner, at time instants $t=j_k h$, with h being the sampling time, and j_k $(k=1,2,3,..)$ integers such that $\{j_1,j_2,j_3,...\}\subseteq\{1,2,3,...\}$ and $j_k<j_{k+1}$. With these assumptions packet losses are contemplated, while out-of-order packets are rejected by the controller or the actuator.

Let us define $t\in[t_k,t_{k+1})$ as the time intervals where the control input applied to the system is constant, where t_k are the time instants when the control signal, corresponding to the plant state at $t=j_k h$, reaches the plant.

Therefore, the control input can be written as:

$$u(t)=Kx(t_k-\tau_{sc}(k)-\tau_{ca}(k)), \quad t\in[t_k,t_{k+1}), \tag{4}$$

where $\tau_{sc}(k)$ and $\tau_{ca}(k)$ are the network induced delays of the data corresponding to the measured plant state at $t=j_k h$, from sensor to controller and from controller to actuator, respectively. The round-trip delay $\tau_{sa}(k)$ can also be defined as $\tau_{sa}(k)=\tau_{sc}(k)+\tau_{ca}(k)$.

Thus the system (1)-(3) under the control law (4) can be rewritten as:

$$\begin{aligned} \dot{x}(t) &= Ax(t)+BKx(t-\tau(t))+B_\omega\omega(t), & (5)\\ z(t) &= Cx(t)+DKx(t-\tau(t)), \quad \forall t\in[t_k,t_{k+1}), & (6)\\ x(t) &= \phi(t), \quad t\in[t_0-\tau_M,t_0], & (7)\end{aligned}$$

where $\tau(t)=t-t_k+\tau_{sc}(k)+\tau_{ca}(k)$ and τ_M is the upper bound of $\tau(t)$ (see Definition 1).

The following assumptions, which are fairly common in the NCS framework, are also made in this paper.

Assumption 1. The sensor is clock-driven. The controller and actuator are event-driven. The actuator is a zero-order holder.

Assumption 2. Two constants $\underline{\tau}_{sa},\overline{\tau}_{sa}\geq 0$, exist such that the following inequality holds:

$$\underline{\tau}_{sa}\leq\tau_{sa}(k)\leq\overline{\tau}_{sa}, \forall k\in\mathbb{N}. \tag{8}$$

Assumption 3. The maximum number of consecutive data dropouts from sensor to actuator is bounded by $n_p\in\mathbb{N}$.

Furthermore, this definition will be needed in the following sections.

Definition 1. Regarding to Assumptions 2 and 3, it is possible to define two constants $\tau_m\geq 0$ and $\tau_M>\tau_m$ such that:

$$\begin{aligned} \tau(t) &\geq \underline{\tau}_{sa}=\tau_m, & (9)\\ \tau(t) &\leq (1+n_p)h+\overline{\tau}_{sa}=\tau_M. & (10)\end{aligned}$$

III. MAIN RESULT

A. Problem Formulation

In this section the optimal control problem with L_2-gain disturbance rejection for Networked Control System is formulated. Next, a general solution is proposed. As we will prove, the design of a suitable controller for the proposed problem can be stated as an optimization problem.

Definition 2. The Optimal L_2-gain Control Problem

Consider that the LTI system described by (5)-(7) is controlled over a communication network, which satisfies (9)-(10). Given:

- A desired level of disturbance attenuation γ.
- A quadratic cost function $J=\int_{t_0}^{\infty}[x^T(t)Qx(t)+u^T(t)Ru(t)]dt$, with $Q,R>0$,

The Optimal L_2-gain problem consists in finding an stabilizing linear controller K such that:

1) The closed-loop system is asymptotically stable with $\omega(t)\equiv 0$,
2) The controller minimizes the cost function J with $\omega(t)\equiv 0$,
3) Under the assumption of zero initial condition, the controlled output $z(t)$ satisfies $\|z(t)\|_2\leq\gamma\|\omega(t)\|_2$ for any nonzero $\omega(t)\in L_2[0,\infty)$.

The following assumptions will be additionally needed to provide a solution to the problem.

