

1-4-2024

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Recommended Citation

Cavagnaro, Catherine (2024) "Modeling Aircraft Takeoffs," *CODEE Journal*: Vol. 17, Article 9.
Available at: <https://scholarship.claremont.edu/codee/vol17/iss1/9>

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Modeling Aircraft Takeoffs

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Keywords: aircraft takeoff distance, ordinary differential equations

Manuscript received on July 12, 2023; published on January 4, 2024.

Abstract: Real-world applications can demonstrate how mathematical models describe and provide insight into familiar physical systems. In this paper, we apply techniques from a first-semester differential equations course that shed light on a problem from aviation. In particular, we construct several differential equations that model the distance that an aircraft requires to become airborne. A popular thumb rule that pilots have used for decades appears to emanate from one of these models. We will see that this rule does not follow from a representative model and suggest a better method of ensuring safety during takeoff. Aircraft safety is definitely a matter of public concern, although it is the FAA (Federal Aviation Administration) that makes the regulations.

1 Introduction

The Federal Aviation Administration requires pilots to perform several pre-flight actions to ensure safety of flight. One of those actions is to ensure that the runway at the departure airport is sufficiently long for a safe departure. Once the pilot has made such a determination, she should have a plan to abort the takeoff and bring the aircraft to a stop in the remaining runway if performance proves insufficient. In this paper, we consider a popular rule in the aviation literature for aborting a takeoff.

2 The Fundamentals of Takeoff Performance

Forces

Four fundamental forces affect the flight of an aircraft: *lift*, *weight*, *thrust* and *drag*, Figure 1. The engine (propeller or jet) generates a *thrust* force approximately parallel to the flight path. Air resistance opposes that motion and creates a *drag* force in the opposite direction. As the plane moves through the air, flow across the curved top of the wing is accelerated and the resulting difference in velocity (and hence, pressure) compared with the bottom of the wing creates a net *lift* force that acts in the vertical direction and

must overcome the *weight* force due to gravity that acts on the aircraft. Before becoming airborne, the runway provides the vertical or *normal* force that counteracts weight.

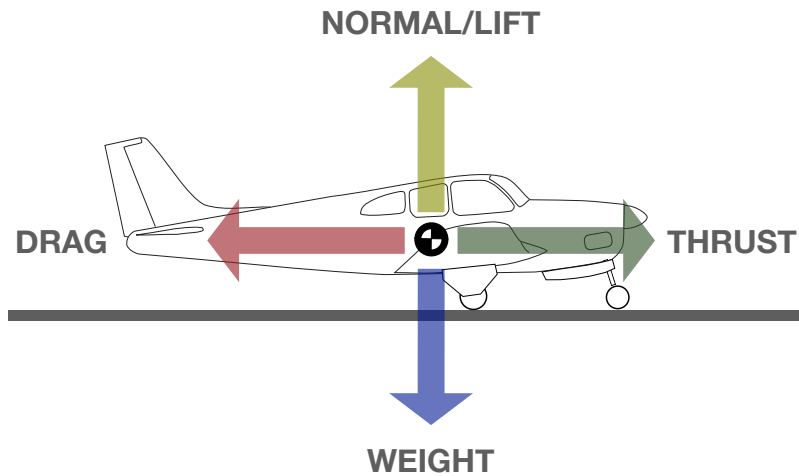


Figure 1: An aircraft on the runway is subject to four fundamental forces: thrust, drag, weight and the normal force from the ground. After lift-off, the lift created by the airplane provides the force that opposes aircraft weight.

Takeoff Distance

As the airplane prepares to fly, the runway provides the normal force that opposes the gravitational force. As the airplane gains velocity during its takeoff run, more of that upward force is borne by the wings and, at the point of liftoff, the runway no longer supplies that force. Each airplane has a *rotation speed* v_r at which the pilot pulls back on the control yoke and the aircraft becomes airborne. Takeoff distance s_r is the horizontal distance from the start of the takeoff roll to the point along the runway at which the airplane leaves the ground, Figure 2.

