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ODEs and Mandatory Voting

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Abstract: This paper presents mathematics relevant to the question whether voting should be mandatory. Assuming a static distribution of voters' political beliefs, we model how politicians might adjust their positions to raise their share of the vote. Various scenarios can be explored using our app at `https://centrism.streamlit.app/`. Abstentions are found to have great impact on the dynamics of candidates, and in particular to introduce the possibility of discontinuous jumps in optimal candidate positions. This is an unusual application of ODEs. We hope that it might help engage some students who may find it harder to connect with the more customary applications from the natural sciences.

1 Introduction

Mandatory voting has been adopted by over 20 of the world's democracies, including Brazil and Australia, and has been discussed in the United States as well [4, 5, 6]. Mathematical modeling may help understand its effects. To accompany this article, we have created a web app hosted at `https://centrism.streamlit.app/` that allows students to experiment with problem parameters and outcomes.[1] The app, though built with Python, has a friendly graphical user interface that requires no knowledge of coding. As an intermediate step between the app and the pure Python scripts, we have also prepared a public Python

[1]The app will take some time to load if you have never used it, or if you have not used it in 24 hours. Please be patient.

workbook in Google Colab that walks through the models discussed in this paper, available at https://annahaensch.com/centrism.html. This workbook is appropriate for a student who has some familiarity with coding. For the curious and Python proficient student the code underlying the web app (as well as sample Python scripts) is also available in the public GitHub repository https://github.com/annahaensch/Centrism. The web app and the paper have the same general structure, but the paper has answers to the discussion questions and additional homework problems for the students.

The organization of the paper is as follows: in Section 2 we introduce the model of voters' beliefs. We assume that the distribution of voters' beliefs does not change over time. In Section 3 we start with a scenario where every voter casts a vote, which is what *nearly* happens when voting is mandatory. Under this assumption, we explore which political candidate wins when (1) candidates' positions are fixed, or (2) candidates are given time to optimize their positions. In the latter case, we assume that each candidate position is governed by an ordinary differential equation, the motion is in the direction of increasing vote share, and the speed of motion is proportional to the steepness of the increase. Briefly, the candidates use *continuous steepest ascent* to maximize their share of the vote. Section 4 has the same outline as Section 3, except now allowing for the possibility that voters choose not to vote when neither political candidate is sufficiently appealing. In Section 5 we take a closer look at what happens from an ODE perspective when voters are allowed to abstain, specifically looking at possible discontinuities in the system. Finally, in Section 6 we summarize our results.

Throughout the paper we provide *in-class discussion questions* and *homework problems* to complement the class discussion. We provide our own thoughts on the in-class discussion questions in Appendix A, and sketches of solutions to selected homework problems in Appendix B.

2 Distribution of Voter Beliefs

Throughout the paper, we assume that the political candidates and voters' political beliefs can be represented on a "left-right" axis. We'll start by seeding a population with political beliefs on the left-right spectrum. To keep things simple, we assume that the belief distribution is a *Gaussian mixture*, that is, it is described by a weighted average of Gaussian densities with different means and variances. As an example, Figure 1 shows the average of two Gaussian probability density functions, with means -1 and 1 and both variances equal to 0.5.

In the distribution of Figure 1, there are two "camps" of voters, a "left-wing" and a "right-wing" camp. Very few people hold the most extreme liberal (less than -3) or extreme conservative (greater than 3) views, many people hold views that are fairly liberal (-1.5 to -0.5) or conservative (1.5 to 0.5), and a few people hold moderate views (-0.5 to 0.5).

Discussion Question 2.1. Does a weighted average of Gaussian densities seem like a reasonable representation of the views of U.S. voters?

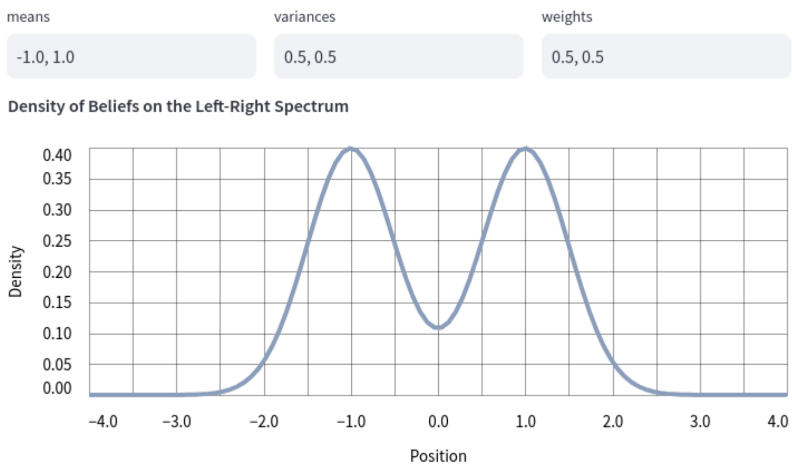

Figure 1: Example of a distribution of voter beliefs

In general, a Gaussian mixture is a density of the form

$$f(x) = \sum_{m=1}^{M} \omega_m \cdot \frac{1}{\sigma_m \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_m}{\sigma_m}\right)^2}, \tag{2.1}$$

with weights $\omega_m \geq 0$ that sum to 1. In our app, you can enter the means and variances that you'd like to use for each of the Gaussian modes. The default is $M = 2$, the smallest value of M for which we find discontinuous dependence of the optimal candidate strategy on voter loyalty (see [3] and compare also Problem 5.2). However, the app allows any M between 1 and 5.

