Ab Initio Thermodynamics Calculation of Beta Decay Rates

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Beta-decay half-lives for the free neutron, for ^{6}He and ^{8}He, and for ^{8}Li are calculated ab initio from geometrical thermodynamics arguments, independently of any quantum mechanics. Half-lives for the decay of ^{8}Be to two alphas and for the disintegration of the tetraneutron are also calculated. The calculated values are close to those experimentally observed.

1. Introduction

We develop our previous quantitative geometrical thermodynamics (QGT) description of the key geometric features associated with the halo isotopes of helium[1] to show how the beta decay rates of these and related nuclear isotopes are also determined by the entropic geometry. Previously we showed that the proton radius, as the appropriate characteristic length scale, determines the holographic calculation of the entropy of both the halo isotopes of helium (^{6}He, ^{8}He) and also the $A = 4n$ series of nuclei (^{4}He, ^{8}Be, ^{12}C, ^{16}O, ^{20}Ne, ^{24}Mg, ^{28}Si, ^{32}S, ^{36}Ar, ^{40}Ca), and that given this characteristic length (and the QGT formalism) the sizes of these nuclei can be accurately calculated ab initio (that is, without any quantum mechanics). Previously we also remarked, without elaboration, that the $^6\text{He}\rightarrow{}^6\text{Li}$ beta decay could be interpreted as adding an extra integer "degree of freedom" (DoF; noting that the QGT treatment depends on understanding the number of DoFs appropriate for each system considered). Here we give the details of this entropic interpretation of beta decay.

In QGT the alpha particle (in its ground state) is treated as a unitary entity (than which exists nothing simpler: that is, at the scale considered, the entity has no internal degrees of freedom). In other words, using Henry More's 17th century terminology[2] in QGT the alpha is considered to be an "atom": not strictly "indivisible" of course, but "indiscerpible", that is, "*not easily torn asunder*"; which is to say that although its binding energy is finite, it is also very high. QGT emphasizes the wholeness of Maximum Entropy entities since it is based on the properties of holomorphism ("holomorphic" is literally "*the shape of wholeness*"). In the modern (rather restricted) usage, "unitary operator" is taken to mean that the operator's reciprocal is given by its adjoint. But here we employ an older and wider usage of *unitary*, applied now to real entities, not mathematical objects; a usage which (as we showed previously in Parker et al.[1]) promotes accurate calculations not only of the matter (and charge) radii of ^{6}He and ^{8}He, but also of the matter radii of the $A = 4n$ series of nuclei (^{4}He ... ^{40}Ca).

This previous QGT treatment (whose formalism was introduced in 2019[3]) was entirely geometrical, describing stable systems (that is, not changing in time), with examples of both DNA and spiral galaxies[3], atomic nuclei[1], fullerenes[4] and black holes[5] Black holes are an interesting case since although they are Maximum Entropy, their entropy production (that is, the rate of change of entropy which is a Noether-conserved quantity[5]) is non-zero: that is, a black hole continues to be the same unitary entity even though it is growing. All black holes are self-similar and very simple, being completely determined by only four scalars: mass, spin, charge, and a scaling factor which for all black holes is the Planck length: Frank Wilczek (2021[6] ch.3 p.73) omits this factor.

These examples underline the scale independence that a valid thermodynamics must have; as is to be expected from its essentially logarithmic nature, which in turn is in accordance with the hyperbolic spacetime of our universe).[3] In addition, we emphasize the application of the holographic principle[7] (and see refs. [1, 3]) to QGT as a means to determine the entropy of a system: the length scale and associated surface area both contributing to the overall *scale-independent* formulation of QGT. We now start to address the case of temporally changing systems, in particular, three systems undergoing beta-decay: i) the free neutron (with a half-life of 608 s); ii) ^{6}He (with a half-life of 807 ms); and ^{8}He (with a half-life of 119 ms). We will show how an ab initio QGT treatment recovers these lifetimes. Until now such temporal quantities have never been obtained from a QGT treatment. We remark that measuring the lifetime of the free neutron turns out to be surprisingly non-trivial: significant experimental effort has been required.

For completeness, we will also sketch a possible QGT treatment of the ^{8}Li and ^{8}Be lifetimes—the latter being a disintegration event, not a beta decay one. Moreover, since in our QGT treatment, we could represent the ^{8}He nucleus as being treated as a clathrate compound of a "tetraneutron" with an alpha guest, and since the (unbound) tetraneutron has recently been observed[8] we also sketch a possible QGT calculation of the free tetraneutron disintegration half-life.

Beta decay is currently being intensively investigated since it is central to the rapid neutron-capture process in stellar

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nucleosynthesis[9] (the so-called "r process") which accounts for about half of the heavy elements (Fe-U) in the solar system. The problem is that this process is not understood in detail, and although there are systematic measurements of beta-decay rates[10,11] many isotopes currently defy measurement (underlining the importance of reliable modeling), and anyway the interpretation of such data is difficult so there are various different approaches in the literature. Nuclear density functional theory (and in particular the "*deformed quasiparticle random-phase approximation*") was used to calculate β-decay properties of neutron-rich Ge, Se, Kr, Sr, Ru, and Pd isotopes[12] specifically with the r-process in mind and recognizing the importance of the nuclear deformations intrinsic to the process[13] Deformed geometries are expected to be amenable to a QGT approach: so far only the simplest case of the spherically symmetrical Buckminsterfullerene (we have already mentioned the scale independence of thermodynamics) has been explicitly treated by QGT[14] but other fullerenes should also be amenable to QGT even though most are aspherical. Mustonen & Engel[15] underline the importance of treating deformation, noting that "*Computational barriers have thus far prevented the production of a beta-decay table for the entire nuclear chart in a fully self-consistent Skyrme mean-field approach that allows deformation.*" Recently Ravlić et al.[16] have used comparable (but extended) methods to include temperature effects, which clearly suggests that a thermodynamics approach would be appropriate (if feasible). Helpfully for us, they point out that "*nuclear β-decay is a fundamental process in atomic nuclei*" playing "*a decisive role in nuclear astrophysics [the r-process] and particle physics as well as for the properties and structure of nuclei.*"

The extensive work cited here makes it very clear that the number of nucleons (odd or even) strongly influences the β-decay process: this is also plain from QGT (including the present work: see **Table** C1 for example) since the QGT treatment of probability itself distinguishes odd and even terms (see §5.4 of Parker & Jeynes 2023[17]). The practical aim of the quantum mechanics work is to understand the r-process, and the nuclear "shell models" have been drivers for understanding these phenomena. In their extensive review of the shell model, Caurier et al.[18] approach it "*as a unified view of nuclear structure*", and they conclude: "*The shell model has been craft and science: one invented model spaces and interactions and forced them on the spectra. Sometimes it worked very well.*" Systematic shell model calculations of specific cases have notably been made by the RIKEN group[19,20] Finally, Ferretti et al.[21] address the β-decay of specific (odd-numbered) isotopes of Rh analytically, using an extension of the Interacting Boson Model, a model that has strong resonances with the Maximum Entropy (QGT) treatment of the neutron halos of ^{6}He and ^{8}He ref. [1]. Nomura[22] uses all the techniques we have mentioned (tied together in a shell model) to analytically investigate the behaviour of the many isotopes of Ge and As (systematically different for the odd- and even-numbered isotopes). This whole field is very active (if rather heuristic), and we expect that QGT will be able to contribute a firm thermodynamics basis for it.

