ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2019, Vol. 40, No. 9, pp. 1242–1249. © Pleiades Publishing, Ltd., 2019.

Differential Inequalities and Univalent Functions

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Received April 3, 2019; revised April 30, 2019; accepted May 8, 2019

Abstract—Let $\mathcal{M}$ be the class of analytic functions in the unit disk $\mathbb{D}$ with the normalization $f(0) = f'(0) - 1 = 0$, and satisfying the condition

$$\left| z^2 \left(\frac{z}{f(z)} \right)'' + f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| \leq 1, \quad z \in \mathbb{D}.$$

Functions in $\mathcal{M}$ are known to be univalent in $\mathbb{D}$. In this paper, it is shown that the harmonic mean of two functions in $\mathcal{M}$ are closed, that is, it belongs again to $\mathcal{M}$. This result also holds for other related classes of normalized univalent functions. A number of new examples of functions in $\mathcal{M}$ are shown to be starlike in $\mathbb{D}$. However we conjecture that functions in $\mathcal{M}$ are not necessarily starlike, as apparently supported by other examples.

DOI: 10.1134/S1995080219090038

Keywords and phrases: *differential inequalities, harmonic mean, subclasses of analytic univalent functions, starlike functions.*

1. INTRODUCTION

Let $\mathcal{H}$ denote the family of analytic functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{A}$ its subclass of normalized functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. Further, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions f univalent in $\mathbb{D}$. Denote by $\mathcal{S}^*$ and $\mathcal{C}$ respectively the subclasses of $\mathcal{S}$ consisting of starlike and convex functions. Functions $f \in \mathcal{S}^*$ map $\mathbb{D}$ onto starlike domains with respect to the origin, while $f \in \mathcal{C}$ whenever $f(\mathbb{D})$ is a convex domain. Analytically, $f \in \mathcal{S}^*$ if $\mathrm{Re}\,(zf'(z)/f(z)) > 0$, while $f \in \mathcal{C}$ if $\mathrm{Re}\,(1 + zf''(z)/f'(z)) > 0$.

Investigations into particular subclasses of $\mathcal{A}$ continued to be of recent interest. These include the class $\mathcal{U}$ consisting of functions $f \in \mathcal{A}$ satisfying

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| \leq 1, \quad z \in \mathbb{D},$$

as well as the class $\mathcal{P}$ of functions $f \in \mathcal{A}$ with

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq 2, \quad z \in \mathbb{D}.$$

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The strict inclusion $\mathcal{P} \subsetneq \mathcal{U} \subsetneq \mathcal{S}$ holds within these classes (see [2, 5, 14] for a proof). There are several generalizations [7] of this result. For recent investigations on $\mathcal{U}$ and its generalization, we refer to [11–13] and the references therein.

In this paper, the phrase $f \in \mathcal{U}$ (respectively, $f \in \mathcal{P}$) in $|z| < r$ means that the defining inequality holds in $|z| < r$ instead of the full disk $|z| < 1$. We also follow this standard convention for other classes. In [8] and [9], the authors discussed the classes $\mathcal{M}$ and $\mathcal{N}$ of functions from $\mathcal{A}$ satisfying respectively the differential inequality

$$|\mathcal{M}_f(z)| \leq 1 \quad \text{and} \quad |\mathcal{N}_f(z)| \leq 1, \quad z \in \mathbb{D},$$

where

$$\mathcal{M}_f(z) = z^2\left(\frac{z}{f(z)}\right)'' + f'(z)\left(\frac{z}{f(z)}\right)^2 - 1 \quad \text{and} \quad \mathcal{N}_f(z) = -z^3\left(\frac{z}{f(z)}\right)''' + f'(z)\left(\frac{z}{f(z)}\right)^2 - 1.$$

These classes are also closely related to the class $\mathcal{U}$ in the sense of the strict inclusions $\mathcal{N} \subsetneq \mathcal{M} \subsetneq \mathcal{P} \subsetneq \mathcal{U}$. A slightly general version of this result is given in [1].

