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How to Intercept a High-Speed Rocket with a Pair of Compasses and a Straightedge?

Yagub N. Aliyev
ADA University

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Yagub N. Aliyev
School of IT and Engineering, ADA University, Baku AZ1008, Azerbaijan

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Abstract: In this paper a nonlinear differential equation arising from an elementary geometry problem is discussed. This geometry problem was inspired by one of the proofs of the first remarkable limit discussed in a typical first semester undergraduate Calculus course. It is known that the involved differential equation can be reduced to Abel's differential equation of the first kind. In this paper the problem was solved using an approximate geometric method which constructs a piecewise linear solution approximation for the curve. The compass tool of GeoGebra was extensively used for these constructions. At the end of the paper, some generalizations are discussed. A new transformation of curves, named "Interception", is introduced and its approximate construction using GeoGebra is described. Some possible applications include geometry, calculus, ordinary differential equations, and military interceptions.

1 Introduction

One of the first theorems that an undergraduate student learns from a Calculus 1 course, sometimes called the First Remarkable Limit, is

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

Most Calculus textbooks provide the following proof based on the inequality

$$\sin\,\theta < \theta < \tan\,\theta \quad \text{for} \quad \theta \in (0, \pi/2)\,.$$

Dividing all the sides of this double inequality by positive number $\sin \theta$ and then tending $\theta \to 0$, we obtain, by the already covered Sandwich Theorem, the required limit. To prove the double inequality, the following standard diagram is used.

In Figure 1, a circle of unit radius with the centre at the point O is drawn, OA and OB are its radii, and the tangent of the circle at the point B intersects the extension of the

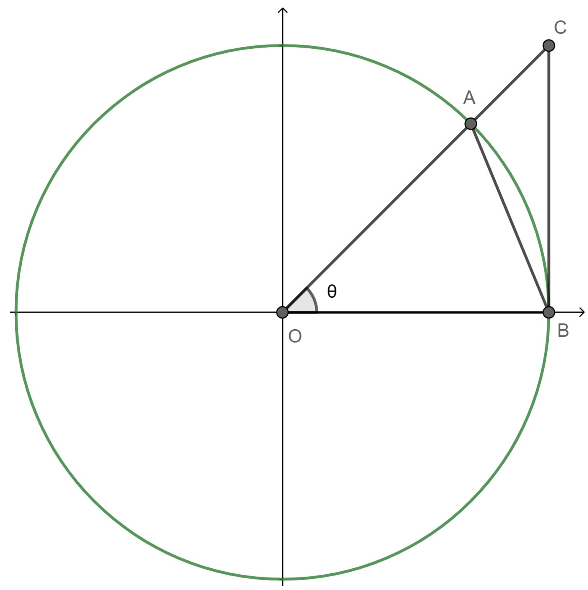

Figure 1: The proof of The First Remarkable Limit

radius OA at the point C. It is easy to see that the area of the triangle AOB is less than the area of the sector AOB, which in turn is less than the area of triangle BOC. By using formulas for the area of triangles and a circular sector, this can be expressed as a double inequality

$$\frac{\sin\,\theta}{2} < \frac{\theta}{2} < \frac{\tan\,\theta}{2} \quad \text{for} \quad \theta \in (0, \pi/2) \quad \text{where} \quad \theta = \angle AOB.$$

The inequality

$$\theta < \tan\,\theta \quad \text{for} \quad \theta \in (\theta, \pi/2)$$

also means that the length of the arc AB is less than the length of the tangent BC, as the line OA rotates counterclockwise around the point O.

So, it is natural to ask, whether it is possible to replace the unit circle with another smooth curve passing through the point B such that now the length of the curve AB is equal to the length of the tangent BC as the line OA rotates counterclockwise around the point O. This would mean that the curve starts at the point $(1, 0)$, so that $r(\theta) = 1$, and, for each $\theta > 0$, the arclength of the curve from $(1, 0)$ to $(r(\theta)\cos\theta, r(\theta)\sin\theta)$ equals the length along the line $x = 1$, from $(1, 0)$ to its intersection with the line through the origin with polar angle θ. In other words, as shown in Figure 2, the length of the line segment BC would be equal to the arclength of BA of the unknown curve.

