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Asymptotic Properties of Solutions for Lanchester Type Models with Time Dependent Coefficients

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Abstract

We consider an ordinary differential system which is a so-called Lanchester's linear law model with time dependent coefficients. We study on asymptotic forms of solutions that decay to a point on the x-axis and y-axis.

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1 Introduction

In this paper, we consider the ordinary differential system of the form :

$$\begin{cases} x'(t) = -a(t)x(t)y(t) \\ y'(t) = -b(t)x(t)y(t) \end{cases} \tag{1.1}$$

where $a(t)$ and $b(t)$ are positive continuous functions on $[0, \infty)$, and satisfy

$$A(t) = \int_0^t a(s)\,ds \to \infty \quad \text{and} \quad B(t) = \int_0^t b(s)\,ds \to \infty \tag{1.2}$$

as $t \to \infty$.

The initial value problem (1.1) with positive initial data

$$x(0) = x_0 > 0 \quad \text{and} \quad y(0) = y_0 > 0 \tag{1.3}$$

has non-negative solutions.

System (1.1) is known as one of Lanchester type models, which describes many phenomena appearing in economics, logistics, biology, and so on.

In [5], F.W.Lanchester first poposed system (1.1) to describe combat situations (see Taylor [9] for a review). System (1.1) is said a model of guerrilla engagements (see [2], [3], [10] and the references cited therein).

There are some mathematically treated reseach works for Lanchester type models (see [4] and [11] for Lanchester linear-law models, and see [6] and [8] for Lanchester square-law models, and also see [1] and [7] for Lanchester models with mixed forces).

First, we consider the special case where $a(t) = \alpha$ and $b(t) = \beta$ for some positive constants $\alpha > 0$ and $\beta > 0$, that is,

$$\begin{cases} x'(t) = -\alpha x(t)y(t) \\ y'(t) = -\beta x(t)y(t) \end{cases} \tag{1.4}$$

with initial data (1.3), then we have that $A(t) = \alpha t$ and $B(t) = \beta t$ in (1.2). Using the exchange ratio $E = \alpha/\beta$ of (1.4), we can easily see that $(x(t) - Ey(t))' = 0$, and hence, $x(t) - Ey(t)$ is a constant value which is denoted by symbol M, that is,

$$x(t) - Ey(t) = x_0 - Ey_0 = M. \tag{1.5}$$

Thus, we have from (1.4) and (1.5) that

$$x'(t) = -\alpha x(t)y(t) = -\beta x(t)(x(t) - M)\,,$$

and moreover, by fundamental calculation we obtain the following representation fomula of solution $(x(t), y(t))$ of (1.4) :

(1) When $M = 0$ (i.e. $x_0 = Ey_0$),

$$x(t) = (x_0^{-1} + \beta t)^{-1} \quad \text{and} \quad y(t) = (y_0^{-1} + \alpha t)^{-1} \tag{1.6}$$

for $t \geq 0$.

(2) When $M \neq 0$ (i.e. $x_0 \neq Ey_0$),

$$x(t) = \frac{M}{1 - (1 - M/x_0)e^{-M\beta t}} = \frac{(x_0/(x_0 + EN))ENe^{-EN\beta t}}{(1 - (x_0/(x_0 + EN))e^{-EN\beta t}} \tag{1.7}$$

and

$$y(t) = \frac{(y_0/(y_0 + M/E))(M/E)e^{-(M/E)\alpha t}}{1 - (y_0/(y_0 + M/E))e^{-(M/E)\alpha t}} = \frac{N}{1 - (1 - (N/y_0))e^{-N\alpha t}} \tag{1.8}$$

where $N = -M/E$ and hence

$$x(t) - M = \frac{(1 - M/x_0)e^{-M\beta t}}{1 - (1 - M/x_0)e^{-M\beta t}} \tag{1.9}$$

and

$$y(t) - N = \frac{(1 - N/y_0)e^{-N\alpha t}}{1 - (1 - N/y_0)e^{-N\alpha t}} \tag{1.10}$$

for $t \geq 0$.

In what follows, "$f(t) \sim g(t)$ as $t \to \infty$" means that $\lim_{t\to\infty} f(t)/g(t) = 1$ for positive functions $f(t)$ and $g(t)$ defined near $+\infty$. Similary, for vector-valued functions "$(f_1(t)\,,\, f_2(t)) \sim (g_1(t)\,,\, g_2(t))$ as $t \to \infty$" means that $f_i(t) \sim g_i(t)$ as $t \to \infty$, $i = 1, 2$.

