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RESEARCH ARTICLE

Simulation of linear elastic structural elements using the Petrov–Galerkin finite element method

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Abstract

In this contribution, it is demonstrated that the mesh sensitivity of linear elastic Reissner–Mindlin finite-element plate formulations can be significantly reduced by using a Petrov–Galerkin-based approach. In contrast to the usual Bubnov–Galerkin method, Petrov–Galerkin methods are generally characterized by the fact that the test function and the trial function are approximated using different shape functions. To provide an overview, established Petrov–Galerkin methods for 2D solid elements, which have been shown to reduce mesh sensitivity, are reviewed first. It is then investigated whether a suitable Petrov– Galerkin plate formulation can be developed. In this context, it is demonstrated that a full Petrov–Galerkin method leads to problems in the treatment of transverse shear locking. However, the proposed partial Petrov–Galerkin method shows the desired mesh-insensitive behavior.

KEYWORDS
finite element method, Petrov-Galerkin method, plate structures, structural mechanics

1 | MOTIVATION

An inherent problem when simulating plate or shell structures using the finite element method is the dependence of the results on the selected mesh. This can be observed, for example, in the case of a clamped square plate, which is subjected to a concentrated load in the middle of the plate. For this common numerical test, there exists an analytical solution for the maximum vertical displacement of the plate [1], which is given by

$$w_{\max} = 0.0056FL^2/D, \tag{1}$$

where $D = Et^3/(12(1-\nu^2))$. Thus, by choosing suitable parameters (length $L = 100$, plate thickness $t = 1$, material constants $E = 10^4$, $\nu = 0.3$, concentrated load $F = 16.3527$), the analytical solution for the maximum vertical displacement can be set to be $w_{\max} = 1$. A numerical solution to this problem can be determined using the finite element method. Due to the symmetry of the problem, it is sufficient to consider only one quarter of the plate. Following Andelfinger [2], this plate quarter is discretized with four finite elements; see Figure 1. Using the well-established Assumed Natural Strain (ANS) formulation by Bathe and Dvorkin [3], the numerical result for the maximum vertical displacement is $w_{\max} = 0.8676$ for a regular mesh.

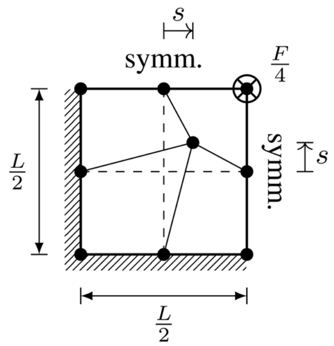

FIGURE 1 The mesh of the clamped square plate example depending on a mesh distortion parameter s.

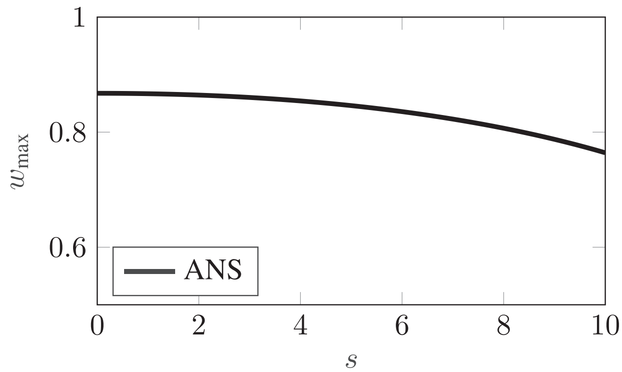

FIGURE 2 Resulting maximum vertical displacement $w_{\max}$ for the clamped square plate example, when employing four-node ANS (ANS) elements.

However, as Figure 2 indicates, changing the element geometry leads to a deterioration of the numerical result. More precisely, a decrease in the computed value for the maximum vertical displacement can be observed with an increasing mesh distortion s. From the problem considered, it is evident that this is a purely numerical phenomenon.

The current contribution investigates to what extent the use of a Petrov–Galerkin method can reduce this influence of the finite element mesh when considering plate structures. In this context, a Petrov–Galerkin method is generally understood as a procedure in which the test function and the trial function are approximated with different shape functions. This is in contrast to the usual Bubnov–Galerkin method, where the same shape functions are used.