It will be shown in Section 2 that this problem is equivalent to a nonlinear differential equation of the first order, and an overview of the literature about the equation will be given there. In Section 3 we will use an approximate method to determine the shape of this curve. In Section 4 we generalize the problem and in Section 5 we give some elementary examples. In Section 6 we mention one possible application of this theory. For diagrams and numerical experiments, we used the GeoGebra Calculator. This paper also intends to motivate professors and students interested in undergraduate research projects, by showing that it is possible to jump from a familiar textbook topic directly to an advanced research problem. The first version of this paper was presented in 2021 at the 5th International Conference on Mathematics: An Istanbul Meeting for World Mathematicians [8].

2 The Differential Equation

We want to find a curve whose polar equation $r = r(\theta)$ satisfies $r(0) = 1$ and the length of its arc in the interval $[0, \theta]$ is equal to the length of the line segment connecting the point on the curve at $\theta = 0$ with the intersection of the ray at angle θ and the tangent line at $\theta = 0$. We want to use polar coordinates because we will be dealing with changing r values. In Figure 2, the length of the line segment BC, which is perpendicular to OB, is equal to the length of the arc $\overset{\frown}{BA}$ of the required curve as OC rotates around O for $\theta \in (-\pi/2, \pi/2)$.

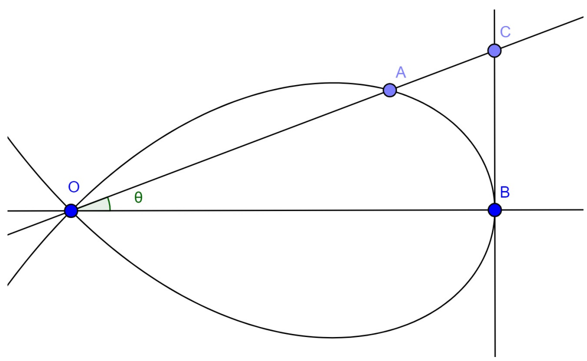

Figure 2: The length of arc $\overset{\frown}{BA}$ is to be equal to the length of the line segment BC.

Using the formula for the length of a curve given in polar form $r = r(\theta)$, we obtain the equation

$$\int_0^\theta \sqrt{r^2 + (r')^2} d\theta = \tan\,\theta,$$

where $\angle COB$ and $|OA| = r(\theta)$. By taking the derivative of both sides with respect to θ we obtain the differential equation

$$\sqrt{r^2 + (r')^2} = \sec^2\theta \tag{2.1}$$

with initial condition $r(\theta) = 1$. One of the solutions of the differential equation is its known solution $r = \sec\theta$, which is the equation of the vertical tangent line $x = 1$. This solution corresponds to the branch $r' = \sqrt{\sec^4\theta - r^2}$. We are interested in the existence of the other solution corresponding to the branch $r' = -\sqrt{\sec^4\theta - r^2}$. Using standard methods of series solutions of ODEs, we can find the first terms of the Maclaurin series of the two solutions:

$$r_1 = 1 + \frac{1}{2}\theta^2 + \frac{5}{24}\theta^4 + \frac{61}{720}\theta^6 + \frac{277}{8064}\theta^8 + O(\theta^{10}),$$
$$r_2 = 1 - \theta^2 - \frac{2}{21}\theta^4 - \frac{1933}{24255}\theta^6 - \frac{6004}{169785}\theta^8 + O(\theta^{10}).$$

Maple 2021 was used to obtain these series. See Figure 3.

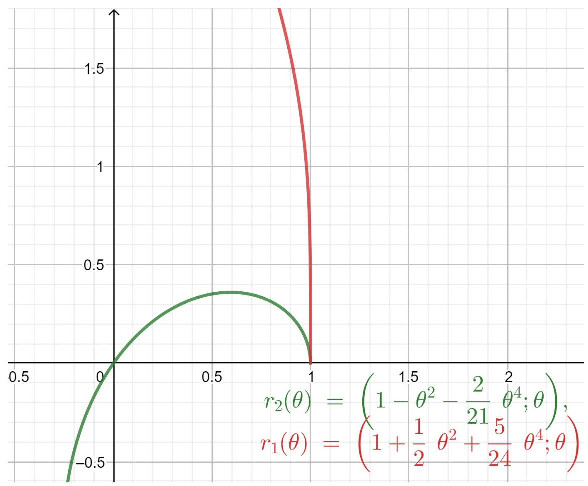

Figure 3: 4th order series approximations of solutions r_1, r_2 for equation 2.1.

