Categorical Torelli Theorems for Fano Threefolds

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Seneliui ir Močiutei

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

Where the work was done in collaboration with others, I have made a significant contribution. In particular, Chapter 6 and and Section 4.5 are joint work with Xun Lin, Zhiyu Liu, and Shizhuo Zhang.

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Abstract

The derived category $\mathrm{D^b}(X)$ of a variety contains a lot of information about X. If X and X' are Fano, then an equivalence $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X')$ implies that X and X' are isomorphic. For prime Fano threefolds X (of Picard rank 1, index 1, and genus $g \geq 6$) the derived category decomposes semiorthogonally as $\langle \mathsf{Ku}(X), \mathcal{E}, \mathcal{O}_X \rangle$, where $\mathcal{E}$ is a certain vector bundle on X. Therefore one can ask whether less data (in particular the Kuznetsov component $\mathsf{Ku}(X)$) than the whole of $\mathrm{D^b}(X)$ determines X isomorphically (or at least birationally).

In this thesis, we focus on this question in the case of ordinary Gushel–Mukai threefolds (genus 6 prime Fano threefolds). We show that $\mathsf{Ku}(X)$ determines the birational class of X which proves a conjecture of Kuznetsov–Perry in dimension 3. We also prove a refined categorical Torelli theorem for oridnary Gushel–Mukai threefolds. In other words, we show that $\mathsf{Ku}(X)$ along with the data of the vector bundle $\mathcal{E}$ is enough to determine X up to isomorphism.

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Chapter 1

Introduction

Given a smooth projective variety X over the complex numbers, one can ask whether topological invariants (for example its singular cohomology) recover it up to isomorphism, or birational equivalence. If they hold, such statements are called *Torelli theorems*, after Ruggiero Torelli who showed [Tor13] that the Jacobian

$$J(C) \coloneqq \operatorname{Pic}^0(C) = \frac{H^0(C, \Omega^1_C)^\vee}{H_1(C, \mathbf{Z})}$$

of an algebraic curve recovers the curve.

Analogous statements in terms of intermediate Jacobians, Hodge isometries between the middle primitive cohomology of varieties, etc., have subsequently been shown to hold (see for example [CG72, Tju70, Don83]).

On the other hand, one can consider the abelian category of coherent sheaves $\operatorname{Coh}(X)$ on X and ask how much of the geometric information of X it recovers. Due to a theorem of Gabriel [Gab62], $\operatorname{Coh}(X)$ reconstructs X up to isomorphism. However, in some sense $\operatorname{Coh}(X)$ is too restrictive to work with (see for example [Tho00]). Furthermore, physics (for example the Homological Mirror Symmetry conjecture [Kon95]) also calls for a looser object of study, the bounded derived category of coherent sheaves $\mathrm{D}^{\mathrm{b}}(X)$ on X. Coherent sheaves get replaced with complexes of coherent sheaves, and they are identified up to quasi-isomorphism (as opposed to up to actual isomorphism like in $\operatorname{Coh}(X)$).

One can then start asking similar questions as before; how much of the geometric information of X does $\mathrm{D}^{\mathrm{b}}(X)$ carry? It turns out that $\mathrm{D}^{\mathrm{b}}(X)$ is no longer a perfect invariant, in the sense that there are examples of derived equivalent varieties which are not even birationally equivalent [Căl07].

Categorical Torelli questions

However, in the setting of Fano varieties (those with ample anti-canonical bundle), Bondal–Orlov showed that $\mathrm{D}^{\mathrm{b}}(X)$ recovers X up to isomorphism [BO01]. Derived categories of Fano varieties also admit semiorthogonal decompositions (roughly a collection of subcategories which generate $\mathrm{D}^{\mathrm{b}}(X)$ and have restrictions on the direction in which morphisms can exist between the objects of the subcategories). For example, certain Fano varieties Y have semiorthogonal decompositions of the form

$$\mathrm{D}^{\mathrm{b}}(Y) = \langle \mathsf{Ku}(Y), \mathcal{O}_Y, \mathcal{O}_Y(H) \rangle$$

where $\mathsf{Ku}(Y)$ is a special subcategory of $\mathrm{D}^{\mathrm{b}}(Y)$ called the *Kuznetsov component of* Y defined by its semiorthogonality to the line bundles $\mathcal{O}_Y$ and $\mathcal{O}_Y(1)$. This subcategory has been extensively studied by Kuznetsov and others in e.g. [Kuz04, Kuz06, Kuz07, Kuz09, Kuz10, KP17, KPS18, Kuz19, KP23].

With this in mind, one can ask whether the Kuznetsov component of Y (in particular, not the whole derived category) determines Y up to isomorphism.

Motivating examples

A particularly geometric motivating example of this occurring is the intersection of two 4-dimensional quadrics in $\mathbf{P}^5$. Such varieties give one of the deformation classes of smooth Picard rank 1 Fano threefolds (the ones of index 2 and degree 4), and to each such Fano threefold we can associate a curve. Indeed, following [Kuz09, Section 4.2] let $Y_4 = Q \cap Q' \subset \mathbf{P}^5$ where Q, Q' are 4-dimensional quadrics. Now consider the pencil of quadrics $\{Q_\lambda\}_{\lambda \in \mathbf{P}^1}$ generated by Q and Q'. Since Y_4 is smooth, the generic Q_λ is smooth, and there are six points $\lambda_1, \dots, \lambda_6 \in \mathbf{P}^1$ where Q_λ is singular. The smooth quadrics Q_λ have two choices of ruling, and the singular ones have one choice of ruling. This gives a double cover $C \to \mathbf{P}^1$ branched in the points $\lambda_1, \dots, \lambda_6$. Then C is a genus 2 curve. This gives an isomorphism between the moduli space of smooth Picard rank 1, index 2, degree 4 Fano threefolds and the moduli space of genus 2 curves.

Furthermore, it is shown in [BO95, Kuz08] that $\mathsf{Ku}(Y_4) \coloneqq \langle \mathcal{O}_Y, \mathcal{O}_Y(H) \rangle^\perp \simeq \mathrm{D}^{\mathrm{b}}(C)$, where C is the curve associated to the Fano threefold Y_4. Therefore, an equivalence of the Kuznetsov components $\mathsf{Ku}(Y_4) \simeq \mathsf{Ku}(Y_4')$ of two such Fano threefolds Y_4, Y_4' gives rise to an equivalence $\mathrm{D}^{\mathrm{b}}(C) \simeq \mathrm{D}^{\mathrm{b}}(C')$ of the associated genus 2 curves C, C'. Bondal–Orlov's Reconstruction Theorem [BO01] gives an isomorphism $C \cong C'$, and as we have seen the curves determine their corresponding associated Fano threefolds, thus we get an isomorphism $Y_4 \cong Y_4'$ and categorical Torelli holds in this case.

For certain other deformation classes of smooth Picard rank 1 Fano threefolds, such a geometric picture no longer exists. However, we still have semiorthogonal decompositions and therefore Kuznetsov components, which we can in these cases regard as "non-commutative curves". Already for cubic threefolds $Y_3 \subset \mathbf{P}^4$, there is no geometric description of $\mathsf{Ku}(Y_3)$. The categorical Torelli question in this case can be tackled by considering Bridgeland stable objects inside $\mathsf{Ku}(Y_3)$. In [BMMS12] (and later [PY22]), the authors realise these moduli spaces with respect to the numerical class of the ideal sheaf of a line as Hilbert schemes of lines on Y_3. An equivalence $\mathsf{Ku}(Y_3) \simeq \mathsf{Ku}(Y_3')$ induces an isomorphism of these moduli spaces, and in turn an isomorphism of the corresponding Hilbert schemes of lines on the cubic threefolds. Since a cubic threefold can be recovered from its Hilbert scheme of lines, categorical Torelli also holds in this case.

We also remark here that the moduli space $M_C(2, \mathcal{L})$ of stable rank 2 vector bundles with fixed determinant on C (the genus 2 curve associated to Y_4) is isomorphic to Y_4, i.e. $M_C(2, \mathcal{L}) \cong Y_4$ (see [New68, NR69]). Furthermore, we have $\mathcal{M}_\sigma(\mathsf{Ku}(Y_4), v) \cong M_C(2, \mathcal{L})$ where $\mathcal{M}_\sigma(\mathsf{Ku}(Y_4), v)$ is a moduli space of Bridgeland stable objects in $\mathsf{Ku}(Y_4)$ with numerical class v. So in a similar fashion to the previous paragraph an equivalence of Kuznetsov components $\mathsf{Ku}(Y_4) \simeq \mathsf{Ku}(Y_4')$ induces an isomorphism $M_C(2, \mathcal{L}) \cong M_{C'}(2, \mathcal{L}')$ and therefore an isomorphism $Y_4 \cong Y_4'$.

The case of Gushel–Mukai threefolds

This thesis focuses on answering the question of whether categorical Torelli holds for a certain deformation class of Fano threefolds known as *Gushel–Mukai threefolds* X, whose derived categories have the semiorthogonal decompositions

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^\vee \rangle \simeq \langle \mathcal{A}_X, \mathcal{Q}^\vee, \mathcal{O}_X \rangle$$

where $\mathcal{E}$ (and $\mathcal{Q}$) is a certain rank 2 (rank 3) vector bundle on X. Like in the cubic threefold case, there is no known geometric description of the Kuznetsov component $\mathcal{A}_X$ so we can also regard it as a non-commutative curve. In the spirit of [BMMS12], we first study moduli spaces of Bridgeland stable objects in $\mathcal{A}_X$ for certain choices of numerical class. We specifically choose so-called (-1)-classes in the numerical Grothendieck group of $\mathcal{A}_X$. There are two of these, x and y, up to sign, and we prove that $\mathcal{M}_\sigma(\mathcal{A}_X, x)$ is isomorphic to the minimal surface $\mathcal{C}_m(X)$ of the Hilbert scheme of conics on X. We also show that $\mathcal{M}_\sigma(\mathcal{A}_X, y)$ is isomorphic to a certain Gieseker moduli space $M_G^X(2,1,5)$ of Gieseker stable sheaves on X.

Now take two Gushel–Mukai threefolds X, X' and suppose there is an equivalence of Kuznetsov components $\mathcal{A}_X \simeq \mathcal{A}_{X'}$. Since an equivalence takes a (-1)-class to a (-1)-class, we get two possible isomorphisms of Bridgeland moduli spaces: $\mathcal{C}_m(X) \cong \mathcal{C}_m(X')$ and $\mathcal{C}_m(X) \cong M_G^{X'}(2,1,5)$. Then results on the birational geometry of Hilbert schemes of conics on Gushel–Mukai threefolds due to [Log82, DIM12] allow us to deduce the following:

Theorem 1.0.1 (= Theorem 6.6.3)**.** *Let X, X' be general ordinary Gushel–Mukai threefolds. Suppose there is an equivalence $\mathcal{A}_X \simeq \mathcal{A}_{X'}$. Then X and X' are birational.*

This proves a conjecture of Kuznetsov–Perry [KP23, Conjecture 1.7] in dimension 3, with the assumption that the threefolds are general within their moduli. Notice that in this case, the Kuznetsov component does not determine the Fano threefold up to isomorphism, only birational equivalence. Indeed, there are birational but non-isomorphic Gushel–Mukai threefolds which have equivalent Kuznetsov components [KP23, Theorem 1.6].

If we further impose the condition that certain extra categorical data is preserved (the *gluing data* of the category $\langle \mathcal{A}_X, \mathcal{Q}^\vee \rangle \subset \mathrm{D}^\mathrm{b}(X)$), then we can recover X up to isomorphism. More precisely:

Theorem 1.0.2 (= Theorem 6.6.2)**.** *Let X, X' be general ordinary Gushel–Mukai threefolds. Suppose there is an equivalence $\Phi\colon \mathcal{A}_X \simeq \mathcal{A}_{X'}$ such that $\Phi(\mathcal{G}) \cong \mathcal{G}'$, where $\mathcal{G}, \mathcal{G}'$ are the gluing data of the categories $\langle \mathcal{A}_X, \mathcal{Q}^\vee \rangle$ and $\langle \mathcal{A}_{X'}, \mathcal{Q}'^\vee \rangle$, respectively. Then X and X' are isomorphic.*

Recall the *period map*

$$\mathcal{P}\colon \mathcal{X} \to \mathcal{A}_{10}, \qquad X \mapsto J(X)$$

from Hodge theory, where $\mathcal{X}$ is the moduli space of Gushel–Mukai threefolds and $\mathcal{A}_{10}$ is the moduli space of dimension 10 principally polarised abelian varieties. Its fibers can be thought of as measuring how badly classical Torelli fails to hold. Debbarre–Iliev–Manivel make the following conjecture:

Conjecture 1.0.3 (Debarre–Iliev–Manivel)**.**

$$\mathcal{P}^{-1}(J(X)) = \mathcal{C}_m(X) \cup M_G^X(2,1,5).$$

Now consider the *categorical period map*

$$\mathcal{P}_{\text{cat}} \colon \mathcal{X} \to \{\mathcal{A}_X\}/\sim, \qquad X \mapsto \mathcal{A}_X.$$

We prove that the fibers of the categorical period map are equal to the conjectural fibers of the classical period map:

Theorem 1.0.4 (= Theorem 6.8.3)**.**

$$\mathcal{P}_{\text{cat}}^{-1}(\mathcal{A}_X) = \mathcal{C}_m(X) \cup M_G^X(2,1,5).$$

Indeed, the precise statement (see Theorem 6.6.3) of Theorem 1.0.1 gives precisely what X' can be after fixing X. It turns out that X' can be either a *conic transformation* or a conic transformations of a *line transformation* of X, both certain types of birational surgery on X. Since $\mathcal{C}_m(X)$ and $M_G^X(2,1,5)$ together parametrise all conic and line transformations of X, the categorical period map fiber statement follows.

The theorem above allows us to restate the Debarre–Iliev–Manivel Conjecture as follows:

Conjecture 1.0.5 (= Conjecture 6.8.6)**.** *If X and X' are Gushel–Mukai threefolds, then*

$$J(X) \cong J(X') \implies \mathcal{A}_X \simeq \mathcal{A}_{X'}.$$

Notation

- We use the symbol $\cong$ for isomorphisms of varieties, vector spaces, groups, and rings. We use the symbol $\simeq$ for birational equivalence of varieties, and for equivalences of categories.

- If F is an object in a triangulated category $\mathcal{T}$, we use $[F]$ to denote its class in the numerical Grothendieck group $\mathcal{N}(\mathcal{T})$.

- Unless otherwise stated, varieties X are smooth Picard rank 1, index 1 Fano threefolds, and varieties Y are smooth Picard rank 1, index 2 Fano threefolds. The notation X_g means that X_g has genus g, and the notation Y_d means Y_d has degree d.

- We denote by $\mathcal{H}^i_\sigma$ the i-th cohomology object with respect to the heart $\mathcal{A}_\sigma$. If $\mathcal{A} = \mathrm{Coh}(X)$, we denote the cohomology objects by $\mathcal{H}^i$ for simplicity.

- Unless otherwise stated, if X is a Fano variety then X' belongs to the same deformation class. All functors associated to X decorated with a dash will be understood to be the analogous functor associated to X'.

- If $F \in \mathrm{D^b}(X)$, then $F(H) \coloneqq F \otimes \mathcal{O}_X(H)$ where H is the ample hyperplane class on X.

- $\mathbb{D}(-) \coloneqq \mathrm{R}\mathcal{H}om(-, \mathcal{O}_X)$ denotes the derived dual functor.

- We use the notation $\hom(-,-) \coloneqq \dim \mathrm{Hom}(-,-)$ and $\mathrm{ext}^i(-,-) = \dim \mathrm{Ext}^i(-,-)$.

Chapter 2

Fano threefolds

A *Fano variety* X is an irreducible and reduced scheme whose anti-canonical divisor $-K_X$ is ample. Fano varieties were originally studied by Gino Fano (see e.g. [Fan29, Fan41]). Restricting to smooth Fano varieties of dimension 3, a detailed modern account of their classification into deformation classes can be found in [Isk99]. These threefolds were originally classified by Fano. Another method using vector bundles was used by Gushel and Mukai [Gus83b, Gus83a, Gus92, Muk89, Muk92].

In this thesis we will only be concerned with smooth Fano threefolds with $\operatorname{Pic} X = \mathbf{Z}$. The *index* of such a Fano threefold is the positive integer i_X such that $-K_X = i_X H$ where H is the ample hyperplane class of X. There are 17 deformation classes of Picard rank 1 Fano threefolds, and they satisfy $1 \leq i_X \leq 4$. When $i_X = 4$ we have $X = \mathbf{P}^3$ and when $i_X = 3$ then X is a quadric in $\mathbf{P}^4$. For this thesis, we restrict to the other 15 deformation classes of index 1 and 2. Such Fano threefolds are classified into their deformation classes by their degrees (equivalently their genera). By the *degree* of a Fano threefold, we mean the number $d_X \coloneqq -K_X^3$. By the *genus* we mean the geometric genus $g = h^0(X, K_X)$.