3 When Voting is Mandatory

From now on, we will assume that there are two political candidates L and R. We denote their positions by l and r, and assume $l < r$. In this Section, we assume that voting is mandatory, i.e. everyone votes. As a reminder, countries like Australia and Brazil make voting mandatory. We will first see who wins the election based only on the initial position of the candidates in Subsection 3.1 and then allow the candidates to move in Subsection 3.2.

3.1 Static Candidates

We assume that every person will vote for the candidate whose position is closest to their beliefs. Therefore the left candidate gets all of the votes to the left of the midpoint between l and r, $(l+r)/2$. The vote shares of the left and right candidates, S_L and S_R, respectively, are

$$S_L = \int_{-\infty}^{\frac{l+r}{2}} f(x)dx \quad \text{and} \quad S_R = \int_{\frac{l+r}{2}}^{\infty} f(x)dx \tag{3.1}$$

If $S_L > S_R$, then L wins, and if $S_R > S_L$, then R wins; see Figure 2. In our app, you can use the slider to pick the positions of the left and the right candidate; see Figure 3.

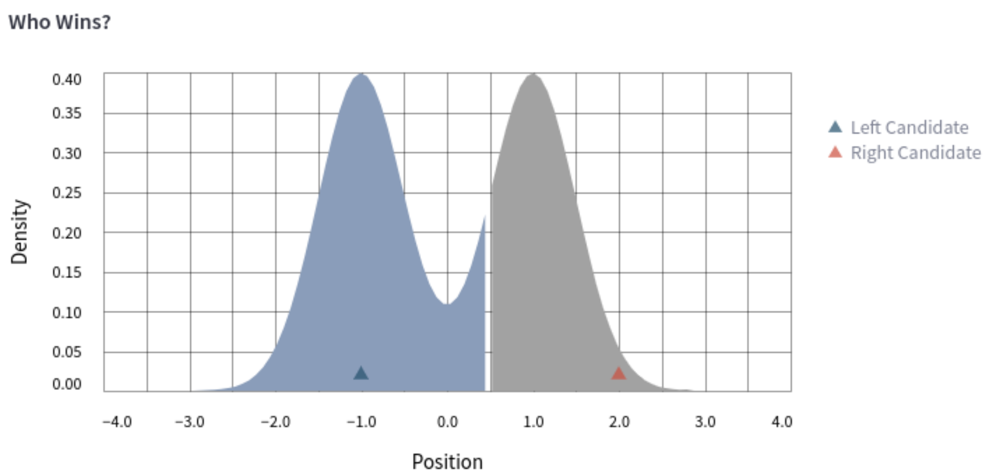

Figure 2: Example of a voter share. Blue shaded area is larger than the gray area, therefore the left candidate wins

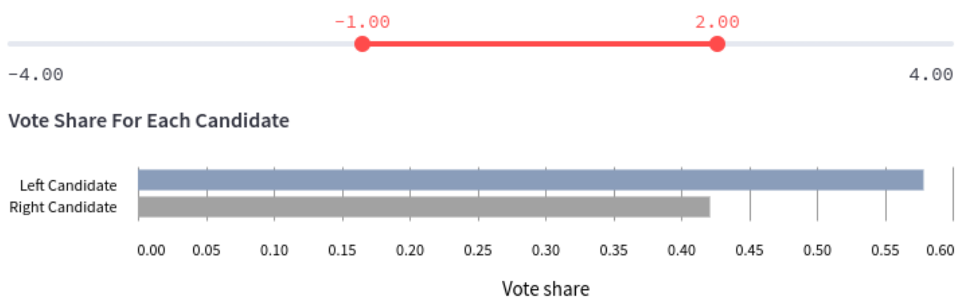

Figure 3: Example of an initial candidate positions

Discussion Question 3.1. This model is making an assumption about what happens when a country adopts a mandatory voting policy. Please explain this assumption in your own words and discuss possible violations.

3.2 Dynamic Candidates

If the right candidate has a fixed position, the left candidate can increase their vote share by moving closer to the right candidate. We will assume that candidates aim to maximize their vote share, although this does not perfectly reflect political reality [11]. To see why it is always in the left candidate's interest to move toward the right candidate, we compute the vote shares S_L and S_R as functions of the left candidate position, l, with $r = 2$ fixed; see Figure 4. As soon as S_L rises above the dashed line, L has more than 50% of the votes and therefore wins.

Discussion Question 3.2. Why does the proportion of the population voting for the left candidate get so close to one, but never quite reach it?