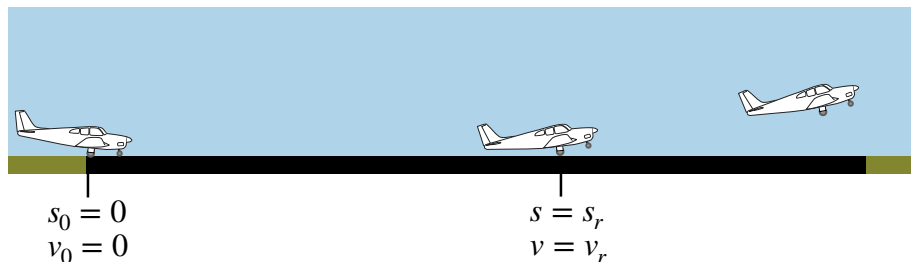


Figure 2: We assume that the aircraft begins the takeoff roll at the beginning of the runway where distance, s and velocity v are both zero. Once the aircraft achieves its rotation speed v_r , the pilot pulls back on the control yoke which makes the nose of the aircraft rotate upward and the aircraft becomes airborne. The distance s_r is called the takeoff distance.

Factors that Affect Takeoff Distance

Takeoff distance can vary wildly with changes in runway or atmospheric conditions, aircraft loading or pilot technique. Increases in altitude, temperature, humidity, a tailwind component, or aircraft weight will all cause takeoff distance to increase. Low atmospheric pressure, poor runway quality or sloppy pilot technique can also extend the ground roll. On the other hand, a headwind component always reduces takeoff distance so pilots will typically take off into the wind. It is the pilot's responsibility to consider all these factors to ensure that the departure runway is long enough for safe operations. Ideally, the runway should provide distance for the takeoff run as well as, assuming an anomaly during the takeoff roll, for the pilot to bring the aircraft to a halt in the remaining runway, should she elect to abort the takeoff. In a typical year, many aviation accidents occur due to a paucity of such planning.

Abort Plans

Guidance about generating abort plans abound in pilot literature. For many decades, authors who write on aviation safety as well as the Federal Aviation Administration have promoted the "70%-50%" rule: If the aircraft has not achieved 70% of its takeoff speed by the 50% of available runway point, the takeoff should be aborted. While the rule only implies that 70% of rotation speed is necessary at the 50% point, in a practical sense, the pilot who continues the takeoff roll is assuming it is sufficient as well.

There are actually two versions of the 70%-50% rule, the first being the one stated above, Figure 3.

How To Use The 50/70 Rule For Your Next Takeoff

By Colin Cutler | 07/02/2022 | [Previous](#) | [Next](#)





[Photo method](#)

Will you lift off before you're out of runway? It's not a question you want to be asking yourself on your takeoff roll, especially as you approach rotation speed.

So what is the 50/70 rule? **It's a general rule for GA aircraft that says if you haven't reached 70% of your takeoff speed by the time you've reached 50% of the length of the runway, you should abort your takeoff.**

Figure 3: In this version of the 70%-50% rule, the halfway point on the runway serves as the abort point.[2]

The second version of the rule asks the pilot to compute the *estimated takeoff distance* s_e using information in the pilot’s operating handbook for the aircraft and abort the takeoff attempt if 70% of takeoff speed is not achieved by 50% s_e , Figure 4. In either case, the pilot uses aircraft performance during the estimated “first half” of the takeoff roll to predict future performance during the “second half.” We will refer to these as the 70%-50%-1 and 70%-50%-2 rules, respectively, when focusing on just one of the rules.

Pattern Precision

The FAA, General Aviation Joint Steering Committee (GAJSC), and industry agree that regular, structured, proficiency training is perhaps the most effective means of reducing GA accidents. Because the traffic pattern involves nearly all piloting tasks, it is a logical choice for a proficiency training environment. Commitment to precision and consistency in pattern operations will yield operational safety benefits throughout the flight task spectrum.