Remark 2.0.1. We make the following remark on notational convention. For this thesis, we will denote index 2 Fano threefolds by Y_d where the subscript d is the degree d_Y of Y_d. We will denote index 1 Fano threefolds by X_g where g denotes the *genus* $g_X \coloneqq \frac{1}{2}d_X + 1$ of X_g.

For index 1 Fano threefolds, we have $2 \leq g \leq 12$ and $g \neq 11$. For index 2 Fano threefolds, we have $1 \leq d \leq 5$. We now list the Fano threefolds of index 1 and 2, and their geometric descriptions. We have taken the table from [Kuz09]. See also [Bel23] which lists details of all of these Fano threefolds.

X_{12}	The zero locus of a global section of the vector bundle $(\Lambda^2\mathcal{U}^\vee)^{\oplus 3}$ on $\mathrm{Gr}(3,7)$, where $\mathcal{U}$ is the tautological bundle.
X_{10}	A linear section of codimension 2 of the minimal compact homogeneous space $G_2\mathrm{Gr}(2,7)$ for the simple algebraic group G_2, inside $\mathbf{P}^{13}$
X_9	A linear section of codimension 3 of the symplectic Lagrangian Grassmannian Plücker embedded into $\mathbf{P}^{13}$
X_8	a linear section of codimension 5 of $\mathrm{Gr}(2,6)$ which is Plücker embedded into $\mathbf{P}^{14}$
X_7	a codimension 7 linear section of the connected component of the orthogonal Lagrangian Grassmannian $\mathrm{OGr}_+(5,10)$ inside $\mathbf{P}^{15}$ via the half-spinor embedding
X_6	a quadric section of a linear section of codimension 2 of $\mathrm{Gr}(2,5)$ which is Plücker embedded into $\mathbf{P}^9$; or the double cover of Y_5 ramified in a quadric
X_5	the intersection of three 5-dimensional quadrics in $\mathbf{P}^6$
X_4	the intersection of a quadric and a cubic in $\mathbf{P}^5$
X_3	a quartic in $\mathbf{P}^4$; or a double cover of a quadric $Q \subset \mathbf{P}^4$ ramified in the intersection of Q with a quartic
X_2	the double cover of $\mathbf{P}^3$ ramified in a sextic

Table 2.1: Fano threefolds of index 1.

Remark 2.0.2. We call the first type of genus 6 index 1 Fano threefolds *ordinary Gushel–Mukai threefolds*, and the second type *special Gushel–Mukai threefolds*. We will discuss both of these types in Section 6.

We write $\mathcal{X}_g$ to mean the moduli stack parametrising the deformation class of genus g Fano threefolds of index 1. We similarly write $\mathcal{Y}_d$ to mean the analogous object parametrising Fano threefolds of index 2 and degree d.

Y_5	a codimension 3 linear section of $\mathrm{Gr}(2,5)$ Plücker embedded into $\mathbf{P}^9$
Y_4	an intersection of two 4-dimensional quadrics in $\mathbf{P}^5$
Y_3	a cubic hypersurface in $\mathbf{P}^4$
Y_2	the double cover of $\mathbf{P}^3$ ramified in a quartic
Y_1	a degree 6 hypersurface is $\mathbf{P}(1,1,1,2,3)$; or the double cover of the cone over a Veronese surface ramified in a sextic

Table 2.2: Fano threefolds of index 2.

Theorem 2.0.3 ([Muk92], [Kuz09, Theorem 2.5])**.** *Let X be a Picard rank 1, index 1 Fano threefold with genus $g = 2s$ where s is a positive integer. Then there exists a unique μ-stable*[1] *vector bundle $\mathcal{E}$ on X of rank 2, with $c_1(\mathcal{E}) = -H$ and $\mathrm{ch}_2(\mathcal{E}) = (s-2)L$. Moreover, $\mathcal{E}$ is exceptional and $H^\bullet(X, \mathcal{E}) = 0$.*

Remark 2.0.4. Mukai's original proof of this theorem has a gap; an upcoming paper of Bayer–Macrì–Kuznetsov fixes this gap.

[1] See Definition 4.1.1 for the definition of μ-stability.

Chapter 3

Derived categories

3.1 Derived categories of coherent sheaves

We briefly recall the derived category of coherent sheaves on a smooth variety X. A detailed and complete account on triangulated categories and derived categories of coherent sheaves on varieties can be found in [Huy06].

Definition 3.1.1. The *Serre functor* $S_{\mathcal{D}}$ of a triangulated category $\mathcal{D}$, when it exists, is the autoequivalence of $\mathcal{D}$ such that there is a functorial isomorphism of vector spaces

$$\mathrm{Hom}_{\mathcal{D}}(A, B) \cong \mathrm{Hom}_{\mathcal{D}}(B, S_{\mathcal{D}}(A))^{\vee}$$

for any $A, B \in \mathcal{D}$.

We will use the following fact frequently:

Example 3.1.2. The Serre functor of $\mathrm{D^b}(X)$ is given by $S_{\mathrm{D^b}(X)}(-) = (- \otimes K_X)[\dim X]$.

3.2 Semiorthogonal decompositions

Definition 3.2.1. Let $\mathcal{D}$ be a triangulated category. We say that $E \in \mathcal{D}$ is an *exceptional object* if $\mathrm{Hom}^{\bullet}(E, E) = \mathbf{C}$. Now let $\{E_1, \dots, E_m\}$ be a collection of exceptional objects in $\mathcal{D}$. We say it is an *exceptional collection* if $\mathrm{Hom}^{\bullet}(E_i, E_j) = 0$ for $i > j$.

Definition 3.2.2. Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a triangulated subcategory. We define the *right orthogonal complement* of $\mathcal{C}$ in $\mathcal{D}$ as the full triangulated subcategory

$$\mathcal{C}^{\perp} = \{X \in \mathcal{D} \mid \mathrm{Hom}(Y, X) = 0 \text{ for all } Y \in \mathcal{C}\}.$$

The *left orthogonal complement* is defined similarly, as

$$^{\perp}\mathcal{C} = \{X \in \mathcal{D} \mid \mathrm{Hom}(X, Y) = 0 \text{ for all } Y \in \mathcal{C}\}.$$

Definition 3.2.3. Let $\mathcal{D}$ be a triangulated category. We say a triangulated subcategory $\mathcal{A} \subset \mathcal{D}$ is *admissible*, if the inclusion functor $i \colon \mathcal{A} \to \mathcal{D}$ has a left adjoint i^* and right adjoint $i^!$.

Definition 3.2.4. Let $\mathcal{D}$ be a triangulated category, and $\mathcal{A}_1, \ldots, \mathcal{A}_m$ be full admissible subcategories of $\mathcal{D}$. We say that $\mathcal{D} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle$ is a *semiorthogonal decomposition* of $\mathcal{D}$ if $\mathcal{A}_j \subset \mathcal{A}_i^{\perp}$ for all $i > j$, and the subcategories $\mathcal{A}_1, \ldots, \mathcal{A}_m$ generate $\mathcal{D}$, i.e. the category resulting from taking all shifts and cones of objects in the categories $\mathcal{C}_1, \ldots, \mathcal{C}_m$ is equivalent to $\mathcal{D}$.

Proposition 3.2.5. *If $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ is a semiorthogonal decomposition, then $\mathcal{D} = \langle S_{\mathcal{D}}(\mathcal{D}_2), \mathcal{D}_1 \rangle = \langle \mathcal{D}_2, S_{\mathcal{D}}^{-1}(\mathcal{D}_1) \rangle$ are also semiorthogonal decompositions.*

3.2.1 Mutations

Let $\mathcal{A} \subset \mathcal{D}$ be an admissible subcategory. Then the *left mutation functor* $\mathbf{L}_{\mathcal{A}}$ through $\mathcal{A}$ is defined as the functor lying in the canonical functorial exact triangle

$$ii^! \longrightarrow \mathrm{id} \longrightarrow \mathbf{L}_{\mathcal{A}}$$

and the *right mutation functor* $\mathbf{R}_{\mathcal{A}}$ through $\mathcal{A}$ is defined similarly, by the triangle

$$\mathbf{R}_{\mathcal{A}} \longrightarrow \mathrm{id} \longrightarrow ii^*.$$

When $E \in \mathrm{D^b}(X)$ is an exceptional object, and $F \in \mathrm{D^b}(X)$ is any object, the left mutation $\mathbf{L}_E F$ fits into the triangle

$$E \otimes \mathrm{Hom}^\bullet(E, F) \longrightarrow F \longrightarrow \mathbf{L}_E F, \tag{3.2.1}$$

and the right mutation $\mathbf{R}_E F$ fits into the triangle

$$\mathbf{R}_E F \longrightarrow F \longrightarrow E \otimes \mathrm{Hom}^\bullet(F, E)^\vee. \tag{3.2.2}$$

Proposition 3.2.6. *Let $\mathcal{D} = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition. Then*

$$S_{\mathcal{B}} = \mathbf{R}_{\mathcal{A}} \circ S_{\mathcal{D}} \quad \textit{and} \quad S_{\mathcal{A}}^{-1} = \mathbf{L}_{\mathcal{B}} \circ S_{\mathcal{D}}^{-1}.$$

Lemma 3.2.7 ([Kuz10, Lemma 2.7])**.** *Let $\mathcal{D} = \langle \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n \rangle$ be a semiorthogonal decomposition with all components being admissible. Then for each $1 \leq k \leq n-1$, there is a semiorthogonal decomposition*

$$\mathcal{D} = \langle \mathcal{C}_1, \ldots, \mathcal{C}_{k-1}, \mathbf{L}_{\mathcal{C}_k}\mathcal{C}_{k+1}, \mathcal{C}_k, \mathcal{C}_{k+2} \ldots, \mathcal{C}_n \rangle$$

and for each $2 \leq k \leq n$ there is a semiorthogonal decomposition

$$\mathcal{D} = \langle \mathcal{C}_1, \ldots, \mathcal{C}_{k-2}, \mathcal{C}_k, \mathbf{R}_{\mathcal{C}_k}\mathcal{C}_{k-1}, \mathcal{C}_{k+1} \ldots, \mathcal{C}_n \rangle.$$

3.3 Group actions on categories

Definition 3.3.1 ([Del97], [Ela14, Definition 3.1])**.** Let $\mathcal{C}$ be a pre-additive $\mathbf{C}$-linear category, and let G be a finite group. A (right) action of G on $\mathcal{C}$ is defined as the following data:

1. a family of autoequivalences $\phi_g \colon \mathcal{C} \to \mathcal{C}$ for all $g \in G$;
2. a family of isomorphisms $\epsilon_{g,h} \colon \phi_g\phi_h \to \phi_{hg}$ for which the diagrams all the diagrams

$$\begin{tikzcd}
\phi_f\phi_g\phi_h \arrow[r, "\epsilon_{g,h}"] \arrow[d, "\epsilon_{f,g}"'] & \phi_f\phi_{hg} \arrow[d, "\epsilon_{f,gh}"] \\
\phi_{gf}\phi_h \arrow[r, "\epsilon_{gf,h}"'] & \phi_{hgf}
\end{tikzcd}$$

are commutative.

Definition 3.3.2 ([Ela14, Definition 3.5])**.** A *G-equivariant object* of $\mathcal{C}$ is a pair (F, ϕ) consisting of an object $F \in \mathcal{C}$ and a collection of isomorphisms $\phi_g \colon F \xrightarrow{\sim} g^*(F)$ for all $g \in G$ such that the diagram

$$\begin{tikzcd}
F \arrow[r, "\phi_h"] \arrow[rrd, "\phi_{gh}"'] & h^*(F) \arrow[r, "h^*(\phi_g)"] & h^*(g^*(F)) \\
& & (gh)^*(F) \arrow[u, "c_{g,h}(F)"']
\end{tikzcd}$$

commutes for all $g, h \in G$. The isomorphisms $\phi = \{\phi_g\}_{g \in G}$ are called the G-*linearisation*. The G-*equivariant category* $\mathcal{C}^G$ of $\mathcal{C}$ is the category whose objects are the G-equivariant objects of $\mathcal{C}$, and morphisms are those between G-invariant objects of $\mathcal{C}$ that commute with the G-linearisations.

3.4 Derived categories of Fano threefolds

We now define one of our main objects of study, the Kuznetsov component of a Fano threefold.

Definition 3.4.1 ([Kuz09])**.** Let $Y \in \mathcal{Y}_d$. The *Kuznetsov component* of Y is defined by the semiorthogonal decomposition

$$\mathrm{D^b}(Y) = \langle \mathsf{Ku}(Y), \mathcal{O}_Y, \mathcal{O}_Y(H) \rangle.$$

Definition and Proposition 3.4.2 ([Kuz09])**.** Let $X \in \mathcal{X}_g$ with even genus $g \geq 6$. Then the vector bundles $\{\mathcal{E}, \mathcal{O}_X\}$ (cf. Theorem 2.0.3) form an exceptional collection, and the *Kuznetsov component* of X is defined by the semiorthogonal decomposition

$$\mathrm{D^b}(X) = \langle \mathsf{Ku}(X), \mathcal{E}, \mathcal{O}_X \rangle.$$

Next we come to the case of the odd genus index 1 Fano threefolds with $g \geq 6$.

Definition and Proposition 3.4.3 ([Muk92], [Kuz06])**.** Let X be a Picard rank 1, index 1 Fano threefold of genus 7 (respectively 9). Then there exist rank 5 (respectively rank 3) vector bundles, both denoted by $\mathcal{E}$, such that the pair $\{\mathcal{E}, \mathcal{O}_X\}$ is an exceptional collection. The *Kuznetsov components* of these Fanos are defined by the semiorthogonal decompositions

$$\mathrm{D^b}(X) = \langle \mathsf{Ku}(X), \mathcal{E}, \mathcal{O}_X \rangle.$$

It is useful to note the Chern characters of the vector bundle $\mathcal{E}$ for each case of the genus:

$$\mathrm{ch}(\mathcal{E}) = \begin{cases} (2, -H, L, \frac{1}{3}P), & g = 6 \\ (5, -2H, 0, P), & g = 7 \\ (2, -H, 2L, \frac{1}{6}P), & g = 8 \\ (3, -H, 0, \frac{1}{3}P), & g = 9 \\ (2, -H, 3L, 0), & g = 10 \\ (2, -H, 4L, -\frac{1}{6}P), & g = 12. \end{cases}$$

We lastly come to the case of the index 1 Fano threefolds with genus $g < 6$.

Definition 3.4.4. Let X be a Picard rank 1, index 1 Fano threefold of genus $g < 6$. The *Kuznetsov components* of these Fanos are defined by the semiorthogonal decompositions

$$\mathrm{D^b}(X) = \langle \mathsf{Ku}(X), \mathcal{O}_X \rangle.$$

We now summarise all of these semiorthogonal decompositions in the following tables. These tables are taken from [BLMS23, p. 24]:

X_g	SOD	$\exists$ SOD	$\exists\ \sigma$
X_{12}	$\mathrm{D^b}(X_{12}) = \langle \mathcal{E}_4, \mathcal{E}_3, \mathcal{E}_2, \mathcal{O} \rangle$	[Kuz09]	[BLMS23]
X_{10}	$\mathrm{D^b}(X_{10}) = \langle \mathrm{D^b}(C_2), \mathcal{E}_2, \mathcal{O} \rangle$	[Kuz06]	[BLMS23]
X_9	$\mathrm{D^b}(X_9) = \langle \mathrm{D^b}(C_3), \mathcal{E}_3, \mathcal{O} \rangle$	[Kuz06]	[JLZ22]
X_8	$\mathrm{D^b}(X_8) = \langle \mathsf{Ku}(X_8), \mathcal{E}_2, \mathcal{O} \rangle$	[Kuz04]	[BMMS12, BLMS23]
X_7	$\mathrm{D^b}(X_7) = \langle \mathrm{D^b}(C_7), \mathcal{E}_5, \mathcal{O} \rangle$	[Kuz06]	[JLZ22]
X_6	$\mathrm{D^b}(X_6) = \langle \mathsf{Ku}(X_6), \mathcal{E}_2, \mathcal{O} \rangle$	[Kuz09]	[BLMS23]
X_5	$\mathrm{D^b}(X_5) = \langle \mathsf{Ku}(X_5), \mathcal{O} \rangle$		[BLMS23]
X_4	$\mathrm{D^b}(X_4) = \langle \mathsf{Ku}(X_4), \mathcal{O} \rangle$		[BLMS23]
X_3	$\mathrm{D^b}(X_3) = \langle \mathsf{Ku}(X_3), \mathcal{O} \rangle$		[BLMS23]
X_2	$\mathrm{D^b}(X_2) = \langle \mathsf{Ku}(X_2), \mathcal{O} \rangle$		[BLMS23]

Table 3.1: Semiorthogonal decompositions of Fano threefolds of index 1. The right-most column indicates whether stability conditions are known to exist on the Kuznetsov component, see Section 4.4. The subscripts of $\mathcal{E}$ mean the rank of $\mathcal{E}$.