2. QGT Formalism

We first exploit the fact that the entropy production Π is a Noether-conserved quantity[5] (just as is the energy E), together with the isomorphism[3] between the (kinematic) Planck constant h and the (entropic) Boltzmann constant k_B. The appropriate relation for the change in entropy production $\Delta\Pi$ isomorphic to the Planck–Einstein relation $\Delta E = hf$ for the change in energy ΔE (where f is the frequency of the emitted radiation) is:

$$\Delta\Pi = \Pi_1 - \Pi_2 = k_B f = \frac{k_B c}{\lambda} \tag{1}$$

where c is the speed of light, and we invoke the (entropic) length scale λ associated with the phenomenon being described, such that $c = f\lambda$ as usual. Appendix A (Equation (A7)) shows how the Entropy Production Π is related to the exponential-decay time constant τ of a process together with the change in the number of DoFs of the system Δ:

$$\Pi = \frac{e^{-\Delta}}{\tau} S_0 \tag{2}$$

where S_0 is the initial (or background) entropy of the system, and we use "degrees of freedom" (DoFs) in the same way as explained previously.[1] From Equation (2) we see that the entropy production is inversely proportional to the temporal decay constant τ (as might be expected), but also varies according to the *negative* exponential of the change in the number of degrees of freedom Δ. This second aspect is interesting (and perhaps non-intuitive) since it implies that the larger the increase in Δ due to a decay event (in effect, the larger the increase in entropy), the lower the associated entropy production Π. That is, an increase in the number of DoFs is analogous to the "energy barrier" seen in quantum-mechanical tunneling (equivalent to the Arrhenius activation energy).

Previously,[1] using the example of the decay of ^{6}He to the stable ^{6}Li, we noted that the observed nuclear radius change after a beta decay event (interpreted by QGT) indicates that the entropy *increases by a single degree of freedom* when a nuclear neutron disintegrates; that is for beta decay the change in the number of degrees of freedom is Δ=+1.

The quantity Δ represents the change in the number of DoFs, but as a physical quantity (quantized by the Boltzmann constant, see Equation (A2)) is closely related to the entropy. The term $e^{-\Delta}$ in Equation (2) looks formally like a probability term of the partition function, and we have previously demonstrated[7] that the Schrödinger Equation is isomorphic to a probability term of the *entropic* Partition Function (defined by path integrals obeying the stationary principle).

More specifically, while the Principle of Least Action allows the argument of the exponential associated with the Schrödinger Equation formalism to be interpreted as a stationary sum-over-histories solution to the physical phenomenon under consideration; so too, for the *entropic* Partition Function, the integral of the *entropic* Hamiltonian also represents a stationary sum-over-histories solution according to the Principle of Least Exertion (isomorphic to the Principle of Least Action[3]). This is a consequence of the Planck constant (the quantum of action) being isomorphic to the Boltzmann constant (the quantum of entropy). Note also that the "exertion" quantity in QGT is equivalent to the quantity that Edwin Jaynes calls "caliber".[23]

Therefore, the quantity Δ can also be interpreted as representing the appropriate entropic sum-over-histories that achieves a particular change in the number of DoFs over the course of a

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Table 1. Half-lives of various systems, measured (Gonzales et al.[28] Audi et al.[29] Duer et al.[8]), and calculated by QGT. Gonzales et al. and Duer et al. give "lifetimes": these are converted to half-lives here. The neutron and tetraneutron ("tn") measurements may have a significantly larger uncertainty than the estimates given (see text). These values are all obtained using a proton radius of $r_p = 0.84087$ fm (Carlson, 2015[30]).

System	Measured half-life [s]	Reference half-life	Calculated half-life [s]	Characteristic length λ
n	608.4 ± 0.4	Gonzales et al. 2021	605.6	Hyperbolic radius $r_p/\ln 2$
^{6}He	0.80692 ± 0.00024	Audi et al. 2017	0.766	Proton diameter $2r_p$
^{8}He	0.1191 ± 0.0012	Audi et al. 2017	0.1125	Proton diameter $2r_p$
^{8}Li	0.83940 ± 0.00036	Audi et al. 2017	0.766	Proton diameter $2r_p$
^{8}Be	$(8.2 \pm 0.4) \times 10^{-17}$	Audi et al. 2017	7.9×10^{-17}	Proton diameter $2r_p$
tn $\equiv$ 4n	$(2.6 \pm 0.6) \times 10^{-22}$	Duer et al. 2022	2.4×10^{-22}	Hyperbolic radius $r_p/\ln 2$

physical process. For the "simple" beta-decay process, the overall change in the number of DoFs is only a single unit and $\Delta = 1$. However, for other processes where the associated energy landscape is more complex, and where the overall change in the number of DoFs is larger, then the number of different pathways is correspondingly larger such that $\Delta > 1$.

Parenthetically, we emphasize that the dimensionless Δ is not conserved, just as action (with units of energy-time) is not a conserved quantity; rather it is the *temporal differential* either of *entropy* or of *action* (that is, either the *entropy production* or the *energy*) that is conserved. Thus, although we are interested in the change in the number of degrees of freedom Δ, this does not arise from needing to account for the appearance or disappearance of DoFs. Rather, we recognize that as a physical process proceeds Δ is not conserved, but can be interpreted as representing the number of pathways for the DoF-change (a "sum-over-histories") for the process to proceed. Appendix B indicates how this might work for the case of the decay of the tetraneutron.

Recapitulating, Equation (1) relies on the isomorphism between the energy quantum hf (representing a photon) and the entropy-production quantum $k_B f$ (where in this case $f \equiv c/\lambda$). It seems that the physical significance of the quantity $k_B c/\lambda$ on the RHS of Equation (1) should be regarded (isomorphically to the photon energy) as a quantized particle of entropy production which we will call the **tropon** (a neologism sharing its etymology with "entropy": both coming from τροπή, "transformation"). The tropon is closely related to the "degrees of freedom" of the system, contributing to the structural (geometrical) properties emphasized by Prigogine in ref. [24].

Just as the photon follows a kinematic pathway that minimizes the action, so the tropon follows an entropy pathway that maximizes the entropy (or in QGT terms, minimizes the exertion).[3] It seems that the quantum of entropy should be the Degree of Freedom (DoF), which can only exist as an integer unit; or, perhaps, half-integer—see Appendix C for a discussion of this point using a comparison of the measured charge radii (from the compilation of Angeli 1999 (ref. [25])) with the values calculated by QGT. That is, the DoF should be considered as some kind of "entropic particle" (and of course, particles are not conserved in general). We have already shown that the energy and entropy production of a system are two sides of the same coin, being Hilbert transforms of each other[26] (see Appendix D for more background on this.)

Whereas the scale (the energy) of the photon is controlled by f (its frequency), the scale of the isomorphic tropon (the "entropic particle" related to the DoF) is controlled by λ (its "wavelength"). That is to say, the quantum entity (the tropon) associated with the entropy production has its scale controlled by the associated scale length λ. In QGT, the scale length determines the scale of the physical process under investigation. Just as in relativity, the frame of reference must be identified to correctly describe the physics, so also in the context of *scale relativity*[27] the scale of the physical process being described must also be identified. In the particular cases described here (mostly involving beta decay), we will show that the proton radius determines the appropriate scale of these nucleonic phenomena.