Assumption 4. Given a continuous quadratic Lyapunov-Krasovskii functional (LKF) $V(t)$, it is assumed that the derivative for $t\in[t_k,t_{k+1})$ can be written in the following way:

$$\dot{V}(t)\leq\xi^T(t)\Xi(K)\xi(t)-z(t)^T z(t)+\gamma^T\omega^T(t)\omega(t), \tag{11}$$

where $\xi(t)\in\mathbb{R}^{n_\xi}$ is an augmented state vector and $\Xi(K)\in\mathbb{R}^{n_\xi\times n_\xi}$ is a symmetric matrix which depends, among others, on the controller matrix.

Assumption 5. The cost function J can be written as:

$$J=\int_{t_0}^{\infty}[\xi^T(t)\Phi(K)\xi(t)]dt, \tag{12}$$

where $\Phi(K)$ is a positive definite matrix of appropriate dimensions which possibly depends on K.

The following lemma is required for further developments in this work

Lemma 1. Suppose that Assumptions 4-5 hold. Then, the Optimal L_2-gain control problem can be solved by finding a controller matrix K such that:

$$\begin{aligned} &\min_K \quad \alpha, & (13)\\ &\text{subject to} \quad \alpha\Xi(K) < -\Phi(K) & (14)\\ &\qquad \alpha > 0, \quad \alpha\in\mathbb{R} & (15)\end{aligned}$$

Proof. In the proof of this lemma, we will demonstrate that a controller which solves the optimization problem (13)-(15) also satisfies all the issues of Definition 2.

1) For $\omega(t)\equiv 0$, considering (11) for $t\in[t_k,t_{k+1})$ the following holds:

$$\dot{V}(t)\leq\xi^T\Xi(K)\xi(t)-z(t)^T z(t). \tag{16}$$

From (16) and regarding to the assumption 5 and the equations (14)-(15), one can obtain that $V(t)$ decrease for $t \in [t_k, t_{k+1})$. Due to $V(t)$ is continuous in $[t_0, \infty)$, then $\dot{V}(x_t) \leq -\rho \|x(t)\|^2$ for a sufficient small $\rho > 0$, which ensure asymptotic stability of system (5)-(7), see e.g. [14].

2) For $\omega(t) \equiv 0$ and the condition (15):

$$\dot{V}(t) \leq \xi^T \Xi(K) \xi(t) < -\xi^T(t) \frac{1}{\alpha} \Phi(K) \xi(t). \tag{17}$$

Integrating both sides of (17) from t_k to $t \in [t_k, t_{k+1})$, yields,

$$V(t) - V(t_k) < -\frac{1}{\alpha} \int_{t_k}^{t} [x^T(s)Qx(s) + u^T(s)Ru(s)]ds.$$

Obviously $\bigcup_{k=1}^{\infty} [t_k, t_{k+1}) = [t_0, \infty)$. Provided that $V(t)$ is continuous in t, one can see that,

$$V(t) - V(t_0) \leq -\int_{t_0}^{t} [x^T(s)Qx(s) + u^T(s)Ru(s)]ds.$$

When $t \to \infty$, the asymptotic stability of the system implies that $V(t) \to 0$, so that,

$$-V(t_0) < -\frac{1}{\alpha} \int_{t_0}^{\infty} [x^T(s)Qx(s) + u^T(s)Ru(s)]ds \Rightarrow J < \alpha V(t_0).$$

The value of $V(t_0)$ depends on the initial condition $\phi(t)$ and it is a measure of its norm. Therefore, minimizing α the cost function J is minimized regardless of the initial conditions.

3) For $\omega \neq 0$ and under zero initial conditions:

$$\dot{V}(t) \leq -z(t)^T z(t) + \gamma^T \omega^T(t) \omega(t). \tag{18}$$

Integrating both sides of (18) and using the same continuity argument as before, one can see that,

$$V(t) - V(t_0) \leq -\int_{t_0}^{t} z^T(s)z(s)ds + \int_{t_0}^{t} \gamma^2 \omega^T(s)\omega(s)ds.$$

Then, letting $t \to \infty$, taking into account that under zero initial condition $V(t_0) = 0$ and the positive definitiveness of the functional, it can be shown that,

$$\int_{t_0}^{\infty} z^T(s)z(s)ds \leq \int_{t_0}^{\infty} \gamma^2 \omega^T(s)\omega(s)ds \Rightarrow \|z(t)\|_2 \leq \gamma \|\omega(t)\|_2$$

□

Lemma 1 proposes a general solution to the proposed control problem. It can be used for different LKFs and for different network constraints.