Y_d	SOD	$\exists$ SOD	$\exists\ \sigma$
Y_5	$\mathrm{D^b}(Y_5) = \langle \mathcal{F}_2(-H), \mathcal{O}(-H), \mathcal{F}_2, \mathcal{O} \rangle$	[Orl91]	[BLMS23]
Y_4	$\mathrm{D^b}(Y_4) = \langle \mathrm{D^b}(C_2), \mathcal{O}(-H), \mathcal{O} \rangle$	[BO95]	[BLMS23]
Y_3	$\mathrm{D^b}(Y_3) = \langle \mathsf{Ku}(Y_3), \mathcal{O}, \mathcal{O}(H) \rangle$		[BMMS12, BLMS23]
Y_2	$\mathrm{D^b}(Y_2) = \langle \mathsf{Ku}(Y_2), \mathcal{O}, \mathcal{O}(H) \rangle$		[BLMS23]
Y_1	$\mathrm{D^b}(Y_1) = \langle \mathsf{Ku}(Y_1), \mathcal{O}, \mathcal{O}(H) \rangle$		[BLMS23]

Table 3.2: Semiorthogonal decompositions of Fano threefolds of index 2

Remark 3.4.5. Fano threefolds $X \in \mathcal{X}_g$ when $g \geq 6$ also have *alternative Kuznetsov components* $\mathcal{A}_X$ which are defined by the semiorthogonal decompositions

$$\mathrm{D^b}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^\vee \rangle.$$

It turns out that $\mathcal{A}_X \simeq \mathsf{Ku}(X)$. Indeed, we have the following lemma.

Lemma 3.4.6. *We have the equivalence* $\Xi \colon \mathsf{Ku}(X) \simeq \mathcal{A}_X$ *given by* $E \mapsto \mathbf{L}_{\mathcal{O}_X}(E \otimes \mathcal{O}_X(H))$*, with inverse given by* $F \mapsto (\mathbf{R}_{\mathcal{O}_X} F) \otimes \mathcal{O}_X(-H)$.

Proof. Using Lemma 3.2.7 and noting that $\mathcal{E}(H) \cong \mathcal{E}^\vee$, we manipulate the semiorthogonal decomposition as follows:

$$\begin{aligned}
\mathrm{D^b}(X) &= \langle \mathsf{Ku}(X), \mathcal{E}, \mathcal{O}_X \rangle \\
&\simeq \langle \mathsf{Ku}(X) \otimes \mathcal{O}_X(H), \mathcal{E}^\vee, \mathcal{O}_X(H) \rangle \\
&\simeq \langle \mathcal{O}_X, \mathsf{Ku}(X) \otimes \mathcal{O}_X(H), \mathcal{E}^\vee \rangle \\
&\simeq \langle \mathbf{L}_{\mathcal{O}_X}(\mathsf{Ku}(X) \otimes \mathcal{O}_X(H)), \mathcal{O}_X, \mathcal{E}^\vee \rangle.
\end{aligned}$$

Now comparing with the definition of $\mathcal{A}_X$, we get $\mathcal{A}_X \simeq \mathbf{L}_{\mathcal{O}_X}(\mathsf{Ku}(X) \otimes \mathcal{O}_X(H))$ and the desired result follows. The reverse direction is similar. □

Definition 3.4.7.

- Denote the left adjoint to the inclusion $\mathsf{Ku}(X) \subset \mathrm{D^b}(X)$ by i^*. We have $i^* = \mathbf{L}_{\mathcal{E}} \mathbf{L}_{\mathcal{O}_X}$;
- Denote the left adjoint to the inclusion $\mathcal{A}_X \subset \mathrm{D^b}(X)$ by pr. We have $\mathrm{pr} = \mathbf{L}_{\mathcal{O}_X} \mathbf{L}_{\mathcal{E}^\vee}$.

3.4.1 Numerical Grothendieck groups of index 1 Fanos

Let $\mathrm{K}_0(\mathcal{D})$ denote the Grothendieck group of a triangulated category $\mathcal{D}$. We have the bilinear Euler form

$$\chi(E, F) = \sum_{i \in \mathbf{Z}} (-1)^i \operatorname{ext}^i(E, F)$$

for $E, F \in K_0(\mathcal{D})$. The *numerical Grothendieck group* is defined to be $\mathcal{N}(\mathcal{D}) := \mathrm{K}_0(\mathcal{D}) / \ker \chi$. We also have [Kuz09, p. 5] $\chi(u, v) = \chi_0(u^* \cap v)$ where $u \mapsto u^*$ is an involution of $\oplus_{i=0}^3 H^i(X, \mathbf{Q})$ given by multiplication with $(-1)^i$ on $H^{2i}(X, \mathbf{Q})$. By Hirzebruch–Riemann–Roch, we have

$$\chi_0(x + yH + zL + wP) = x + \frac{17}{6} y + \frac{1}{2} z + w. \tag{3.4.1}$$

Kuznetsov components

When $g \geq 6$ and g is even, by [Kuz09, p. 5] we know that the numerical Grothendieck group $\mathcal{N}(\mathsf{Ku}(X_{2g-2}))$ is a rank two integral lattice and generated by

$$\mathcal{N}(\mathsf{Ku}(X_{2g-2})) = \langle v := 1 - \frac{g}{2}L + \frac{g-4}{4}P, w := H - \frac{3g-6}{2}L + \frac{7g-40}{12}P\rangle$$

with Euler form given by

$$\begin{bmatrix} 1-\frac{g}{2} & -\frac{g}{2} \\ 3-g & 1-g \end{bmatrix}.$$

When $g = 7$, the Todd class of X is given by $\mathrm{td}(X) = 1+\frac{1}{2}H+3L+P$. Using Hirzebruch-Riemann-Roch theorem, one can verify with a direct computation that the numerical Grothendieck group is a rank two integral lattice generated by

$$\mathcal{N}(\mathsf{Ku}(X_7)) = \langle v := 2 - 5L + \frac{1}{2}P, w := H - 6L\rangle$$

with Euler form given by

$$\begin{bmatrix} -6 & -5 \\ -7 & -6 \end{bmatrix}.$$

When $g = 9$, the Todd class of X is given by $\mathrm{td}(X) = 1 + \frac{1}{2}H + \frac{10}{3}L + P$. The numerical Grothendieck group is a rank two integral lattice generated by

$$\mathcal{N}(\mathsf{Ku}(X_9)) = \langle v := 1 - 3L + \frac{1}{2}P, w := H - 8L + \frac{2}{3}P\rangle$$

with Euler form given by

$$\begin{bmatrix} -2 & -3 \\ -5 & -8 \end{bmatrix}.$$

Alternative Kuznetsov components of Gushel–Mukai threefolds

In the $g = 6$ case, we also give a description of $\mathcal{N}(\mathcal{A}_{X_6})$ which will be useful later on in the thesis. As in [Kuz09, Proposition 3.9], it follows from a straightforward computation that

$$\mathcal{N}(\mathcal{A}_{X_6}) = \langle x := 1 - 2L, y := H - 4L - \frac{5}{6}P\rangle.$$

with Euler form given by

$$\begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix}.$$

Remark 3.4.8. It is straightforward to check that the (-1)-classes of $\mathcal{N}(\mathcal{A}_{X_6})$ are x and $2x-y$, up to sign.

3.4.2 Numerical Grothendieck groups of index 2 Fanos

Similarly, $1 \leq d \leq 5$, by [Kuz09, pp. 5-6] we know that the numerical Grothendieck group $\mathcal{N}(\mathsf{Ku}(Y_d))$ is a rank two integral lattice and generated by

$$\mathcal{N}(\mathsf{Ku}(Y_d)) = \langle s \coloneqq 1 - L, t \coloneqq H - \frac{d}{2}L + \frac{d-6}{6}P\rangle$$

with Euler form given by

$$\begin{bmatrix} -1 & -1 \\ 1-d & -d \end{bmatrix}.$$

Chapter 4

Bridgeland stability conditions

Bridgeland stability was introduced by Bridgeland in [Bri07] as a generalisation of the notion of slope/Gieseker stability of sheaves, to the world of complexes of objects in triangulated categories.

4.1 Classical notions of stability

Before we outline the construction of Bridgeland stability conditions on triangulated categories, we recall the classical notions of slope stability and Gieseker stability. For this section, let X be a smooth projective threefold with H an ample divisor.

Definition 4.1.1 ([Mum62, Tak72])**.**

1. Let $E \in \mathrm{Coh}(X)$. Then its *slope* is defined as

$$\mu(E) \coloneqq \begin{cases} \frac{H^2 \cdot \mathrm{ch}_1(E)}{H^3 \mathrm{rk}(E)}, & \mathrm{rk}(E) \neq 0 \\ 0, & \text{else} \end{cases}$$

2. We say $E \in \mathrm{Coh}(X)$ is μ-(semi)stable if for any non-trivial proper subsheaf $F \subset E$ we have $\mu(F) < (\leq) \mu(E/F)$.

One construct moduli spaces of stable vector bundles on curves using μ-stability, but to construct moduli spaces of sheaves on higher-dimensional varieties we require the notion of Gieseker stability. For $E \in \mathrm{Coh}(X)$ define its *Hilbert polynomial* to be $P(E, m) \coloneqq \chi(\mathcal{O}_X, E(mH)) = \sum_{i=0}^{3} \alpha_i(E) m^i$, and let

$P_2(E,m) \coloneqq \sum_{i=1}^{3} \alpha_i(E)m^i$. Since we do not use it explicitly in this thesis, we refer the reader to [BBF+22, Definition 4.2] for a definition of the partial order $\preceq$ that we will use to define Gieseker stability below.

Definition 4.1.2 ([BBF+22, Definition 4.3])**.**

1. The sheaf E is *Gieseker-(semi)stable* if for all non-trivial proper subsheaves $F \subset E$ we have $P(F,m) \prec (\preceq) P(E,m)$.

2. The sheaf E is 2-*Gieseker-(semi)stable* if for all non-trivial proper subsheaves $F \subset E$ we have $P_2(F,m) \prec (\preceq) P_2(E,m)$.

4.2 Weak stability conditions

For background on abelian categories, Grothendieck groups, t-structures and hearts see e.g. [MS17]. Let $\mathcal{D}$ be a triangulated category, and $\mathrm{K}_0(\mathcal{D})$ its Grothendieck group. Fix a surjective morphism $v\colon \mathrm{K}_0(\mathcal{D}) \to \Lambda$ to a finite rank lattice. For this thesis, we let Λ be the numerical Grothendieck group $\mathcal{N}(\mathcal{D})$.

Definition 4.2.1 ([Bri07, Lemma 3.2])**.** The *heart of a bounded t-structure* on $\mathcal{D}$ is an abelian subcategory $\mathcal{A} \subset \mathcal{D}$ such that the following conditions are satisfied:

1. for any $E, F \in \mathcal{A}$ and $n < 0$, we have $\mathrm{Hom}(E, F[n]) = 0$;

2. for any object $E \in \mathcal{D}$ there exist objects $E_i \in \mathcal{A}$ and maps

$$0 = E_0 \xrightarrow{\phi_1} E_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_m} E_m = E$$

   such that $\mathrm{Cone}(\phi_i) = A_i[k_i]$ where $A_i \in \mathcal{A}$ and the k_i are integers such that $k_1 > k_2 > \cdots > k_m$.

Definition 4.2.2 ([BLMS23, Definition 2.2])**.** Let $\mathcal{A}$ be an abelian category and $Z\colon \mathrm{K}_0(\mathcal{A}) \to \mathbf{C}$ be a group homomorphism such that for any $E \in \mathcal{A}$ we have $\Im Z(E) \geq 0$ and if $\Im Z(E) = 0$ then $\Re Z(E) \leq 0$. Then we call Z a *weak stability function* on $\mathcal{A}$. If furthermore we have for $0 \neq E \in \mathcal{A}$ that $\Im Z(E) = 0$ implies that $\Re Z(E) < 0$, then we call Z a *stability function* on $\mathcal{A}$.

Definition 4.2.3 ([BLMS23, Definition 2.3])**.** A *weak stability condition* on $\mathcal{D}$ is a pair $\sigma = (\mathcal{A}, Z)$ where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$, and $Z\colon \Lambda \to \mathbf{C}$ is a group homomorphism such that

1. the composition $Z \circ v\colon \mathrm{K}_0(\mathcal{A}) \cong \mathrm{K}_0(\mathcal{D}) \to \mathbf{C}$ is a weak stability function on $\mathcal{A}$. From now on, we write $Z(E)$ rather than $Z(v(E))$.

Much like the slope in classical μ-stability, we can define a *slope* μ_σ for σ using Z. For any $E \in \mathcal{A}$, set

$$\mu_\sigma(E) := \begin{cases} -\frac{\Re Z(E)}{\Im Z(E)}, & \text{if } \Im Z(E) > 0 \\ +\infty, & \text{otherwise.} \end{cases}$$

We say an object $0 \neq E \in \mathcal{A}$ is σ-(semi)stable if $\mu_\sigma(F) < \mu_\sigma(E/F)$ (respectively $\mu_\sigma(F) \leq \mu_\sigma(E/F)$) for all proper subobjects $F \subset E$.

2. any object $E \in \mathcal{A}$ has a Harder–Narasimhan filtration in terms of σ-semistability defined above.

3. there exists a quadratic form Q on $\Lambda \otimes \mathbf{R}$ such that $Q|_{\ker Z}$ is negative definite, and $Q(E) \geq 0$ for all σ-semistable objects $E \in \mathcal{A}$. This is known as the *support property*.

If the composition $Z \circ v$ is a stability function, then σ is a *stability condition* on $\mathcal{D}$.

Let us briefly give an alternative (equivalent) interpretation of a weak Bridgeland stability condition $\sigma = (\mathcal{A}, Z)$. We take the following definitions from [PY22, p. 5] and [MS17, Definition 5.5]:

Definition 4.2.4. The *phase* of an object $E \in \mathcal{A}$ is defined to be

$$\phi(E) := \frac{1}{\pi} \arg(Z(E)) \in (0, 1].$$

If $Z(E) = 0$ then $\phi(Z(E)) = 1$ and $\phi(E[n]) := \phi(E) + n$.

Definition 4.2.5 ([Bri07])**.** A *slicing* $\mathcal{P}$ of the triangulated category $\mathcal{D}$ is a collection of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for $\phi \in \mathbf{R}$ such that

1. for $\phi \in (0, 1]$, the subcategory $\mathcal{P}(\phi)$ is given by the zero object and all σ-semistable[1] objects of phase ϕ;

2. for $\phi + n$ with $\phi \in (0, 1]$, we set $\mathcal{P}(\phi + n) := \mathcal{P}(\phi)[n]$.

[1]Where σ-stability is defined using Z in the same way as in Definition 4.2.3.

3. If $\phi_1 > \phi_2$ and $A \in \mathcal{P}(\phi_1)$, $B \in \mathcal{P}(\phi_2)$, then $\mathrm{Hom}(A, B) = 0$;

4. For all $E \in \mathrm{D^b}(X)$ there are real numbers $\phi_1 > \cdots > \phi_m$, objects $E_i \in \mathrm{D^b}(X)$ for $i = 1, \dots, m$ and a collection of triangles

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{m-1} \arrow[r] \arrow[d] & E_m = E \arrow[d] \\
 & A_1 \arrow[ul, dashed] & A_2 \arrow[ul, dashed] & & A_{m-1} \arrow[ul, dashed] & A_m \arrow[ul, dashed]
\end{tikzcd}$$

where $A_i \in \mathcal{P}(\phi_i)$.

The last property in the above definition is called the *Harder–Narasimhan filtration* of E. For an object $E \in \mathrm{D^b}(X)$, we write $\phi^-(E) := \phi_m$ and $\phi^+(E) := \phi_1$.

In what follows, we will interchange between $\sigma = (\mathcal{A}, Z)$ and $\sigma = (\mathcal{P}, Z)$ where $\mathcal{A} = \mathcal{P}((0, 1])$.

4.3 Tilt-stability conditions

Let $\sigma = (\mathcal{A}, Z)$ be a weak stability condition on a triangulated category $\mathcal{D}$. Now consider the following subcategories of $\mathcal{A}$, where $\langle - \rangle$ denotes the extension closure:

$$\begin{aligned}
\mathcal{T}^\mu_\sigma &= \langle E \in \mathcal{A} \mid E \text{ is } \sigma\text{-semistable with } \mu_\sigma(E) > \mu \rangle \\
\mathcal{F}^\mu_\sigma &= \langle E \in \mathcal{A} \mid E \text{ is } \sigma\text{-semistable with } \mu_\sigma(E) \leq \mu \rangle.
\end{aligned}$$

Then it is a result of [HRS96] that

Proposition 4.3.1. *The abelian category $\mathcal{A}^\mu_\sigma := \langle \mathcal{T}^\mu_\sigma, \mathcal{F}^\mu_\sigma[1] \rangle$ is the heart of a bounded t-structure on $\mathcal{D}$.*

We call $\mathcal{A}^\mu_\sigma$ the *tilt* of $\mathcal{A}$ around the torsion pair $(\mathcal{T}^\mu_\sigma, \mathcal{F}^\mu_\sigma)$. Let X be an n-dimensional smooth projective complex variety. Tilting can be applied to the weak stability condition $(\mathrm{Coh}(X), Z_H)$[2] to form the once-tilted heart $\mathrm{Coh}^\beta(X)$, where $Z_H(E) := -\mathrm{ch}_1(E)H^{n-1} + \mathrm{i}\,\mathrm{rk}(E)H^n$ for any $E \in \mathrm{Coh}(X)$. Define for $E \in \mathrm{Coh}^\beta(X)$

$$Z_{\alpha,\beta}(E) = \frac{1}{2}\alpha^2 H^n \mathrm{ch}^\beta_0(E) - H^{n-2}\mathrm{ch}^\beta_2(E) + \mathrm{i}H^{n-1}\mathrm{ch}^\beta_1(E). \tag{4.3.1}$$

[2] Note that this coincides with the usual notion of μ-stability.