As already discussed, an entity may be described as unitary *at a given scale*. It is clear that the number of DoFs associated with any physical entity is a function of the scale used to view the entity: for a unitary entity the number of DoFs is minimum; while at a smaller scale (in general, associated with higher energies) the number of DoFs will increase as the finer granularity exposes the existence of additional entities (the alpha has four nucleons for example). This again emphasizes that Δ (like entropy) is not conserved but varies according to scale. Thus, adopting the proton radius as the length scale in this analysis intrinsically determines the number of DoFs appropriate to the systems considered. Just as studying the energetic (kinematic) behavior of an atomic system requires photons of appropriate frequencies for effective spectroscopy; so, too, the "correct" length scale must be adopted in order to achieve a useful analysis of the entropic behavior of a given entity.

The half-life $t_{½}$ of a radioactive process is related to the associated (exponential) time constant τ via $t_{½} = \tau \ln 2$, so that from Equation (2) the entropy production associated with a beta-decay nuclear disintegration is simply given by:

$$\Pi = \frac{e^{-\Delta_\beta}}{\tau} S_0 = \frac{\ln 2 e^{-\Delta_\beta}}{t_{1/2}} S_0 \tag{3}$$

where Δ_β represents the (quantized) increase in entropy (+1; $\Delta_\beta = +1$) due to the beta-decay.

For the radioactive system progressions that we consider here, the initial number of (relative) degrees of freedom of a system is given by Δ_1, while for the neutron-based systems the background (constant of integration) entropy is determined by

Table 2. Entropy and degrees of freedom (DoFs) associated with various neutron-based radioactive systems.

System	Relative DoFs Δ_1	S_0	$e^{-\Delta_1}$	$S_1 = S_0\, e^{-\Delta_1}$	β-decay and/or disintegration DoFs Δ	$S_2 = S_1\, e^{-\Delta}$	System Product	Relative DoFs Δ_2
n	−1	e^8	e^1	e^9	+1	e^8	p	0
^{6}He	6	e^8	e^{-6}	e^2	+1	e^1	^{6}Li	7
^{8}He	7	e^8	e^{-7}	e^1	+1	e^0	^{8}Li	8
^{8}Li	8	e^8	e^{-8}	$e^0 = 1$	+1, −2	$e^{-1+2} = e^1$	$2\times\alpha$	6
^{8}Be	9	e^8	e^{-9}	e^{-1}	+35.5	$e^{-36.5}$	$2\times\alpha$	N/A
tn ≡ 4n	4	e^8	e^{-4}	e^4	+51.5	$e^{-47.5}$	$4\times$ n	N/A

S_0 (which is related, but not necessarily equal, to the background entropy), and which has a number of degrees of freedom given heuristically by $\Delta_0 \equiv -8$, so that S_0 is given by $\ln S_0 = -\Delta_0 = 8$. The initial entropy state S_1 is:

$$S_1 = S_0\, e^{-\Delta_1} \tag{4a}$$

The end entropy of a system S_2 obtained by a single beta-decay process from initial system S_1 is:

$$S_2 = S_1 e^{-\Delta_\beta} = S_0 e^{-\left(\Delta_1+\Delta_\beta\right)} = e^{-\left(\Delta_0+\Delta_1+\Delta_\beta\right)} \tag{4b}$$

Table 1 shows a summary of the unstable systems that we analyze using QGT as the underlying theory to calculate the expected half-lives. **Table 2** collects the various parameters needed to calculate the expected half-lives.

3. n→p

The simplest case is the decay of a free neutron to a proton: we assume a (relative) initial number of degrees of freedom $\Delta_1 = -1$ (see Table 2). Since the resulting proton is unconditionally stable, the entropy production associated with it is zero, $\Pi 2 = 0$. Then from Equation (1), the entropy production of the free neutron Π_n is:

$$\Pi_1 \equiv \Pi_n = \frac{S_2}{t_{1/2}/\ln 2} = \frac{e^{+8}}{t_n/\ln 2} = \frac{k_B c}{\lambda} \tag{5}$$

where $t_{1/2}$ is the half-life of the entity, t_n is the half-life of the free neutron, and where $\lambda = r_p/\ln 2$ with the proton radius given by $r_p = 0.84087$ fm[30] is in this case taken to be the proton radius obtained from meson scattering (which is consistent with the new value from the "proton charge radius experiment"[31]). Thus, t_n is:

$$t_n = e^8 \ln 2\, \frac{\lambda}{k_B c} = 605\ \text{s} \tag{6}$$

which is within 0.5% of the half-life value of 608 s reported recently (2021) by the UCNτ collaboration.[28]

Interestingly, Wanpeng Tan's 2023 review[32] of the various measurements of the free neutron half-life concludes that its real uncertainty is larger than might be expected from the claimed uncertainty of the various experimental results. Even apparently equivalent methods yield values that differ much more than the estimated uncertainties. Tan calls this the "Neutron Lifetime Anomaly" and assigns an evaluated combined uncertainty of ≈1% to the (nominally) high-precision measurements. In particular, he lists a wide variety of measurements with hugely differing values, including a magnetic trap ("ultracold neutron"; UCN) measurement of the lifetime 707 ± 20 s (equivalent to a half-life of 490 ± 14 s) at NIST by Craig Huffer (see ref. [33]). This latter used a Ioffe-type magnetic trap, but following these anomalously low results the measurements were discontinued.

4. ^{6}He→^{6}Li

A similar analysis applies for the ^{6}He→^{6}Li beta decay, where ^{6}Li is absolutely stable ($\Pi_2 = 0$). Previously,[1] we demonstrated that for ^{6}He the (relative) initial number of degrees of freedom $\Delta_1 = 6$ (see Table 2) and therefore the entropy production is:

$$\Pi_1 \equiv \Pi_{6\text{He}} = \frac{S_2}{t_{1/2}/\ln 2} = \frac{e^{+1}}{t_{6\text{Li}}/\ln 2} \tag{7}$$

Again using Equation (1) and now taking λ as the proton diameter (as for the QGT treatment of the helium isotopes in ref. [1]), the calculated half-life of ^{6}He is:

$$t_{6\text{Li}} = e^1 \ln 2\, \frac{\lambda}{k_B c} = 765\ \text{ms} \tag{8}$$

which is within about 5% of the measured 807 ms (Tilley et al., 2002[34]).

5. ^{8}He→^{8}Li

For the ^{8}He→^{8}Li beta decay, the resulting entropy production is non-zero ($\Pi_2 > 0$) since the ^{8}Li is not stable. Previously,[1] we demonstrated that for ^{8}He the (relative) initial number of degrees of freedom is given by $\Delta_1 = 7$ (see Table 2). The number of degrees of freedom associated with ^{8}Li is less clear. One measurement[35] of the ^{8}Li nucleus determines a neutron radius (i.e., matter radius) of 2.68 ± 0.09 fm which is consistent with the ^{8}Li nucleus having 8 degrees of freedom, $\Delta_2 = 8$, and with beta decay causing a +1 increase in the nuclear DoFs. However, further interpretation of these measurements[36] determine the radius of the ^{8}Li valence neutron as 2.58 ± 0.48 fm, corresponding to $\Delta_2 = 7.5$ DoFs. Therefore, we assume that the resulting ^{8}Li nucleus immediately after the decay of ^{8}He has a radius of