B. A Solution to the optimal L_2-gain control problem

In this section the previous result is applied to a particular LKF and a solution to the optimal L_2-gain control problem is given. Particularly, we use the following LKF:

$$V(t) = x^T(t)Px(t) + \int_{t-\tau_m}^{t} x^T(s)Q_1x(s)ds + \int_{t-\tau_M}^{t} x^T(s)Q_2x(s)ds$$
$$+ \int_{-\tau_M}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z_1\dot{x}(s)dsd\theta + \int_{-\tau_M}^{-\tau_m}\int_{t+\theta}^{t} \dot{x}^T(s)Z_2\dot{x}(s)dsd\theta. \tag{19}$$

where all the involved matrices are required to be definite positive.

The following theorem offers a particular solution for the proposed problem.

Theorem 1. Given scalars $\tau_m, \tau_M, \gamma, \varepsilon > 0$ and the weighting matrices Q and R, if the matrices $X, \tilde{Q}_1, \tilde{Q}_2, \tilde{Z}_1, \tilde{Z}_2 > 0$ and any matrices $Y, \tilde{N}_i, \tilde{M}_i, \tilde{S}_i$, $(i = 1, 2)$ of appropriate dimensions solve the following optimization problem for the two vertices of the polytope $\tau(t)$ defined by (9)-(10),

$$\begin{array}{cc} & min \quad \alpha \\ \text{subject to} & (20), \end{array}$$

then, the optimal L_2-gain controller for the system (5)-(7) with a control network satisfying Assumptions 1-3 is given by $K = YX^{-1}$.

Proof. To prove the previous theorem it suffices to show that the derivative of the LKF (19) can be written in the form (11) and that the aforementioned optimization problem is equivalent to that in (13)-(15).

Taking the time derivative of $V(t)$ along the trajectory of (5) yields that, for $t \in [t_k, t_{k+1})$:

$$\begin{aligned} \dot{V}(t) &= 2x^T(t)P\dot{x}(t) + x^T(t)(Q_1 + Q_2)x(t) - x^T(t-\tau_m)Q_1x(t-\tau_m) \\ &- x^T(t-\tau_M)Q_2x(t-\tau_M) + \dot{x}^T(t)(\tau_M Z_1 + \Delta\tau Z_2)\dot{x}(t) \\ &- \int_{t-\tau_M}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds - \int_{t-\tau_M}^{t-\tau_m} \dot{x}^T(s)Z_2\dot{x}(s)ds. \end{aligned} \tag{21}$$

The augmented state vector is defined as: $\xi^T(t) = \left[x^T(t),\, x^T(t-\tau(t)),\, x^T(t-\tau_m),\, x^T(t-\tau_M),\, \omega^T(t)\right]$. Then, the following null terms are added to the right-hand side of (21):

$$\begin{aligned} 0 &= 2\xi^T(t)\bar{N}\left[x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t} \dot{x}(s)ds\right], \\ 0 &= 2\xi^T(t)\bar{S}\left[x(t-\tau(t)) - x(t-\tau_M) - \int_{t-\tau_M}^{t-\tau(t)} \dot{x}(s)ds\right], \\ 0 &= 2\xi^T(t)\bar{M}\left[x(t-\tau_m) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t-\tau_m} \dot{x}(s)ds\right], \\ 0 &= \gamma^2\omega^T(t)\omega(t) - \gamma^2\omega^T(t)\omega(t), \\ 0 &= \xi^T(t)\bar{C}\bar{C}^T\xi(t) - z^T(t)z(t). \end{aligned}$$

where:

$$\begin{array}{c} \bar{N}^T = [N_1^T \quad N_2^T \quad 0 \quad 0 \quad 0]\ ,\ \bar{M}^T = [M_1^T \quad M_2^T \quad 0 \quad 0 \quad 0]; \\ \bar{S}^T = [S_1^T \quad S_2^T \quad 0 \quad 0 \quad 0]\ ,\ \bar{C}^T = [C \quad DK \quad 0 \quad 0 \quad 0]. \end{array}$$