Proposition 4.3.2 ([BMT13, BMS16]). *Let $\alpha > 0$ and $\beta \in \mathbf{R}$. The pair $\sigma_{\alpha,\beta} = (\mathrm{Coh}^{\beta}(X), Z_{\alpha,\beta})$ defines a weak stability condition on $\mathrm{D}^{\mathrm{b}}(X)$. The quadratic form Q is given by the discriminant*

$$\Delta_H(E) = (H^{n-1}\mathrm{ch}_1(E))^2 - 2H^n\mathrm{ch}_0(E)H^{n-2}\mathrm{ch}_2(E).$$

The stability conditions $\sigma_{\alpha,\beta}$ vary continuously as $(\alpha, \beta) \in \mathbf{R}_{>0} \times \mathbf{R}$ varies. Furthermore, for any $v \in \Lambda_H^2$ there is a locally finite wall-and-chamber structure on $\mathbf{R}_{>0} \times \mathbf{R}$ controlling stability of objects with class v.

Weak stability conditions of the above form are called *tilt-stability conditions*. We now state a useful lemma which relates 2-Giesesker-stability and tilt-stability.

Lemma 4.3.3 ([BMS16, Lemma 2.7], [BBF+22, Proposition 4.8]). *Let $E \in \mathrm{D}^{\mathrm{b}}(X)$.*

1. *Let $\beta < \mu(E)$. Then $E \in \mathrm{Coh}^{\beta}(X)$ is $\sigma_{\alpha,\beta}$-(semi)stable for $\alpha \gg 0$ if and only if $E \in \mathrm{Coh}(X)$ and E is 2-Gieseker-(semi)stable.*

2. *If $E \in \mathrm{Coh}^{\beta}(X)$ is $\sigma_{\alpha,\beta}$-semistable for $\beta \geq \mu(E)$ and $\alpha \gg 0$, then $\mathcal{H}^{-1}(E)$ is a torsion free μ-semistable sheaf and $\mathcal{H}^0(E)$ is supported in dimension not greater than one. If $\beta > \mu(E)$ and $\alpha > 0$, then $\mathcal{H}^{-1}(E)$ is also reflexive.*

4.3.1 Finding solutions for walls in tilt-stability

In this section, we describe a way of finding (potential) walls in tilt-stability with respect to objects in the derived category with a given truncated Chern character. This is similar to the method used in e.g. [PY22, Proposition 4.1] to find walls for certain objects. Let $M \in \mathrm{Coh}^{\beta}(X)$ be the object in question, and let its truncated Chern character be $\mathrm{ch}_{\leq 2}(M) = (m_0, m_1 H, \frac{m_2}{d} H^2)$, where d is the degree of X.

Assume there is a short exact sequence $0 \to E \to M \to F \to 0$ which makes M strictly semistable. We can assume that E and F are tilt-semistable using the existence of Harder–Narasimhan or Jordan–Hölder filtrations. Then the following conditions must be satisfied:

1. $\mathrm{ch}_{\leq 2}(M) = \mathrm{ch}_{\leq 2}(E) + \mathrm{ch}_{\leq 2}(F)$;

2. $\mu_{\alpha,\beta}(E) = \mu_{\alpha,\beta}(M) = \mu_{\alpha,\beta}(F)$;

3. $\Delta_H(E) \geq 0$ and $\Delta_H(F) \geq 0$;

4. $\Delta_H(E) \leq \Delta_H(M)$ and $\Delta_H(F) \leq \Delta_H(M)$.

Since $E, F \in \mathrm{Coh}^{\beta}(X)$, we also must have $\mathrm{ch}_1^{\beta}(E) \geq 0$ and $\mathrm{ch}_1^{\beta}(F) \geq 0$. Solving the system of inequalities above gives an even number of solutions of $(e_0, e_1, e_2) \in \mathbf{Z}^3$; half of them are solutions for the destabilising subobject E, and the other half are the corresponding quotients F.

4.3.2 Stronger BG inequalities

In this subsection, we state stronger Bogomolov–Gieseker (BG) style inequalities, which hold for tilt-semistable objects. The first is a stronger version of Proposition 4.3.2, which was proved by Chunyi Li in [Li18, Proposition 3.2] for Fano threefolds of Picard rank one.

Lemma 4.3.4 (Stronger BG I)**.** *Let X be an index 1 prime Fano threefold with degree d, and $E \in \mathrm{D}^{\mathrm{b}}(X)$ a $\sigma_{\alpha,\beta}$-stable object where $\alpha > 0$. Let $k := \lfloor \mu(E) \rfloor$. Then we have:*

$$\frac{H \cdot \mathrm{ch}_2(E)}{H^3 \cdot \mathrm{ch}_0(E)} \leq \max\left\{ k\mu_H(E) - \frac{k^2}{2}, \frac{1}{2}\mu_H(E)^2 - \frac{3}{4d}, (k+1)\mu_H(E) - \frac{(k+1)^2}{2} \right\}.$$

Moreover, if the equality holds, then E has rank one or two.

The second is due to Naoki Koseki and Chunyi Li. It is based on [Kos22, Lemma 4.2, Theorem 4.3], however for our purposes we quote a reformulation for Fano threefolds from [JLZ22]. Chunyi Li also sent us a similar inequality from his upcoming paper [Li23].

Lemma 4.3.5 (Stronger BG II)**.** *Let X_{2g-2} be an index 1 Fano threefold of degree $d = 2g - 2$, and $E \in \mathrm{Coh}^0(X)$ be a $\sigma_{\alpha,0}$-semistable object for some $\alpha > 0$ with $|\mu_H(E)| \in [0, 1]$ and $\mathrm{rk}(E) \geq 2$. Then*

$$\frac{H \cdot \mathrm{ch}_2(E)}{H^3 \cdot \mathrm{ch}_0(E)} \leq \max\left\{ \frac{1}{2}\mu_H(E)^2 - \frac{3}{4d}, \mu_H(E)^2 - \frac{1}{2}|\mu_H(E)| \right\}.$$

4.4 Stability conditions on Kuznetsov components

In this section we recall the construction of Bridgeland stability conditions on the subcategory $\mathsf{Ku}(X) \subset \mathrm{D}^{\mathrm{b}}(X)$ due to [BLMS23].

4.4.1 Double-tilted stability conditions

As in [BLMS23], we pick a weak stability condition $\sigma_{\alpha,\beta}$ and tilt the once-tilted heart $\mathrm{Coh}^{\beta}(X)$ with respect to the tilt slope $\mu_{\alpha,\beta}$ (i.e. the slop with respect to the tilt stability function (4.3.1)) and some second tilt parameter μ. One gets a torsion pair $(\mathcal{T}^{\mu}_{\alpha,\beta}, \mathcal{F}^{\mu}_{\alpha,\beta})$ and another heart $\mathrm{Coh}^{\mu}_{\alpha,\beta}(X)$ of $\mathrm{D}^{\mathrm{b}}(X)$. Now "rotate" the stability function $Z_{\alpha,\beta}$ by setting

$$Z^{\mu}_{\alpha,\beta} := \frac{1}{u} Z_{\alpha,\beta}$$

where $u \in \mathbf{C}$ such that $|u| = 1$ and $\mu = -\frac{\Re u}{\Im u}$.

Proposition 4.4.1 ([BLMS23, Proposition 2.15]). *The pair $(\mathrm{Coh}^{\mu}_{\alpha,\beta}(X), Z^{\mu}_{\alpha,\beta})$ defines a weak stability condition on $\mathrm{D}^{\mathrm{b}}(X)$.*

For example, if we choose $\mu = 0$, we have

$$Z^{0}_{\alpha,\beta}(E) = H^{n-1}\mathrm{ch}_1^{\beta}(E) + \mathfrak{i}(H^{n-2}\mathrm{ch}_2^{\beta}(E) - \frac{1}{2}\alpha^2 H^n \mathrm{ch}_0^{\beta}(E)).$$

The slope of an object E with respect to this central charge will be denoted $\mu^{0}_{\alpha,\beta}(E)$.

We next state a result which gives a criterion for checking when weak stability conditions on a triangulated category can be used to induce stability conditions on a subcategory.

Proposition 4.4.2 ([BLMS23, Proposition 5.1]). *Let $\mathcal{D}$ be a triangulated category with an exceptional collection $\{E_1, \dots, E_m\}$, and let $\mathcal{D}_2$ be the category generated by the exceptional collection. Consider the resulting semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$. Let $(\mathcal{A}, Z)$ be a weak stability condition on $\mathcal{D}$ such that for all $i = 1, \dots, m$:*

1. *$E_i \in \mathcal{A}$;*
2. *$S_{\mathcal{D}}(E_i) \in \mathcal{A}[1]$; and*
3. *$Z(E_i) \neq 0$.*

Assume further that $Z_1 := Z|_{\mathrm{K}_0(\mathcal{A}_1)}$ (where $\mathcal{A}_1 := \mathcal{A} \cap \mathcal{D}_1$) is a stability function. Then $\sigma_1 = (\mathcal{A}_1, Z_1)$ is a stability condition on $\mathcal{D}_1$.

Each criterion of this proposition can be checked for $\mathsf{Ku}(X) \subset \mathrm{D}^{\mathrm{b}}(X)$ to give stability conditions on $\mathsf{Ku}(X)$, as we will see in the next section.

4.4.2 Stability conditions on Kuznetsov components

Let $\mathcal{A}(\alpha,\beta) := \mathrm{Coh}^{\mu}_{\alpha,\beta}(X) \cap \mathsf{Ku}(X)$ and $Z(\alpha,\beta) := Z^{\mu}_{\alpha,\beta}|_{\mathsf{Ku}(X)}$. Furthermore, let $0 < \epsilon \ll 1$, $\beta = -1+\epsilon$ and $0 < \alpha < \epsilon$. Also impose the following condition on the second tilt parameter μ:

$$\mu_{\alpha,\beta}(\mathcal{E}(-H)[1]) < \mu_{\alpha,\beta}(\mathcal{O}_X(-H)[1]) < \mu < \mu_{\alpha,\beta}(\mathcal{E}) < \mu_{\alpha,\beta}(\mathcal{O}_X). \tag{4.4.1}$$

Then we get the following theorem.

Theorem 4.4.3 ([BLMS23, Theorem 6.9]). *Let X be a Fano threefold of genus $6, 8, 10$ or 12, and let ϵ, α, β and μ be parameters as above. Then the pair $\sigma(\alpha,\beta) = (\mathcal{A}(\alpha,\beta), Z(\alpha,\beta))$ defines a Bridgeland stability condition on $\mathsf{Ku}(X)$.*

Remark 4.4.4. In slicing notation, we will write $\sigma(\alpha,\beta) = (\mathcal{P}(\alpha,\beta), Z(\alpha,\beta))$ where

$$\mathcal{P}(\alpha,\beta)((0,1]) = \mathcal{A}(\alpha,\beta)$$

as in [PY22, Section 3.3].

In this thesis, we fix $\mu = 0$, i.e. $\sigma(\alpha,\beta) := \sigma^0_{\alpha,\beta}|_{\mathsf{Ku}(X)}$. The same definition also applies when we replace $\mathsf{Ku}(X)$ by the alternative Kuznetsov component $\mathcal{A}_X$.

Proposition 4.4.5 ([JLZ22, Proposition 4.9]). *Let $X = X_g$ be a prime Fano threefold of index 1 and even genus $6 \leq g \leq 12, g \neq 11$, and $\mathcal{E}_g$ the vector bundles defined in Section 3.4. Then $\sigma(\alpha,\beta) := (\mathcal{A}(\alpha,\beta), Z^0_{\alpha,\beta}|_{\mathsf{Ku}(X)})$ is a stability condition for (α,β) listed below:*

- *$g = 6$: $\beta = -\frac{9}{10}, 0 < \alpha < 1+\beta$,*
- *$g = 8$: $\beta = -\frac{22}{25}, 0 < \alpha < 1+\beta$,*
- *$g = 10$: $\beta = -\frac{22}{25}, 0 < \alpha < 1+\beta$,*
- *$g = 12$: $\beta = -\frac{21}{25}, 0 < \alpha < 1+\beta$.*

Proof. One can check that $\mathcal{E}_g, \mathcal{E}_g(-H)[1], \mathcal{O}_X, \mathcal{O}_X(-H)[1] \in \mathrm{Coh}^{\beta}(X)$, and that they satisfy

$$\mu_{\alpha,\beta}(\mathcal{E}(-H)[1]) < \mu_{\alpha,\beta}(\mathcal{O}_X(-H)[1]) < 0 < \mu_{\alpha,\beta}(\mathcal{E}) < \mu_{\alpha,\beta}(\mathcal{O}_X)$$

for each (α,β) listed above.

Since $g \geq 6$ is even, from [PR23, Proposition 3.2] we know that $\sigma(\alpha,\beta)$ is a stability condition for (α,β) as above. □

The result above with appropriate choices of (α, β) is also true for the odd genus cases by [JLZ22, Appendix A.1], but since we do not require these cases for this thesis we omit them.

4.4.3 Stability conditions on the alternative Kuznetsov component of a Gushel–Mukai threefold

For a Gushel–Mukai threefold X, the focus of this thesis, consider $\mathcal{A}_X \subset \mathrm{D}^{\mathrm{b}}(X)$ and set $\mathcal{A}_{\mathrm{alt}}(\alpha, \beta) := \mathrm{Coh}^0_{\alpha,\beta}(X) \cap \mathcal{A}_X$ and $Z_{\mathrm{alt}}(\alpha, \beta) := Z^0_{\alpha,\beta}|_{\mathcal{N}(\mathcal{A}_X)}$. Then analogously to Proposition 4.4.5 we have:

Proposition 4.4.6 ([JLLZ21, Theorem 4.10])**.** *Let X be a Gushel–Mukai threefold. Then $\sigma_{\mathrm{alt}}(\alpha, \beta) := (\mathcal{A}_{\mathrm{alt}}(\alpha, \beta), Z_{\mathrm{alt}}(\alpha, \beta))$ is a Bridgeland stability condition for (α, β) in*

$$V := \{(\alpha, \beta) \mid -\frac{1}{10} < \beta < 0,\ 0 < \alpha < -\beta\}.$$

Proof. By [BLMS23, Theorem 6.9] and [PR23, Proposition 3.2]. □

4.5 Serre-invariance of stability conditions

We now introduce the notion of Serre-invariance of stability conditions on Kuznetsov components. This becomes very important when one uses equivalences of Kuznetsov components to induce (iso)morphisms between the corresponding moduli spaces of stable objects. We will see this later on in Chapter 6.

Definition 4.5.1. Let σ be a stability condition on the Kuznetsov component $\mathsf{Ku}(X)$. It is called *Serre-invariant* if $S_{\mathsf{Ku}(X)}(\sigma) = \sigma \cdot g$ for some $g \in \widetilde{\mathrm{GL}}^+(2, \mathbf{R})$.

Theorem 4.5.2. *The stability conditions from Propositions 4.4.5 and 4.4.6 are Serre-invariant.*

Proof. For the cases when $g \geq 6$ is even, Serre-invariance follows from [PR23, Theorem 3.18]. □

4.5.1 Serre-invariant stability conditions on Gushel–Mukai threefolds

In this section, we state a few Serre-invariance related results specifically for Gushel–Mukai threefolds. Here, we work with the alternative Kuznetsov component $\mathcal{A}_X$ as it will be more convenient for us to use later on, but everything written below holds for $\mathsf{Ku}(X)$ too (see Lemma 3.4.6).

Note that by [KP18, Proposition 2.6], there is a natural involutive autoequivalence functor of $\mathcal{A}_X$, denoted by $\tau_{\mathcal{A}}$. It is related to the Serre functor as follows:

$$S_{\mathcal{A}_X} = \tau_{\mathcal{A}}[2].$$

We will discuss this in more detail in Section 6.3.1.

Section 4.5.1 and Section 4.5.2 are joint work with Xun Lin, Zhiyu Liu, and Shizhuo Zhang, and are taken from the paper [JLLZ21].