Predictable Patterns

How many times have you been on approach where you get behind the aircraft a little or are not set up properly, and wind up landing a little long? Does it matter? Yes, it does. Airport traffic pattern operations are an essential part of every flight. But sometimes we take those routine movements for granted, and we can get a little sloppy.

In addition to helping you execute a safe and stable approach, precise pattern flying makes you sharper in other flight procedures. It can also improve your confidence and reassure your passengers. Let’s go back to our example. You may think it doesn’t matter if you land long, but what if your runway is compromised and your landing distance is much shorter? What if your aircraft has a problem and you need to carry out a forced landing?

Preparing for Pattern Precision


It all starts with knowing your aircraft, its performance parameters, and the required research

to determine what sort of takeoff and landing performance you will get.

By documenting these numbers and then comparing them with your actual performance, you will be able to create predictable expectations. Don’t forget to calculate performance based off of your predicted flying weight.

Here’s some rules of thumb to consider when computing your takeoff calculations:

- If you have a fixed pitch prop, add 15% to your calculated takeoff distance for each 1,000 foot increase in density altitude, up to 8,000 feet.
- For constant speed props, add 12% per 1,000 feet of density altitude, up to 6,000 feet.
- **(50/70 Rule)** When planning takeoff from short unobstructed runways, establish a landmark at 50% of your calculated takeoff distance. When reaching that landmark, you should be at 70% of your rotation speed. If not, abort the takeoff and reduce weight or wait for more favorable wind and temperature conditions.
- **(30/70 Rule)** If you must clear obstructions on takeoff, you’ll need to have 70% of your rotation speed by the time you’ve traveled 30% of your available takeoff distance.



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Figure 4: This version of the 70%-50% rule makes the abort point at 50% of the estimated takeoff distance, or $s_e/2$. Note that this publication also provides a modified 70%-30% rule for takeoffs over an obstacle off the departure end of the runway. But without specifying the height of the obstacle, no such rule can make sense, [4].

In this paper, we consider models for takeoff performance to ascertain the origin of the 70%-50% rule, determine its efficacy and explore possibilities for an improved abort plan.

3 Modeling Aircraft Takeoff Distance

We restrict our attention to forces in the horizontal direction and assume that the force due to gravity is balanced by the normal force provided by the ground during the takeoff roll. Consider a general aviation airplane that is moved forward by a piston engine that drives a propeller. The rotation speed will be 70 knots (nautical miles per hour) or 117 ft/s, a value typical of the general aviation fleet like the author's Beechcraft Bonanza she calls *Niky*.

Model 1: Constant Acceleration

Let's assume that the thrust produced by the engine provides a constant acceleration a , air resistance is negligible and no other forces are at play. Using Newton's second law, we observe that the force, or rate of change of momentum of the system, is equal to the mass times the acceleration of the aircraft so that $F = m \frac{dv}{dt} = ma$. Then integrating $\frac{dv}{dt} = a$ yields $v(t) = at + v_0$ where v_0 is the initial velocity. Since the aircraft starts from rest, $v(t) = at$ is the model for velocity. Integrating once more and setting the initial position to be zero, the model for position is $s(t) = \frac{a}{2}t^2$.

For example, if $a = 5ft/s^2$ then the aircraft will reach its rotation speed $v_r = 117ft/s$ after 23 seconds when the aircraft has traveled $s_r = 1361ft$ down the runway. This model predicts that, after 222 seconds, the aircraft will reach the speed of sound and the velocity will increase without bound. Of course this is unrealistic for any airplane. Indeed, *Niky* cannot travel anywhere close to the speed of sound as is evidenced by her pilot's operating handbook, Figure 5. The dose of realism this model needs is that air resistance plays a role in aircraft performance.

Model 2: Constant Acceleration and Air Resistance

We'll continue to assume that the engine creates constant acceleration and assume that the air resistance force is proportional to velocity. For general aviation aircraft like *Niky*, this is a common and reasonable assumption. (Note that for bodies moving with a higher velocity, air resistance proportional to v^2 may be warranted.)