With these terms, equation (21) can be rewritten as:

$$\begin{aligned} \dot{V}(t) &= \xi^T(t)(\Gamma + \bar{C}\bar{C}^T)\xi(t) + \dot{x}^T(t)(\tau_M Z_1 + \Delta\tau Z_2)\dot{x}(t) \\ &- \int_{t-\tau(t)}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds - 2\xi^T(t)\bar{N}\int_{t-\tau(t)}^{t} \dot{x}(s)ds \\ &- \int_{t-\tau(t)}^{t-\tau_m} \dot{x}^T(s)Z_2\dot{x}(s)ds - 2\xi^T(t)\bar{M}\int_{t-\tau(t)}^{t-\tau_m} \dot{x}(s)ds \\ &- \int_{t-\tau_M}^{t-\tau(t)} \dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)ds - 2\xi^T(t)\bar{S}\int_{t-\tau_M}^{t-\tau(t)} \dot{x}(s)ds \\ &+ \gamma^2\omega^T(t)\omega(t) - z^T(t)z(t). \end{aligned} \tag{22}$$

Now, using the well-known upper bound for the inner product of two vectors:

$$-a^TXa - 2b^Ta \leq b^TX^{-1}b, \quad X > 0,$$

the following upper bounds for the integral terms in (22) can be found:

$$-\int_{t-\tau(t)}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds - 2\xi^T(t)\bar{N}\int_{t-\tau(t)}^{t} \dot{x}(s)ds$$
$$\leq (\tau(t) + \varepsilon)\xi^T(t)\bar{N}Z_1^{-1}\bar{N}^T\xi(t),$$

$$\begin{bmatrix} \tilde{\Gamma} & (\tau(t)+\varepsilon)\tilde{N} & (\tau(t)+\varepsilon-\tau_m)\tilde{M} & (\tau_M+\varepsilon-\tau(t))\tilde{S} & \tau_M\tilde{A} & \Delta\tau\tilde{A} & \tilde{C} & \tilde{Q} & \tilde{R} \\ * & -(\tau(t)+\varepsilon)\tilde{Z}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -(\tau(t)+\varepsilon-\tau_m)Z_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -(\tau_M+\varepsilon-\tau(t))(\tilde{Z}_1+\tilde{Z}_2) & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\tau_M X\tilde{Z}_1^{-1}X & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\Delta\tau X\tilde{Z}_2^{-1}X & 0 & 0 & 0 \\ * & * & * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & * & * & -\alpha Q^{-1} & 0 \\ * & * & * & * & * & * & * & * & -\alpha R^{-1} \end{bmatrix} < 0, \qquad (20)$$

where

$$\begin{aligned} \tilde{N}^T &= [\tilde{N}_1^T \quad \tilde{N}_2^T \quad 0 \quad 0 \quad 0]; \\ \tilde{M}^T &= [\tilde{M}_1^T \quad \tilde{M}_2^T \quad 0 \quad 0 \quad 0]; \\ \tilde{S}^T &= [\tilde{S}_1^T \quad \tilde{S}_2^T \quad 0 \quad 0 \quad 0]; \\ \tilde{C}^T &= [CX^T \quad DY \quad 0 \quad 0 \quad 0]; \\ \tilde{A}^T &= [AX \quad BY \quad 0 \quad 0 \quad B_\omega]; \end{aligned} \qquad \tilde{\Gamma} = \begin{bmatrix} \theta_{11} & \theta_{12} & \tilde{M}_1 & -\tilde{S}_1 & B_\omega \\ * & \theta_{22} & \tilde{M}_2 & -\tilde{S}_2 & 0 \\ * & * & -\tilde{Q}_1 & 0 & 0 \\ * & * & * & -\tilde{Q}_2 & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix}, \qquad \begin{aligned} \tilde{Q}^T &= [X \quad 0 \quad 0 \quad 0 \quad 0]; \\ \tilde{R}^T &= [0 \quad Y \quad 0 \quad 0 \quad 0]; \\ \theta_{11} &= AX + XA^T + \tilde{Q}_1 + \tilde{Q}_2 + \tilde{N}_1 + \tilde{N}_1^T; \\ \theta_{12} &= BY - \tilde{N}_1 + \tilde{S}_1 - \tilde{M}_1 + \tilde{N}_2^T; \\ \theta_{22} &= -\tilde{N}_2 - \tilde{N}_2^T + \tilde{S}_2 + \tilde{S}_2^T - \tilde{M}_2 - \tilde{M}_2^T; \end{aligned}$$