In [10], Obradović, and Ponnusamy discussed "harmonic mean" of two univalent analytic functions. These are functions F of the form

$$F(z) = \frac{2f(z)g(z)}{f(z) + g(z)}, \tag{1}$$

or equivalently,

$$\frac{1}{F(z)} - \frac{1}{z} = \frac{1}{2}\left[\left(\frac{1}{f(z)} - \frac{1}{z}\right) + \left(\frac{1}{g(z)} - \frac{1}{z}\right)\right], \tag{2}$$

where $f, g \in \mathcal{S}$. In particular, the authors in [10] determined the radius of univalency of F, and proposed the following two conjectures.

Conjecture 1. (a) *The function F defined by (1) is not necessarily univalent in $\mathbb{D}$ whenever $f, g \in \mathcal{S}$ such that $((f(z) + g(z))/z) \neq 0$ in $\mathbb{D}$.*

(b) *The function F defined by (1) is univalent in $\mathbb{D}$ whenever $f, g \in \mathcal{C}$ such that $((f(z) + g(z))/z) \neq 0$ in $\mathbb{D}$.*

The authors in [10] showed that whenever $f, g \in \mathcal{U}$, then the function F defined by (1) belongs to $\mathcal{U}$ in the disk $|z| < \sqrt{(\sqrt{5} - 1)/2} \approx 0.78615$.

While Conjecture 1 remains open, the aim of this paper is to show that Conjecture 1 **(a)** does not hold when the class $\mathcal{S}$ is replaced by $\mathcal{U}$. Indeed, it does not hold true even for the classes $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{P}$. The second objective of the paper is to consider several examples in examining starlikeness of functions in the classes $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{P}$. We conclude with a conjecture that functions in the class $\mathcal{M}$ are not necessarily starlike in $\mathbb{D}$.

2. ON THE HARMONIC MEAN OF UNIVALENT FUNCTIONS

Theorem 1. *Let $f, g \in \mathcal{U}$ satisfy $[f(z) + g(z)]/z \neq 0$ for $z \in \mathbb{D}$. Then the function F given by (1) also belongs to the class $\mathcal{U}$.*

Proof. From (2), it readily follows from the triangle inequality that the function F satisfies

$$\left|F'(z)\left(\frac{z}{F(z)}\right)^2 - 1\right| = \left|-z^2\left(\frac{1}{F(z)} - \frac{1}{z}\right)'\right| \leq \frac{1}{2}\left|-z^2\left(\frac{1}{f(z)} - \frac{1}{z}\right)'\right|$$

$$+ \frac{1}{2}\left|-z^2\left(\frac{1}{g(z)} - \frac{1}{z}\right)'\right| = \frac{1}{2}\left|f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| + \frac{1}{2}\left|g'(z)\left(\frac{z}{g(z)}\right)^2 - 1\right| < 1.$$

Thus $F \in \mathcal{U}$. □

Moreover, we see that Theorem 1 holds true if the class $\mathcal{U}$ is replaced by the class $\mathcal{M}$.

Theorem 2. *Suppose $f, g \in \mathcal{M}$ satisfy $[f(z) + g(z)]/z \neq 0$ for $z \in \mathbb{D}$. Then the function F given by (1) also belongs to the class $\mathcal{M}$.*

Proof. Now

$$f'(z)\left(\frac{z}{f(z)}\right)^2-1=-z^2\left(\frac{1}{f(z)}-\frac{1}{z}\right)'.$$

Using this equality, it follows that

$$\mathcal{M}_f(z)=z^2\left[\left(\frac{z}{f(z)}\right)''-\left(\frac{1}{f(z)}-\frac{1}{z}\right)'\right]=z^2\left[\left(\left(\frac{z}{f(z)}\right)'-\frac{1}{f(z)}+\frac{1}{z}\right)'\right]$$

$$=z^2\left[\left(z\left(\frac{1}{f(z)}\right)'+\frac{1}{z}\right)'\right]=z^2\left[\left(z\left(\frac{1}{f(z)}-\frac{1}{z}\right)'\right)'\right]=z^3\left(\frac{1}{f(z)}-\frac{1}{z}\right)''+z^2\left(\frac{1}{f(z)}-\frac{1}{z}\right)'.$$