Proposition 4.5.3 ([JLLZ21, Proposition 4.12])**.** *Let σ be a Serre-invariant stability condition on $\mathcal{A}_X$. Then*

1. *the homological dimension of the heart of σ is 2.*
2. $\mathrm{ext}^1(A, A) \geq 2$ *for every non-trivial object A in the heart of σ.*

Proof. Let A, B be objects in the heart $\mathcal{A}$ of σ. Then $\mathrm{Hom}(A, B[i]) = 0$ for $i < 0$. Note that the phases of the semistable factors of $\tau_{\mathcal{A}}(A)$ are in the interval $(0, 1)$, and the phases of the semistable factors of $B[i]$ are in $(i, i+1)$. Then $\mathrm{Hom}(A, B[i]) \cong \mathrm{Hom}(B[i], \tau_{\mathcal{A}}(A)[2]) = 0$ if $i \geq 3$. This proves (1). For (2), note that $\chi(A, A) \leq -1$ for all non-zero objects $A \in \mathcal{A}$, so the result follows. □

Proposition 4.5.4 ([JLLZ21, Proposition 4.14])**.** *Let X be a Gushel–Mukai threefold and E an object in $\mathsf{Ku}(X)$ such that $\mathrm{ext}^1(E, E) = 2$ or 3 and $\chi(E, E) = -1$. Then E is σ-stable for every Serre-invariant stability condition σ on $\mathcal{A}_X$.*

Proof. The proof is the same as in [Zha21, Corollary 4.15]. □

4.5.2 Uniqueness of Serre-invariant stability conditions

In this section, we show that all Serre-invariant stability conditions on $\mathsf{Ku}(Y_d)$ and $\mathsf{Ku}(X_{4d+2})$ (or $\mathcal{A}_{X_{4d+2}}$) are in the same $\widetilde{\mathrm{GL}}^+(2, \mathbf{R})$-orbit (which is what we mean by *uniqueness*) for each $d \geq 2$.

Lemma 4.5.5 ([JLLZ21, Lemma 4.15])**.** *Let σ' be a Serre-invariant stability condition on $\mathsf{Ku}(Y_d)$ where $d \geq 2$. Then the heart of σ' has homological dimension at most 2.*

Proof. When $d = 2$, this follows from the same argument as in Proposition 4.5.3. When $d = 3$, this follows from [PY22, Lemma 5.10]. When $d = 4$ and 5, since $\mathsf{Ku}(Y_4) \simeq \mathrm{D}^{\mathrm{b}}(C_2)$ and $\mathsf{Ku}(Y_5) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{Rep}(K(3)))$ where C_2 is a genus 2 smooth curve and $\mathrm{Rep}(K(3))$ is the category of representations of the 3-Kronecker quiver ([KPS18, p. 173]), then in these two cases the heart has homological dimension 1. □

Lemma 4.5.6 ([JLLZ21, Lemma 4.16]). *Let σ be a Serre-invariant stability condition on $\mathsf{Ku}(Y_d)$ where $d \geq 2$. If E and F are two σ-semistable objects with phases $\phi(E) < \phi(F)$, then $\mathrm{Hom}(E, F[2]) = 0$.*

Proof. When $d = 4$ and 5, this follows from the fact that the heart of σ has homological dimension 1. When $d = 2$ and 3, this is by [PY22, Section 5, Section 6]. □

Lemma 4.5.7 (Weak Mukai Lemma, [PY22, Lemma 5.12]). *Let $F \to E \to G$ be an exact triangle in $\mathsf{Ku}(Y_d)$ such that $\mathrm{Hom}(F, G) = 0$ and such that the phases of all the σ'-semistable factors of F are greater than the phases of the σ'-semistable factors of G. Then we have*

$$\mathrm{ext}^1(F, F) + \mathrm{ext}^1(G, G) \leq \mathrm{ext}^1(E, E).$$

Lemma 4.5.8 ([JLLZ21, Lemma 4.18]). *Let σ be a Serre-invariant stability condition on $\mathsf{Ku}(Y_d)$ where $d \geq 2$. Assume that there is a triangle $F \to E \to G$ of $E \in \mathsf{Ku}(Y_d)$ such that the phases of all the σ-semistable factors of F are greater than the phases of the σ-semistable factors of G. Then we have $\mathrm{ext}^1(F, F) < \mathrm{ext}^1(E, E)$ and $\mathrm{ext}^1(G, G) < \mathrm{ext}^1(E, E)$.*

Proof. Since $\phi^-(F) > \phi^+(G)$, we have $\mathrm{Hom}(F, G) = 0$. By [PY22, Lemma 5.11] we have that there do not exist non-zero objects A in the heart of $\mathcal{A}$ such that $\mathrm{Ext}^1(A, A) = 0$ or $\mathbf{C}$. Thus $\mathrm{ext}^1(A, A) \geq 2$ and by the Weak Mukai Lemma 4.5.7 the result follows. □

Let $\sigma = \sigma(\alpha, -\frac{1}{2})$ and $Y := Y_d$ where $d \geq 2$. As shown in [PY22, Section 4], the moduli spaces $\mathcal{M}_\sigma(\mathsf{Ku}(Y), -s)$ and $\mathcal{M}_\sigma(\mathsf{Ku}(Y), t - s)$ are non-empty. Let $A, B \in \mathcal{A}(\alpha, -\frac{1}{2})$ with $[A] = -s$, $[B] = t - s$ be σ-stable objects. We denote the phase with respect to $\sigma = \sigma(\alpha, -\frac{1}{2})$ by $\phi(-)$.

Now let $\sigma_1 = (\mathcal{A}_1, Z_1)$ be any Serre-invariant stability condition on $\mathsf{Ku}(Y)$. By [PY22, Remark 5.14], there is a $T = (t_{ij})_{1 \leq i,j \leq 2} \in \mathrm{GL}^+(2, \mathbf{R})$ such that $Z_1 = T \cdot Z(\alpha, -\frac{1}{2})$. Since A is stable with respect to every Serre-invariant stability condition by [PY22, Lemma 5.13], we can assume $A[m] \in \mathcal{A}_1$ for some $m \in \mathbf{Z}$. Let $\sigma_2 = \sigma \cdot \tilde{g}$ for $\widetilde{g} := (g, T) \in \widetilde{\mathrm{GL}}^+(2, \mathbf{R})$ such that $A[m] \in \mathcal{A}_2$ and $Z_2 = Z_1$. Then we have $\phi_1(A) = \phi_2(A)$ and $\mathcal{A}_2 = \mathcal{P}(\alpha, -\frac{1}{2})((g(0), g(0) + 1])$.

Lemma 4.5.9 ([JLLZ21, Lemma 4.19])**.** *Fix the notation as above. Then A and B are σ_1-stable with phase $\phi_1(A) = \phi_2(A)$ and $\phi_1(B) = \phi_2(B)$.*

Proof. The stability of A and B is from [PY22, Lemma 5.13]. By definition of σ_2, we know $\phi_1(A) = \phi_2(A)$ and $\phi_2(B) < \phi_2(A) < \phi_2(B) + 1$. Also, from [PY22, Remark 4.8] we know $\phi_1(B) < \phi_1(A) = \phi_2(A) < \phi_1(B) + 1$. Thus $\phi_1(B) = \phi_2(B)$. $\square$

Recall that the numerical Grothendieck group of a Gushel–Mukai threefold X is $\mathcal{N}(\mathsf{Ku}(X)) = \langle v, w \rangle$, a rank two lattice generated by $v = 1 - 3L + \frac{1}{2}P = [I_C]$ and $w = H - 6L + \frac{1}{6}P$, where C is a twisted cubic on X.

Lemma 4.5.10 ([JLLZ21, Lemma 4.21])**.** *Let X be a smooth Gushel–Mukai threefold and $A'', B'' \in \mathsf{Ku}(X)$ be two σ-stable objects of numerical class $[A''] = -(3v - 2w)$ and $[B''] = v$, where σ is a Serre-invariant stability condition. Then we have $\phi_\sigma(B'') < \phi_\sigma(A'') < \phi_\sigma(B'') + 1$.*

Proof. Let X be a Gushel–Mukai threefold. Let A'', B'' and B' be (-2)-class σ-stable objects in $\mathsf{Ku}(X)$ with respect to a Serre-invariant stability condition σ. In particular, let $B'' := i^*(I_C)$ where[3] $I_C \notin \mathsf{Ku}(X)$. Thus $i^*(I_C) \cong i^*(G)$, where G is the twisted derived dual of a line L such that $L \cup C = Z(s)$, and where s is a section of $\mathcal{E}^\vee$. Note that G is given by the triangle

$$\mathcal{O}_X(-H)[1] \to G \to \mathcal{O}_L(-2). \tag{4.5.1}$$

Let $A'' := i^*(I_L)$ and $B' := i^*(I_D) = I_D$, where D is a twisted cubic with an irreducible component L and $I_D \in \mathsf{Ku}(X)$. Note that $[B''] = [B'] = v$. Then the result follows Lemma 4.5.11, Lemma 4.5.12, and Lemma 4.5.13, all of which we prove below. $\square$

Lemma 4.5.11 ([JLLZ21, Lemma 4.22])**.** *We have* $\mathrm{Hom}(A'', B''[1]) \neq 0$.

Proof. By adjunction, we have $\mathrm{Hom}(i^*(I_L), i^*(G)[1]) \cong \mathrm{Hom}(I_L, i^*(G)[1])$. Note that $i^*(G)$ fits into the exact triangle

$$G \to i^*(G) \to \mathcal{E}$$

[3]Recall the notation for left adjoint (projection) functors from Definition 3.4.7

by [Zha21, Proposition 5.3]. Next apply $\mathrm{Hom}(I_L, -)$ to the above triangle to get the exact sequence

$$\cdots \to \mathrm{Hom}(I_L, \mathcal{E}) \to \mathrm{Ext}^1(I_L, G) \to \mathrm{Ext}^1(I_L, i^*(G)) \to \mathrm{Ext}^1(I_L, \mathcal{E}) \to \cdots .$$

It is clear that $\mathrm{Hom}(I_L, \mathcal{E}) = 0$ and $\mathrm{Ext}^1(I_L, \mathcal{E}) \cong \mathrm{Ext}^2(\mathcal{E}^\vee, I_L) = 0$ so we get $\mathrm{Ext}^1(I_L, i^*(G)) \cong \mathrm{Ext}^1(I_L, G)$. Applying $\mathrm{Hom}(I_L, -)$ to the triangle (4.5.1) defining G, we get a long exact sequence

$$\cdots \to \mathrm{Ext}^i(I_L, \mathcal{O}_X(-H)[1]) \to \mathrm{Ext}^i(I_L, G) \to \mathrm{Ext}^i(I_L, \mathcal{O}_L(-2)) \to \cdots .$$

By Serre duality, we have $\mathrm{Ext}^i(I_L, \mathcal{O}_X(-H)) = \mathrm{Ext}^{3-i}(\mathcal{O}_X, I_L) = 0$ for all i. Then we have $\mathrm{Ext}^1(I_L, G) \cong \mathrm{Ext}^1(I_L, \mathcal{O}_L(-2))$. By the adjunction associated to the embedding $j\colon L \to X$, we get $\mathrm{Ext}^1(I_L, \mathcal{O}_L(-2)) \cong \mathrm{Ext}^1(j^* I_L, \mathcal{O}_L(-2)) \cong \mathrm{Ext}^1(\mathcal{N}_{L|X}, \mathcal{O}_L(-2))$. As the normal bundle of L in X is either $\mathcal{N}_{L|X} = \mathcal{O}_L \oplus \mathcal{O}_L(-1)$ or $\mathcal{O}_L(1) \oplus \mathcal{O}_L(-2)$, we get

$$\mathrm{Ext}^1(I_L, \mathcal{O}_L(-2)) \cong \mathrm{Ext}^1(\mathcal{O}_L \oplus \mathcal{O}_L(1), \mathcal{O}_L(-2)) = \mathbf{C}^3$$

or $\mathrm{Ext}^1(I_L, \mathcal{O}_L(-2)) \cong \mathrm{Ext}^1(\mathcal{O}_L(-1) \oplus \mathcal{O}_L(2), \mathcal{O}_L(-2)) = \mathbf{C}^3$. □

Lemma 4.5.12 ([JLLZ21, Lemma 4.23])**.** *We have* $\mathrm{Hom}(B', A'') \neq 0$.

Proof. Applying $\mathrm{Hom}(I_D, -)$ to the triangle $\mathcal{E}^{\oplus 2} \to I_L \to i^*(I_L)$, we get an exact sequence

$$0 \to \mathrm{Hom}(I_D, \mathcal{E}^{\oplus 2}) \to \mathrm{Hom}(I_D, I_L) \to \mathrm{Hom}(I_D, i^*(I_L)) \to \cdots .$$

Note that $\mathrm{Hom}(I_D, \mathcal{E}) = 0$, thus $\hom(I_D, i^*(I_L)) \geq \hom(I_D, I_L)$. Since $L \subset D$ is an irreducible component of D, $\hom(I_D, I_L) = 1$. Then the result follows. □

Lemma 4.5.13 ([JLLZ21, Lemma 4.24])**.** *The twisted cubics C and D, and a line L as in Lemma 4.5.10 exist.*

Proof.

1. Let X be a special Gushel–Mukai threefold, $\pi\colon X \to Y_5$ the double cover and $\mathcal{B} \subset Y_5$ the branch locus. Let $c = l_1 \cup l_2 \cup l_3$ be a twisted cubic on Y_5 such that each l_i is tangent to $\mathcal{B}$. Note that l_1 is in the conic $l_1 \cup l_2$; pulling back to X via π, we get a twisted cubic C such that $L \cup C = \pi^{-1}(l_1 \cup l_2)$ and $\tau(L) \subset C$. On the other hand, l_1 is in $c = l_1 \cup l_2 \cup l_3$; it is a twisted cubic triple tangent to $\mathcal{B}$, and pulling back to X we get a twisted cubic D and $I_D \in \mathsf{Ku}(X)$. Note that $L \subset D$.

2. If X is an ordinary Gushel–Mukai threefold, the locus of irreducible twisted cubics has dimension ≤ 2, and the locus of twisted cubics that are in $\mathsf{Ku}(X)$ has dimension 3. Thus there exists a twisted cubic D that contains a line L and $I_D \in \mathsf{Ku}(X)$. On the other hand, since $\mathrm{Hom}(\mathcal{E}, I_L) = \mathbf{C}^2$, the locus of twisted cubics C such that $C \cup L$ is the zero locus of a section of $\mathcal{E}^\vee$ is parametrized by $\mathbf{P}^1$, where $I_C \notin \mathsf{Ku}(X)$. Choose one such twisted cubic C.

$\square$

Theorem 4.5.14 ([JLLZ21, Theorem 4.20])**.** *All Serre-invariant stability conditions on $\mathsf{Ku}(X)$ are in the same $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$-orbit. Here $X \coloneqq X_{2d+2}$ or Y_d for $d \geq 2$.*

Proof. Fix the notation to be the same as after Lemma 4.5.8. We are going to show that $\sigma_1 = \sigma_2$. Since $\mathsf{Ku}(X_7)$, $\mathsf{Ku}(X_9)$ and $\mathsf{Ku}(X_{10}) \simeq \mathsf{Ku}(Y_4)$ are equivalent to the bounded derived categories of some smooth curves of positive genus, the results for these three cases follow from [Mac07, Theorem 2.7]. The results for $\mathsf{Ku}(X_8)$ and $\mathsf{Ku}(X_{12})$ are from the results for $\mathsf{Ku}(Y_3)$ and $\mathsf{Ku}(Y_5)$ and the equivalences $\mathsf{Ku}(Y_d) \simeq \mathsf{Ku}(X_{2d+2})$, where $d \geq 3$ (see [Kuz09]). Thus we only need to prove this for Y_d when $d \geq 2$ and $X \coloneqq X_6$.

We first prove this for Y_d when $d \geq 2$. Let $E \in \mathcal{A}(\alpha, -\frac{1}{2})$ be a σ-semistable object with $[E] = as + bt$. First we are going to show that if E is σ_1-semistable, then $\phi_2(E) = \phi_1(E)$. Note that we have the following relations:

1. $\chi(E, A) = a + (d-1)b$, $\chi(A, E) = a + b$; and $\mu^0_{\alpha,-\frac{1}{2}}(E) > \mu^0_{\alpha,-\frac{1}{2}}(A) \iff b < 0$
2. $\chi(E, B) = -b$, $\chi(B, E) = -[(d-2)a + (d-1)b]$; and $\mu^0_{\alpha,-\frac{1}{2}}(E) > \mu^0_{\alpha,-\frac{1}{2}}(B) \iff a + b < 0$.

From the definition of $\sigma = \sigma(\alpha, -\frac{1}{2})$-stability we have $a \leq 0$. When $a = 0$, by the definition of a stability condition we have $b < 0$. Thus in the case $b > 0$ we always have $a < 0$. Note that by the proof of Lemma 4.5.9, we have $\phi_2(B) < \phi_2(A)$ and both of them lie in the interval $(g(0), g(0) + 1]$.