2.68 fm ($\Delta_2 = 8$) but that it quickly relaxes to a lower energy state corresponding to a radius of 2.58 fm and also with a lower geometrical entropy given by $\Delta_2 = 7.5$. That is to say, the ^{8}Li nucleus loses 0.5 DoF as it relaxes to a lower energy state, prior to decaying to ^{8}Be (see Appendix C on half-integer DoFs). In which case, the entropy production of the ^{8}He nucleus (as it decays to the initially higher energy state ^{8}Li nucleus with $\Delta_2 = 8$) is given by:

$$\Pi_1 \equiv \Pi_{8\text{He}} = \frac{S_2}{t_{1/2}/\ln 2} = \frac{S_0 e^{-(\Delta_1+\Delta_\beta)}}{t_{8\text{He}}/\ln 2} = \frac{e^0}{t_{8\text{He}}/\ln 2} \tag{9}$$

where

$$\Delta_2 = \Delta_1 + \Delta_\beta \tag{10}$$

In order to calculate the entropy production associated with the decay of ^{8}Li (from its assumed lower energy state with DoFs given by $\Delta_2 = 7.5$) we also need a clear understanding of the degrees of freedom associated with the resulting product ^{8}Be, in order to calculate the change in DoFs during the decay process toward stability. In particular, in previous work[1] we have asserted that the number of degrees of freedom associated with the ^{8}Be nucleus is $\Delta_3 \equiv \Delta_{8\text{Be}} = 6$; although QGT theory for beta decay suggests that the DoFs of ^{8}Be should be one more (+1) compared with the DoFs of ^{8}Li, ($\Delta_{8\text{Be}} = 9$, $\Delta_{8\text{Li}} = 8$, as given in Table 2). However, the high instability of ^{8}Be (which rapidly decays to two alphas, see following two sections §§6,7) suggests that its previously assigned[1] value $\Delta_3 \equiv \Delta_{8\text{Be}} = 6$ DoFs is appropriate for when the ^{8}Be nucleus is essentially discernible as two (highly stable) alphas. This means that when a ^{8}Li nucleus (at the lower energy state with assumed matter radius of 2.58 fm and $\Delta_2 = 7.5$) beta decays to increase its DoFs to 8.5, then as part of the decay process toward stability, 2.5 DoFs must be lost such that the resulting ^{8}Be nucleus (or, in effect, two alphas) ends up with $\Delta_3 = 6$ DoFs. For the 3rd stable product (^{8}Be, or equivalently two alphas) of the decay sequence $^8\text{He}\rightarrow{}^8\text{Li}\rightarrow{}^8\text{Be}$ the resulting entropy is therefore given by:

$$S_3 = S_0 e^{-(\Delta_2+\Delta_\beta-2.5)} \tag{11}$$

where $\Delta_2 = 7.5$, and 2.5 DoFs (explicit in Equation (11)) are additionally lost in the decay (disintegration) process. Then the entropy production of ^{8}Li is given by:

$$\Pi_2 \equiv \Pi_{8\text{Li}} = \frac{S_3}{t_{1/2}/\ln 2} = \frac{S_2 e^{-(\Delta_\beta-2.5)}}{t_{8\text{Li}}/\ln 2} = \frac{e^{+1.5}}{t_{8\text{Li}}/\ln 2} \tag{12}$$

Given the half-life of ^{8}Li is known to be $t_{8\text{Li}} = 839$ ms,[36] the entropy production of ^{8}Li is therefore:

$$\Pi_2 = \Pi_{8\text{Li}} = \frac{e^{1.5}}{0.839/\ln 2} \tag{13}$$

and the half-life of the ^{8}He nucleus can be calculated (using Equation (1)) via:

$$\Pi_1 = \frac{k_B c}{\lambda} + \Pi_2 \rightarrow \frac{e^0}{t_{8\text{He}}/\ln 2} = \frac{k_B c}{\lambda} + \frac{e^{1.5}}{0.839/\ln 2} \tag{14}$$

$$t_{8\text{He}} = \left(\frac{k_B c}{\lambda \ln 2} + \frac{e^{1.5}}{0.839}\right)^{-1} = 112 \text{ ms} \tag{15}$$

which is within about 6% of the measured 119 ms (Tilley et al. 2004[37]).

6. $^8\text{Li}\rightarrow 2\times\alpha$

We analyze the decay of ^{8}Li using QGT by ignoring the intermediate (highly excited) ^{8}Be state. In this particular analysis, we do not directly consider the issues that may be involved with an entropic *anti*-Planck–Einstein process, where the end state of a process may finish at a *higher* entropy production level than at the outset. Rather, here we assume that, as a natural phenomenon, the decay of ^{8}Li describes the progression of moving from a lower stability (high entropy production) to a greater stability (lower entropy production) state according to the 2nd Law. From that perspective, we make the approximation that the end-state of ^{8}Li is assumed to directly reach the two (highly stable) alpha particles, so that we ignore the intermediate (highly excited) ^{8}Be state. (We explicitly consider this more complex ^{8}Be decay in the next section.)

In this regard, it is also important to note that whereas the ^{8}Be (as the natural decay product from ^{8}Li) is expected to have nine DoFs, the two alpha particles (that we are assuming here to be the end product of the decay of ^{8}Li) between themselves consist of only six DoFs (i.e., three DoFs each). Thus, in this approximate analysis of the decay of ^{8}Li we also need to take into account the overall loss of three DoFs from the system. In which case, the overall change in DoFs as ^{8}Li decays to the two alphas is $\Delta = +1 - 3 = -2$: the first +1 due to the beta decay, and then the "disintegration" loss of three DoFs in order to account for the six DoFs associated with the two alphas.

In our previous work,[1] and as indicated in Table 2, we demonstrated that for ^{8}Li the (relative) initial number of degrees of freedom $\Delta_1 = 8$ so that its entropy production is:

$$\Pi_1 = \Pi_{8\text{Li}} = \frac{S_2}{t_{1/2}/\ln 2} = \frac{e^{+2}}{t_{8\text{Li}}/\ln 2} \tag{16}$$

Again using Equation (1) and taking λ as the proton diameter, and assuming the absolute stability of the resulting two alpha such that $\Pi_2 \equiv \Pi_\alpha = 0$ the calculated half-life of ^{8}Li is therefore approximately given by:

$$t_{8\text{Li}} = \frac{e^2 \ln 2}{\left(\frac{k_B c}{\lambda} + \Pi_2\right)} \approx e^2 \ln 2 \frac{\lambda}{k_B c} = 2.080 \text{ s} \tag{17}$$

much higher than the observed value of 839 ms. But, if we take account of excited states in the decay process and ascribe an additional 0.5 DoF to each of the resultant alphas (such that each alpha ends up with 3.5 DoFs in an excited state, which then subsequently relaxes by the loss of a half-integer DoF into the stable 3 DoFs state) then for the decay rate of ^{8}Li we only need to account for the loss of two DoFs from the system. In which case, $\Delta = +1 - 2 = -1$, such that:

$$t_{8\text{Li}} \approx e^1 \ln 2 \frac{\lambda}{k_B c} = 0.765 \text{ s} \tag{18}$$

which is the same half-life as determined for ^{6}He. Equation (18) gives a value much closer to the observed half-life for ^{8}Li than does Equation (17). Thus it is clear that, when it progresses directly to the two stable alphas, our approximate determination of the half-life of ^{8}Li is too large (with the ^{8}Li calculated to be more stable than it really is); whereas if additional intermediate excited states are assumed in its decay progression, then the ^{8}Li becomes less stable and is calculated to have a shorter (and more realistic) half-life.

7. ^{8}Be→2 × α

The unstable ^{8}Be (which quickly decays into two alphas) can also be analyzed using QGT. In this case, the final decay of ^{8}Be to two alphas is not a beta-decay process, but is rather a process resulting from it being particle-unstable, and where the final "product" and "ejected" particles are indistinguishable. The conventional kinematic (electronic–photonic) analog to this disintegration process is equivalent to a *two-photon* decay of an electron from a high energy state to a low (stable) energy state, where two identical photons are emitted. This so-called "spontaneous parametric down-conversion in non-linear optics" has been observed by Hayat et al.[38] and Ota et al.[39] Whether or not the two photons are in reality created simultaneously, this process can be represented as a sequential one in which the high energy electron emits a first photon and enters a virtual (unstable) intermediate state (exactly midway between the initial and final states), before emitting a second (identical) photon and reaching its final low energy state.