$$\begin{aligned} &-\int_{t-\tau(t)}^{t-\tau_m} \dot{x}^T(s)Z_2\dot{x}(s)ds - 2\xi^T(t)\bar{M}\int_{t-\tau(t)}^{t-\tau_m}\dot{x}(s)ds \\ &\qquad \le (\tau(t)-\tau_m+\varepsilon)\xi^T(t)\bar{M}Z_2^{-1}\bar{M}^T\xi(t), \\ &-\int_{t-\tau_M}^{t-\tau(t)} \dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)ds - 2\xi^T(t)\bar{S}\int_{t-\tau_M}^{t-\tau(t)}\dot{x}(s)ds \\ &\qquad \le (\tau_M-\tau(t)+\varepsilon)\xi^T(t)\bar{S}(Z_1+Z_2)^{-1}\bar{S}^T\xi(t). \end{aligned} \qquad (23)$$

where the constant $\varepsilon \in \mathbb{R}^+$ have been introduced in the bounding terms for design convenience. Then, combining (22) with (23), it can be shown that, for $t \in [t_k, t_{k+1})$,

$$\begin{aligned} \dot{V}(t) \quad &\le \quad \xi^T(t)\Big(\Gamma + (\tau(t)+\varepsilon)\bar{N}Z_1^{-1}\bar{N}^T + (\tau(t)-\tau_m+\varepsilon)\bar{M}Z_2^{-1}\bar{M}^T \\ &+ \quad (\tau_M-\tau(t)+\varepsilon)\bar{S}(Z_1+Z_2)^{-1}\bar{S}^T + \bar{C}\bar{C}^T + \bar{A}\tau_M Z_1\bar{A}^T \\ &+ \quad \bar{A}\Delta\tau Z_2\bar{A}^T\Big)\xi(t) + \gamma^2\omega^T(t)\omega(t) - z^T(t)z(t). \end{aligned} \qquad (24)$$

where $\bar{A}^T = [A \quad BK \quad 0 \quad 0 \quad B_\omega]$. So that, if we define $\Xi(K)$ as:

$$\begin{aligned} &\Xi(K) \triangleq \Gamma + (\tau(t)+\varepsilon)\bar{N}Z_1^{-1}\bar{N}^T + (\tau(t)-\tau_m+\varepsilon)\bar{M}Z_2^{-1}\bar{M}^T + \qquad (25) \\ &(\tau_M-\tau(t)+\varepsilon)\bar{S}(Z_1+Z_2)^{-1}\bar{S}^T + \bar{C}\bar{C}^T + \bar{A}\tau_M Z_1\bar{A}^T + \bar{A}\Delta\tau Z_2\bar{A}^T, \end{aligned}$$

it yields that the derivative of the LKF (19) can be written as in (11). It remains to prove that the optimization problem given in Lemma 1 is analogous to that in Theorem 1. Considering the equation (15) in Lemma 1 and the Assumption 5, yields,

$$\alpha\Xi(K) < -\Phi(K) \Leftrightarrow \Xi(K) - \bar{K}^T\frac{-R}{\alpha}\bar{K} - \bar{I}^T\frac{-Q}{\alpha}\bar{I} < 0 \qquad (26)$$

where $\bar{K}^T = [0 \quad K \quad 0 \quad 0 \quad 0]^T$ and $\bar{I}^T = [I \quad 0 \quad 0 \quad 0 \quad 0]^T$. From equation (25) and (26), applying Schur complements leads to a matrix inequality with the same structure as (20). To obtain (20) is sufficient to introduce the definitions $X = P^{-1}$, $\tilde{Q}_i = XQ_iX$, $\tilde{M}_i = XM_iX$, $\tilde{N}_i = XN_iX$, $\tilde{S}_i = XS_iX$, $\tilde{Z}_i = XZ_iX$, $i = 1,2$ and pre- and post-multiply the matrix inequality by $diag = [X,X,X,X,I,X,X,X,I,I,I,I]$.