In view of (2), this means that $\mathcal{M}_F(z)=\frac{1}{2}\left(\mathcal{M}_f(z)+\mathcal{M}_g(z)\right)$, and use of the triangle inequality yields the desired result. □

Theorem 3. *Let $f,g\in\mathcal{N}$ satisfy $[f(z)+g(z)]/z\neq 0$ for $z\in\mathbb{D}$. Then the function F given by (1) also belongs to the class $\mathcal{N}$.*

Proof. As in the proof of Theorem 2, we see that

$$\mathcal{N}_f(z)=-z^2\left[z\left(\left(\frac{z}{f(z)}\right)'\right)''+\left(\frac{1}{f(z)}-\frac{1}{z}\right)'\right]$$

$$=-z^2\left[z\left(\frac{1}{f(z)}-\frac{1}{z}f'(z)\left(\frac{z}{f(z)}\right)^2\right)''+\left(\frac{1}{f(z)}-\frac{1}{z}\right)'\right]$$

$$=-z^2\left[z\left(z\left(\frac{1}{f(z)}-\frac{1}{z}\right)'+\frac{1}{f(z)}-\frac{1}{z}\right)''+\left(\frac{1}{f(z)}-\frac{1}{z}\right)'\right]$$

$$=-z^4\left(\frac{1}{f(z)}-\frac{1}{z}\right)'''-3z^3\left(\frac{1}{f(z)}-\frac{1}{z}\right)''-z^2\left(\frac{1}{f(z)}-\frac{1}{z}\right)'.$$

Thus relation (2) gives $\mathcal{N}_F(z)=\frac{1}{2}\left(\mathcal{N}_f(z)+\mathcal{N}_g(z)\right)$, and the proof of theorem readily follows. □

Finally, it is also readily shown that the above theorem holds true for the class $\mathcal{P}$.

3. EXAMPLES AND A CONJECTURE

It is known that functions in the class $\mathcal{U}$ are not necessarily starlike. There are a number of examples displaying functions in $\mathcal{U}$ that are not starlike in $\mathbb{D}$, see for instance [6]. However, is $\mathcal{M}\subset\mathcal{S}^*$? This section discusses the latter problem.

Example 3. To present a one-parameter family of functions in $\mathcal{M}$ that are also starlike, consider the function f given by $z/f(z)=1+(1-\alpha)z+\alpha z^m$, where $\alpha\in(0,1)$ and $m\in\mathbb{N}\backslash\{1\}=\{2,3,\ldots\}$ are such that $\alpha(m-1)^2=1$. Then $z/f(z)\neq 0$ in $\mathbb{D}$ and

$$\sum_{k=2}^{\infty}(k-1)^2|b_k|=(m-1)^2\alpha=1,$$

and therefore, $f\in\mathcal{M}$.

Next, we show that f is starlike whenever $m>1$ is an odd integer. Now, a simple calculation shows

$$\frac{zf'(z)}{f(z)}=\frac{1-\alpha(m-1)z^m}{1+(1-\alpha)z+\alpha z^m}.$$

With $z=e^{i\theta}$, then

$$\frac{e^{i\theta}f'(e^{i\theta})}{f(e^{i\theta})}=\frac{A(\theta)+iB(\theta)}{|1+(1-\alpha)e^{i\theta}+\alpha e^{im\theta}|^2},$$

where

$$A(\theta) = 1 + (1-\alpha)\cos\theta - \alpha(m-2)\cos(m\theta) - \alpha(1-\alpha)(m-1)\cos(m-1)\theta - \alpha^2(m-1).$$

Note that $A(\theta) = A(-\theta)$. As $\alpha = 1/(m-1)^2$, the expression for $A(\theta)$ reduces to

$$A(\theta) = 1 - \frac{1}{(m-1)^3} - \frac{m(m-2)}{(m-1)^2}D(\theta), \quad \text{where} \quad D(\theta) = -\cos\theta + \frac{1}{m}\cos(m\theta) + \frac{\cos(m-1)\theta}{m-1}.$$