Applying Newton's law, the differential equation

$$F = m \frac{dv}{dt} = ma - \gamma v$$

is a first-order, linear differential equation with constant coefficients that may be solved by separating the variables or, by using the integrating factor $\mu = e^{\gamma t/m}$, as we do here. Then

$$\frac{dv}{dt} + \frac{\gamma}{m}v = a \text{ implies that } v(t) = \frac{ma}{\gamma} \left(1 - e^{-\gamma t/m}\right).$$

**BEECHCRAFT
Debonair C33A
Bonanza E33A**

**Section V
Performance**

CRUISE SPEEDS

ASSOCIATED CONDITIONS:
AVERAGE CRUISE WEIGHT
TEMPERATURE

3100 LBS
STANDARD DAY (ISA)

EXAMPLE:

PRESSURE ALTITUDE 11500 FT
POWER SETTING FULL THROTTLE
2500 RPM

TRUE AIRSPEED 167 KNOTS

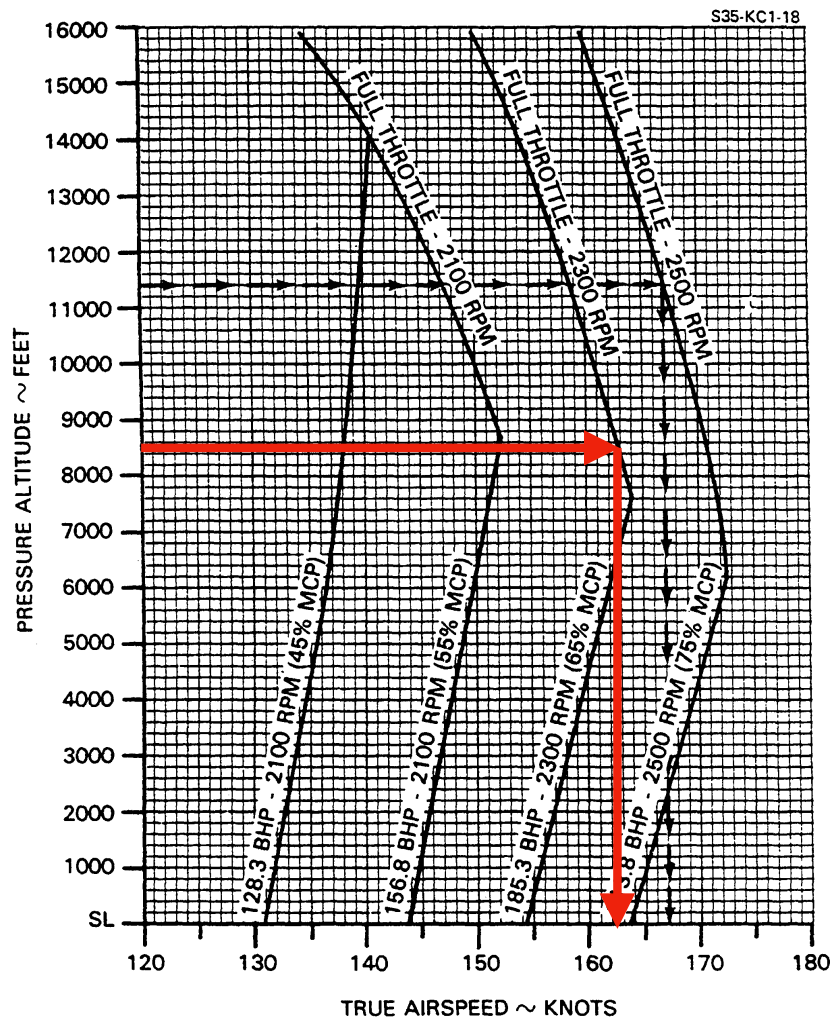


Figure 5: Given a cruise pressure altitude and a power setting, this chart provides the attendant cruise true airspeed for a Beechcraft model E33. For example, while flying at 8500 feet pressure altitude and using 65% maximum power, this aircraft will achieve a true airspeed of 162 knots. This is the analog of terminal velocity that, say, a skydiver achieves during free fall after jumping from an airplane in flight, [1].