□

Remark. The scalar parameter $\varepsilon > 0$ needs to be introduced in order to make feasible the LMIs appearing in the next section. Otherwise, some null matrices appears in the diagonal of the LMIs. It is worth to mention that this modification does not introduce any conservatism, since $\varepsilon > 0$ can be chosen as small as necessary, i.e., $\varepsilon \to 0^+$.

C. Algorithm for Controller Design

Notice that (20) is not an LMI. There is procedure (see [15]) which let us to address the nonlinear matrix inequality (20) by introducing some new matrix variables. First, define two variables T_1, T_2 such that,

$$X\tilde{Z}_1^{-1}X \ge T_1 > 0, \quad X\tilde{Z}_2^{-1}X \ge T_2 > 0, \qquad (27)$$

which is equivalent to:

$$\begin{bmatrix} -T_1^{-1} & X^{-1} \\ X^{-1} & -\tilde{Z}_1^{-1} \end{bmatrix} \le 0, \quad \begin{bmatrix} -T_2^{-1} & X^{-1} \\ X^{-1} & -\tilde{Z}_2^{-1} \end{bmatrix} \le 0. \qquad (28)$$

Now, introducing some new variables,

$$\bar{X} = X^{-1}, \quad \bar{T}_i = T_i^{-1}, \quad \bar{Z}_i = \tilde{Z}_i^{-1} \quad i = 1,2, \qquad (29)$$

equation (28) can be rewritten as,

$$\begin{bmatrix} -\bar{T}_1 & \bar{X} \\ \bar{X} & -\bar{Z}_1 \end{bmatrix} \le 0, \quad \begin{bmatrix} -\bar{T}_2 & \bar{X} \\ \bar{X} & -\bar{Z}_2 \end{bmatrix} \le 0. \qquad (30)$$

This way, instead of using the original condition (20), the following nonlinear minimization problem involving LMI conditions, can be stated as:

$$\text{Minimize } \mathbf{Tr}\left(\bar{X}X + \bar{T}_1T_1 + \bar{T}_2T_2 + \bar{Z}_1\tilde{Z}_1 + \bar{Z}_2\tilde{Z}_2\right)$$

subject to

$$\left\{ \begin{array}{lll} \tilde{\Lambda} < 0, & \begin{bmatrix} -\bar{T}_1 & \bar{X} \\ \bar{X} & -\bar{Z}_1 \end{bmatrix} \le 0, & \begin{bmatrix} -\bar{T}_2 & \bar{X} \\ \bar{X} & -\bar{Z}_2 \end{bmatrix} \le 0. \\ \begin{bmatrix} X & I \\ I & \bar{X} \end{bmatrix} \ge 0, & \begin{bmatrix} T_1 & I \\ I & \bar{T}_1 \end{bmatrix} \ge 0, & \begin{bmatrix} T_2 & I \\ I & \bar{T}_2 \end{bmatrix} \ge 0, \end{array} \right. \qquad (31)$$

where $\tilde{\Lambda}$ is the matrix required to be definite negative in (20), but substituting the elements $X\tilde{Z}_i^{-1}X$ by T_i, $i = 1,2$. Regarding to the equations (27), it is immediate that, if $\tilde{\Lambda} < 0$, then (20) holds. The minimization problem is introduced to force (29). When the LMIs in the second row of the restrictions (31) saturate, the optimum is reached and (31) holds.

In order to solve the aforementioned minimization problem (31) the algorithm introduced in [15] can be implemented. Theorem 2.1 in [15] proved the convergence of this algorithm.

IV. APPLICATION OF THE OPTIMAL L_2-GAIN CONTROLLER

This section describes simulations on a particular problem to show the performance of the optimal L_2-gain controller.