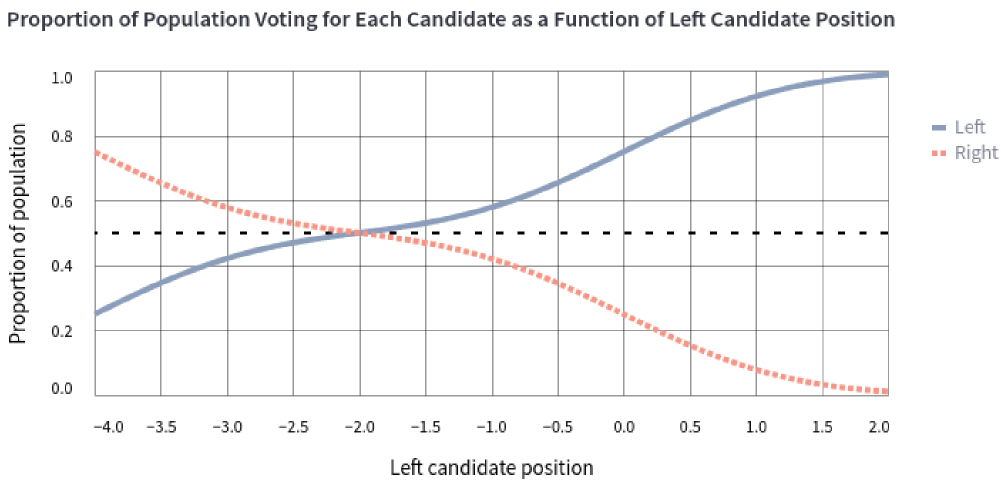

Figure 4: Proportion of population voting for each candidate as a function of left candidate position

Discussion Question 3.3. Do the proportions S_L and S_R always sum to one? Why or why not?

Discussion Question 3.4. Where does the left candidate have to be on the spectrum in order to win the election?

From the previous discussion we've seen that candidates might want to change their position on the political spectrum in order to get more votes. Some candidates will do this more eagerly than others. Let's include a measure of **candidate opportunism** into our model, and the larger this measure is, the more eagerly (i.e. more rapidly) a candidate will move to enlarge their share of the vote. If both candidates are opportunistic, they might for instance follow a system of ordinary differential equations of the form

$$\frac{dl}{dt} = \alpha \frac{\partial S_L}{\partial l} \quad \text{and} \quad \frac{dr}{dt} = \beta \frac{\partial S_R}{\partial r}, \tag{3.2}$$

where t is time. If S_L and S_R are given by (3.1), then (3.2) means

$$\frac{dl}{dt} = \frac{\alpha}{2} f\left(\frac{l+r}{2}\right) \quad \text{and} \quad \frac{dr}{dt} = -\frac{\beta}{2} f\left(\frac{l+r}{2}\right).$$

This is called *steepest ascent* — each candidate moves in the direction in which their vote share increases, given the other candidate's position. The parameters $\alpha \geq 0$ and $\beta \geq 0$ measure the degrees of opportunism of L and R, respectively. In our app you can use sliders to pick values for α and β. We will pick values 1 and 0.2 for α and β respectively; see Figure 5. We then plot the candidates' movements governed by eqs. 3.2; see Figure 6.

The candidates will eventually meet at some collision point. In Figure 7 we show the vote shares for each candidate at this point.

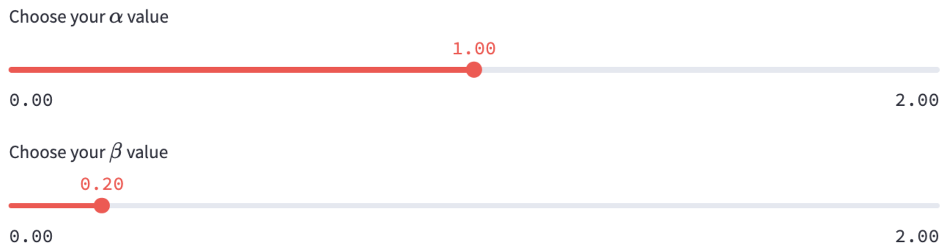

Figure 5: Example of values for α and β in our app.