In this case, as time increases the velocity approaches the terminal velocity of ma/γ ft/s so this version of the model is a significant improvement.

For example, let the mass of the airplane be $m = 81$ slug and $\gamma = 1/2$ slug/s. Then the aircraft will reach its rotation speed at a point 1507 feet along the runway.

Model 3: Varying Acceleration and Air Resistance

Constant thrust, therefore constant acceleration, is characteristic of an aircraft equipped with jet engines. A propeller-driven aircraft, on the other hand, produces constant *power*, or the product of thrust and velocity, during its takeoff roll. So thrust, and therefore acceleration, is inversely proportional to velocity. As the takeoff roll proceeds, acceleration decreases. In refining the model further, replace the constant acceleration a with a term k/v .

Applying Newton's law once more, the differential equation $F = m \frac{dv}{dt} = mk/v - \gamma v$ can be rearranged into

$$\frac{dv}{dt} + \frac{\gamma}{m}v = \frac{k}{v}$$

which is a Bernoulli differential equation of the form $v' + av = bv^n$ with $n = -1$. Multiplying both sides by $(1 - n)v^{-n} = 2v$ gives

$$2vv' + \frac{2\gamma}{m}v^2 = 2k.$$

Substituting $u = v^2$ results in the first-order linear differential equation $u' + \frac{2\gamma}{m}u = 2k$ which may be solved using the integrating factor $\mu = e^{2\gamma t/m}$. The solution becomes

$$v(t) = \sqrt{\frac{km}{\gamma} \left(1 - e^{-2\gamma t/m}\right)}.$$

For example, if $k = 375 \text{ ft}^2/\text{s}^3$, the aircraft will rotate 1637 feet along the runway.

Model 4: Adding Realism and Complexity

As with many physical phenomena, the takeoff distance model can be made more realistic with added complexity. For example, we have assumed that the runway provides all of the force in the upward direction during the takeoff roll. In reality, that load continually shifts to the wings as the lift they provide increases. Rolling resistance is a form of drag that will, therefore, decrease during the takeoff roll. We invite the interested reader to develop a new model that factors in this decrease in drag or to consider other models for takeoff distance in the literature, Figure 6.

4 Exploring the 70%–50% Rule

The author learned about the 70%–50% rule as a student pilot and surmised that the rule must emanate from a mathematical model. But years of searching myriad references produced no information about its origin or the model that created it. So we will review the above models to determine what $X\% - 50\%$ rule might be indicated.

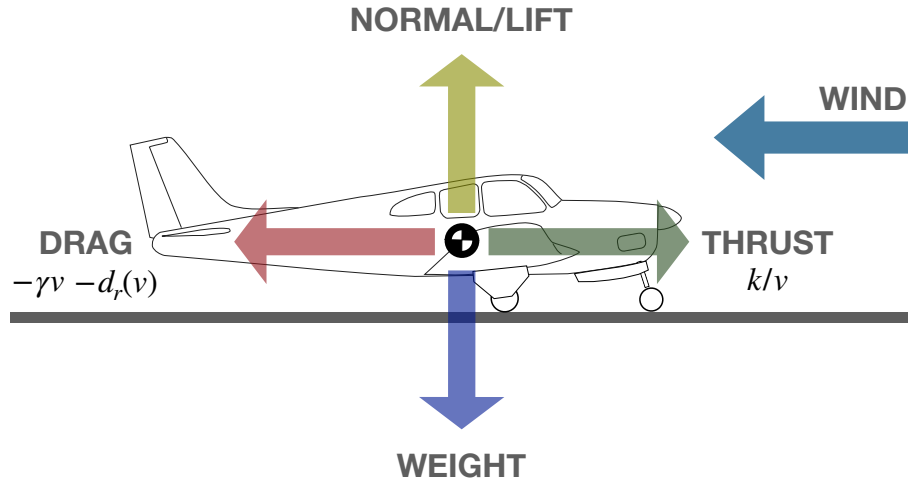


Figure 6: The horizontal forces can be made more realistic by incorporating rolling resistance $d_r(v)$ that decreases as velocity increases. Thrust which, in the above models is assumed to be constant, is better approximated by a term k/v . In particular, note that this means that acceleration is lower during the second half of the takeoff roll. Finally, a headwind/tailwind will have a profound effect on takeoff distance.