- Assume that $b > 0$ and $a + b > 0$. Then $\mu^0_{\alpha,-\frac{1}{2}}(E) < \mu^0_{\alpha,-\frac{1}{2}}(B) < \mu^0_{\alpha,-\frac{1}{2}}(A)$ and hence $\phi_2(E) < \phi_2(B) < \phi_2(A)$. We also have $\chi(E, A) > 0$. Thus by Lemma 4.5.6 we know $\mathrm{Hom}(E, A[2]) = 0$. Thus $\chi(E, A) = \hom(E, A) - \hom(E, A[1]) > 0$ implies $\hom(E, A) > 0$, and therefore $\phi_1(E) < \phi_1(A)$.

Also from $\chi(B,E) < 0$ and Lemma 4.5.5 we have $\phi_1(B) - 1 < \phi_1(E)$. Then we have $\phi_1(B) - 1 < \phi_1(E) < \phi_1(A)$. But by Lemma 4.5.9 we know $\phi_1(B) = \phi_2(B)$, $\phi_1(A) = \phi_2(A)$. Also, from the definition of σ_2 we have $|\phi_2(B) - \phi_2(A)| < 1$ and $|\phi_2(A) - \phi_2(E)| < 1$. Thus $\phi_2(E) - \phi_1(E) = 0$ or 1. But if $\phi_2(E) = \phi_1(E) + 1$, then $\phi_2(B) - 1 = \phi_1(B) - 1 < \phi_2(E) < \phi_1(B) = \phi_2(B)$. This implies $1 = \phi_1(B) - \phi_1(B) + 1 > \phi_2(E) - \phi_1(B) + 1 = \phi_1(E) - \phi_1(B) + 2$, which is impossible since $\phi_1(B) - 1 < \phi_1(E)$. Thus we have $\phi_1(E) = \phi_2(E)$.

- Assume that $b > 0$ and $a+b < 0$. Then $\mu^0_{\alpha,-\frac{1}{2}}(B) < \mu^0_{\alpha,-\frac{1}{2}}(E) < \mu^0_{\alpha,-\frac{1}{2}}(A)$ and hence $\phi_2(B) < \phi_2(E) < \phi_2(A)$. Since $\chi(A,E) < 0$ and $\chi(E,B) < 0$, from Lemma 4.5.5 we know $\hom(A, E[1]) > 0$ and $\hom(E, B[1]) > 0$, hence $\phi_1(A) - 1 < \phi_1(E) < \phi_1(B) + 1$. This means $|\phi_1(E) - \phi_2(E)| = 0$ or 1. But $|\phi_1(E) - \phi_2(E)| = 1$ is impossible since $\phi_1(B) = \phi_2(B) < \phi_2(E) < \phi_2(A) = \phi_1(A)$. Therefore we have $\phi_1(E) = \phi_2(E)$.

- Assume that $b < 0$. Then $\mu^0_{\alpha,-\frac{1}{2}}(B) < \mu^0_{\alpha,-\frac{1}{2}}(A) < \mu^0_{\alpha,-\frac{1}{2}}(E)$ and hence $\phi_2(B) < \phi_2(A) < \phi_2(E)$. Since $\chi(E,A) < 0$, from Lemma 4.5.5 we have $\hom(E, A[1]) > 0$ and $\phi_1(E) < \phi_1(A) + 1$. By Lemma 4.5.6, $\mu^0_{\alpha,-\frac{1}{2}}(B) < \mu^0_{\alpha,-\frac{1}{2}}(E)$ and $\chi(B,E) > 0$, we know that $\hom(B,E) > 0$. Thus $\phi_1(B) < \phi_1(E) < \phi_1(A) + 1$. Hence $\phi_1(E) - \phi_2(E) = 0$ or 1. But since $\mu^0_{\alpha,-\frac{1}{2}}(A) < \mu^0_{\alpha,-\frac{1}{2}}(E)$, we have $\phi_2(A) = \phi_1(A) < \phi_2(E)$. Thus $\phi_1(A) < \phi_2(E) < \phi_1(A) + 1$. Then $\phi_1(E) - \phi_2(E) = 1$ is impossible since $\phi_1(E) < \phi_1(A) + 1$. Therefore we have $\phi_1(E) = \phi_2(E)$.

- When $b = 0$, we have $[E] = -a \cdot [A]$. Hence $\chi(E,A) = \chi(A,E) < 0$ and we have $\phi_1(A) - 1 \leq \phi_1(E) \leq \phi_1(A) + 1$. But $\mu_1(E) = \mu_1(A)$, so we know $\phi_1(E) - \phi_1(A)$ is an integer. Thus $\phi_1(E) = \phi_1(A) \pm 1$. But from the definition of a stability function, we have $Z_1(E[\pm 1]) = -Z_1(A)$. Thus $\phi_1(E) = \phi_1(A) = \phi_2(E)$.

- When $a + b = 0$, we have $[E] = -a \cdot [B]$. Hence $\chi(E,B) = \chi(B,E) < 0$ and we have $\phi_1(B) - 1 \leq \phi_1(E) \leq \phi_1(B) + 1$. But $\mu_1(E) = \mu_1(B)$, so we know $\phi_1(E) - \phi_1(B)$ is an integer. Thus $\phi_1(E) = \phi_1(B) \pm 1$. But from the definition of a stability function, we have $Z_1(E[\pm 1]) = -Z_1(B)$. Thus $\phi_1(E) = \phi_1(B) = \phi_2(E)$.

Next we show that $E \in \mathcal{A}_2$ is σ_2-semistable if and only if $E \in \mathcal{A}_1$ is σ_1-semistable. We prove this by induction.

If $\mathrm{ext}^1(E, E) < 2$, this is by [PY22, Section 5]. Now assume this is true for all $E \in \mathcal{A}_2$ σ_2-semistable such that $\mathrm{ext}^1(E, E) < N$.

When $E \in \mathcal{A}_2$ is σ_2-semistable and has $\mathrm{ext}^1(E, E) = N$, assume otherwise that E is not σ_1-semistable. Let A_0 be the first HN-factor of E with respect to σ_1 and A_n be the last one. Then $\phi_1(A_0) > \phi_1(A_n)$. By Lemma 4.5.8, $\mathrm{ext}^1(A_0, A_0) < N$ and $\mathrm{ext}^1(A_n, A_n) < N$. Thus A_0 and A_n are σ_2-semistable by the induction hypothesis and $\phi_2(A_0) > \phi_2(A_n)$ by the results above. Since $\mathrm{Hom}(A_0, E)$ and $\mathrm{Hom}(E, A_n)$ are both non-zero, we know that $\phi_2(A_0) \leq \phi_2(E)$ and $\phi_2(E) \leq \phi_2(A_n)$, which implies $\phi_2(A_0) \leq \phi_2(A_n)$ and gives a contradiction. Thus E is σ_1-semistable. When $E \in \mathcal{A}_1$ is σ_1-semistable, the same argument shows that $E \in \mathcal{A}_2$ is also σ_2-semistable.

Since every object in the heart is the extension of semistable objects, we have $\mathcal{A}_1 = \mathcal{A}_2$. And from $Z_1 = Z_2$, we know that $\sigma_1 = \sigma_2 = \sigma \cdot \widetilde{g}$. Hence σ_1 is in the orbit of $\sigma = \sigma(\alpha, -\frac{1}{2})$.

For a Gushel–Mukai threefold X_6, the result follows from Lemma 4.5.10 and a similar argument as the previous index 2 cases. □

Remark 4.5.15. The idea of the proof of Theorem 4.5.14 was first explained to us by Arend Bayer. In [Zha21, Proposition 4.21], Zhang made an attempt to prove this statement but the argument was incomplete. Here, we fill the gaps and give a uniform argument for all $\mathsf{Ku}(Y_d)$ and $\mathsf{Ku}(X_{4d+2})$ when $d \geq 2$.

In the paper [FP23, Theorem 3.1], the authors prove the uniqueness of Serre-invariant stability conditions for a general triangulated category satisfying a list of very natural assumptions; these categories include Kuznetsov components of all of the Fano threefolds that we consider.

Chapter 5

Categorical Torelli questions

Before we begin reviewing and investigating categorical Torelli questions, we very briefly review the Hodge-theoretic construction of the intermediate Jacobian of a complex smooth projective variety (Fano threefolds in our case). We also briefly touch on the classical notion of a Torelli theorem to motivate the categorical perspective.

5.1 Intermediate Jacobians, period maps, and classical Torelli theorems

Definition 5.1.1. The *intermediate Jacobian* of a Fano threefold is the complex torus

$$J(X) := \frac{H^{1,2}(X)}{H^3(X, \mathbf{Z})} \cong \frac{H^{2,1}(X)^\vee}{H_3(X, \mathbf{Z})}.$$

The intermediate Jacobian comes with a principal polarisation, and it is a generalisation of the Jacobian variety $J(C) := H^0(\Omega^1_C)^\vee / H_1(C, \mathbf{Z})$ of a curve. Note here that $H_1(C, \mathbf{Z})$ is embedded in $H^0(\Omega^1_C)^\vee$ via the map $\omega \mapsto \int_\gamma \omega$ where γ is a closed path in C. The homology $H_3(X, \mathbf{Z})$ is embedded analogously inside $H^{2,1}(X)^\vee$.

In [Tor13], Torelli showed that $J(C)$ considered as a principally polarised abelian variety determines C up to isomorphism. The same question can be asked for the intermediate Jacobian of a Fano threefold. In the celebrated papers [CG72, Tju70], it is proved that the intermediate Jacobian of a cubic threefold

$Y \subset \mathbf{P}^4$ considered as a principally polarised abelian variety determines the cubic threefold up to isomorphism. This can be restated as the *period map*

$$\mathcal{P}\colon \mathcal{Y}_3 \to \mathcal{A}_5, \quad Y \mapsto J(Y)$$

being injective, where $\mathcal{A}_5$ is the moduli space of dimension 5 principally polarised abelian varieties.

On the other hand, there are ordinary Gushel–Mukai threefolds which are non-isomorphic but have isomorphic intermediate Jacobians [DIM12]. So Torelli does not hold and the associated period map $\mathcal{P}\colon \mathcal{X}_6 \to \mathcal{A}_{10}$ has non-trivial fibers. The form that these fibers take is still a conjecture (see Conjecture 6.8.1).

As we will see later on in this thesis, analogous categorical statements can be made, and these may end up shedding some light on the classical Hodge-theoretic situation (see Section 6.8).

5.2 Categorical Torelli questions

For Fano threefolds, one can ask the following natural questions:

Questions 5.2.1.

1. Does $\mathsf{Ku}(X)$ determine the isomorphism class of X? In other words, for Fano threefolds X and X' of the same deformation family, does $\mathsf{Ku}(X) \simeq \mathsf{Ku}(X')$ imply $X \cong X'$? We call such an implication a *categorical Torelli theorem.*

2. Does $\mathsf{Ku}(X)$ determine the birational equivalence class of X? In other words, for Fano threefolds X and X' of the same deformation family, does $\mathsf{Ku}(X) \simeq \mathsf{Ku}(X')$ imply X is birational to X'? We call such an implication a *birational categorical Torelli theorem.*

3. What extra data along with $\mathsf{Ku}(X)$ is required to identify X within its birational equivalence class? We call such a statement a *refined categorical Torelli theorem.*

Remark 5.2.2. Similar questions can be asked for other (non-Fano) varieties, providing they admit a semiorthogonal decomposition with a non-trivial piece. We will touch on these questions in the next Section 5.3.

5.3 Related work

In this section, we give an overview of which categorical Torelli-type statements are known outside of the work in this thesis. We note here the fantastic overview [PS22] on this topic.

5.3.1 Higher genus index 1 Fano threefolds

In [JLZ22], via a uniform argument we prove refined categorical Torelli theorems for Fano threefolds of Picard rank 1, index 1 and genus $g \geq 6$. We show that $\langle \mathcal{O}_X \rangle^\perp = \langle \mathsf{Ku}(X), \mathcal{E} \rangle \subset \mathrm{D}^\mathrm{b}(X)$ (see Table 3.1 for the relevant semiorthogonal decompositions) is precisely the data required to recover X. More precisely, we show that given an equivalence of Kuznetsov components $\mathsf{Ku}(X) \simeq \mathsf{Ku}(X')$ preserving the gluing objects associated to the subcategories $\langle \mathsf{Ku}(X), \mathcal{E} \rangle$ and $\langle \mathsf{Ku}(X'), \mathcal{E}' \rangle$, the threefolds X and X' are isomorphic. We do this by recovering X as a *Brill–Noether locus* of objects inside the moduli space of Bridgeland stable objects in $\mathsf{Ku}(X)$. The moduli space is with respect to the numerical class of the projection of a skyscraper sheaf into $\mathsf{Ku}(X)$, and the Brill–Noether locus is defined using the gluing data of $\langle \mathsf{Ku}(X), \mathcal{E} \rangle$. An equivalence of Kuznetsov components preserving the gluing data induces an ismorphism of the Brill–Noether loci, and thus of X and X'.

5.3.2 Index 2 Fano threefolds

In this section we summarise what is known about categorical Torelli theorems for Fano threefolds Y_d. Before we do, we note recent work of [FLZ23] which provides two new ways of proving categorical Torelli for $\mathsf{Ku}(Y_d)$ when $2 \leq d \leq 4$.

One of the methods is to show that the gluing object (in the same sense as Section 6.1) is uniquely/canonically determined in the Kuznetsov component [FLZ23, Theorem 1.1]. A categorical Brill–Noether theorem [FLZ23, Theorem 1.2] (in the same sense as Section [JLZ22]) is proved which proves categorical Torelli.

The second method is to show that any Fourier–Mukai equivalence of Kuznetsov components lifts to an equivalence of their derived categories [FLZ23, Theorem 1.4]. Categorical Torelli then follows by Bondal–Orlov's Reconstruction Theorem [BO01].

Y_5. This Fano threefold is rigid, i.e. it has no moduli (see e.g. [Bel23]). Therefore there is no Torelli theorem to prove here.

Y_4. Recall that the derived category of Y_4 has the semiorthogonal decomposition $\mathrm{D^b}(Y_4) = \langle \mathrm{D^b}(C_2), \mathcal{O}(-H), \mathcal{O} \rangle$, i.e. $\mathsf{Ku}(Y_4)$ is the derived category of a genus 2 curve. This case is discussed in detail in the introduction (see the section "Motivating Examples"). In short, since Y_4 can be recovered from the associated curve C_2, categorical Torelli follows.

Y_3. Categorical Torelli for the cubic threefold $Y_3 \subset \mathbf{P}^4$ was first proved in [BMMS12, Theorem 1.1]. Since this paper was written before the uniform construction of stability conditions on Kuznetsov components [BLMS23], the authors used the fact that $\mathsf{Ku}(Y_3)$ is equivalent to an admissible subcategory of $\mathrm{D^b}(\mathbf{P}^2, \mathcal{B})$ to induce stability conditions on $\mathsf{Ku}(Y_3)$. Here $\mathrm{D^b}(\mathbf{P}^2, \mathcal{B})$ is the bounded derived category of the abelian category $\mathrm{Coh}(\mathbf{P}^2, \mathcal{B}_0)$ of right coherent $\mathcal{B}_0$-modules, where $\mathcal{B}_0$ is the even part of the Clifford algebra on corresponding to a certaain conic fibration of $\mathbf{P}^2$ related to a strict transform of Y_3. They then showed that the moduli space of stable objects in $\mathsf{Ku}(Y_3)$ of numerical class the ideal sheaf of a line in Y_3 (with respect to the aforementioned stability condition) is isomorphic to the Hilbert scheme of lines on Y_3. Classical results imply that Y_3 is determined by the Hilbert scheme of lines on it; indeed the intermediate Jacobian $J(Y_3)$ is the Albanese variety of the Hilbert scheme of lines on Y_3 and $J(Y_3)$ determines Y_3 by the Torelli theorem for cubic threefolds due to [CG72, Tju70]. Hence, categorical Torelli follows.

Another proof of categorical Torelli for cubic threefolds was provided in [PY22, Theorem 5.17]. In this paper, the authors showed that given a Serre-invariant stability condition as constructed in [BLMS23], the moduli space of stable objects with respect to the same numerical class as in the previous paragraph is isomorphic to the same Hilbert scheme as before. Categorical Torelli then follows from the same argument.

Yet another proof of categorical Torelli was provided in [BBF+22]. In this paper, the authors considered the moduli space of Bridgeland stable (with respect to a Serre invariant stability condition) objects with respect to the class of the projection of the skyscraper sheaf into $\mathsf{Ku}(Y_3)$. They showed that the Bridgeland moduli space is isomorphic to a Gieseker moduli space with respect to the same numerical class. They also showed that the Gieseker moduli space is the blow-up of the theta divisor of $J(Y_3)$ in the unique singular point of the theta divisor, with the exceptional divisor being Y_3 itself. Assuming an equivalence of Kuznetsov components, due to the Serre invariance of the stability conditions the Bridgeland moduli spaces end up being isomorphic. Thus the aforementioned Gieseker moduli spaces are isomorphic, and since they uniquely determine Y_3, this gives a proof of categorical Torelli for $\mathsf{Ku}(Y_3)$.