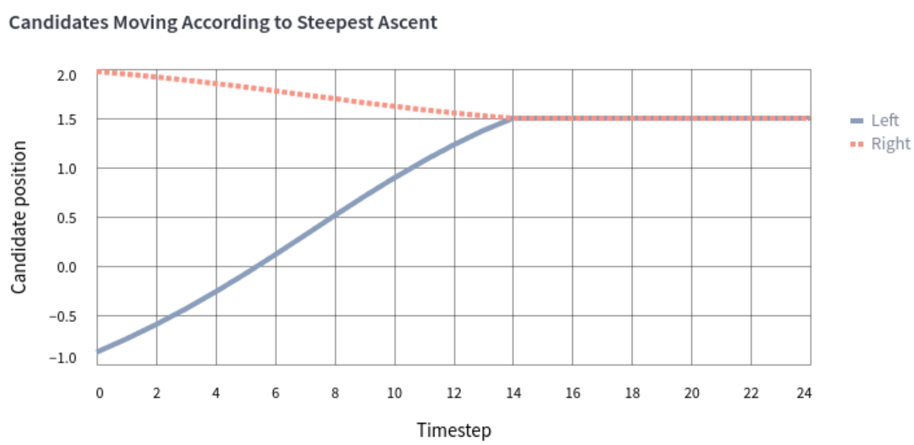

Figure 6: Candidates moving according to steepest ascent

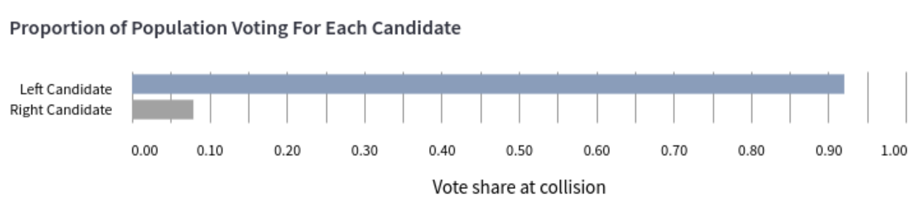

Figure 7: Proportion of population voting for each candidate

Discussion Question 3.5. Suppose that the left candidate is far more opportunistic than the right candidate. Who will win? Which position will they end up at?

Discussion Question 3.6. Is there ever a way for the less eager candidate to win?

Discussion Question 3.7. The model proposed here doesn't allow the candidates to cross over each other. When the candidates reach each other, we simply stop simulating, and pretend that both candidates will from then on stay at the position at which they met. We might attempt to allow L to cross over R. Since L, as soon as they cross over R, becomes the more right-leaning candidate, we might define

$$S_L(l,r) = \begin{cases} \int_{-\infty}^{(l+r)/2} f(x)dx & \text{if } l < r, \\ \\ \int_{(l+r)/2}^{\infty} f(x)dx & \text{if } l > r. \end{cases}$$

The mathematical definition of $S_L(l,r)$ for $l = r$ would be a bit unclear. Similarly we would define S_R distinguishing the two cases $l < r$ and $l > r$. Can you make sense of this mathematically? Does it make sense politically?

3.3 Suggested homework questions

Problem 3.8. Suppose f is as in Figure 1. Let $l = -1$ and $r = 1.5$, and assume that $\alpha = \beta = 0$, so the candidates don't change positions with time. Who wins?

Problem 3.9. In the previous problem, let $\alpha = 1$, $\beta = 0$. In a finite amount of time, l will become equal to r. When that happens, we assume that they stop moving. Who wins then?

Problem 3.10. Repeat Problem 3.9 with $\alpha = \beta = 1$.

Problem 3.11. In Figure 6, do l and r meet each other *tangentially* or at a non-zero angle? That is, is $\frac{dl}{dt} = \frac{dr}{dt} = 0$ at the time when l and r meet, or not? (Answer this question using mathematics, not just looking at the plot.)

Problem 3.12. Prove: If $\alpha = \beta = 0$, so the candidate positions don't move, the winner is the candidate whose position is closer to the median. (The median of the voter belief distribution is the number m with $\int_{-\infty}^{m} f(x)dx = \frac{1}{2}$.) This is a special case of the *median voter theorem* [2, 7, 12].

Problem 3.13. (Programming problem) Try out what happens with the definitions (A.1) and (A.2) in Appendix A when the two candidates collide.

Problem 3.14. Would the answer to problem 3.11 change if (A.1) and (A.2) in Appendix A were the definitions of S_L and S_R?

Figure 8: Slider with the set value of $\gamma = 3$

4 When Voting is Not Mandatory

In the United States, voting is not mandatory. If a voter doesn't feel strongly about either candidate they might choose to stay home. Therefore, it's not always in a candidate's best interest to be overly opportunistic, since they might risk alienating their base. To account for this, we'll add a measure of *voter loyalty* to our model. In the voting literature, voter alienation is a more common term; see [13] but also [2] and [12]. A high level of voter loyalty means that a voter is likely to stick with a candidate even as their position drifts, as long as that candidate is still the candidate closest to the voter's position.

We denote by $g(z)$ the probability that a voter will still go vote if the candidate closest to them differs from their views by z. We assume

$$g(z) = e^{-z/\gamma} \tag{4.1}$$

where $\gamma > 0$ is the model parameter measuring voter loyalty. Other choices of g would be possible, but we always take g to be decreasing, differentiable, with $g(0) = 1$. We express the left and right candidate share of votes as a function of position as

$$S_L = S_L(l,r) = \int_{-\infty}^{\frac{l+r}{2}} f(x)g(|l-x|)dx \quad \text{and} \quad S_R = S_R(l,r) = \int_{\frac{l+r}{2}}^{\infty} f(x)g(|r-x|)dx. \tag{4.2}$$

Again, L wins if $S_L > S_R$, and R wins if $S_R > S_L$.