An outline for the rest of this paper is as follows: First, as a basis for the plate formulations, established Petrov–Galerkin formulations for 2D solid elements are presented in Section 2. It is highlighted how these formulations manage to achieve a lower mesh sensitivity than conventional methods. Subsequently, in Section 3, the key ideas are applied to plate formulations. In this context, arising problems are addressed, and finally, a proposal for a suitable Petrov– Galerkin plate formulation is made.

2 | PETROV–GALERKIN 2D SOLID ELEMENTS

In this section, two established Petrov–Galerkin finite-element formulations for 2D solids are briefly reviewed.

2.1 | The eight-node element by Rajendran and Liew [4]

The first finite element formulation that, to the best knowledge of the authors, addressed and substantially improved the problem of mesh dependency is the eight-node element by Rajendran and Liew [4]. The key idea of their formulation is to use so-called metric shape functions, denoted here by M_i^e. A more detailed specification of metric shape functions is given in the next section. These metric shape functions are used to approximate the displacement field $\vec{u}$ instead of the usual serendipity shape functions, denoted here by N_i. The virtual displacement field $\delta\vec{u}$, however, is still interpolated using serendipity shape functions. Thus,

$$\delta\vec{u}^{\mathrm{h},e} = \sum_{i=1}^{8} N_i(\vec{\xi})\,\delta\vec{u}_i^e, \qquad \vec{u}^{\mathrm{h},e} = \sum_{i=1}^{8} M_i^e(\vec{x})\,\vec{u}_i^e. \tag{2}$$

In contrast to the serendipity shape functions, the metric shape functions are not constructed in the reference element but, at least in this work, in the physical element in terms of global Cartesian coordinates. This is motivated by the objective to fulfill the so-called *completeness requirement*

$$\sum_{i=1}^{n} M_i^e x_i^p y_i^q = x^p y^q, \qquad p, q = 0, 1, 2, \dots, \tag{3}$$

which is given here in its general form. The authors realized that given polynomial displacement fields can be reproduced exactly only if the employed shape functions satisfy this condition. Therefore, they derived the metric shape functions from it. As a consequence, their eight-node element is able to reproduce quadratic displacement fields exactly, even when the element is distorted. Interestingly, the usual serendipity shape functions also satisfy (3) in the case of undistorted elements. Accordingly, in this case, the metric shape functions are identical to the serendipity shape functions.

Numerical investigations verify that the proposed Petrov–Galerkin formulation shows a significantly lower mesh sensitivity than the usual Bubnov–Galerkin formulation. However, Ooi et al. [5] could show that metric shape functions, which are set up in the physical domain, are not objective. As a remedy, Xie et al. [6] proposed to construct the metric shape functions in terms of so-called skew coordinates $\vec{\bar{\xi}}$. This skew coordinates are defined by the element-wise affine mapping

$$\vec{\bar{\xi}} = \left[\bar{\xi}, \bar{\eta}\right]^{\mathrm{T}} = \vec{J}_0^{-1}(\vec{x} - \vec{x}_0), \tag{4}$$

where $\vec{x}_0$ denotes the physical coordinates of the element center and $\vec{J}_0$ is the usual Jacobian matrix evaluated at the element center. It can be shown that these skew coordinates are frame-indifferent. Accordingly, it has become established that the metric shape functions are no longer constructed in terms of Cartesian coordinates but in terms of skew coordinates.

2.2 | The four-node element by Pfefferkorn and Betsch [7]

Pfefferkorn and Betsch [7] succeeded in developing a four-node element with similar properties as the eight-node element introduced in the previous section. To achieve this, they developed a Petrov–Galerkin formulation of the well-known enhanced assumed strain (EAS) method. Inspired by the eight-node element, they approximated the displacement field using metric shape functions in terms of skew coordinates and the virtual displacements using Lagrangian shape functions. More precisely,

$$\delta\vec{u}^{\mathrm{h},e} = \sum_{i=1}^{4} N_i(\vec{\xi})\,\delta\vec{u}_i^e, \qquad \vec{u}^{\mathrm{h},e} = \sum_{i=1}^{4} M_i^e(\vec{\bar{\xi}})\,\vec{u}_i^e. \tag{5}$$