Under a Fourier–Mukai assumption on the equivalence, [BT16, Corollary 3.1] also proved the categorical Torelli theorem for cubic threefolds (see more details on this in the next Y_2 section).

Y_2. Categorical Torelli for the quartic double solid Y_2 was first considered in [BT16, Corollary 3.1]. Using the language of non-commutative motives, they showed that a Fourier–Mukai equivalence of Kuznetsov components induces an isomorphism of intermediate Jacobians as principally polarised abelian varieties. By the classical Torelli theorem for quartic double solids due to [Voi88, Deb90], the categorical Torelli theorem (with the Fourier–Mukai assumption) follows.

Remark 5.3.1. Using the same technique, [BT16] also proved categorical Torelli theorems (with a Fourier–Mukai assumption) for intersections of two even dimensional quadrics, and intersections of three odd dimensional quadrics; in other words whenever a classical Torelli theorem in terms of the intermediate Jacobian holds, for a variety whose Kuznetsov component is the orthogonal to an exceptional collection (which implies their $\mathbf{J}(\mathsf{Ku}(Y)^{\perp,\mathrm{dg}}) = 0$ condition; cf. [BT16, Theorem 2.4]).

The same categorical Torelli theorems follow from Perry's construction of the intermediate Jacobian of the Kuznetsov component [Per22].

A proof which does not require the Fourier–Mukai assumption on the equivalence was given in [APR22]. The authors studied moduli spaces of stable objects in $\mathsf{Ku}(Y_2)$ of a certain numerical class (related to the projection of the skyscraper sheaf into $\mathsf{Ku}(Y_2)$ by the Serre functor of $\mathsf{Ku}(Y_2)$) with respect to Serre invariant stability conditions. They described the irreducible components of these moduli spaces (by tracking changes of objects during wall-crossing to the Bridgeland moduli spaces, from Gieseker moduli spaces of the same numerical class). One of the irreducible components is Y_2 itself, and the authors showed that given an equivalence of Kuznetsov components, the component Y_2 is sent to the corresponding component Y_2' thus proving categorical Torelli.

Y_1. The categorical Torelli problem for $\mathsf{Ku}(Y_1)$ is currently open. One cannot use the moduli-theoretic approach from [APR22] due to the homological dimension of the heart of stability conditions on $\mathsf{Ku}(Y_1)$ being too large. Furthermore, we do not know whether Serre invariant stability conditions on $\mathsf{Ku}(Y_1)$ are unique.

5.3.3 Cubic fourfolds and other hypersurfaces

For hypersurfaces of projective space satisfying certain numerical conditions, in [HR19], the authors relate the "twisted" Hochschild cohomology of the Kuznetsov component $\mathrm{HH}^*(\mathsf{Ku}(Y), (1))$ to the *Jacobian ring*

$$\mathrm{Jac}(Y) \coloneqq \mathbf{C}[y_0, \dots, y_n]/(\partial_0 y_0, \dots, \partial y_n)$$

of Y. When $Y \subset \mathbf{P}^5$ is a cubic fourfold (and other hypersurfaces satisfying a numerical condition on degree and dimension), the map between $\mathrm{HH}^*(\mathsf{Ku}(Y), (1))$ and $\mathrm{Jac}(Y)$ ends up being an isomorphism. Therefore, assuming that we have an equivalence of Kuznetsov components which commutes with the twist functor (1), this induces an isomorphism of the Jacobian rings of the cubic fourfolds. Then by the Mather–Yau Theorem [Don83] the two cubic fourfolds are isomorphic.

In [Pir22], the author generalises the results of [HR19] by dropping a numerical assumption required in their paper.

In [LZ23], the authors prove categorical Torelli theorems for a range of hypersurfaces in projective space by relating the Serre algebra of the Kuznetsov component to the Jacobian ring, and then using the Mather–Yau Theorem as in previous paragraphs. This allows them to drop that the equivalence commutes with the degree shift functor (1) as it needs to do in [HR19, Pir22].

In [BLMS23, Theorem A.1], the authors prove a categorical Torelli theorem for cubic fourfolds using Bridgeland moduli spaces of stable objects in Kuznetsov components of cubic fourfolds.

5.3.4 X_2

Recall that X_2 is a cover of $\mathbf{P}^3$ branched in a sextic hypersurface Z. In [DJR23], we relate the equivariant Kuznetsov component of X_2 to the middle primitive cohomology of Z. More precisely, we show that the orthogonal (inside the topological K-theory of the equivariant Kuznetsov component of X) to the algebraic K-theory of the equivariant Kuznetsov component of X, is isomorphic to the middle primitive cohomology of Z. In this case, an equivalence of Kuznetsov components descends to an equivalence between the equivariant Kuznetsov components. We show that this equivariant equivalence induces a Hodge isometry between the middle primitive cohomologies of the respective branch divisors, which by Donagi's Torelli theorem for hypersurfaces [Don83] implies that the branch divisors are isomorphic. This gives the categorical Torelli theorem.

In [LZ23, 6. Appendix], the authors can adapt their methods to weighted projective hypersurfaces, and thus also show a categorical Torelli theorem for X_2 considered as a hypersurface in $\mathbf{P}(1,1,1,1,3)$.

The upcoming work [LPS23] also treats this case with independent methods.

5.3.5 Enriques surfaces

The derived category of an Enriques surface determines it up to isomorphism ([BM01] in characteristic 0 and [HLT21] in positive characteristic). Furthermore, derived categories of Enriques surfaces admit (non-full) length 10 exceptional collections. Thus, one can ask whether categorical Torelli holds in this case too.

In [LNSZ21], the authors prove a categorical Torelli theorem for generic Enriques surfaces by starting with an equivalence between the Kuznetsov components, and showing that one can uniquely extend this equivalence, exceptional-by-exceptional to an equivalence of the whole derived categories. The aforementioned derived Torelli result is then used to complete the proof of categorical Torelli. Uniquely extending this equivalence through the semiorthogonal decomposition relies on the projections of the exceptional objects into $\mathsf{Ku}(X)$ being *spherical objects*. The non-generic case is proved in the sequel paper [LSZ22].

Chapter 6

Categorical Torelli for Gushel–Mukai threefolds

We begin by studying the categorical Torelli question for ordinary Gushel–Mukai threefolds. In the paper [KP23, Theorem 1.6], the authors show that birational but non-isomorphic Gushel–Mukai threefolds (in their language, *period partners*) have equivalent Kuznetsov components. Therefore, it makes sense to ask (2) and (3) from Questions 5.2.1.

We first recall the definition of an n-dimensional Gushel–Mukai variety X. They are given as $X = \mathrm{Cone}(\mathrm{Gr}(2,5)) \cap \mathbf{P}^{n+4} \cap Q$ where $2 \leq n \leq 6$, $\mathrm{Cone}(\mathrm{Gr}(2,5))$ is the cone over the Plücker embedded Grassmannian, and $Q \subset \mathbf{P}^{n+4}$ is a quadric hypersurface. Projecting from the vertex of the cone gives a morphism $X \to \mathrm{Gr}(2,5)$ and a vector bundle $\mathcal{E}$ associated to this morphism. Gushel–Mukai varieties have semiorthogonal decompositions [KP18, Proposition 2.3]

$$\mathrm{D^b}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^\vee, \mathcal{O}_X(H), \mathcal{E}^\vee(H), \dots, \mathcal{O}_X((n-3)H), \mathcal{E}^\vee((n-3)H) \rangle. \tag{6.0.1}$$

Remark 6.0.1. We make a remark on notation. Take for example a Gushel–Mukai threefold ($n = 3$). The semiorthogonal decomposition (6.0.1) above gives the semiorthogonal decomposition $\mathrm{D^b}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^\vee \rangle$ whereas the semiorthogonal decomposition from Table 3.1 gives $\mathrm{D^b}(X) = \langle \mathsf{Ku}(X), \mathcal{E}, \mathcal{O}_X \rangle$. It is important to note that while these two Kuznetsov components are equivalent by Lemma 3.4.6, i.e. $\mathsf{Ku}(X) \simeq \mathcal{A}_X$, they are *not* equal. However, for the purposes of this thesis (categorical Torelli statements etc.), they can be considered the same since they carry the same data up to equivalence.

Therefore, for the rest of the thesis we set some notational convention. When X is an index 1 Fano threefold and we write $\mathsf{Ku}(X)$ we mean the right orthogonal to $\langle \mathcal{E}, \mathcal{O}_X \rangle$. We call $\mathsf{Ku}(X)$ the *Kuznetsov component*. When we write $\mathcal{A}_X$ we mean the right orthogonal to $\langle \mathcal{O}_X, \mathcal{E}^\vee \rangle$, and we call $\mathcal{A}_X$ the *alternative Kuznetsov component*.

For the rest of Section 6 we will work with the alternative Kuznetsov component $\mathcal{A}_X$.

Remark 6.0.2. Recall the rank 2 vector bundle $\mathcal{E}$ on Gushel–Mukai threefolds X. In the ordinary case, it is the restriction of the tautological bundle on the Grassmannian $\mathrm{Gr}(2,5)$ which the Gushel–Mukai threefold lives in. There is a tautological short exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_X^{\oplus 5} \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is the tautological quotient bundle. It is the pullback of the tautological quotient bundle on $\mathrm{Gr}(2,5)$ to X.

Kuznetsov and Perry make the following conjecture regarding the Kuznetsov components of Gushel–Mukai varieties:

Conjecture 6.0.3 ([KP23, Conjecture 1.7])**.** *Suppose X and X' are Gushel–Mukai varieties of the same dimension such that there is an equivalence $\mathsf{Ku}(X) \simeq \mathsf{Ku}(X')$. Then $X \simeq X'$.*

In Section 6.6.3, we give a proof of this conjecture for the case of general (in their moduli) ordinary Gushel–Mukai threefolds.

Chapter 6 is joint work with Xun Lin, Zhiyu Liu, and Shizhuo Zhang, and is taken from the paper [JLLZ21].

6.1 Gluing data

In this section, we study the projection $i^!(\mathcal{E})$ of $\mathcal{E}$ into the Kuznetsov component along the right adjoint functor of the inclusion $i\colon \mathsf{Ku}(X) \to \langle \mathsf{Ku}(X), \mathcal{E} \rangle$. We call this object the *gluing data/object*, because we have the following *gluing category* description (see e.g. [KL15, Definition 2.4, Lemma 2.5]) of the two term semiorthogonal decomposition $\langle \mathsf{Ku}(X), \mathcal{E} \rangle \subset \mathrm{D^b}(X)$:

$$\langle \mathsf{Ku}(X), \mathcal{E} \rangle \simeq \{(E_1, E_2, \varphi) \mid E_1 \in \mathsf{Ku}(X), E_2 \in \langle \mathcal{E} \rangle, \varphi\colon E_1 \mapsto i^!(E_2)\}$$

In other words, we see that the two term semiorthogonal decomposition $\langle \mathsf{Ku}(X), \mathcal{E} \rangle$ is essentially determined by $\mathsf{Ku}(X)$ and the object $i^!(\mathcal{E})$.

6.1.1 For $\mathsf{Ku}(X)$

Lemma 6.1.1 ([JLLZ21, Lemma 5.1]). *The projection object $i^!(\mathcal{E})$ is given by $\mathbf{L}_{\mathcal{E}} \mathcal{Q}(-H)[1]$. It is a two-term complex with cohomologies*

$$\mathcal{H}^i(i^!(\mathcal{E})) = \begin{cases} \mathcal{Q}(-H), & i = -1 \\ \mathcal{E}, & i = 0 \\ 0, & i \neq -1, 0. \end{cases}$$

Proof. Indeed, by e.g. [Kuz10, p. 4] we have the exact triangle

$$ii^!(\mathcal{E}) \to \mathcal{E} \to \mathbf{L}_{\mathsf{Ku}(X)} \mathcal{E} \to .$$

But note that $\langle \mathsf{Ku}(X), \mathcal{E} \rangle = \langle S_{\mathcal{D}}(\mathcal{E}), \mathsf{Ku}(X) \rangle = \langle \mathbf{L}_{\mathsf{Ku}(X)} \mathcal{E}, \mathsf{Ku}(X) \rangle$. The first equality is by Proposition 3.2.5 and the second is by Lemma 3.2.7. Therefore the triangle above becomes $ii^!(\mathcal{E}) \to \mathcal{E} \to S_{\mathcal{D}}(\mathcal{E})$. To find $S_{\mathcal{D}}(\mathcal{E})$ explicitly, note that $S_{\mathcal{D}} \cong \mathbf{R}_{\mathcal{O}_X(-H)} \circ S_{\mathrm{D^b}(X)}$. Since $\mathbf{R}_{\mathcal{O}_X(-H)} \mathcal{E}(-H) \cong \mathcal{Q}(-H)[-1]$, we have $S_{\mathcal{D}}(\mathcal{E}) \cong \mathcal{Q}(-H)[2]$. So the triangle above becomes

$$ii^!(\mathcal{E}) \to \mathcal{E} \to \mathcal{Q}(-H)[2].$$

Applying $i^* = \mathbf{L}_{\mathcal{E}}$ to the triangle and using the fact that $i^* i \cong \mathrm{id}$ and $i^* \mathcal{E} = 0$ gives $i^!(\mathcal{E}) \cong \mathbf{L}_{\mathcal{E}} \mathcal{Q}(-H)[1]$, as required. Taking the long exact sequence with respect to $\mathcal{H}^*$ gives the cohomology objects. $\square$

Remark 6.1.2. Since $\hom(\mathcal{E}, \mathcal{Q}(-H)[2]) = 1$, the object $i^!(\mathcal{E})$ is the unique object that lies in the non-trivial triangle

$$\mathcal{Q}(-H)[1] \to i^!(\mathcal{E}) \to \mathcal{E}. \tag{6.1.1}$$

Lemma 6.1.3 ([JLLZ21, Lemma 5.3]). *Let X be a Gushel–Mukai threefold. Then we have*

1. $\mathrm{Hom}^\bullet(\mathcal{Q}(-H), \mathcal{E}) = \mathrm{Hom}^\bullet(\mathcal{E}, \mathcal{Q}^\vee) = \mathbf{C}^2$ *when X is ordinary.*
2. $\mathrm{Hom}^\bullet(\mathcal{Q}(-H), \mathcal{E}) = \mathrm{Hom}^\bullet(\mathcal{E}, \mathcal{Q}^\vee) = \mathbf{C}^3 \oplus \mathbf{C}[-1]$ *when X is special.*
3. $\mathrm{Hom}^\bullet(\mathcal{E}, \mathcal{Q}(-H)) = \mathbf{C}[-2]$.

Proof. When X is ordinary, this follows from the Koszul resolution of $X \subset \mathrm{Gr}(2,5)$ and the Borel–Bott–Weil Theorem. When X is special, note that $\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}_Y(-1)$. Then the result follows from the projection formula and [San14, Lemma 2.14, Proposition 2.15]. $\square$

Lemma 6.1.4 ([JLLZ21, Lemma 5.4]). *Let X be a Gushel–Mukai threefold. Then we have*

- $\mathrm{Hom}^\bullet(i^!(\mathcal{E}), i^!(\mathcal{E})) = \mathbf{C} \oplus \mathbf{C}^2[-1]$ *when X is ordinary.*
- $\mathrm{Hom}^\bullet(i^!(\mathcal{E}), i^!(\mathcal{E})) = \mathbf{C} \oplus \mathbf{C}^3[-1] \oplus \mathbf{C}[-2]$ *when X is special.*

Hence $i^!(\mathcal{E})$ is stable with respect to every Serre-invariant stability condition on $\mathsf{Ku}(X)$.

Proof. The first statement follows from applying $\mathrm{Hom}(-, \mathcal{E})$ to triangle (6.1.1) and Lemma 6.1.3, and also the fact that $\mathrm{Hom}^\bullet(i^!(\mathcal{E}), i^!(\mathcal{E})) = \mathrm{Hom}^\bullet(i^!(\mathcal{E}), \mathcal{E})$ which is by adjunction. The last statement follows from Proposition 4.5.4. $\square$

6.1.2 For $\mathcal{A}_X$

In this section, we compute the analogous gluing object for $\mathcal{A}_X$. Recall that we have the semiorthogonal decompositions

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^\vee \rangle \simeq \langle \mathcal{A}_X, \mathcal{Q}^\vee, \mathcal{O}_X \rangle$$

where the second equivalence is because $\mathbf{L}_{\mathcal{O}_X}\mathcal{E}^\vee \cong \mathcal{Q}^\vee[1]$. Let $\mathcal{D} = \langle \mathcal{A}_X, \mathcal{Q}^\vee \rangle$ and let $i \colon \mathcal{A}_X \to \mathcal{D}$ be the inclusion functor[1]. Because $\mathcal{A}_X \subset \mathcal{D}$ is an admissible subcategory, i has a left adjoint i^* and a right adjoint $i^!$. The aim of this section is to compute and describe the image $i^!(\mathcal{Q}^\vee)$.