Notice that as $\gamma \to \infty$, $g(z)$ (for a fixed z) tends to 1. Therefore the factors $g(|l-x|)$ and $g(|r-x|)$ in (4.2) disappear in the limit as $\gamma \to \infty$, and Section 3 can be said to be about the special case $\gamma = \infty$. In other words, we think of mandatory voting as the special case of (enforced) infinite loyalty. This is an idealization of course, since in reality voter participation is not a hundred percent even when voting is mandatory.

4.1 Static Candidates

Using the slider in our app, you can pick a value for γ. Remember, a greater value of γ means that voters are more *loyal*, i.e., more likely to stick with their candidate even as the candidate moves away from them. We will go with the default value $\gamma = 3$; see Figure 8. In Figure 9 we see the proportion of the population voting for each candidate as a function of position with finite voter loyalty.

Discussion Question 4.1. Do the values of the two functions shown in Figure 9 always sum to 1?

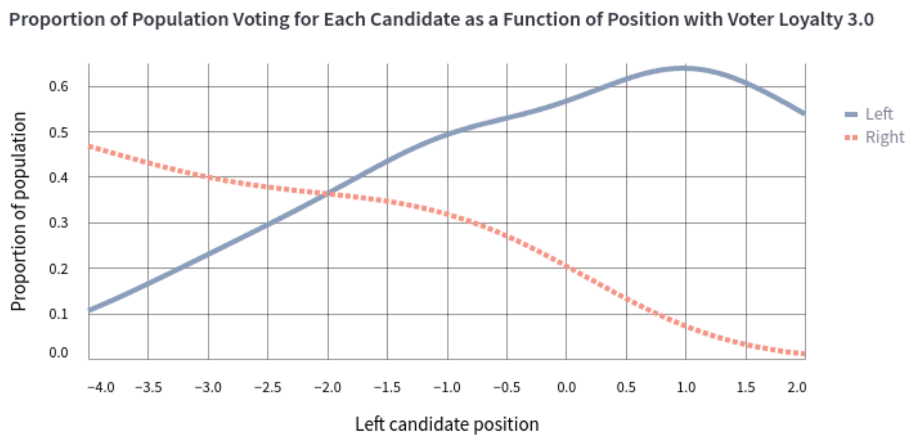

Figure 9: Proportion of population voting for each candidate as a function of position with voter loyalty $\gamma = 3$.

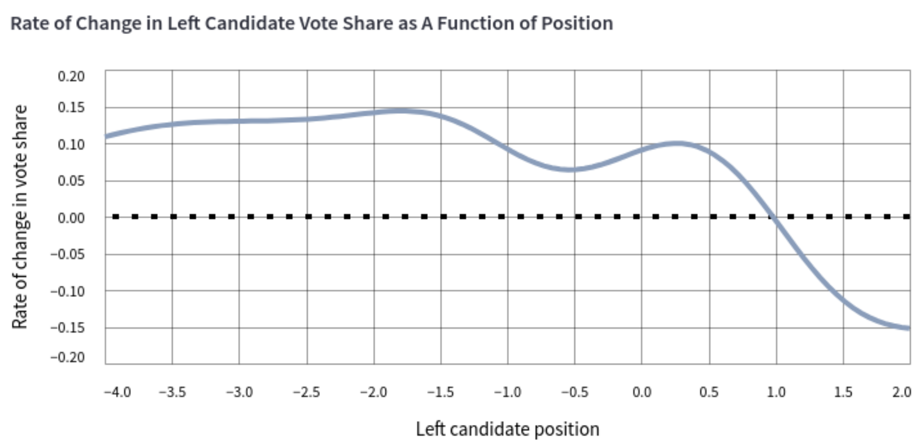

Figure 10: $\frac{\partial S_L}{\partial l}$, as a function of position when $\gamma = 3.0$

Discussion Question 4.2. If the right candidate stands at $r = 2$, where does the left candidate have to be on the spectrum in order to win the election? Why?

Discussion Question 4.3. Why is it possible for the left candidate to win with less that 50% of the vote?

4.2 Dynamic Candidates

Next, we will look at how the share of votes changes as a function of candidate position, with the introduction of finite voter loyalty, assuming again equations 3.2. We see in Figure 10 the derivative of S_L with respect to l, keeping $r = 2$ fixed.

Discussion Question 4.4. What does it mean for points on this graph to be below zero?

Discussion Question 4.5. Looking at this graph, where are the fixed points and which of these are stable?

4.3 Homework problems

Problem 4.6. Assume the distribution of Figure 1. Assume people easily abandon their candidates, so γ is small; try $\gamma = 0.2$ for instance. Suppose that the right candidate is positioned at $r = 1$. Where should the left candidate position themselves in order to win the election?

Problem 4.7. (Programming problem) Pick a different function g that still makes intuitive sense. Then, implement all the steps we have covered so far in the app and compare with $g(z) = e^{-z/\gamma}$. Use `https://github.com/annahaensch/Centrism`. for this problem.