Here, the metric shape functions M_i^e can be computed by

$$\begin{bmatrix} M_1^e & M_2^e & M_3^e & M_4^e \end{bmatrix} = \begin{bmatrix} 1 & \bar{\xi} & \bar{\eta} & \bar{\xi}\bar{\eta} \end{bmatrix} \begin{bmatrix} 1 & \bar{\xi}_1 & \bar{\eta}_1 & \bar{\xi}_1\bar{\eta}_1 \\ 1 & \bar{\xi}_2 & \bar{\eta}_2 & \bar{\xi}_2\bar{\eta}_2 \\ 1 & \bar{\xi}_3 & \bar{\eta}_3 & \bar{\xi}_3\bar{\eta}_3 \\ 1 & \bar{\xi}_4 & \bar{\eta}_4 & \bar{\xi}_4\bar{\eta}_4 \end{bmatrix}^{-1}, \tag{6}$$

where $\vec{\bar{\xi}}_i = [\bar{\xi}_i, \bar{\eta}_i]^{\mathrm{T}}$ denotes the position vector of node i in skew coordinates. Note that, as shown in (6), the metric shape functions must be computed separately for each element. This, of course, adds some additional numerical cost compared to the computation of the usual Lagrangian shape functions.

The enhanced assumed strain fields were then chosen in a way that led to a locking-free and mesh-distortion-insensitive element. See [7] for more details. The authors were also able to transfer their concept to the 3D setting. In addition, they recently succeeded in applying their approach to nonlinear problems [8]. This illustrates the wide applicability of the presented Petrov–Galerkin method based on metric shape functions.

3 | PETROV–GALERKIN PLATE ELEMENTS

In this section, it is analyzed how the Petrov–Galerkin method introduced in Section 2 can be applied to plate formulations.

The scope of this work is thereby limited to plate formulations, which are formulated on the basis of the Reissner–Mindlin theory. This can be justified, among other reasons, by the fact that for these formulations, the mesh-dependent behavior described in Section 1 can be observed. Furthermore, these are the formulations in which Lagrangian or serendipity shape functions are commonly used. As a basis for the finite element discretization serves the equation

$$\delta W = \underbrace{\int_\Omega \delta\vec{\kappa}^{\mathrm{T}} \vec{E}_B \vec{\kappa}\,\mathrm{d}A + \int_\Omega \delta\vec{\gamma}^{\mathrm{T}} \vec{E}_S \vec{\gamma}\,\mathrm{d}A}_{\delta W_{\mathrm{int}}} + \delta W_{\mathrm{ext}} = 0, \tag{7}$$

which corresponds to the principle of virtual work. The internal part δW_{int} is thereby split into a bending part and a shear part. The curvatures

$$\vec{\kappa} = \vec{D}_B\,\vec{u} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} w \\ \Theta_x \\ \Theta_y \end{bmatrix} \tag{8}$$

in the bending part are only dependent on the rotations $\vec{\Theta} = [\Theta_x, \Theta_y]^{\mathrm{T}}$. The transverse shear strain

$$\vec{\gamma} = \vec{D}_S\,\vec{u} = \begin{bmatrix} \frac{\partial}{\partial x} & 1 & 0 \\ \frac{\partial}{\partial y} & 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ \Theta_x \\ \Theta_y \end{bmatrix}, \tag{9}$$

on the other hand, depends not only on the rotations but also on the vertical displacement w.

3.1 | An eight-node Petrov–Galerkin element

To show that the Petrov–Galerkin method also has advantageous properties with respect to mesh dependency when simulating plate structures, an eight-node element will first be considered, just as in the 2D solid formulations. For this purpose, inspired by the eight-node element of Rajendran and Liew (cf. Section 2.1), both the vertical displacement and the rotations are approximated using metric shape functions. More precisely,

$$w^{\mathrm{h},e} = \sum_{i=1}^{8} M_i^e(\vec{\bar{\xi}})\, w_i^e, \qquad \vec{\Theta}^{\mathrm{h},e} = \sum_{i=1}^{8} M_i^e(\vec{\bar{\xi}})\, \vec{\Theta}_i^e. \tag{10}$$