Immediately, we can obtain from (1.6)–(1.10) the following decay properties of solution $(x(t), y(t))$ of (1.4) :

(i) When $M = 0$, $(x(t), y(t)) \to (0, 0)$ as $t \to \infty$ and

$$(x(t)\,,\, y(t)) \sim ((\beta t)^{-1}\,,\, (\alpha t)^{-1}) \quad \text{as} \quad t \to \infty\,. \tag{1.11}$$

(ii) When $M > 0$, $(x(t), y(t)) \to (M, 0)$ as $t \to \infty$ and

$$(\log(x(t) - M)\,,\, \log y(t)) \sim (-M\beta t\,,\, -(M/E)\alpha t) \quad \text{as} \quad t \to \infty \tag{1.12}$$

and $x(t) - M = O(e^{-M\beta t})$ and $y(t) = O(e^{-(M/E)\alpha t})$.

(iii) When $N = -M/E > 0$, $(x(t), y(t)) \to (0, N)$ as $t \to \infty$ and

$$(\log x(t)\,,\, \log(y(t) - N)) \sim (-EN\beta t\,,\, -N\alpha t) \quad \text{as} \quad t \to \infty \tag{1.13}$$

and $x(t) = O(e^{-EN\beta t})$ and $y(t) - N = O(e^{-N\alpha t})$.

Remark. When the time dependent coefficients $a(t)$ and $b(t)$ in (1.1) satisfy $a(t)/b(t) = \text{const} > 0$ for $t \geq 0$, we can obtain the similar representation formula of solution $(x(t), y(t))$ of (1.1) replaced αt and βt in (1.6)–(1.10) by $A(t)$ and $B(t)$, respectively.

In [4], Ito, Ogiwara and Usami have derived the following asymptotic forms (1.14) and (1.15) of solution $(x(t), y(t))$ of (1.1) decaying to the origin $(0, 0)$, like (1.11) for (1.1) with constant coefficients :

(i) If $a(t)$ and $b(t)$ satisfy (1.2) and $\lim_{t\to\infty} a(t)/b(t) = \text{const} > 0$, then

$$(x(t)\,,\, y(t)) \sim (B(t)^{-1}\,,\, A(t)^{-1}) \quad \text{as} \quad t \to \infty\,. \tag{1.14}$$

(ii) If $a(t)$ and $b(t)$ are of class C^1 and satisfy

$$\left(\frac{a(t)}{b(t)}\right)' \leq 0 \quad \text{for large } t$$

and

$$\lim_{t\to\infty} \frac{a(t)B(t)}{A(t)b(t)} = k > 0 \quad \text{and} \quad \lim_{t\to\infty} \left(\frac{a(t)B(t)}{A(t)b(t)}\right)' \frac{B(t)}{b(t)} = 0\,,$$

then

$$(x(t)\,,\, y(t)) \sim (kB(t)^{-1}\,,\, k^{-1}A(t)^{-1}) \quad \text{as} \quad t \to \infty\,. \tag{1.15}$$

However, there is no known research work related to asymptotic forms of solutions of (1.1) decaying to a point other than the origin $(0,0)$, like (1.12) and (1.13) for (1.4) with constant coefficients.

The notations we use in this paper are standard. Positive constants will be denoted by C and will change from line to line.

2 Results

We will give asymptotic forms of solutions of (1.1) decaying to a point on the x-axis and y-axis.

Theorem 2.1 *Let E, M and N be constants. Assume that $a(t)$ and $b(t)$ satisfy* (1.2) *and*

$$\lim_{t\to\infty} \frac{a(t)}{b(t)} = E > 0\,. \tag{2.1}$$

Then, we have the following :

(i) *For $M > 0$, every solution $(x(t), y(t))$ of* (1.1) *decaying to $(M, 0)$ has the asymptotic form*

$$(\log(x(t) - M)\,,\, \log y(t)) \sim (-MB(t)\,,\, -(M/E)A(t)) \quad \textit{as} \quad t \to \infty\,. \tag{2.2}$$

(ii) *For $N > 0$, every solution $(x(t), y(t))$ of* (1.1) *decaying to $(0, N)$ has the asymptotic form*

$$(\log x(t)\,,\, \log(y(t) - N)) \sim (-ENB(t)\,,\, -NA(t)) \quad \textit{as} \quad t \to \infty\,. \tag{2.3}$$