Consider a vehicle tracking system intended to keep a constant distance between two moving objects. More precisely, assume a master vehicle that moves at unknown time-varying speed, which is followed by a slave vehicle trying to keep a constant distance from the first.

The equations of the dynamical system can be written as follows:

$$\begin{aligned} e(t) &= p_1(t) - p_2(t) - l_r, \\ \dot{e}(t) &= v_1(t) - v_2(t) = y(t), \\ \dot{y}(t) &= a_1(t) - a_2(t). \end{aligned}$$

where $p(t)$, $v(t)$ and $a(t)$ denote positions, velocities and accelerations; and $e(t)$ and $y(t)$ denote the errors in position and velocity, respectively. Moreover, forces equilibrium yields:

$$F_i(t) - F_{a,i}(t) - F_{r,i}(t) = m_i a_i(t), \quad i = 1,2.$$

Where F_i is the motor force, $F_{a,i}$ is the aerodynamical drag and $F_{r,i}$ is the rolling friction, given by:

$$F_a(t) = \frac{1}{2} c_a A_T \rho_{aire} v^2(t), \quad F_r(t) = c_r m g \cos(\alpha(t)).$$

where,

- c_a, c_r: aerodynamic and tire-road drag coefficients, respectively.
- A_T: vehicle's aerodynamic cross-section.
- ρ_{air}: air density.
- m: vehicle mass.
- g: gravity constant.
- $\alpha(t)$: road slope angle.

For the sake of simplicity, both vehicles will be assumed with identical characteristics. The linearization of the aerodynamical force using a Taylor serie expansion yields the following dynamic equations for the system:

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} \int e(t) \\ e(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -c_3 c_0 \end{pmatrix}\begin{pmatrix} \int e(t) \\ e(t) \\ y(t) \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ 0 \\ -c3 \end{pmatrix} F_2(t) + \begin{pmatrix} 0 \\ 0 \\ c3 \end{pmatrix} F_1(t) \end{aligned} \tag{32}$$

with $c_0 = \rho A_T c_d$ and $c_3 = 1/m$.

Assume that the slave vehicle receives position and speed from the master vehicle at a sampling rate h. The objective is computing a control action $F_2(t)$, such that the distance between both vehicles remains as close as possible to a constant set point l_r. The controller is implemented on the slave vehicle, though dynamic variables from the master vehicle are required (position, speed). This data is nonetheless subject to delays and losses due to unreliable data links, or hardware limitations (bus access policy).

Thus, the control signal $F_2(t)$ can be obtained as

$$F_2(t) = K \begin{pmatrix} \int e(t - \tau(t)) \\ e(t - \tau(t)) \\ y(t - \tau(t)) \end{pmatrix}$$

where transmission delays are implicit through the term $\tau(t)$.

Simulations are performed assuming system parameters as $m = 1200\ Kg$, $A_T = 1.5\ m^2$, $c_a = 0.15$, $c_r = 0.015$ and a nominal speed $v_n = 120\ Km/h$.

Network induced delays are assumed to lay in the range from $0ms$ to $200ms$, and a sample period $h = 500ms$.

The following simulations consider an inter-vehicle setpoint distance of 20 meters, with a master vehicle moving at constant speed of $120 Km/h$, except around $t = 45s$ where the vehicle slows down temporarily before recovering the nominal speed.

In Figure 1 the behavior of the proposed L_2-gain control is compared for different disturbance attenuation levels when network-induced delays are considered. It appears as expected that deviations from the setpoint are reduced as parameter γ is decreased. This is achieved at the cost of more aggressive control actions that must be properly weighted to not exceed actuator limits. The trade-off between the L_2-gain of the system and the optimality can be also checked in the Figure 1. The controller with free γ results in a cost $J_1 = 659.1$, the controller with $\gamma < 1.8$ achieves a cost $J_2 = 670.9$ and the controller with $\gamma < 0.8$ has a cost $J_3 = 2478$.