To show starlikeness, that is, $f \in \mathcal{S}^*$, it suffices to show that $A(\theta) \geq 0$ for $0 \leq \theta \leq \pi$. First we prove the assertion for the case $m = 3$, while the general case is obtained separately. Setting $m = 3$, $A(\theta)$ reduces to

$$A(\theta) = \frac{7}{8} - \frac{3}{4}\left[-\cos\theta + \frac{1}{3}\cos 3\theta + \frac{1}{2}\cos 2\theta\right],$$

and from the identities $\cos 2\theta = 2\cos^2\theta - 1$ and $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$,

$$A(\theta) = \frac{1}{4}(5 + 6\cos\theta - 4\cos^3\theta - 3\cos^2\theta) = \frac{1}{4}(1+\cos\theta)^2(5 - 4\cos\theta),$$

which shows that $A(\theta) \geq 0$. Thus, the function $f_3(z)$ given by

$$f_3(z) = \frac{z}{1 + \frac{3}{4}z + \frac{1}{4}z^3} = \frac{4z}{(1+z)(4 - z + z^2)},$$

is starlike in $\mathbb{D}$.

Next, we proceed to prove starlikeness for the general case. This requires more computations. First,

$$\begin{aligned} D'(\theta) &= \sin\theta - \sin(m\theta) - \sin(m-1)\theta = \sin\theta - 2\sin\frac{(2m-1)\theta}{2}\cos\frac{\theta}{2} \\ &= 2\cos\frac{\theta}{2}\left[\sin\frac{\theta}{2} red - \sin\frac{(2m-1)\theta}{2}\right] = 4\cos\frac{\theta}{2}\cos\frac{m\theta}{2}\sin\frac{(m-1)\theta}{2}. \end{aligned}$$

We need to show that $A(\theta) \geq 0$ for $0 \leq \theta \leq \pi$. It is convenient to set $m = 2n+1$, $n \geq 2$ so that

$$D'(\theta) = 4\cos\frac{\theta}{2}\cos\frac{(2n+1)\theta}{2}\sin n\theta, \quad n \geq 2,$$

where $D(\theta)$ takes the form

$$D(\theta) = -\cos\theta + \frac{1}{2n+1}\cos(2n+1)\theta + \frac{1}{2n}\cos(2n\theta).$$

Clearly, $D'(\theta) = 0$ for $\theta = 0, \pi$, and the critical points of $D(\theta)$ in the open interval $(0, \pi)$ are given by

$$\begin{cases} \theta_j = \dfrac{(2j-1)\pi}{2n+1} & \text{for } j = 1, 2, \ldots, n, \\ \theta_j' = \dfrac{j\pi}{n} & \text{for } j = 1, 2, \ldots, n-1, \end{cases}$$

$n \geq 2$. Moreover, for each $n \geq 2$,

$$\begin{cases} \cos\dfrac{(2n+1)\theta}{2} > 0 & \text{for } 0 < \theta < \theta_1, \\ (-1)^j\cos\dfrac{(2n+1)\theta}{2} > 0 & \text{for } \theta_j < \theta < \theta_{j+1} \text{ and for } j = 1, 2, \ldots, n, \\ (-1)^{j-1}\sin n\theta > 0 & \text{for } \theta_{j-1}' < \theta < \theta_j' \text{ and for } j = 1, 2, \ldots, n. \end{cases}$$

In view of the above inequalities and after a careful scrutiny, it follows that

$$D'(\theta) \begin{cases} = 0 & \text{for } \theta = 0, \theta_j, \theta_j' \text{ for } j = 1, 2, \ldots, n, \\ > 0 & \text{for } \theta \in (0, \theta_1) \cup (\theta_j', \theta_{j+1}) \text{ for } j = 1, 2, \ldots, n-1, \\ < 0 & \text{for } \theta \in (\theta_j, \theta_j') \text{ for } j = 1, 2, \ldots, n, \end{cases}$$

where $0<\theta_1<\theta_1'<\theta_2<\cdots<\theta_j<\theta_j'<\theta_{j+1}<\cdots<\theta_n<\theta_n'=\pi$. Therefore,

$$D(\theta)\le \max\left\{D(0), D\left(\theta_j\right), D\left(\theta_j'\right):\ j=1,2,\dots,n\right\}.$$