Problem 4.8. Assume that S_L is defined as in equation (4.2). Show that for all l and r, $l < r$

$$\frac{\partial S_L}{\partial l}(l, r) = -\frac{1}{2} f\left(\frac{l+r}{2}\right) g\left(\frac{r-l}{2}\right) + \int_{-\infty}^{(l+r)/2} f'(x) g(|l-x|) dx \tag{4.3}$$

The point is that the derivatives that appear in (3.2) can be evaluated analytically. (Of course, the integral in (4.3) must still be evaluated numerically in the numerical solution of (3.2).)

Hints: Break the integral in (4.2) into two pieces, one from $-\infty$ to l and one from l to $(l+r)/2$. Use the fundamental theorem of calculus, the chain rule, and integration by parts. Assume that $g(0) = 1$, g is differentiable and decreasing, and $f(x) \to 0$ as $x \to -\infty$.

Problem 4.9. (Real life examples) Can you think of political candidates who have appeared to adjust their positions to enhance their chances of winning elections?

Problem 4.10. (Public Policy) What are some public policy initiatives that could increase voter participation?

5 Discontinuities

Changes in the political environment can have a discontinuous impact on candidate strategy. In Figure 11, we illustrate this point with an example: Here L starts at $l = -1$ and moves, but R sits still at $r = 2$. The voter distribution is that of Figure 1. We plot the final position of L as a function of γ. As you can see in Figure 11, slight changes in γ can cause abrupt substantial changes in candidate movements [3].

To understand how this discontinuity arises, look at Figures 10 (where $\gamma = 3$) and 12 (where $\gamma = 1.7$). As γ rises, the “knee" in Figure 12 rises, and when it rises above the horizontal axis, the final position of L abruptly jumps to the fixed point on the right. This is an example of a *saddle-node bifurcation* [1, 15]. In a saddle-node bifurcation, two fixed points collide as a parameter is changed, and annihilate each other. Thinking of the parameter change in the opposite direction, one might also say that two fixed points appear “out of the blue sky", and the bifurcation is then also called a *blue sky bifurcation.*

If you want to try to generate this graph with different values, check out the script `optimal_left_position.py` in our Github repository.

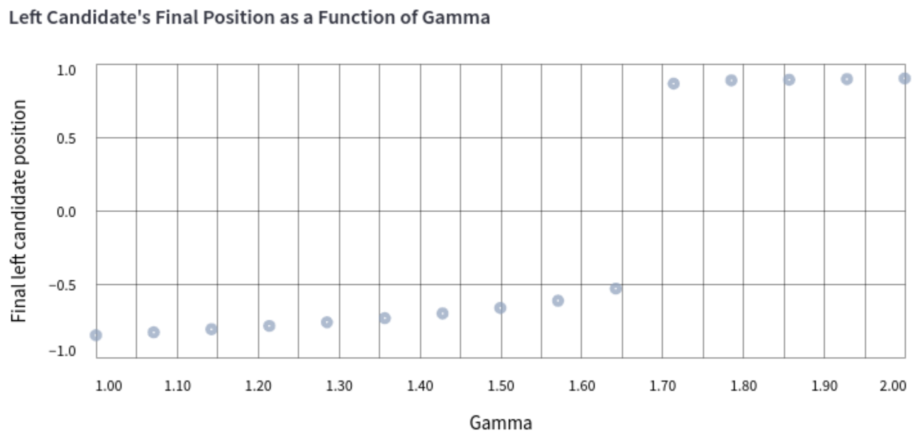

Figure 11: Left candidate's final position as a function of gamma

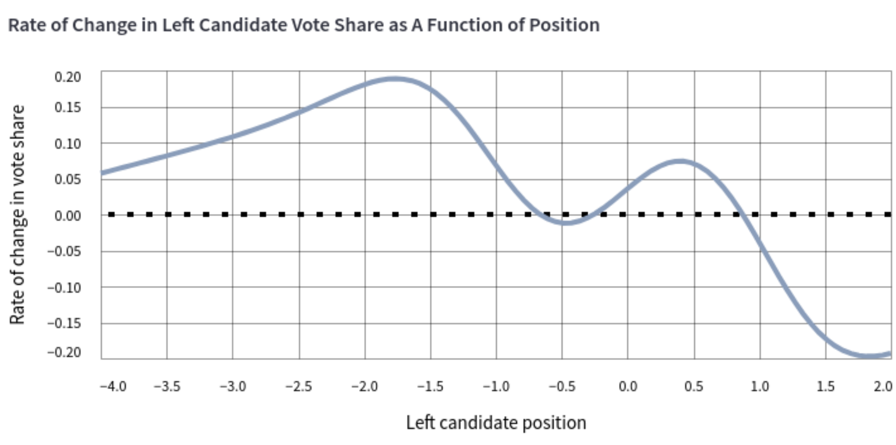

Figure 12: Rate of change in left candidate vote share as a function of position with $\gamma = 1.7$

5.1 Homework problem

Problem 5.1. To illustrate how saddle-node bifurcations give rise to discontinuities, consider the simple example

$$\frac{dx}{dt} = (x^2 - \gamma)(1 - x) \tag{5.1}$$

where $\gamma \in (-1, 1)$ is a parameter. (a) Plot the right-hand side of (5.1) as a function of x for $\gamma = -0.5$, $\gamma = -0.1$, $\gamma = 0$, $\gamma = 0.1$, $\gamma = 0.5$. (b) Let $x(0) = -1$. Compute and plot $x_\infty = \lim_{x\to\infty} x(t)$ as a function of γ. (Hint: No calculations to speak of are needed here.)