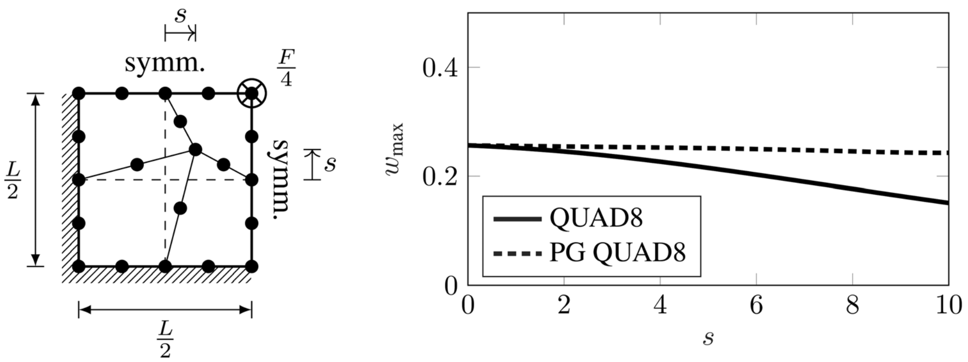

FIGURE 3 Resulting maximum vertical displacement $w_{\max}$ for the clamped square plate example, when employing eight-node Bubnov–Galerkin (QUAD8) or eight-node Petrov–Galerkin (PG QUAD8) elements.

For the approximation of the virtual displacement and the virtual rotations, serendipity shape functions are employed. Accordingly,

$$\delta w^{\mathrm{h},e} = \sum_{i=1}^{8} N_i(\vec{\xi})\,\delta w_i^e, \qquad \delta\vec{\Theta}^{\mathrm{h},e} = \sum_{i=1}^{8} N_i(\vec{\xi})\,\delta\vec{\Theta}_i^e. \tag{11}$$

Examining the clamped plate example using this approach, two main insights can be gained from the results depicted in Figure 3. On the one hand, it is obvious that, also for plate structures, the Petrov–Galerkin element shows a significantly lower mesh dependency than the corresponding Bubnov–Galerkin element. On the other hand, the results show that the vertical displacement is greatly underestimated, as the analytical solution is $w_{\max} = 1$.

This phenomenon is well known for plate structures under the name *transverse shear locking*. It is an artificial stiffening effect that occurs due to parasitic terms in the transverse shear strains $\vec{\gamma}^{\mathrm{h},e}$. It is known that this artificial stiffening effect can be significantly reduced by applying the so-called p-method. This refers to an increase of the order of the shape functions. However, this method possesses some disadvantages; the main disadvantage is the sharply increasing numerical complexity due to the additional nodes. Therefore, this method will not be discussed further, even though it would be possible to implement it as a Petrov–Galerkin method. However, other methods were developed for Bubnov–Galerkin formulations that eliminate transverse shear locking even for four-node elements reliably. The next section takes a closer look at one of these methods.

3.2 | A four-node Petrov–Galerkin ANS element

As pointed out in Section 3.1, the elimination of transverse shear locking is a major challenge when designing new plate elements. For Bubnov–Galerkin plate formulations, it has been shown that the ANS method of Bathe and Dvorkin [3] has outstanding properties in this respect. This is achieved by interpolating each of the transverse shear strain components between two sampling points. More precisely, the interpolation

$$\begin{aligned} \tilde{\gamma}_{\xi z}^{\mathrm{h},e} &= \frac{1}{2}(1+\eta)\,\gamma_{\xi z}^{\mathrm{h},e}|_{\vec{\xi}=\vec{\xi}^A} + \frac{1}{2}(1-\eta)\,\gamma_{\xi z}^{\mathrm{h},e}|_{\vec{\xi}=\vec{\xi}^C} \\ \tilde{\gamma}_{\eta z}^{\mathrm{h},e} &= \frac{1}{2}(1+\xi)\,\gamma_{\eta z}^{\mathrm{h},e}|_{\vec{\xi}=\vec{\xi}^D} + \frac{1}{2}(1-\xi)\,\gamma_{\eta z}^{\mathrm{h},e}|_{\vec{\xi}=\vec{\xi}^B} \end{aligned} \tag{12}$$

eliminates the parasitic terms in the transverse shear strains.

This procedure can be applied analogously when designing a four-node Petrov–Galerkin ANS element. However, numerical investigations show that this approach does not lead to satisfactory results in general. This can be seen, for instance, from the clamped square plate example (cf. Figure 4). To be more precise, a different type of mesh dependency can be observed, which is presumably due to the fact that the use of metric shape functions no longer allows a reliable elimination of the locking-causing terms via this interpolation. Interestingly, a similar behavior can be observed when the Petrov–Galerkin method is applied to the closely related ANS element by Hughes and Tezduyar [9].