Since

$$D(0)=-1+\frac{1}{2n+1}+\frac{1}{2n}=-\frac{2n}{2n+1}+\frac{1}{2n},\quad D(\pi)=1-\frac{1}{2n+1}+\frac{1}{2n}=\frac{2n}{2n+1}+\frac{1}{2n}>0,$$

then $D(0)\le D(\pi)$. Moreover,

$$\begin{aligned}D\left(\theta_j\right)&=-\cos\theta_j+\frac{1}{2n+1}\cos(2j-1)\pi+\frac{1}{2n}\cos(2n+1-1)\theta_j\\&=-\cos\theta_j-\frac{1}{2n+1}-\frac{1}{2n}\cos\theta_j=-\left(\frac{2n+1}{2n}\right)\cos\theta_j-\frac{1}{2n+1},\end{aligned}$$

and

$$\begin{aligned}D\left(\theta_j'\right)&=-\cos\theta_j'+\frac{1}{2n+1}\cos(2n+1)\frac{j}{n}\pi+\frac{1}{2n}\cos(2j\pi)\\&=-\left(1-\frac{1}{2n+1}\right)\cos\theta_j'+\frac{1}{2n}=-\frac{2n}{2n+1}\cos\theta_j'+\frac{1}{2n}.\end{aligned}$$

We deduce that $D\left(\theta_j\right)\le D(\pi)$ and $D\left(\theta_j'\right)\le D(\pi)$ holds for each $j=1,2,\dots,n$. Thus, $D(\theta)\le D(\pi)$ for $\theta\in[0,\pi]$. This observation shows that

$$A(\theta)\ge A(\pi)=1-\frac{1}{8n^3}-\frac{(2n+1)(2n-1)}{4n^2}\left(\frac{2n}{2n+1}+\frac{1}{2n}\right)=0\ \text{ for } \theta\in[0,\pi].$$

Hence $\mathrm{Re}(e^{i\theta}f'(e^{i\theta})/f(e^{i\theta}))\ge 0$, which implies that f is starlike in $\mathbb{D}$. Summarizing, for each $n\ge 1$, the function f_n given by

$$\frac{z}{f_n(z)}=1+\left(1-\frac{1}{4n^2}\right)z+\frac{1}{4n^2}z^{2n+1},$$

belongs $\mathcal{M}$, and f_n is starlike in $\mathbb{D}$.

Example 4. Consider

$$f(z)=\frac{z}{\phi(z)},\quad \phi(z)=1+\left(1-\frac{\zeta(5)}{\zeta(3)}\right)z+\frac{1}{\zeta(3)}\sum_{n=2}^{\infty}\frac{z^n}{(n-1)^5}.$$

We may rewrite ϕ as

$$\phi(z)=1+\left(1-\frac{\zeta(5)}{\zeta(3)}\right)z+\frac{1}{\zeta(3)}\frac{z^2}{4!}\int_0^1\frac{(\log(1/t))^4dt}{1-tz}.$$

It is a simple exercise to see that $\phi(z)\ne 0$ in $\mathbb{D}$ and $f\in\mathcal{M}$. The Mathematica software is used to display the image of the unit disk under f as shown in Figure 1. It apparently displays that $f(\mathbb{D})$ is a starlike domain.