Problem 5.2. (Programming problem) Start by reproducing our Figure 11. Then, run the code with a unimodal f. How does the left candidate's final position as a function of gamma change now? Can you give an intuitive explanation of the result?

6 Discussion

Mandatory voting would raise the voter loyalty γ. Our model suggests that in a two-party system, that may incentivize politicians to adopt centrist, compromising positions, in agreement with [14]. Surprisingly, the optimal candidate position can depend on γ discontinuously, jumping to the center abruptly as γ increases.

However, mandatory voting would surely affect f as well, since with voluntary voting, some population groups tend to participate less than others [9]. We note that γ and f can also be affected by opening hours of polling stations, get-out-the-vote efforts [8], making Election Day a public holiday, etc. Mathematics cannot tell us whether such public policy measures are desirable, but it can shed light on their potential consequences.

7 Acknowledgment

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Appendix A: Some thoughts on the discussion questions

Discussion Question 2.1. Although US politics are often said to be deeply polarized, the distribution of ideological views on the left-right spectrum may still best be viewed as unimodal [10]. Most people are ideologically moderate and hold a mix of liberal and conservative views. Ideology within each major party is approximately normally distributed, where the Democratic party has a more liberal mean than the Republican party, but there is a fair amount of overlap between the two distributions.

Nonetheless, many analysts are interested in how polarization affects political systems, and how candidates would respond to an electorate, or a group of potential donors, that was more polarized than the US public.

Discussion Question 3.1. The model assumes that everyone in the population of eligible voters actually casts a ballot if they are required to by law, and that they vote for the candidate who has an ideological position most similar to their own position.

In fact, a much higher proportion of the population tends to vote in countries where voting is mandatory, but not 100%. Some mandatory voting laws are not enforced, and often penalties for not voting are minimal. People can also typically avoid penalties by casting blank ballots if they prefer not to choose a candidate. Voters do typically select candidates that are closer to them ideologically, but may have reasons for preferring a different candidate. In general, it is useful to think about what would happen in a simple setting even if it does not perfectly describe the real world.

Discussion Question 3.2. The left candidate maximizes their votes by positioning themselves just barely to the left of the right candidate, thereby winning the support of all voters to the left of the right candidate, but still not the support of those voters who are to the right of the right candidates.

Discussion Question 3.3. They do. When voting is mandatory, we assume that everyone votes for one of the two candidates.

Discussion Question 3.4. If the distribution of voter views is symmetric and the right candidate is fixed at position 2, the left candidate will win as soon as they position themselves to the right of -2. They just need to be slightly more moderate than the right candidate.

Discussion Question 3.5. If the left candidate is opportunistic, they are willing to move faster toward positions that will get them more votes. They will quickly start adopting more conservative positions. This will allow them to win the election, but they will end up adopting conservative positions to do so. (They will adopt more conservative positions than they would need to adopt in order to win.)

Discussion Question 3.6. The less eager candidate can only win if they start off at a more moderate position closer to the center. Even then they might lose if their opponent is sufficiently opportunistic and not too much more ideologically extreme.

Discussion Question 3.7. S_L and S_R would now in general be discontinuous at $l = r$, which makes the meaning of the partial derivatives in (3.2) questionable. If a politician very slightly to the left of their opponent shifts a bit to the right and is now very slightly to the right of their opponent, the voters almost certainly would *not* suddenly all swap their allegiances. Faced with two candidates of very similar views, the voters will not decide based on who is closest to their views any more, but based on other factors — who gives the more rousing speeches, who has the better haircut, and so on.

One might overcome some of these issues by defining S_L and S_R like this:

$$S_L(l,r) = \int_{-\infty}^{\infty} f(x)s((x-l)^2-(x-r)^2)dx, \qquad (A.1)$$

$$S_R(l,r) = \int_{-\infty}^{\infty} f(x)s((x-r)^2-(x-l)^2)dx, \qquad (A.2)$$

where $s(z)$, $z \in \mathbb{R}$, is a function that is about 1 for $z < 0$ and about 0 for $z > 0$, but transitions from 1 to 0 smoothly, say with $s(0) = 1/2$. Now (3.2) is defined for all l and r.

Discussion Question 4.1. They do not, since not everybody votes.

Discussion Question 4.2. With a symmetric distribution of ideology in the population and an equal level of loyalty for liberal and conservative members of the population, the left candidate still needs to be to the right of -2 in order to win.