In Figure 3, we see the approximation for $r_1 = 1/\cos\theta = \sec\theta$, which is the vertical line $x = 1$. But it is not so clear what r_2 is. We can try to solve 2.1 explicitly. See Appendix A for one such attempt. Trying to use all the involved substitutions in Appendix A to obtain the solution of 2.1 as a closed formula would not be easy. Therefore, we will focus on the approximate solutions to find the shape of the curve.

It is worthwhile to note that similar problems have also been discussed in the context of interception of high-speed targets by beam rider missiles [5]. The cases that we consider model the situation when *the target and the missile have the same speed.* A similar question with boundary condition $r(0) = 0$ was discussed in [7] and it would be interesting to find a parametric representation of the current problem similar to the formula (5) in [7]. Similar questions were considered by [3], [2], [4].

3 Approximate geometric solution

Let us draw the rays $\theta_1 = \pi/n$, $\theta_2 = 2\pi/n$, $\theta_3 = 3\pi/n, \ldots, \theta_n = n\pi/n, \ldots$, for the given n. Denote the intersections of these rays with the vertical line $r = 1/\cos\theta$ by $C_1, C_2, C_3, \ldots$. Let us construct the points $A_1, A_2, A_3, \ldots$ on the rays $OC_1, OC_2, OC_3, \ldots$, respectively, so that

$$BA_1 = BC_1, \quad A_1A_2 = C_1C_2, \quad A_2A_3 = C_2C_3, \quad \ldots$$

We can use the compass tool of GeoGebra for this purpose. To construct the point A_1 we draw circle with radius BC_1 at the centre B and denote its second intersection with the line OC_1 by A_1. Similarly, to find the point A_2, we draw circle with radius C_1C_2 at the centre A_1, and denote its intersection with the line OC_2 farthest from C_2 by A_2. The other points $A_3, A_4, \ldots$ are constructed in the same way, as shown in Figure 4.

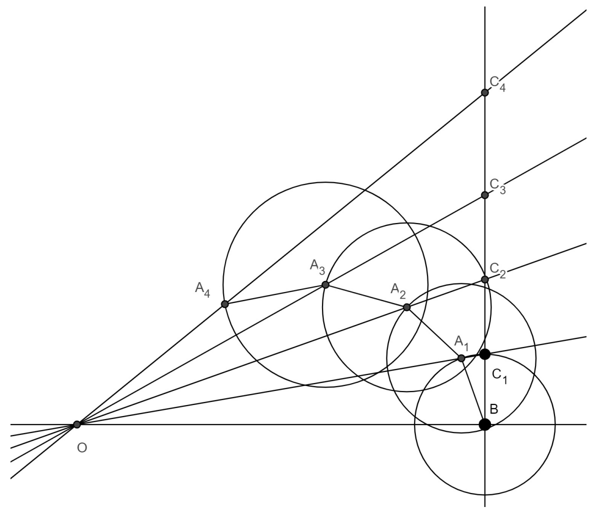

Figure 4: The construction of the piecewise linear solution.

It is now obvious that the length of the line segment $BC_1C_2C_3 \ldots C_n$ is equal to the length of the piecewise linear solution $BA_1A_2A_3 \ldots A_n$ and as $n \to \infty$ this piecewise linear solution approaches the required curve. It is noteworthy that all these constructions can be done using only an unmarked ruler and a pair of compasses. One can also observe that as the point C_1 moves, the locus of each of the points $A_1, A_2, A_3, \ldots, A_n$ can be interpreted as approximations of the required curve. We used one of these approximations to draw the curve in Figure 2. Each intersection of a given circle with its outer ray is used as the centre of the next circle. The centre of each next circle is taken at the point obtained in the previous construction, as in Figure 5.