Example 5. It is illustrative to present a general example showing that functions in $\mathcal{U}$ do not necessarily belong to $\mathcal{S}^*$. For $n\ge 3$, consider the function

$$f_n(z)=\frac{z}{1+ibz+(1/(n-1))e^{2i\beta}z^n}.$$

For $|b|\le(n-2)/(n-1)$ and β a real number, then

$$\mathrm{Re}\left(\frac{z}{f_n(z)}\right)>1-|b|-\frac{1}{n-1}\ge 0,$$

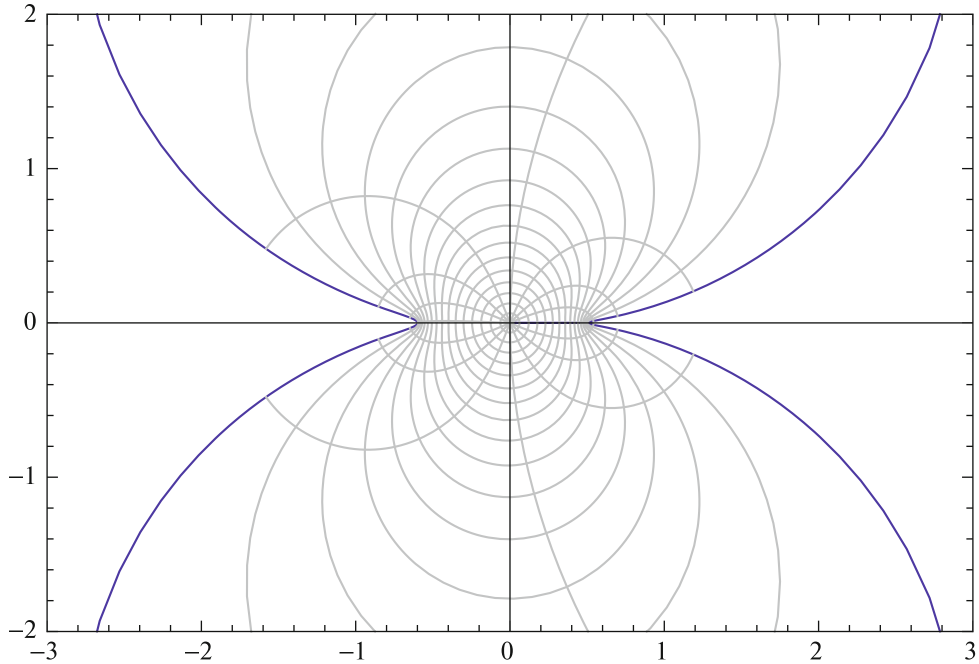

Fig. 1.

and

$$\left|\left(\frac{z}{f_n(z)}\right)^2 f_n'(z)-1\right| = \left|-e^{2i\beta}z^n\right| < 1 \quad \text{for} \quad z\in\mathbb{D},$$

so that $f_n \in \mathcal{U}$ for each $n \geq 3$. On the other hand, f_n is not in $\mathcal{S}^*$ when $0 < b \leq (n-2)/(n-1)$ and $0 < \beta < \arctan(b(n-1)/(n-2))$. This follows on account that

$$\mathrm{Re}\left(\frac{zf_n'(z)}{f_n(z)}\right)\Big|_{z=1} = \frac{[(2(n-2)/(n-1))\sin\beta - 2b\cos\beta]\sin\beta}{|1+ib+(e^{2i\beta}/(n-1))|^2} < 0.$$

Example 6. Consider the function f defined by $z/f(z) = 1+(1-\alpha)z+\alpha z^m$, where $\alpha \in (0,1)$ and $m \geq 3$ is an odd integer such that $\alpha m(m-1) = 2$. Then $z/f(z) \neq 0$ in $\mathbb{D}$ and

$$\left|\left(\frac{z}{f(z)}\right)''\right| = |\alpha m(m-1)z^{m-2}| < \alpha m(m-1) = 2,$$

and therefore, $f \in \mathcal{P}$. As in Example 3,

$$\mathrm{Re}\left(\frac{e^{i\theta}f'(e^{i\theta})}{f(e^{i\theta})}\right) = \frac{A(\theta)}{|1+(1-\alpha)e^{i\theta}+\alpha e^{im\theta}|^2},$$

where

$$A(\theta) = 1+(1-\alpha)\cos\theta - \alpha(m-2)\cos(m\theta) - \alpha(1-\alpha)(m-1)\cos(m-1)\theta - \alpha^2(m-1).$$