Proof. (i) Let $M > 0$ and $(x(t), y(t)) \to (M, 0)$ as $t \to \infty$. By L'Hospital's rule, we have from (2.1) that

$$\begin{aligned} \lim_{t\to\infty} \frac{x(t) - M}{y(t)} &= \lim_{t\to\infty} \frac{x'(t)}{y'(t)} = \lim_{t\to\infty} \frac{-a(t)x(t)y(t)}{-b(t)x(t)y(t)} \\ &= \lim_{t\to\infty} \frac{a(t)}{b(t)} = E\,, \end{aligned} \tag{2.4}$$

and hence, we obtain from (2.1) and (2.4) that

$$\begin{aligned} \lim_{t\to\infty} \frac{\log(x(t) - M)}{-MB(t)} &= \lim_{t\to\infty} \frac{(x(t) - M)^{-1}x'(t)}{-Mb(t)} \\ &= \lim_{t\to\infty} \frac{(x(t) - M)^{-1}(-a(t)x(t)y(t))}{-Mb(t)} \\ &= \lim_{t\to\infty} \frac{a(t)}{b(t)} \frac{y(t)}{x(t) - M} \frac{x(t)}{M} = 1\,, \end{aligned}$$

which implies $\log(x(t)-M) \sim -MB(t)$ as $t \to \infty$.

On the other hand, by L'Hospital's rule again, we have from (2.1) that

$$\lim_{t\to\infty} \frac{\log y(t)}{-(M/E)A(t)} = \lim_{t\to\infty} \frac{y(t)^{-1}y'(t)}{-(M/E)a(t)} = \lim_{t\to\infty} \frac{y(t)^{-1}(-b(t)x(t)y(t))}{-(M/E)a(t)}$$
$$= E \lim_{t\to\infty} \frac{b(t)}{a(t)} \frac{x(t)}{M} = 1\,,$$

which implies $\log y(t) \sim -(M/E)A(t)$ as $t \to \infty$.

(ii) Next, let $N > 0$ and $(x(t), y(t)) \to (0, N)$ as $t \to \infty$. By L'Hospital's rule, we have from (2.1) that

$$\lim_{t\to\infty} \frac{\log x(t)}{-ENB(t)} = \lim_{t\to\infty} \frac{x(t)^{-1}x'(t)}{-ENb(t)} = \lim_{t\to\infty} \frac{x(t)^{-1}(-a(t)x(t)y(t))}{-ENb(t)}$$
$$= \frac{1}{E} \lim_{t\to\infty} \frac{a(t)}{b(t)} \frac{y(t)}{N} = 1\,,$$

which implies $\log x(t) \sim -ENB(t)$ as $t \to \infty$.

On the other hand, by L'Hospital's rule again, we have from (2.1) that

$$\lim_{t\to\infty} \frac{x(t)}{y(t)-N} = \lim_{t\to\infty} \frac{x'(t)}{y'(t)} = \lim_{t\to\infty} \frac{a(t)}{b(t)} = E\,, \tag{2.5}$$

and hence, we obtain from (2.1) and (2.5) that

$$\lim_{t\to\infty} \frac{\log(y(t)-N)}{-NA(t)} = \lim_{t\to\infty} \frac{(y(t)-N)^{-1}y'(t)}{-Na(t)}$$
$$= \lim_{t\to\infty} \frac{(y(t)-N)^{-1}(-b(t)x(t)y(t))}{-Na(t)}$$
$$= \lim_{t\to\infty} \frac{b(t)}{a(t)} \frac{x(t)}{y(t)-N} \frac{y(t)}{N} = 1\,,$$

which implies $\log(y(t)-N) \sim -NA(t)$ as $t \to \infty$. □

Theorem 2.2 *Let M be a constant. Assume that $a(t)$ and $b(t)$ are of class C^1 and satisfy* (1.2) *and*

$$\left(\frac{a(t)}{b(t)}\right)' \le 0 \quad \textit{for large } t\,. \tag{2.6}$$

Then, for $M > 0$, every solution $(x(t), y(t))$ of (1.1) *decaying to $(M, 0)$ has*

$$x(t) - M = O(e^{-MB(t)})\,. \tag{2.7}$$

In addition, if there exists a positive constant K such that

$$\lim_{t\to\infty} \frac{a(t)^2 e^{MB(t)}}{b(t)^2 e^{KA(t)}} = const > 0\,, \tag{2.8}$$

then

$$y(t) = O(e^{-KA(t)})\,. \tag{2.9}$$

Proof. Since $x(t) \to M$ as $t \to \infty$ and $(-x(t))' = (a(t)/b(t))(-y(t))'$, it follows that

$$\begin{aligned} x(t) - M &= \int_t^\infty (-x(s))'\,ds = \int_t^\infty \frac{a(s)}{b(s)}(-y(s))'\,ds \\ &= \frac{a(t)}{b(t)}y(t) + \int_t^\infty \left(\frac{a(s)}{b(s)}\right)' y(s)\,ds \end{aligned}$$