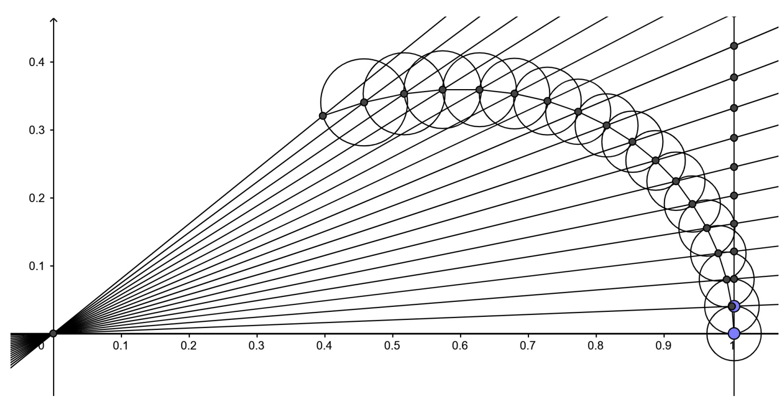

Figure 5: The piecewise linear solution approximates the curve.

Note that for $\theta = \theta_0 \approx 0.9235 \approx 52.9°$ the curve $r = r(\theta)$ passes through the point O. It would be interesting to find out how the constant θ_0 is related to the other constants in mathematics such as e or π. For the constant θ_0 we can show that

$$1 = \theta_0 \sec^2 \theta_0 - \theta_0^2 \sec^2 \theta_0 \tan \theta_0 - \frac{4 \tan^4 \theta_0 - \tan^2 \theta_0 - 1}{6} \theta_0^3 + \dots .$$

where an arbitrary number of terms of the series on the right-hand side can be calculated (see Appendix B).

4 Generalizations and "Interception"

Let us now replace the vertical line BC by an arbitrary differentiable curve $r = \Phi(\theta)$ passing through the point $B(1, 0)$. Then the differential equation becomes

$$\sqrt{r^2 + (r')^2} = \phi(\theta),$$

where $\phi(\theta) = \sqrt{\Phi(\theta)^2 + (\Phi'(\theta))^2}$. One of the solutions is obviously $r_1 = \Phi(\theta)$. It is interesting to find the other solution $r_2 = r_2(\theta)$. As the case $\Phi(\theta) = 1/\cos\theta$ suggests, it is not always easy to do this analytically. So, it is reasonable to have an approximate method for the solution. The approximate method described in Section 3 can be applied here again with obvious modifications, as in Figure 6. We just need to take the points $C_1, C_2, C_3, \dots, C_n$ on the curve $r = \Phi(\theta)$ and measure the distances BC_1, C_1C_2, $C_2C_3, \dots$ using the piecewise linear solution approximation $BC_1C_2C_3 \dots C_n \dots$ of the curve $r = \Phi(\theta)$. As in the previous case we use the compass to construct the points $A_1, A_2, A_3, \dots$ on the rays $OC_1, OC_2, OC_3, \dots$, respectively, so that

$$BA_1 = BC_1, A_1A_2 = C_1C_2, A_2A_3 = C_2C_3, \dots$$

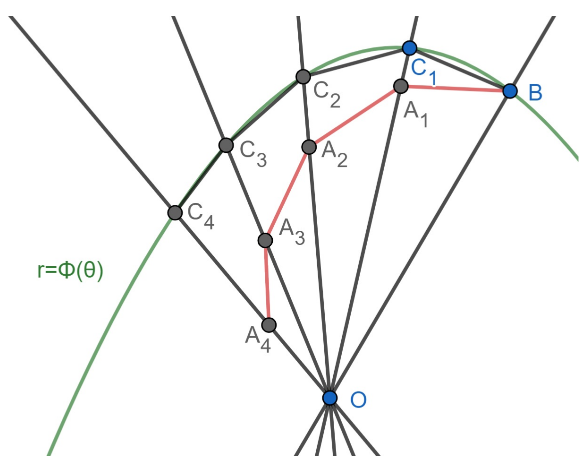

Figure 6: Approximate construction of "The Interception Curve"

The piecewise linear solution $BA_1A_2A_3 \dots A_n$ approximates $r_2 = r_2(\theta)$ and as $n \to \infty$ this piecewise linear solution approaches the curve $r_2 = r_2(\theta)$. We will call the process of obtaining the curve $r_2 = r_2(\theta)$ from the given curve $r_1 = \Phi(\theta)$ as Interception. The reasons for the choice of this name is the connection with mentioned applications [5], [6], [7]. The considered approximate construction can be interpreted in the following way. Suppose that $r = \Phi(\theta)$ in Figure 6 is the trajectory of the target which is detected first at the point C_4. The initial position of the intercepting missile is A_4. Then going back in the construction we find A_3, A_2, A_1, and finally B, where the interception occurs.