In the same way as the frequency of the emitted photons is half that of the overall energy gap between the initial and final states, so the entropic frequency of the associated entropic Planck–Einstein relation must also be halved. Thus, we employ the following equation:

$$\Pi_{8\mathrm{Be}} - \Pi_\alpha = \frac{1}{2}\,\frac{k_\mathrm{B}c}{\lambda} \tag{19}$$

where $\Pi_{8\mathrm{Be}}$ is the entropy production of the ^{8}Be nucleus, while Π_α is the entropy production of a resulting alpha particle, which is zero (being unconditionally stable): $\Pi_\alpha = 0$. As before, the wavelength λ is taken as the proton diameter. Thus from Equation (19):

$$\Pi_{8\mathrm{Be}} = \frac{e^{-\Delta}}{\tau_{8\mathrm{Be}}} S_1 = \frac{1}{2}\,\frac{k_\mathrm{B}c}{\lambda} \tag{20}$$

where $\tau_{8\mathrm{Be}}$ is the time constant of the ^{8}Be decay, the quantity Δ represents as usual the change in the number of degrees of freedom associated with the decay event, while S_1 is the initial entropy of the ^{8}Be nucleus.

From previous work[1] (see Table 2 here), we know that the initial entropy of the ^{8}Be nucleus following beta-decay from ^{8}Li means that it is associated with $\Delta_1 = 9$ DoFs. Thus, using the same background entropy $S_0 = e^8$ the initial entropy of the ^{8}Be nucleus is given by:

$$S_1 = e^{-\Delta_1} S_0 = e^{-1} \tag{21}$$

However, an ^{8}Be nucleus with nine DoFs is in a highly excited and unstable state, and needs to shed DoFs in order to gain a more stable configuration. We now consider the change in entropy Δ associated with the decay of ^{8}Be that we require in Equation (20). Considering Equation (19) we start with an ^{8}Be nucleus with nine DoFs and end with two alpha particles (^{4}He nuclei) with three DoFs each. Thus Equation (19) is associated with the loss of six DoFs. But to argue that only three DoFs in total are lost is false reasoning. Rather, the process of Equation (19) indeed describes the loss of six DoFs, and moreover, there are also *two different (and indistinguishable) ways* in which those six DoFs are lost, since the two resulting alphas are identical and indistinguishable (see also the discussion of symmetries and indistinguishabilities in Appendix B). Then, according to the required permutational logic of the entropic process for the indistinguishable allowable pathways, the number of DoFs that are associated with the process must *multiply*, so that the required overall effective change in the number of DoFs is given by:

$$\Delta = (-6) \times (-6) = (-6)^2 = 36 \tag{22a}$$

Of course, this means that the total number of DoFs is necessarily not a conserved quantity in the decay process; not surprisingly, since it is the entropy production that is conserved.[5] Due to the relative stability of the resulting alpha(s) system we reduce the overall DoF change by a half DoF (the same adjustment is made to account for the relaxation of the excited state of ^{8}Li on the decay of ^{8}He (§5), while we also discuss the existence and significance of half-integer DoFs in Appendix C) such that:

$$\Delta = 36 - 0.5 = 35.5 \tag{22b}$$

Note, we also invoke such a "half DoF" for the tetraneutron calculation of §8 and as discussed in Appendix B. Using the half-life of ^{8}Be given by $t_{1/2} = \tau \ln 2$, and substituting for S_1 (Equation (21)) and Δ (Equation (22b)), Equation (20) yields:

$$t_{8\mathrm{Be}} = e^{-36.5} \ln 2 \frac{2\lambda}{k_\mathrm{B}c} = 7.92 \times 10^{-17}\ \mathrm{s} \tag{23}$$

This is indistinguishable from the empirically found half-life of $(8.2 \pm 0.4) \times 10^{-17}$ s for ^{8}Be (see Table 1).

8. tn→4 × n (Tetraneutron Decay)

From the principle of holomorphic pairing as explained for the case of ^{8}He (see Parker et al. 2022[1]), we assume that a tetraneutron (tn) is composed of two dineutrons (dn), each of which is composed, in turn, of two neutrons. Therefore, we assume that when a tetraneutron disintegrates, it is via the intermediate dineutrons:

$$\mathrm{tn} \rightarrow 2 \times \mathrm{dn} \rightarrow 4 \times \mathrm{n} \tag{24}$$

and consequently, that there are two sets of entropic Planck–Einstein equations to consider, each using the same analogous "two-photon decay" argumentation as used in §7 for the disintegration of ^{8}Be:

$$\Pi_{\rm tn} - \Pi_{\rm dn} = \frac{1}{2}\ \frac{k_B c}{\lambda} \tag{25}$$

$$\Pi_{\rm dn} - \Pi_{\rm n} = \frac{1}{2}\ \frac{k_B c}{\lambda} \tag{26}$$

where we assume the same proton-based "entropic bandgap" being filled by a two-particle process, and hence the factor ½. From the previous analysis of the beta-decay of a neutron (Equation (5)), the entropy production of a neutron is given by $\Pi_{\rm n} = k_{\rm B}c/\lambda$, giving the entropy production of the dineutron (dn) from Equation (26)

$$\Pi_{\rm dn} = \frac{3}{2}\frac{k_{\rm B}c}{\lambda} \tag{27}$$

Assuming the standard relations for the entropy production, the half-life, and the change in DoFs (Equation (2)):

$$\Pi_{\rm dn} = \frac{e^{-\Delta}}{\tau_{\rm dn}} S_0 e^{-\Delta_1} = \frac{e^{-\Delta}}{t_{\rm dn}/\ln 2} S_0 e^{-\Delta_1} \tag{28}$$

In which case, for a dineutron with $\Delta_1 \equiv \Delta_{\rm dn} = 3$, the half-life of the dineutron is given by:

$$t_{\rm dn} = \ln 2 S_0 e^{-\Delta_1} e^{-\Delta}\ \frac{2}{3}\frac{\lambda}{k_{\rm B}c} \tag{29}$$

We now only need to calculate the change in DoFs Δ associated with the disintegration of a dineutron. Considering the entropy production of a tetraneutron, we substitute Equation (27) into Equation (25)

$$\Pi_{\rm tn} = 2\frac{k_{\rm B}c}{\lambda} \tag{30}$$

Therefore, employing Equation (2) for the entropy production we have:

$$\Pi_{\rm tn} = \frac{e^{-\Delta}}{\tau_{\rm tn}} S_0 e^{-\Delta_1} = \frac{e^{-\Delta}}{t_{\rm tn}/\ln 2} S_0 e^{-\Delta_1} \tag{31}$$

such that for a tetraneutron with $\Delta_1 \equiv \Delta_{\rm tn}$, the half-life of the tetraneutron $t_{\rm tn}$ is given by:

$$t_{\rm tn} = \ln 2 S_0 e^{-\Delta_{\rm tn}} e^{-\Delta}\ \frac{\lambda}{2k_{\rm B}c} \tag{32}$$

Using $\Delta_{\rm tn} = 4$ and $\Delta = 51.5$ (see Appendix B) for the change in the number of DoFs as a tetraneutron disintegrates, and reverting back to the hyperbolic proton radius for the length scale of this purely neutron-based system $\lambda = r_p/\ln 2$, its half-life is therefore:

$$t_{\rm tn} = e^{-47.5}\ln 2\ \frac{\lambda}{2k_{\rm B}c} = 2.39 \times 10^{-22}\ {\rm s} \tag{33}$$

This is indistinguishable from the value $(2.6 \pm 0.6) \times 10^{-22}$ s recently quoted for the half-life of the tetraneutron[8] (see Table 1).