Model 1

For modeling aircraft takeoffs, expressing velocity as a function of time is not as helpful as velocity as a function of distance. In this case, $v(s) = \sqrt{2as}$. If s_r is the distance at the time of rotation, then the rotation speed is $v_r = v(s_r) = \sqrt{2as_r}$. This model suggests that the airspeed at the halfway point is $v(s_r/2) = \sqrt{as_r} = v_r/\sqrt{2} \approx 0.707v_r$.

Model 2

We can write distance as a function of velocity for this model as $s(v) = -\frac{m^2 a}{\gamma^2} - \frac{m}{\gamma} v$. In the example for which $a = 5$, $m = 81.25$ and $\gamma = 1/2$, the model predicts a velocity of 84 ft/s at the halfway point of the takeoff roll. This is approximately $0.718v_r$.

Model 3

The model that incorporates air resistance as well as diminishing acceleration during the takeoff roll and $k = 375$ predicts that the velocity will be 94 ft/s or $0.807v_r$.

Identifying the Model

Note that all models predict that the airspeed needs to be in excess of 70% v_r at the 50% mark in order to reach full rotation speed at the 100% point. So the model corresponding to the 70%-50% rule appears to be Model 1. Of course, it really should be the 71%-50% rule since truncating to 70% does not err on the side of safety. Of course, Model 1 is the one *least* representative of takeoff performance in a general aviation aircraft.

5 Other Problems

Problems with the 70%-50% rule do not end with a non-representative model. We discuss just a few below.

No Conditions

The author has never seen either version of this rule associated with conditions for its use. For example, around her home airport on a typical spring day, Niky's takeoff roll is approximately 1000 feet. But, in applying the 70%-50%-1 version on a 10,000 foot runway, if the airplane can't achieve 50 knots by 5000 feet down the runway, there are serious problems. The two versions of the rule become the same for short runways where the expected takeoff distance is what is available.

70%-50%-1: Cutting it Close

Suppose that Model 1 accurately represents the forces at play early in the takeoff roll. Assuming that the pilot saw 70% of the takeoff speed at the 50% mark of the runway, the pilot would continue the takeoff and the wheels would leave the ground in the last inch of the runway. With no margin for safety, this rule is unacceptable.

70%-50%-2: Aborting too Often

This version of the rule requires that the pilot use the operating handbook for the airplane and compute a takeoff distance. Temperature, pressure altitude, aircraft weight and headwind component are considered in a chart like Figure 7 to arrive at an estimate of the takeoff distance s_e . The pilot would then find a landmark a distance of 50% s_e along the runway and use that as an abort point. But aircraft engines become less efficient as they wear and the takeoff distance chart assumes a new engine. So it's likely that the airplane will fail to achieve 70% of its rotation speed by this point which will result in unnecessarily aborted takeoffs since there are no runway length conditions cited in this version of the rule.

Winds Matter

The lift generated by an airplane's wing depends on its velocity through the air so, while the airplane is on the ground, any headwind component provides essentially free lift. Imagine, as a thought experiment that a 50 knot wind blows straight down the runway. Then the aircraft has achieved 70% of its rotation speed with the engine turned off. In this way, airspeed early in the takeoff roll can be a poor predictor of future aircraft performance. Verifying a model would require performing takeoff runs in a no-wind condition as winds can lull a pilot into believing engine performance is better than it is, Figure 6.

6 A Simpler and Safer Rule

Adherents of the 70%-50% rule cling firmly to it, despite the fact that its origins and proof of efficacy remain shrouded in mystery. This is a shame because there is a more effective rule that is much easier for pilots to apply. Use temperature, pressure altitude, weight and wind information along with a chart like Figure 7 (found in the pilot's operating handbook) to compute the expected takeoff roll s_e . Use a multiplier of s_e based on engine age and recent performance information to arrive at a more realistic estimate, say, $s_g = 1.2s_e$. Use a similar chart to calculate the stopping distance s_s . Only attempt the takeoff if the runway available is greater than $s_g + s_s$ and abort the takeoff if the airplane has not lifted off by s_g , see [3].