for large t, and from (2.6) that there exists $t_1 > 0$ such that

$$y(t) \geq \frac{b(t)}{a(t)}(x(t) - M) \quad \text{for } t \geq t_1\,.$$

Then we have

$$x'(t) = -a(t)x(t)y(t) \leq b(t)x(t)(x(t) - M)$$

for $t \geq t_1$. Solving this differential inequality of a separeble type on $[t_1, t]$, we obtain

$$\frac{1}{M}\log\frac{x(t)}{x_1}\frac{x_1 - M}{x(t) - M} \geq B(t) - B_1$$

and

$$x(t) \leq \frac{M}{1 - (1 - M/x_1)e^{-M(B(t)-B_1)}} \leq Ce^{-MB(t)}\,,$$

where we use symbols $x_1 = x(t_1)$ and $B_1 = B(t_1)$, and hence,

$$x(t) - M \leq \frac{(1 - M/x_1)e^{-M(B(t)-B_1)}}{1 - (1 - M/x_1)e^{-M(B(t)-B_1)}} \leq Ce^{-MB(t)} \tag{2.10}$$

for $t \geq t_1$, which implies (2.7).

On the other hand, since $y(t) \to 0$ as $t \to \infty$ and $(-y(t))' = (b(t)/a(t))\cdot(-(x(t) - M))'$, it follows that

$$\begin{aligned} y(t) &= \int_t^\infty (-y(s))'\,ds = \int_t^\infty \frac{b(s)}{a(s)}(-(x(s) - M))'\,ds \\ &= \frac{b(t)}{a(t)}(x(t) - M) + \int_t^\infty \left(\frac{b(s)}{a(s)}\right)'(x(s) - M)\,ds \end{aligned} \tag{2.11}$$

for large t. Here, since it follows from (2.8) that

$$0 \le \frac{b(t)}{a(t)} e^{-MB(t)} = \frac{a(t)}{b(t)} \frac{b(t)^2}{a(t)^2} e^{-MB(t)} \le Ce^{-KA(t)} \quad \text{for larte } t\,, \tag{2.12}$$

and from $(b(t)/a(t))' = -(b(t)/a(t))^2(a(t)/b(t))' \ge 0$ for large t and (2.8) that

$$\begin{aligned} 0 &\le \int_t^\infty \left(\frac{b(s)}{a(s)}\right)' e^{-MB(s)}\, ds \\ &= -\frac{b(t)}{a(t)} e^{-MB(t)} + \int_t^\infty \frac{b(s)}{a(s)} Mb(s) e^{-MB(s)}\, ds \\ &\le -\frac{b(t)}{a(t)} e^{-MB(t)} + C\int_t^\infty Ka(s) e^{-KA(s)}\, ds \\ &= -\frac{b(t)}{a(t)} e^{-MB(t)} + C\int_t^\infty (-e^{-KA(s)})'\, ds \\ &\le Ce^{-KA(t)} \quad \text{for large } t\,, \end{aligned} \tag{2.13}$$

we obtain from (2.10)–(2.13) that

$$y(t) \le Ce^{-KA(t)} + Ce^{-KA(t)} \quad \text{for large } t\,,$$

which implies (2.9). □

Remark. (i) When $a(t) = \alpha > 0$ and $b(t) = \beta > 0$, we see that $(a(t)/b(t))' = 0$ and the limit value of (2.8) is $\alpha^2/\beta^2 > 0$ by taking $K = M\beta/\alpha$.

(ii) When $a(t) = (1+t)^{-1}$ and $b(t) = (e+t)^{-1}$, we see that $(a(t)/b(t))' < 0$ and the limit value of (2.8) is $e^{-M} > 0$ by taking $K = M$.

By the similar argument of Theorem 2.2 we have the following theorem.

Theorem 2.3 *Let N be a constant. Assume that $a(t)$ and $b(t)$ are of class C^1 and satisfy* (1.2) *and*

$$\left(\frac{b(t)}{a(t)}\right)' \le 0 \quad \text{for large } t\,.$$

Then, for $N > 0$, every solution $(x(t), y(t))$ of (1.1) *decaying to $(0, N)$ has*

$$y(t) - N = O(e^{-NA(t)})\,.$$

In addition, if there exists a positive constant K such that

$$\lim_{t\to\infty} \frac{b(t)^2 e^{NA(t)}}{a(t)^2 e^{KB(t)}} = const > 0\,,$$

then

$$x(t) = O(e^{-KB(t)})\,.$$

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