9. Discussion

To date quantitative geometrical thermodynamics (QGT) has been used to demonstrate the effect of the Second Law of Thermodynamics on Maximum Entropy systems, that is, systems stable in time. Normally one would not think that a stable entity such as the alpha particle would be subject to thermodynamics, but we have demonstrated that it is the *geometry* of the entity that embodies thermodynamics. Since Thermodynamics is fundamental it applies to everything. Curiously, the QGT treatment of alpha particles and of black holes is very similar: even though the entropy production of alphas is zero (trivially reversible) and the entropy production of black holes is positive (trivially irreversible) they are both "Maximum Entropy" systems: and in QGT we have demonstrated that reversibility may be treated commensurately with irreversibility.[26]

We have now demonstrated that there exists a coherent QGT account of beta decay, particularly the fact that beta decay is associated with an increase in the number of system degrees of freedom. We have already shown that the QGT formalism can account for the *chirality* of both DNA (see Parker & Walker 2010[40] and Appendix A of Parker & Jeynes 2019[3]) and also the fullerenes,[4] and therefore, we would also in the future expect QGT to also provide a new description of the well-known chirality of beta decay.[41]

Bohr was able to deploy the Planck–Einstein relation in the simplest electronic system represented by the hydrogen atom to derive the Balmer series but calculations for heavier atoms rapidly become intractable. This is not only because of the difficulties of many-body systems as such, but also because the possible number of energy states also rapidly increases. We expect similar considerations in these entropic calculations. Decay paths are complicated in detail and our present very simplistic approach is hardly likely to capture much of this complication. The decay of ^{8}He to ^{8}Li is complex, as is the further decay of ^{8}Li to ^{8}Be (which is not particle-stable). Therefore, the half-lives of ^{8}He and ^{8}Li are obtained here to within ≈5% of empirically determined values (and that we have shown good reasons from QGT to explain their similarity) seems encouraging.

It is worth emphasizing that the discussion of the "neutron mass anomaly" by Tan[32] indicates that independent neutron lifetime measurements differ by far more than the claimed uncertainties of individual measurements. Recent work by Kegel et al.[42] also cast doubt on the reliability of current theory even for (relatively) low energies. This puts the experimental uncertainties of Table 1 into a wider perspective. It may even be that our ab initio thermodynamics approach can contribute significantly to our interpretation of the measurements themselves.

It is notable in Table 1 that different characteristic lengths are required for the neutron systems (the free neutron and the tetraneutron) compared to the systems containing protons. The latter use the same characteristic length as for the QGT calculation of the sizes of the entities,[1] but the former use the corresponding "hyperbolic radius", so-called because the ln 2 factor involves the exponential function which is central to the "hyperbolic space" formalism of QGT.[3] We should also point out that 1/ln 2 is just the ratio of the decay lifetime and the half-life.

Both *energy* and *entropy production* are quantized, with "degrees of freedom" (DoFs) being associated with a quantum of entropy production and having properties of an entropic quantum object; it seems clear that just as there is a particle that expresses the energy quantization (the photon) there should be a particle that expresses the entropy production quantization (which we have called the "*tropon*"). Moreover, it has become clear that in a consistent model the Hilbert transform of the energy representation of the photon should be the entropy production representation of the tropon (and vice versa, see Parker & Jeynes 2023[26] and Appendix D). (Of course, in the energy representation there are many particles other than photons: at present it is an open question whether these also have their entropic counterparts.) DoFs are key formal descriptors of any geometrical (QGT) entity, and therefore intrinsic to issues of geometrical structure and order (as pointed out powerfully by Ilya Prigogine[24]).

It is also worth commenting that the form of Equation (2), with Δ appearing as an exponent, shows that this representation of "degrees of freedom" can account for the extremely large (many decades) dynamic range for observed decay rates that are seen in radioactive phenomena. Thus, half-lives ranging from 10^{-24} s to 10^{30} s "only" require a range for Δ of ≈124. The rigorous calculation of Δ as an entropic quantity using a sum-of-histories formalism is the work of future research, but here we show explicitly that an informal approach may be adequate for the simplest processes (such as beta decay).

We have mentioned that *scale invariance* is a necessary property of thermodynamics (see ref. [26]): this has also recently been proved to be an intrinsic property of QGT.[43]

It is necessary to set up a sophisticated mathematical apparatus to implement QGT, and so far this has been used to give a fundamental thermodynamics interpretation only of simple systems (without using any quantum mechanics or general relativity): various simple nuclei, DNA, Buckminsterfullerene, spiral galaxies, and black holes. Black holes are examples of Maximum Entropy entities with non-zero entropy production: as they grow they (ontologically) *remain* black holes. Here for the first time we now extend the QGT treatment to an *ontological change*: beta decay changes one entity into another at a characteristic rate (although we have no wish here to get into any metaphysical discussions: ontological and epistemological questions are addressed elsewhere).[44] QGT certainly applies to everything in principle, but has so far been applied only to very few things. We hope that its development will be rapid.

10. Conclusion

Treating the alpha particle as a unitary entity (than which exists nothing simpler), and the neutron halos of the ^{6}He and ^{8}He nuclei also as unitary entities, and using the Noether-conservation of entropy production: we show that the characteristic length λ associated with the holographic calculation of the entropy of the decay products of these entities is (approximately) a function of the size of the proton. This yields a decay rate accurate at about ½% for the free neutron beta-decay (n→p), for which λ is the hyperbolic proton radius; and rates close to those observed for the beta-decay of both ^{6}He (^{6}He→^{6}Li) and the more complicated ^{8}He (^{8}He→^{8}Li), for which λ is simply the proton diameter as before.

Just as energetic processes are mediated by quantum mechanical particles (like photons), entropic processes are mediated by *tropons* (thermodynamic "particles") whose entropy production is determined by how much change is needed in the number of degrees of freedom to transform the start-state of the process into the end-state. The quantum of energy is determined by Planck's constant, and the quantum of entropy production is determined by Boltzmann's constant; particles (of whatever kind) are not conserved, but rather it is both *energy* and *entropy production* that are Noether-conserved (and are very strongly related, being representable as Hilbert transforms of each other[26]).

Characterizing the decay processes with a troponic (QGT) formalism, we obtain half-lives for six simple processes which are close to the experimentally observed values (see Table 1): in the simplest case of the free neutron decay the QGT values are very close; that is, within the combined uncertainty of the experiments (according to one evaluation). It may be that these analytical thermodynamic methods will prove able to aid the interpretation of these (difficult) measurements.

The QGT formalism initially[3] treated only Maximum Entropy systems in thermal equilibrium (that is, with zero entropy production): spiral galaxies were approximated as exemplars of the double logarithmic spiral (which was treated rigorously showing a necessarily non-zero entropy production). The rigorous QGT formalism was only subsequently extended explicitly to black holes[5] which are Maximum Entropy systems even though they are *not* in thermal equilibrium since they necessarily have non-zero (positive) entropy production.