The transformation preserves the distances on the curves. It can be generalized for space curves. It would be interesting to find an analogue of this transformation for surfaces in space.

5 Examples of elementary functions for interception

It would be satisfying to see some examples of elementary functions $r_1 = \Phi(\theta)$ for which r_2 is again an elementary function. First, note that if

$$r_1^2 + (r_1')^2 = r_2^2 + (r_2')^2$$

then

$$(r_2 - r_1)(r_2 + r_1) = -(r_2' - r_1')(r_2' + r_1').$$

Let us denote $r_2 - r_1 = 2x$, $r_2 + r_1 = 2y$. Then we obtain $yx = -y'x'$. The last equality can also be rewritten as $x/x' = -y'/y$ or $(\ln y)' = -1/(\ln x)'$. From a geometrical point of view this means that the solutions r_1, r_2 of the differential equation $r^2 + (r')^2 = \phi(\theta)^2$, can be represented as $r_1 = y - x$, $r_2 = y + x$, where the functions $x = x(\theta)$, $y = y(\theta)$ have the nice property that the tangent lines of the functions $\ln x(\theta)$ and $\ln y(\theta)$ are perpendicular to each other at an arbitrary point θ. The equality $(\ln y)' = -1/(\ln x)'$ can be the starting point to find infinitely many elementary examples of such r_1, r_2. Using the easily verifiable fact that

$$(\ln \sin\theta)' = -1/(\ln\cos\theta)',$$

we obtain (see Figure 7) the pair of circles

$$r_1 = \cos\theta - \sin\theta, r_2 = \cos\theta + \sin\theta.$$

Note that the equality of the arcs BA and BC follows easily from the elementary properties of inscribed angles and the fact that these circles have the same radius.

Exercise 1. Show that if $x = \theta$, then $y = e^{-\theta^2/2}$. Draw the graphs of r_1 and r_2.

Exercise 2. Show that if $x = \sqrt{\theta}$, then $y = e^{-\theta^2}$. Draw the graphs of r_1 and r_2.

Exercise 3. Show that if $x = 1/\sqrt{\theta}$, then $y = e^{\theta^2}$.

In the last exercise, the obtained curves $r_1 = e^{\theta^2} + \frac{1}{\sqrt{\theta}}, r_2 = e^{\theta^2} - \frac{1}{\sqrt{\theta}}$ do not intersect (see Figure) but they still have the property that their arcs between two arbitrary lines $f : \theta = \theta_1$ and $g : \theta = \theta_2$, have the same length (length of arc AD = length of arc BC). The transformation "Interception" that was described in the previous section should be modified for such cases. In this case instead of the point B we should take two points: B on $r_1 = \Phi(\theta)$ and B' on r_2, and then construct the piecewise linear solution $B'A_1A_2A_3 \ldots A_n$ so that

$$B'A_1 = BC_1, \quad A_1A_2 = C_1C_2, \quad A_2A_3 = C_2C_3, \quad \ldots$$

Note that although two such piecewise linear solution approximations $B'A_1A_2A_3 \ldots A_n$, are possible, in Figure 9, only one is drawn.

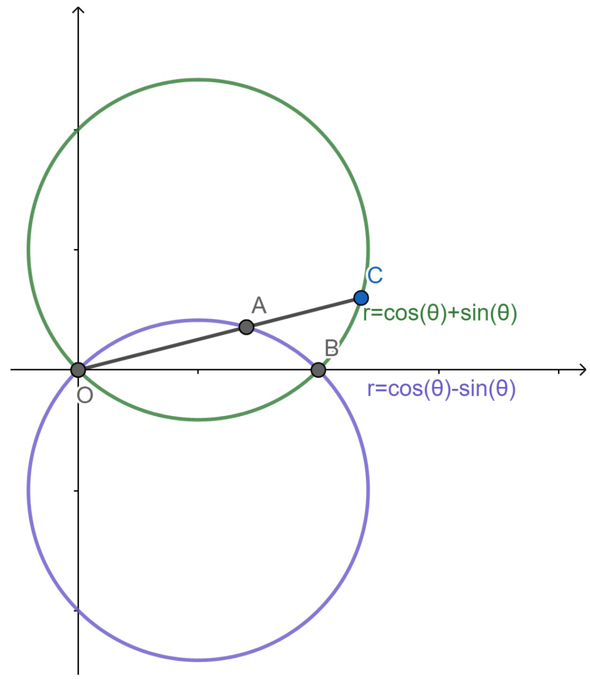

Figure 7: Circles as an example of Interception when $x = \sin\theta$.