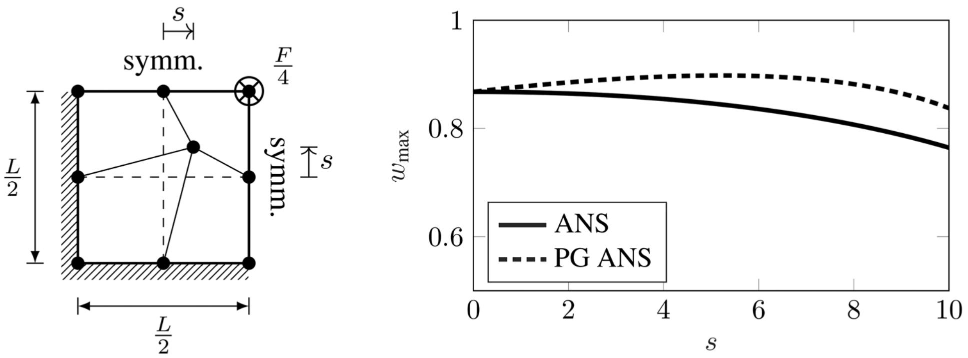

FIGURE 4 Resulting maximum vertical displacement w_{max} for the clamped square plate example, when employing four-node Bubnov–Galerkin ANS (ANS) or four-node Petrov–Galerkin ANS (PG ANS) elements.

3.3 | A four-node partial Petrov–Galerkin ANS element

In this section, another four-node Petrov–Galerkin-based plate formulation is proposed. It is inspired by a property of the ANS interpolation (12) by Bathe and Dvorkin [3]. Namely, it can be shown that their interpolation has no effect on the vertical displacement when Lagrangian shape functions are used, but only on the rotations. More specifically,

$$\begin{aligned}
\tilde{\gamma}^{\mathrm{h},e}_{\xi z} &= \frac{1}{2}\frac{\partial w^{\mathrm{h},e}}{\partial \xi} + \frac{1}{2}(1+\eta)\left[\frac{\partial x^{\mathrm{h},e}}{\partial \xi}\Theta^{\mathrm{h},e}_x + \frac{\partial y^{\mathrm{h},e}}{\partial \xi}\Theta^{\mathrm{h},e}_y\right]\Bigg|_{\vec{\xi}=\vec{\xi}^A} + \frac{1}{2}(1-\eta)\left[\frac{\partial x^{\mathrm{h},e}}{\partial \xi}\Theta^{\mathrm{h},e}_x + \frac{\partial y^{\mathrm{h},e}}{\partial \xi}\Theta^{\mathrm{h},e}_y\right]\Bigg|_{\vec{\xi}=\vec{\xi}^C}, \\
\tilde{\gamma}^{\mathrm{h},e}_{\eta z} &= \frac{1}{2}\frac{\partial w^{\mathrm{h},e}}{\partial \eta} + \frac{1}{2}(1+\xi)\left[\frac{\partial x^{\mathrm{h},e}}{\partial \eta}\Theta^{\mathrm{h},e}_x + \frac{\partial y^{\mathrm{h},e}}{\partial \eta}\Theta^{\mathrm{h},e}_y\right]\Bigg|_{\vec{\xi}=\vec{\xi}^D} + \frac{1}{2}(1-\xi)\left[\frac{\partial x^{\mathrm{h},e}}{\partial \eta}\Theta^{\mathrm{h},e}_x + \frac{\partial y^{\mathrm{h},e}}{\partial \eta}\Theta^{\mathrm{h},e}_y\right]\Bigg|_{\vec{\xi}=\vec{\xi}^B}.
\end{aligned} \tag{13}$$

This representation of the ANS interpolation motivates another reasonable approach. Specifically, a partial Petrov–Galerkin method. In contrast to the full Petrov–Galerkin method (10), in this case, only the vertical displacement is approximated with metric shape functions, whereas the rotations are approximated using the standard Lagrangian shape functions. To be more precise,

$$w^{\mathrm{h},e} = \sum_{i=1}^{4} M^e_i(\vec{\xi})\, w^e_i, \quad \vec{\Theta}^{\mathrm{h},e} = \sum_{i=1}^{4} N_i(\vec{\xi})\, \vec{\Theta}^e_i. \tag{14}$$

The metric shape functions can again be computed using (6). In accordance with (13), the ANS interpolation is only applied to the rotations. Thus, the transverse shear locking is eliminated in the conventional way, whereas at the same time, an improved formulation is obtained due to the better approximation of the vertical displacement. Evaluating the clamped plate example with this formulation reveals a widely mesh-insensitive and locking-free behavior (cf. Figure 5).