Here for the first time, we analytically treat temporal processes in which one system transforms into another. QGT is a general theory capable in principle of interpreting any process; here we start to show how this works in some simple cases. Previously,[1] we gave an analytical treatment of the halo nuclei of helium (^{6}He, ^{8}He) as semi-unitary entities, deriving their nuclear sizes (and ignoring their instability); here we show how to analytically treat the processes of (for example) ^{6}He→^{6}Li and ^{8}He→^{8}Li to derive the lifetimes of the initial state.

Even if the present simple treatment is clearly approximate in parts, it is important to point out that QGT is i) rigorous, and ii) independent of quantum mechanics. Furthermore, it is iii) arguably more fundamental than quantum mechanics since it manifestly also applies to cosmic phenomena, and there is not yet any theory of quantum gravity that commands a consensus.

Appendix A

Relation of Entropy Production to DoFs

The entropy production Π is given by:

$$\Pi = \frac{\partial S}{\partial t} \tag{A1}$$

where S is the system entropy and t is the time. The entropy itself can be expressed using Boltzmann's formula as:

$$S = k_B \ln W = \Delta k_B \tag{A2}$$

where $\Delta \equiv \ln W$ represents the system's effective number of degrees of freedom (DoFs) and W is the overall number of permutations. Taking the time derivative of Equation (A2) we find:

$$\frac{\partial S}{\partial t} = \frac{\partial W}{\partial t}\,\frac{1}{W}\,k_B = \frac{\partial W}{\partial t}\,e^{-\frac{S}{k_B}}\,k_B = \frac{\partial W}{\partial t}\,e^{-\Delta}k_B \tag{A3}$$

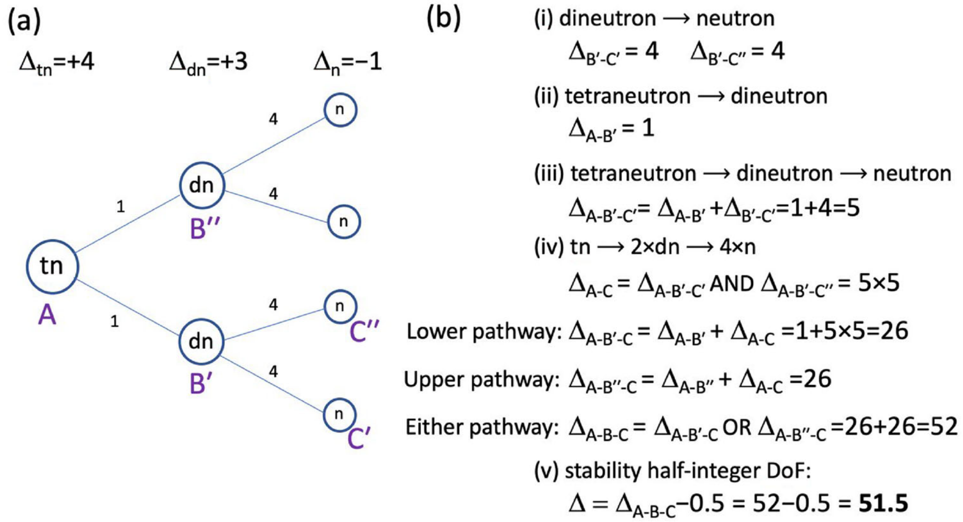

Figure B1. Sum over pathways for tetraneutron decay. Left-hand side: Pathways for tetraneutron disintegration into constituent dineutrons and neutrons; Right-hand side: Calculation of change in DoFs ($\Delta = 51.5$) according to the disintegration pathways of a tetraneutron.

Thus, we can express the entropy production as:

$$\Pi = \frac{\partial W}{\partial t}\, k_{\mathrm{B}} e^{-\Delta} \tag{A4}$$

In the context of the radioactive decay of an individual particle we assume that, in general, the entropy production is constant, both before and after the decay event; that is to say, the quantity $\partial W/\partial t$ is a constant. In which case, employing the exponential-decay time constant τ as part of the definition of $\partial W/\partial t$ we write:

$$\frac{\partial W}{\partial t}\, k_{\mathrm{B}} = \frac{\ln W_0}{\tau}\, k_{\mathrm{B}} \tag{A5}$$

We also need to understand the physical basis for the constant W_0 which represents a system constant related to the size of the Hilbert space (or the number of constituent parts of the system under consideration). We define the entropic quantity:

$$S_0 \equiv k_{\mathrm{B}} \ln W_0 \tag{A6}$$

to be *related to* (but not necessarily the same as!) the background entropy, which can be regarded as a constant of integration due to the logarithmic nature of entropy, and which may take an arbitrary value. Then the entropic version of the Planck–Einstein relation is:

$$\Pi = \frac{e^{-\Delta}}{\tau}\, S_0 = \frac{k_B c}{\lambda} \tag{A7}$$

or:

$$\ln\left(\frac{e^{-\Delta}}{\tau}\right) + \ln S_0 = \ln \frac{k_{\mathrm{B}} c}{\lambda} \tag{A8}$$

We note the analogy of Equation (A8) with the work function aspect of the Planck–Einstein relation:

$$E - W = hf \tag{A9}$$

such that the quantity $\ln S_0$ (related to the background entropy) is seen to be equivalent to the work function and therefore varies for different systems. For the neutron-based systems we're interested in, we empirically assume that $S_0 = e^8$ J/K as before.

Returning to Equation (A7), we express the differences between initial (A) and final (B) entropy productions following a decay (entropy increase) event (taking the kinematic Planck–Einstein relation as our template) as:

$$\Pi_{\mathrm{A}} - \Pi_{\mathrm{B}} = \frac{e^{-\Delta_{\mathrm{A}}}}{\tau_{\mathrm{A}}} S_0 - \frac{e^{-\Delta_{\mathrm{B}}}}{\tau_{\mathrm{B}}} S_0 = \frac{k_{\mathrm{B}} c}{\lambda} \tag{A10}$$

$$\left(\frac{e^{-\Delta_{\mathrm{A}}}}{\tau_{\mathrm{A}}} - \frac{e^{-\Delta_{\mathrm{B}}}}{\tau_{\mathrm{B}}}\right) S_0 = \frac{k_B c}{\lambda} \tag{A11}$$

Appendix B

Δ for Tetraneutron Decay to 4 × Neutrons

For the tetraneutron, we obtain the change Δ in the number of degrees of freedom (DoFs) as $\Delta = 51.5$ using the following calculation (see **Figure B1**):

$$\Delta = 2 \times \left[(1+4)^2 + 1\right] - 0.5 = 51.5 \tag{B1}$$

In QGT the tetraneutron is treated as Buckminsterfullerene (C_{60}),[4] that is a (holomorphic) shell with an (empty) interior. It is well-known that fullerenes may incorporate guest atoms or molecules in their hollow interiors which often serve to stabilize them. Such cage-like structures are called "clathrate" and have many applications. In particular, the neutron halo of ^{8}He is treated in QGT as a holomorphic pair of holomorphic neutron pairs ref. [1], and here we simply treat the tetraneutron as an ^{8}He without the alpha guest. Previously,[1] we showed that a holomorphic pair of (identical) entities have one extra DoF relative to that of the entity itself ($\Delta = +1$), and also that the clathrate structure has an extra three DoFs for the guest entity, which entity may or may not be present: for ^{8}He the guest (the alpha) is present but for the tetraneutron it is not present. From this perspective, it is clear that the number of DoFs associated with the tetraneutron is $\Delta_1 \equiv \Delta_{\mathrm{tn}} = 4$; this being three fewer than the DoFs associated with ^{8}He (which has $\Delta_1 \equiv \Delta_{8\mathrm{He}} = 7$, as per Table 2). Likewise, the dineutron (whose holomorphic pairing comprises a tetraneutron) should have one less DoF as compared with the tetraneutron's number of DoFs, such that for the dineutron $\Delta_1 \equiv \Delta_{\mathrm{dn}} = 3$. This is again consistent with a clathrate ^{6}He (with DoFs $\Delta_1 \equiv \Delta_{6\mathrm{He}} = 6$) absent its alpha particle guest.