Discussion Question 4.3. Candidates only need a majority of votes cast, not the votes of a majority of eligible voters. If many voters abstain, candidates don't need a very large share of eligible voters to win.

Discussion Question 4.4. The candidate is losing votes as they move to the right.

Discussion Question 4.5. Fixed points are points at which the graph crosses the dashed line. Fixed points are stable if the graph has a negative slope (moving from positive values to negative values when going left to right) at the fixed point. Moving a bit to the left, the candidate would prefer to come back to the fixed point because they have more votes to gain. Moving a bit to the right, the candidate would start to lose votes and would want to come back to the fixed point.

Appendix B: Sketches of solutions for selected problems

Of course one learns by thinking things through, not by reading other people's answers. We don't want to spoil that, but we do want to help a reader who feels that they need a bit of help to get started. We therefore sketch answers to some of the homework questions here; don't read our sketches unless you feel you need to.

Problem 3.10. L will win. The reason is that the situation is completely symmetric, except that L has the advantage of starting closer to the median of the distribution (which is 0 in Figure 1) than R. Can you state this argument more precisely?

Problem 3.11. They meet non-tangentially. Think about what are $\frac{dl}{dt}$ and $\frac{dr}{dt}$ at the moment when l and r meet.

Problem 3.12. Let m be the median of the distribution. If $m < (l+r)/2$, then

$$S_L = \int_{-\infty}^{(l+r)/2} f(x)dx > \int_{-\infty}^{m} f(x)dx = \frac{1}{2}.$$

(We assume $f(x) > 0$ everywhere, which is the case for a Gaussian mixture.) So if $m < (l+r)/2$, then L wins. Similarly if $m > (l+r)/2$, then r wins. Now convince yourself that $m < (l+r)/2$ means precisely the same as $|m-l| < |m-r|$, and $m > (l+r)/2$ means $|m-r| < |m-l|$.

In the median voter theorem [2, 7, 12], the assumption is, as in this paper, that voter and candidate views are situated on a one-dimensional axis, and voters support the candidate closest to them. The general statement of the theorem is that among $n \geq 2$ candidates, the one closest to the median of the voter distribution is the *Condorcet candidate*, that is, would beat any of the other candidates in a two-person runoff.

Problem 3.14. The derivatives $\frac{\partial S_L}{\partial l}$ and $\frac{\partial S_R}{\partial r}$ are

$$\frac{\partial S_L}{\partial l} = -2\int_{-\infty}^{\infty} f(x)s'((x-l)^2-(x-r)^2)(x-l)dx$$

and

$$\frac{\partial S_R}{\partial r} = -2\int_{-\infty}^{\infty} f(x)s'((x-l)^2-(x-r)^2)(x-r)dx.$$

This implies (make sure you see why) that for $l = r$,

$$\frac{\partial S_L}{\partial l} = \frac{\partial S_R}{\partial r} = -2s'(0)(\mu - l) = -2s'(0)(\mu - r),$$

where $\mu = \int_{-\infty}^{\infty} f(x)xdx$ is the mean voter position. What does this imply about the question? Do equations (3.2) drive l and r away from μ, or towards μ?

Problem 4.6. Think about which votes L will get when $l = -1$, and which votes L will get when l is nearly 1. Why is it better for l to be positioned at -1 than at 1?

Problem 4.8. This is a somewhat sophisticated calculus exercise. Here are some general calculus formulas that are useful for it:

$$\frac{d}{dl}\int_a^b h(x,l)dx = \int_a^b \frac{\partial h}{\partial l}(x,l)dx$$

(differentiation under the integral sign),

$$\frac{d}{dl}\int_{a(l)}^{b(l)} h(x)dx = h(b(l))b'(l) - h(a(l))a'(l)$$

(this follows from the fundamental theorem of calculus and the chain rule), and the combination of these two:

$$\frac{d}{dl}\int_{a(l)}^{b(l)} h(x,l)dx = h(b(l),l)b'(l) - h(a(l),l)a'(l) + \int_{a(l)}^{b(l)} \frac{\partial h}{\partial l}(x,l)dx.$$

Problem 4.10. How about making Election Day a Sunday?

Problem 5.1. (b) $x_\infty = 1$ if $-1 < \gamma < 0$, and $x_\infty = -\sqrt{\gamma}$ if $0 \leq \gamma < 1$. So the dependence of x_∞ on γ is discontinuous at $\gamma = 0$.

Problem 5.2. There is no discontinuity with the unimodal f. The intuitive explanation is as follows. Suppose first the distribution is bimodal. There is therefore a substantial "left-wing camp". As L moves to the right, they lose left-wing votes, but gain right-wing votes. For large γ, the gain always outweighs the loss until l reaches r. As γ drops, a window of values of l is suddenly created in which L is best off moving to the left to recover some of the lost left-wing votes. This is the reason for the discontinuity. It requires a substantial "left-wing camp" of voters, in other words, a bimodal distribution.