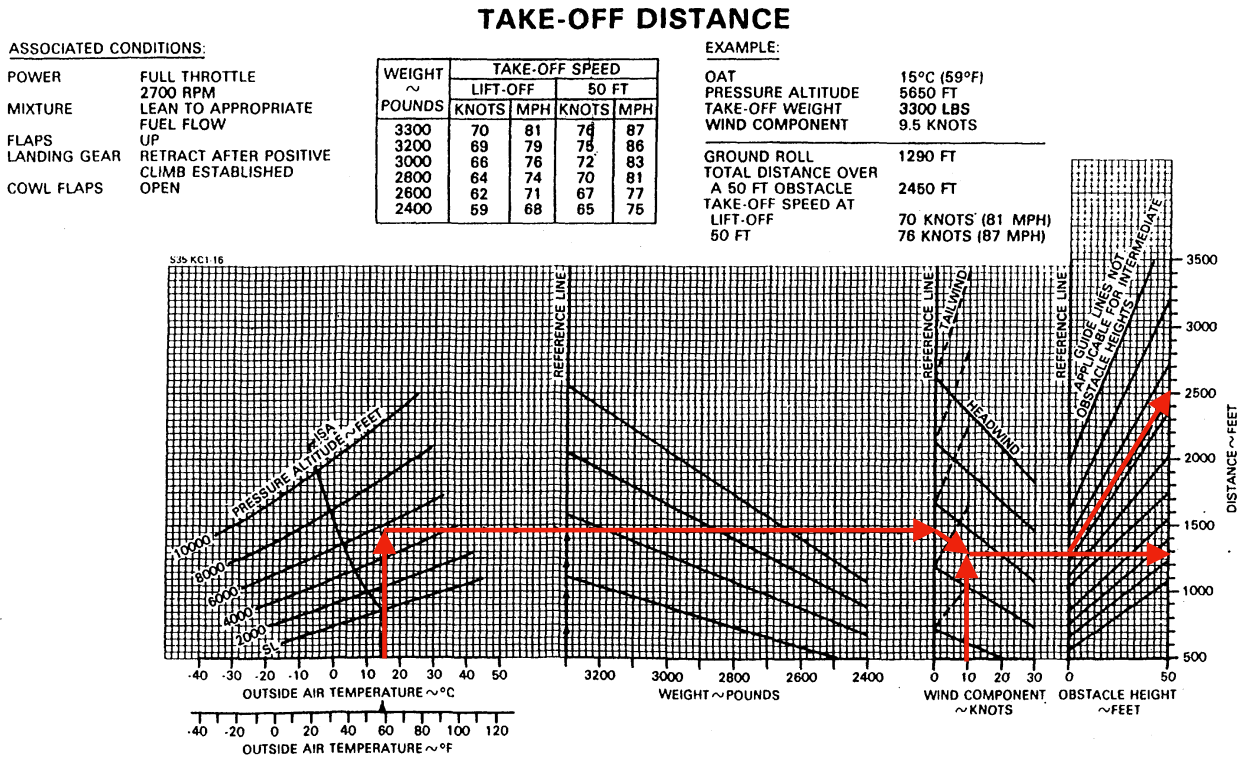


Figure 7: Given pressure altitude, temperature, aircraft weight, headwind component, this chart estimates the takeoff distance s_e for a Beechcraft model E33. For example, if the airport is at a pressure altitude of 5500 feet and the temperature is 15° C and the aircraft, at maximum gross weight of 3300 pounds, takes off into a 10 knot headwind, the estimated rotation distance is 1300 feet and after 2500 feet down the runway, it should be 50 feet in the air. It assumes good runway conditions, a certain aircraft configuration and a new engine that produces fully rated power. As engine power degrades over time and with use, these estimated values can become optimistic [1].

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