In Figure B1 we regard the disintegration process of the tetraneutron as proceeding via the "dineutron". The "dineutron" is regarded as a

Table C1. Nuclear radii measured, and calculated holographically by quantitative geometrical thermodynamics (QGT).

Isotope	A	QGT DoF Δ	Measured fm	Calculated fm	Isotope	A	QGT DoF Δ	Measured fm	Calculated fm
^{4}He	4	3	1.68	1.64	^{3}H	3	2.5	1.76	1.50
^{8}Be	8	6	(2.2)	2.32	^{7}Li	7	5.5	2.40	2.23
^{12}C	12	7	2.47	2.51	^{11}B	11	6.5	2.41	2.42
^{16}O	16	8	2.71	2.68	^{15}N	15	7.5	2.61	2.60
^{20}Ne	20	9	3.01	2.85	^{19}F	19	8.5	2.90	2.77
^{24}Mg	24	10	3.06	3.00	^{23}Na	23	9.5	2.99	2.92
^{28}Si	28	11	3.12	3.15	^{27}Al	27	10.5	3.05	3.08
^{32}S	32	12	3.25	3.29	^{31}P	31	11.5	3.18	3.22
^{36}Ar	36	13	3.39	3.42	^{35}Cl	35	12.5	3.37	3.36
^{40}Ca	40	14	3.48	3.55	^{39}K	39	13.5	3.44	3.49

Left-hand side: Helium series from Parker et al. 2022[1] (Δ is "degrees of freedom" DoF); Right-hand side: Helium series less one proton with Δ_{RHS} given by $\Delta_{LHS}-0.5$. Measured RMS charge radii for both LHS and RHS from Angeli 1999.[25] Calculated radii from Equation (9a) of Parker et al. 2022[1] (see also ibid. Table 2).

holomorphic pair of neutrons and the neutron is regarded as having $\Delta_n = -1$ (see Table 2). Thus the "dineutron" has $\Delta_{dn} = 3$ (= −1+1+3). Then the tetraneutron is a holomorphic pair of "dineutrons": $\Delta_{tn} = 4$ (= 3+1).

The change in the number of DoFs associated with the disintegration of the tetraneutron is calculated by consideration of the various potential entropic pathways leading to the final four neutrons according to the principle of least exertion. The calculus of overall change in DoFs has to consider the fact that the product particles (two intermediate "dineutrons", and finally four neutrons) of the disintegration are essentially identical and indistinguishable from each other.

The calculus of the DoF change is based on the indistinguishability of the various product particles, and the resulting specific symmetries. Then to count the pathways we sum partial Δs for the logical OR function, and take the product of the partial Δs for the logical AND function (see Figure B1).

Then, referring to Figure B1, (i) notes that when a dineutron disintegrates into one of the neutrons the change in DoFs is 4, while (ii) notes that the change in DoFs when the tetraneutron disintegrates into one of the dineutrons is simply 1. Similarly (iii) extends (ii) for the single pathway of a tetraneutron disintegrating into a neutron, there is a change in the number of DoFs of 5 (= 1+4) for that pathway, as indicated. Then to count pathways, (iv), we consider the full number of possibilities, given the various symmetries and indistinguishabilities. Finally, (v) taking the relative stability of the overall tetraneutron structure into account, a half-integer DoF is removed.

Appendix C

Half-Integer Degrees of Freedom

We have assumed that tropons and DoFs are closely related, and therefore that DoFs should be quantized. Of course, DoFs are intrinsically quantized since it seems to make little sense to speak of "fractional DoFs". Nevertheless, we have made use of "half-DoFs" in the main text. Thermodynamics is scale-independent but specific DoFs only apply at a certain given scale so that at different scales different DoFs come into play. Previously,[1] we treated the alpha particle as a "unitary entity" (than which exists nothing simpler) at a scale length given by the proton radius: at this scale the internal structure of the alpha (with four nucleons) is ignored. But on a smaller scale more DoFs will come into play.

Often, a DoF is associated with the quantum of entropy, the Boltzmann constant k_B (in isomorphism to the quantum of kinematics being the reduced Planck constant $\hbar$), e.g., as discussed in ref. [1]. However, just as the half-integer Planck constant $\hbar/2$ is frequently seen in quantum mechanics (e.g., the half-integer spin of fermions) so we also observe the presence of the half-integer Boltzmann constant in QGT. In Appendix 2 of our previous work (Parker et al. 2022[1]) we discuss the application of the quantity $k_B/2$ in the context of DoFs; in which case, it becomes apparent that the concept of a "half-integer DoF" also becomes appropriate. It is also in this context that we see that the identification of a DoF with a tropon is perhaps overly simplistic; particularly in the assignation of either integer k_B or half-integer $k_B/2$ values to a DoF.

Here we show that, in effect, half-integer DoFs are also explicitly required to account plausibly for the radii of nuclei not in the "helium series" ($A = 4n$: ^{4}He ... ^{40}Ca). **Table C1** (left side) shows the results obtained previously for the helium series[1] to be compared with the same nuclei with one proton less (right side), where the nuclear radii are calculated with QGT using a ½DoF less. Clearly, tritium's location doesn't fit well in this series (although a calculated radius of 1.78 fm for DoF = 3.5 is noteworthy), and ^{7}Li also looks out of place, but the other nuclei seem to fit as well as the corresponding nuclei in the helium series.

Appendix D

The Hilbert Transform Relation of *Tropon* to *Photon*

In our previous work ref. [26] we have shown that the spectral representations of the energy E (Hamiltonian) and entropy production ($\Pi \equiv dS/dt$) of a system are Hilbert transforms of each other. That is to say, the spectral representations of energy and entropy production are given by the Fourier transform of the respective temporal variation of energy and entropy production. The conservation of energy (1st Law) is axiomatic, while we have shown that the conservation of entropy production[5] arises as a consequence of Noether's theorem and the associated entropic Euler–Lagrange equation describing the principle of least exertion (or maximum entropy, in its analogous and more familiar guise.) It's also worth noting that Parseval's theorem as applied to the real and imaginary components of the Hilbert transform (in effect, the energy and entropy production of the system) also implies the respective conservation of these physical quantities.

In the context of this present paper, we note that a temporal change in the energy of a system, e.g., due to the emission or absorption of a photon by an atom, intrinsically implies a finite (i.e., not a trivially zero) Hilbert transform relationship between the spectral representations of the system's energy and entropy production. This in turn also implies that a change in energy of a system (e.g., as represented by the addition or loss of a quantum of energy, such as a photon, $\Delta E = hf$) inevitably requires an associated change $\Delta\Pi$ in the entropy production of the system. It is equally clear that the change in entropy production is associated with a quantum of entropy production given by $\Delta\Pi = k_B f = k_B c/\lambda$, where λ defines the length scale of the process. From this perspective, it is reasonable to associate the change in entropy production with an entropic quantum particle that we call the tropon. As an entropic particle, a *tropon* can be considered to contribute toward the structural or geometrical properties of a system, particularly as the system evolves over time in structure and shape.

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Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

All new data are generated analytically and are fully presented in the article.

Keywords

halo isotopes of helium, quantitative geometrical thermodynamics, scale invariance, tropons

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