Another numerical example that is often used in the investigation of new plate formulations is the clamped circular plate. This example shall also be presented here, as it illustrates that the proposed partial Petrov–Galerkin method yields better results, not only for the clamped square plate example.

For this example, there exists again an analytical solution for the maximum vertical displacement [1]. It is given by

$$w_{\text{max}} = \frac{qR^4}{64D}. \tag{15}$$

Therefore, by choosing suitable parameters (radius $R = 5$, plate thickness $t = 0.1$, material constants $E = 10.92 \times 10^5$, $\nu = 0.3$, uniform load $q = 10.24$), the analytical solution for the maximum vertical displacement is again $w_{\text{max}} = 1$. Inspired by a work of Hughes and Tezduyar [9], a quarter of the plate is discretized with three finite elements to solve the problem numerically; see Figure 6. It can be seen that the geometry of the problem forces the use of distorted elements. Performing a simulation using the four-node Bubnov–Galerkin ANS element returns a maximum vertical displacement

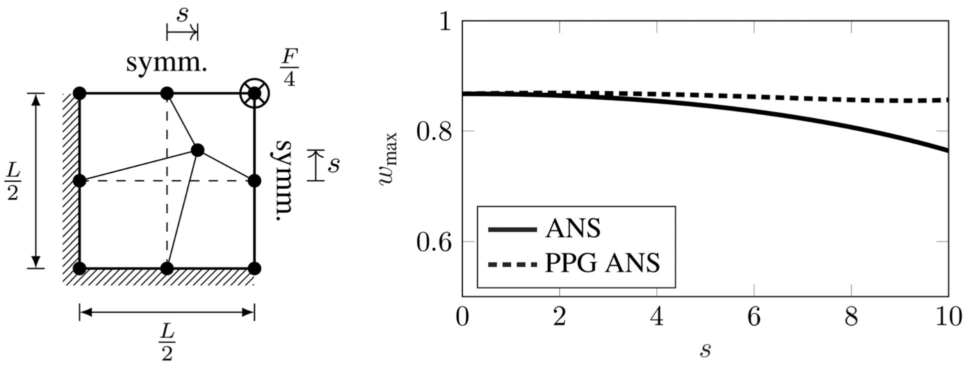

FIGURE 5 Resulting maximum vertical displacement w_{max} for the clamped square plate example, when employing four-node Bubnov–Galerkin ANS (or four-node partial Petrov–Galerkin ANS (PPG ANS) elements.

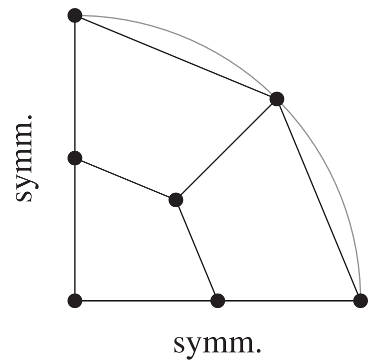

FIGURE 6 The mesh used in the circular plate example.

of $w_{max} = 0.9286$. However, when using the proposed partial Petrov–Galerkin ANS element, the result is even better, as $w_{max} = 0.9676$ in this case.

4 | CONCLUSION

In this contribution, the applicability of the Petrov–Galerkin method to structural elements, more specifically plates, has been investigated. First, it has been recalled that Petrov–Galerkin formulations can be developed for 2D solid elements, which have improved properties with respect to mesh dependency compared to the conventional Bubnov–Galerkin formulations. It has been indicated that this is essentially due to the use of metric shape functions. Afterward, it has been shown that a significant improvement regarding mesh dependency can also be achieved for plate structures. More precisely, it has been demonstrated that a full Petrov–Galerkin method leads to problems in the treatment of transverse shear locking. However, the proposed partial Petrov–Galerkin method circumvents this problem and shows the desired mesh-insensitive behavior. The improved properties in terms of mesh sensitivity have been illustrated by numerical examples.

These results motivate the authors to investigate the presented partial Petrov–Galerkin method in more detail, and to examine transferability to shell formulations.

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