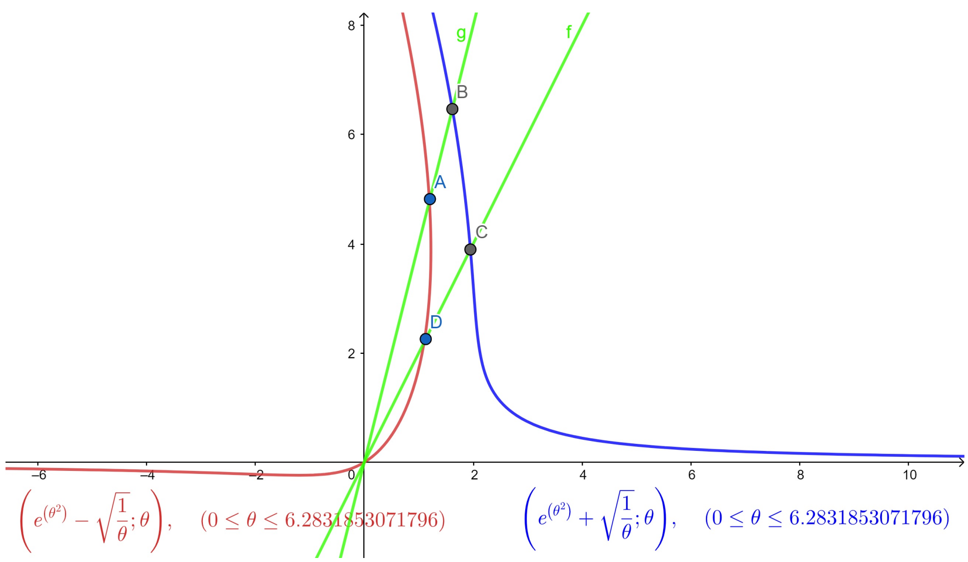

Figure 8: Non-intersecting example of Interception, when $x = 1/\sqrt{\theta}$ (Exercise 3).

6 Conclusion

The study of curves and their properties has a long history dating back to the time of the ancient Greeks. Modern mathematics supplied the theory of curves with analytical tools and an abstract viewpoint. Although it is not a mainstream research topic today, for undergraduate research projects and expository papers the theory of curves can be a source of inspiration and motivation. In the current paper one interesting curve was studied in

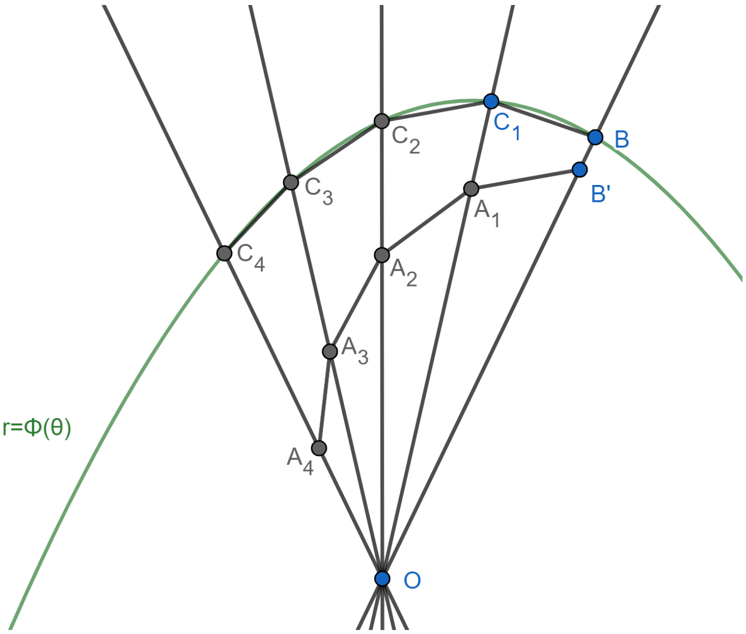

Figure 9: More general Interception.

detail. Its approximate shape was drawn as a piecewise linear function. This construction was done with the help of the compass tool of GeoGebra. After this, a generalization was discussed and a new transformation (named as Interception) preserving the distances on the curves was introduced. Elementary function examples for which Interception gives again an elementary function are given in Section 5.