Similarly, the performance of the system can be modulated acting on the weighting policy in matrices Q and R. Figure 2 shows the effect of this trade-off. In both cases an attenuation factor $\gamma < 1.8$, and a weighting matrix $Q = diag(100, 100, 0.01)$ are chosen. Matrix R is set in one case $R = 10^{-5}$ prioritizing tracking performance, and $R = 10^{-2}$ assuming more stringent control action constraints. As expected, tracking error is reduced in the first case, since control effort is limited in the latest case using the weighting matrix R.

These simulations show how the method can be used to adjust different performance requirements and disturbance rejection capabilities for network control systems, where parameter γ together with matrices Q and R can be tuned for such purpose.

V. CONCLUSIONS

This paper deals with the problem of designing optimal control laws for network control systems with L_2-gain disturbance rejection constraints.

Linear time invariant systems are considered to be controlled through a communication network that imposes delays and data losses in the communications.

The problem, also referred as mixed H_2/H_∞, is derived based on a Lyapunov-Krasovskii approach, considering a polytopic covering of the time-delay range. Solution for the problem is provided in terms of a set of LMIs together with an optimization procedure.

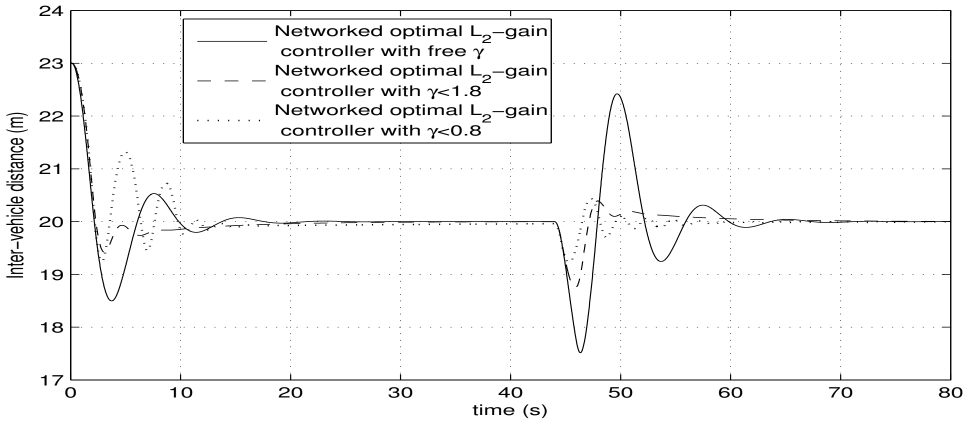

Fig. 1. System response with network effects for various disturbance attenuation levels

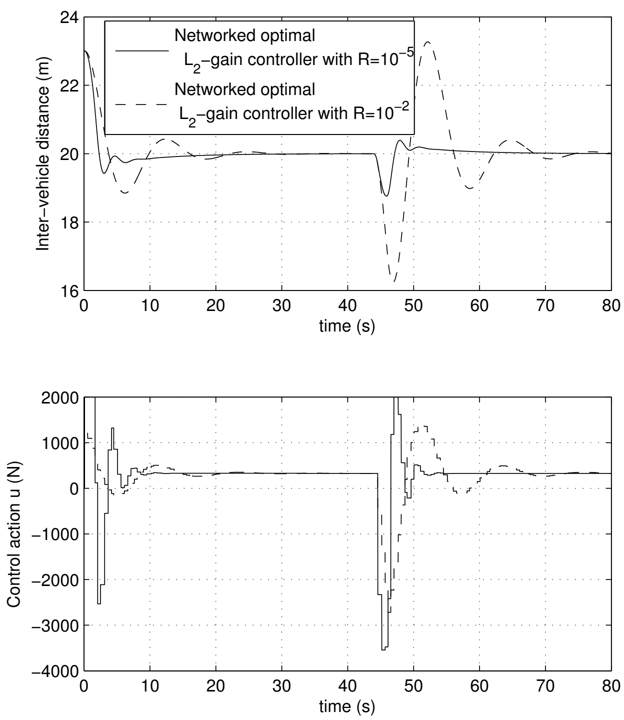

Fig. 2. Trading off performance vs. control effort. Effect of the weighting policy on *R* and *Q*

Simulation results show the effectiveness of the method to trade-off with a unified design methodology, performance, control effort and disturbance rejection capabilities.

VI. ACKNOWLEDGMENTS

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