Substituting $\alpha = 2/(m(m-1))$ and $m = 2n+1$ ($n \geq 1$), the last expression for $A(\theta)$ reduces to

$$A(\theta) = 1 - \frac{2}{n(2n+1)^2} + \frac{2n-1}{n(2n+1)}D(\theta), \tag{3}$$

where

$$D(\theta) = (n+1)\cos\theta - \cos(2n+1)\theta - \frac{2(n+1)}{2n+1}\cos 2n\theta.$$

Table 1. Values of $A(\theta)$ for certain choices of θ

n	value of $A(\theta)$	n	value of $A(\theta)$
1	-0.0258011	8	-0.000243709
2	-0.0103986	9	-0.000154718
3	-0.00437311	10	-0.0000989276
4	-0.00211511	11	-0.0000628326
5	-0.00113174	12	-0.0000388937
6	-0.00064961	13	-0.000022708
7	-0.00039145	14	-0.0000116051

To prove that f is not starlike in $\mathbb{D}$, it suffices to show that $A(\theta) < 0$ for some $\theta \in (-\pi, \pi)$. In the case of $m = 3$ (i.e. $n = 1$), it is a simple exercise to see that

$$A(\theta) = \frac{1}{9}(1 + \cos\theta)(11 + 4\cos\theta - 12\cos^2\theta),$$

which is clearly negative for θ near π. Indeed, substituting $\cos\theta = -8/9$ or $\theta_0 = 6\pi/7$, it can be verified that $A(\theta) \approx -55/2187 < 0$, and $A(\theta_0) \approx -0.25811 < 0$. Thus, the function

$$f_3(z) = \frac{z}{1 + \frac{2}{3}z + \frac{1}{3}z^3} = \frac{3z}{(1+z)(3 - z + z^2)}$$

belongs to $\mathcal{P}\backslash\mathcal{S}^*$.

To do away the problem for some other values of n, we proceed as follows. Set

$$\theta = \frac{2(2n+1)\pi}{4n+3} \quad \text{and} \quad \phi = \frac{\pi}{2(4n+3)}$$

so that $\phi = (\pi - \theta)/2$. Then $\cos\theta = -\cos 2\phi = 2\sin^2\phi - 1$, $\cos(2n+1)\theta = -\cos 2(2n+1)\phi = -\sin\phi$, and $\cos 2n\theta = \cos 4n\phi = \sin 3\phi = 3\sin\phi - 4\sin^3\phi$. Thus, $A(\theta)$ given by (3) can be simplified leading to

$$A(\theta) = 1 - \frac{2}{n(2n+1)^2} - \frac{2(2n-1)(n+1)}{2n(2n+1)} + \frac{2n-1}{n(2n+1)}\left[2(n+1)\sin^2\phi - \frac{4n+5}{2n+1}\sin\phi + \frac{8(n+1)}{2n+1}\sin^3\phi\right].$$

It is seen from the computer algebra system Mathematica that $A(\theta) < 0$ for $n = 1, 2, \ldots, 15$. For easy reference, Table 1 lists the values of $A(\theta)$ for $n = 1, 2, \ldots, 14$.

Thus, we conclude that the above procedure helps us to show that for each $n \in \{1, 2, \ldots, 14\}$, the function f_n given by

$$\frac{z}{f_n(z)} = 1 + \left(1 - \frac{1}{n(2n+1)}\right)z + \frac{1}{n(2n+1)}z^{2n+1}$$

is not starlike in $\mathbb{D}$. By a minor modification in the choice of θ, one can show that f_n is not starlike for some $n \geq 15$ although it is not clear whether f_n is starlike for larger values of n.

The ideas and the motivations behind the above examples lead to the following

Conjecture . *The class $\mathcal{M}$ is not contained in $\mathcal{S}^*$.*

FUNDING

R.M. Ali gratefully acknowledged support from a Universiti Sains Malaysia research university grant 1001/PMATHS/8011101. The work of the third author is supported by Mathematical Research Impact Centric Support of DST, India (MTR/2017/000367).

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

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