The discussed topics also have some connections with problems related to laser-beam riding interception (hence the name of the curve) of high-speed missiles in technology [5],[6]. The method discussed in the current paper can have some applications outside of mathematics.

7 Backmatter

Appendix A

It was noted in [1], Part C (Part 3 in Russian Translation), Sect. 1.370 that a differential equation of the form

$$r^2 + (r')^2 = f^2(x)$$

can always be transformed into the form

$$fu' + f' \tan u = \pm f,$$

using the substitution $r = f(x)\sin u(x)$ (see [1], Sect. 1.370), which in turn can be transformed into the equation

$$fu' + gu^3 + hu^2 + gu + h = 0,$$

using the substitution $u(x) = \tan y$ (see [1], Sect. 1.202). The last equation is a type I Abel equation (see [1], Sect. 4.10, [3], Sect. 4-1). The special case

$$y^2 + (y')^2 = \frac{a^2}{\cos^4 x},$$

was discussed in [3] in relation to one kinematics problem which is a dilational version of our problem (See also [1], Sect. 1.460). A solution expressed through some integrals for the last equation was given in [4]. (See also [1], Sect. 1.460). One can follow the following steps:

1. Use the substitution $y' = y \cot u$ to obtain $y \cos^2 x = \pm a \sin u$. Then differentiating and excluding y and y' we obtain $u' + 2 \tan u \tan x = 1$. See [1], Sect. 1.460.

2. Use substitutions $\eta(\xi) = \tan u$, $\xi = \tan x$ to obtain Abel's equation $(\xi^2 + 1)\eta' = (\eta^2 + 1)(1 - 2\xi\eta)$. See [1], Sect. 1.81.

3. Use substitution $\xi^4 \eta(\xi) = (\xi^2 + 1)z + \xi^3$ to obtain again Abel's equation $\xi^7 z' + 2(\xi^2 + 1)z^3 + 5\xi^3 z^2 = 0$. See [1], Sect. 1.151.

4. Use substitution $v = 1/z$ to obtain $\xi^7 v v' = 2(\xi^2 + 1) + 5\xi^3 v$. See [1], Sect. 1.185.

5. Use substitution $\xi w = \xi^3 v + 1$ to obtain linear equation $\frac{d\xi}{dw} - \frac{\xi w}{2(w^2+1)} + \frac{1}{2(w^2+1)} = 0$, which can be solved using integrals. See [1], Sect. 1.185.

Appendix B

Let $r(\theta_0) = 0$. Then by the differential equation $r'(\theta) = -\sqrt{\sec^4 \theta - r(\theta)^2}$ and therefore $r'(\theta_0) = -\sec^2 \theta_0$. By taking the first, second and higher order derivatives of $\sqrt{r^2 + (r')^2} = \sec^2 \theta$ and substituting $\theta = \theta_0$, we obtain

$$r''(\theta_0) = -2 \sec^2 \theta_0 \tan \theta_0, r'''(\theta_0) = 4 \tan^4 \theta_0 - \tan^2 \theta_0 - 1, \ldots.$$

By Taylor's formula

$$r(\theta) = r(\theta_0) + r'(\theta_0)(\theta - \theta_0) + \frac{r''(\theta_0)}{2!}(\theta - \theta_0)^2 + \frac{r'''(\theta_0)}{3!}(\theta - \theta_0)^3 + \ldots.$$

Substituting $\theta = 0$, and noting $r(0) = 1$ we obtain

$$1 = \theta_0 \sec^2 \theta_0 - \theta_0^2 \sec^2 \theta_0 \tan \theta_0 - \frac{4 \tan^4 \theta_0 - \tan^2 \theta_0 - 1}{6} \theta_0^3 + \ldots.$$

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