Lemma 6.1.5 ([JLLZ21, Lemma 5.5]). *The object $i^!(\mathcal{Q}^\vee)$ is given by $\mathbf{L}_{\mathcal{Q}^\vee}\mathcal{E}[1]$. It is a two-term complex with cohomologies*

$$\mathcal{H}^i(i^!(\mathcal{Q}^\vee)) = \begin{cases} \mathcal{E}, & i = -1 \\ \mathcal{Q}^\vee, & i = 0 \\ 0, & i \neq -1, 0. \end{cases}$$

Proof. By the same argument as Lemma 6.1.1. $\square$

[1]By abuse of notation, we use i for the analogous inclusion to the $\mathsf{Ku}(X)$ case.

Similarly to the $\mathsf{Ku}(X)$ gluing data case, $i^!(\mathcal{Q}^\vee)$ lies in the triangle

$$\mathcal{E}[1] \to i^!(\mathcal{Q}^\vee) \to \mathcal{Q}^\vee. \tag{6.1.2}$$

Remark 6.1.6. The gluing object in $\mathsf{Ku}(X)$ can be identified with the one in $\mathcal{A}_X$ via the equivalence Ξ from Lemma 3.4.6. Indeed, applying Ξ to the triangle (6.1.1) gives the triangle (6.1.2) shifted by $[1]$, i.e. $\Xi(i^!(\mathcal{E})) \cong i^!(\mathcal{Q}^\vee)[1]$.

Remark 6.1.7. Later in Section 6.3, we will see that we in fact have $i^!(\mathcal{Q}^\vee) \cong \mathrm{pr}(I_C)$ where $\mathrm{pr}\colon \mathrm{D^b}(X) \to \mathcal{A}_X$ is the projection functor, and $C \subset X$ is a conic such that $I_C \notin \mathcal{A}_X$ (see Proposition 6.3.2).

6.2 Conics on Gushel–Mukai threefolds

In this section, we summarise the different types of conics which appear on ordinary Gushel–Mukai threefolds X, and the geometry of these conics. A *conic* means a closed subscheme $C \subset X$ with Hilbert polynomial $p_C(t) = 1 + 2t$, and a *line* means a closed subscheme $L \subset X$ with Hilbert polynomial $p_L(t) = 1 + t$. Denote their Hilbert schemes by $\mathcal{C}(X)$ and $\Gamma(X)$, respectively. All the results in this section are from [DIM12, Ili94, Log82].

6.2.1 Conics on ordinary Gushel–Mukai threefolds

Recall that X is a quadric section of a codimension 2 linear section of $\mathrm{Gr}(2,5) = \mathrm{Gr}(2, V_5)$ where V_5 is a 5-dimensional complex vector space. Denote by V_i an i-dimensional vector subspace of V_5.

There are two types of 2-planes in $\mathrm{Gr}(2,5)$; *σ-planes* are given set-theoretically as $\{V_2 \mid V_1 \subset V_2 \subset V_4\}$, and *$\rho$-planes* are given by $\{V_2 \mid V_2 \subset V_3\}$.

Definition 6.2.1 ([DIM12, p. 5])**.**

- A conic $C \subset X$ is called a *τ-conic* if the 2-plane $\langle C \rangle$ is not contained in $\mathrm{Gr}(2, V_5)$, there is a unique $V_4 \subset V_5$ such that $C \subset \mathrm{Gr}(2, V_4)$, the conic C is reduced and if it is smooth, the union of corresponding lines in $\mathbf{P}(V_5)$ is a smooth quadric surface in $\mathbf{P}(V_4)$.

- A conic $C \subset X$ is called a *σ-conic* if the 2-plane $\langle C \rangle$ spanned by C is an σ-plane, and if there is a unique hyperplane $V_4 \subset V_5$ such that $C \subset \mathrm{Gr}(2, V_4)$ and the union of the corresponding lines in $\mathbf{P}(V_5)$ is a quadric cone in $\mathbf{P}(V_4)$.

- A conic $C \subset X$ is called a *ρ-conic* if the 2-plane $\langle C \rangle$ spanned by C is a ρ-plane, and the union of corresponding lines in $\mathbf{P}(V_5)$ is this 2-plane.

Lemma 6.2.2 ([JLLZ21, Lemma 6.2])**.** *Let X be an ordinary Gushel–Mukai threefold and $C \subset X$ a conic.*

1. *If C is a τ-conic, then $\mathrm{Hom}^\bullet(\mathcal{E}, I_C) = \mathbf{C}$ and $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, I_C) = 0$.*
2. *If C is a ρ-conic, then $\mathrm{Hom}^\bullet(\mathcal{E}, I_C) = \mathbf{C}^2 \oplus \mathbf{C}[-1]$ and $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, I_C) = 0$.*
3. *If C is a σ-conic, then $\mathrm{Hom}^\bullet(\mathcal{E}, I_C) = \mathbf{C}$ and $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, I_C) = \mathbf{C}[-1] \oplus \mathbf{C}[-2]$.*

Proof. Note that if $\mathrm{hom}(\mathcal{E}, I_C) = a$, then $C \subset \mathrm{Gr}(2, 5-a) \cap X$. Since for any conic, there is some V_4 such that $C \subset \mathrm{Gr}(2, V_4)$, we have $\mathrm{hom}(\mathcal{E}, I_C) \geq 1$.

If $\mathrm{hom}(\mathcal{E}, I_C) \geq 2$, then $C \subset \mathrm{Gr}(2,3)$ which is a ρ-plane. For a τ-conic, C is not in $\mathrm{Gr}(2,5)$ and for a σ-conic, $\langle C \rangle$ is a σ-plane, so for these two types of conics $\mathrm{Hom}(\mathcal{E}, I_C) = \mathbf{C}$ and for ρ-conics, $\mathrm{hom}(\mathcal{E}, I_C) \geq 2$. But if $\mathrm{hom}(\mathcal{E}, I_C) \geq 3$ then $C \subset \mathrm{Gr}(2,2)$ which is impossible, hence $\mathrm{Hom}(\mathcal{E}, I_C) = \mathbf{C}^2$ for ρ-conics. Now the result on Ext groups follows from applying $\mathrm{Hom}(\mathcal{E}, -)$ to the short exact sequence $0 \to I_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$ and the fact that $\chi(\mathcal{E}, I_C) = 1$.

Applying $\mathrm{Hom}(\mathcal{E}^\vee, -)$ to the short exact sequence $0 \to I_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$ gives $\mathrm{Hom}(\mathcal{E}^\vee, I_C) = \mathrm{Ext}^3(\mathcal{E}^\vee, I_C) = 0$. Since $\chi(\mathcal{E}^\vee, I_C) = 0$, we only need to compute $\mathrm{Ext}^1(\mathcal{E}^\vee, I_C)$. So apply $\mathrm{Hom}(-, I_C)$ to the dualised tautological short exact sequence $0 \to \mathcal{Q}^\vee \to \mathcal{O}_X^{\oplus 5} \to \mathcal{E}^\vee \to 0$. Since $\mathrm{Hom}^\bullet(\mathcal{O}_X, I_C) = 0$ we get $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) \cong \mathrm{Ext}^1(\mathcal{E}^\vee, I_C)$. Similarly to the first paragraph of the proof, if $\mathrm{hom}(\mathcal{Q}^\vee, I_C) = a$ then $C \subset \mathrm{Gr}(2-a, 5-a) \cap X$. Thus $\mathrm{hom}(\mathcal{Q}^\vee, I_C) \leq 1$ for any conic C. Note that $\mathrm{hom}(\mathcal{Q}^\vee, I_C) = 1$ if and only if C is contained in the zero locus of a global section of $\mathcal{Q}$. But such a zero locus is a σ-plane in $\mathrm{Gr}(2,5)$. Hence, $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) = 0$ for τ and ρ-conics, and $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) = \mathbf{C}$ for σ-conics. The result follows. $\square$

We now discuss the geometry of the Hilbert scheme of conics on X, which we denote by $\mathcal{C}(X)$.

Theorem 6.2.3 ([Log82, Section 0], [DIM12])**.** *The Hilbert scheme $\mathcal{C}(X)$ is an irreducible projective surface. If X is general, then $\mathcal{C}(X)$ is furthermore smooth.*

There is a unique ρ-conic on X, denoted by c_X, and there is a line $L_\sigma \subset \mathcal{C}(X)$ of σ-conics on X [DIM12, Section 5.1].

Lemma 6.2.4 ([DIM12, p. 16]). *The only rational curve in $\mathcal{C}(X)$ is L_σ. Furthermore, there exists a surface $\mathcal{C}_m(X)$ and a map $\mathcal{C}(X) \to \mathcal{C}_m(X)$ which contracts L_σ to a point $[\pi]$. If X is general, then $\mathcal{C}_m(X)$ is the minimal surface of $\mathcal{C}(X)$.*

Lemma 6.2.5 ([DIM12, Section 5.2]). *Let X be general. Then there is an involution on $\mathcal{C}_m(X)$ switching the points $[c_X]$ and $[\pi]$.*

Remark 6.2.6. In the paper [DIM12], their $F_g(X)$ is our $\mathcal{C}(X)$, and their $F_m(X)$ is our $\mathcal{C}_m(X)$.

The following theorem reconstructs a general ordinary Gushel–Mukai threefold from its Hilbert scheme of conics. It was originally proved in [Log82, Theorem 7.7] and later reproved in [DIM12, Theorem 9.1].

Theorem 6.2.7 (Logachev's Reconstruction Theorem). *Let X and X' be general ordinary Gushel–Mukai threefolds. Then $\mathcal{C}(X) \cong \mathcal{C}(X')$ implies $X \cong X'$.*

6.2.2 Conics and line transforms

For this section we follow [DIM12, Section 6.1]. Let X be a general ordinary Gushel–Mukai threefold, and let $c \neq c_X$ be a smooth conic on X. Let $\pi_c \colon \mathbf{P}^7 \dashrightarrow \mathbf{P}^4$ be the projection away from the 2-plane $\langle c \rangle$. Let $\epsilon \colon \widetilde{X} \to X$ be the blow up of X at c with exceptional divisor E. The composition $\pi_c \circ \epsilon \colon \widetilde{X} \to \mathbf{P}^4$ is the morphism $\phi_{|-K_{\widetilde{X}}|}$ associated to the linear system $|-K_{\widetilde{X}}|$, and it has Stein factorisation

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "\phi_{|-K_{\widetilde{X}}|}"] \arrow[dr, "\phi"'] & & \mathbf{P}^4 \\ & \overline{X} \arrow[ur] & \end{tikzcd}$$

Since the conditions in [Isk99, Theorem 1.4.15] are all satisfied, there exists a $(-E)$-*flop*

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "f"] \arrow[dr, "\phi"'] & & \widetilde{X}_c \arrow[dl, "\phi_c"] \\ & \overline{X} & \end{tikzcd}$$

A study of the properties of $-K_{\widetilde{X}_c}$ shows that there is a contraction $\epsilon_{c'} \colon \widetilde{X}_c \to X_c$, where X_c is an ordinary Gushel–Mukai threefold, and $\epsilon_{c'} \colon \widetilde{X}_c \to X_c$ is the blow-up of X_c in a smooth conic c' with exceptional divisor $E' = -2K_{\widetilde{X}_c} - f(E)$. In summary, there exists a commutative diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[rr, "f"] \arrow[dd, "\epsilon"'] \arrow[dr, "\phi"] & & \widetilde{X}_c \arrow[dd, "\epsilon_{c'}"] \arrow[dl, "\phi_c"'] \\
& \overline{X} & \\
X \arrow[rr, "\psi_c"] \arrow[ur, "\pi_c"] & & X_c \arrow[ul, "\pi_{c'}"']
\end{tikzcd}$$

where $\psi_c \colon X \to X_c$ is the *elementary transformation of* X *along the conic* c. Note that the elementary transformation of X_c along the conic c' is $\psi_c^{-1} \colon X_c \to X$.

Remark 6.2.8. A similar flopping procedure can be done to construct the *elementary transformation of* X *along the line* L, which we denote as $\psi_L \colon X \to X_L$ (see [DIM12, § 6.2]).

Conic transforms can be defined for any conic $c \subset X$. Such an X_c is called the *period partner* of X in [DK18], and the line transforms are called the *period duals*. We now list some important results about conic and line transforms below.

Theorem 6.2.9 ([DIM12, Theorem 6.4])**.** *Let X be a general ordinary Gushel–Mukai threefold, and let $c \subset X$ be any conic. Then one can give a general ordinary Gushel–Mukai threefold $X_c \simeq X$, such that $\mathcal{C}(X_c)$ is isomorphic to $\mathcal{C}_m(X)$ blown up at the point $[c] \in \mathcal{C}_m(X)$, where $\mathcal{C}_m(X)$ is the minimal surface of $\mathcal{C}(X)$.*

Proposition 6.2.10 ([DIM12, Theorem 6.4, Remark 7.2])**.** *Let X be a general ordinary Gushel–Mukai threefold. Then the isomorphism classes of conic transforms of X are parametrized by the surface $\mathcal{C}_m(X)/\iota$.*

Theorem 6.2.11 ([KP23, Theorem 1.6])**.** *Let X be a general ordinary Gushel–Mukai threefold. Then the Kuznetsov components of all conic transforms and line transforms of X are equivalent to $\mathcal{A}_X$.*

6.2.3 Conics on special Gushel–Mukai threefolds

Let X be a special Gushel–Mukai threefold. Recall that X is a double cover $X \to Y$ of a degree 5 Fano threefold Y with branch locus a quadric hypersurface $\mathcal{B} \subset Y$. When X is general, $\mathcal{B}$ is a smooth K3 surface of Picard rank 1 and degree 10. Recall that Y is a codimension 3 linear section of $\mathrm{Gr}(2,5)$. Let $\mathcal{V}$ be the tautological quotient bundle on Y. We recall some properties of $\mathcal{C}(X)$ from [Ili94].

Theorem 6.2.12 ([Ili94, Proposition 2.1.2])**.** *Let X be a special Gushel–Mukai threefold. Then $\mathcal{C}(X)$ has two components $\mathcal{C}_1$ and $\mathcal{C}_2$. One of the components $\mathcal{C}_2 \cong \Sigma(Y) \cong \mathbf{P}^2$ parametrizes preimages of lines on Y. Moreover, when X is general, $\mathcal{C}(X)$ is smooth away from $\mathcal{C}_1 \cap \mathcal{C}_2$.*

The following lemma will be useful in computations; it is similar to Lemma 6.2.2.

Lemma 6.2.13 ([JLLZ21, Lemma 6.12])**.** *Let X be a special Gushel–Mukai threefold and C a conic on X. Then $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, I_C) \neq 0$ if and only if C is the preimage of a line on Y. In this case $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, I_C) = \mathbf{C}[-1] \oplus \mathbf{C}[-2]$, and such a family of conics is parametrized by the Hilbert scheme of lines $\Sigma(Y) \cong \mathbf{P}^2$ on Y.*

Proof. Recall from the proof of Lemma 6.2.2 that $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) \cong \mathrm{Ext}^1(\mathcal{E}^\vee, I_C)$, so $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, I_C) \neq 0$ if and only if $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) \neq 0$. The image of a non-zero map $\mathcal{Q}^\vee \to I_C$ is the zero locus of a section s of $\mathcal{Q}$, which is the preimage of the zero locus of a section of $\mathcal{V}$. By [San14, Lemma 2.18], the zero locus of a section of $\mathcal{V}$ is either a line or a point. Thus the zero locus of a section of $\mathcal{Q}$ is either the preimage of a line on Y which is a conic on X, or a zero-dimensional closed subscheme of length two. But this zero locus contains a conic $C \subset X$, so $C = Z(s)$ is the preimage of a line on Y and the map $\mathcal{Q}^\vee \to I_C$ is surjective. In particular, such conics are exactly the preimages of lines on Y, and are parametrized by $\Sigma(Y) \cong \mathbf{P}^2$. □

6.3 The Hilbert scheme of conics as a Bridgeland moduli space

In this section, we construct the moduli space of σ-stable objects of the (-1)-class $-x$ in the alternative Kuznetsov component $\mathcal{A}_X$ of a Gushel–Mukai threefold X.

Proposition 6.3.1 ([JLLZ21, Proposition 7.1])**.** *Let $C \subset X$ be a conic on a Gushel–Mukai threefold X. Then $I_C \notin \mathcal{A}_X$ if and only if*

1. *C is a σ-conic when X is ordinary. In particular, such a family of conics is parametrized by the line L_σ.*

2. *C is the preimage of a line on Y when X is special. In particular, such a family of conics is parametrized by the Hilbert scheme of lines $\Sigma(Y) \cong \mathbf{P}^2$ on Y.*

Moreover, we have a short exact sequence

$$0 \to \mathcal{E} \to \mathcal{Q}^\vee \to I_C \to 0.$$

Proof. Recall that the projection functor $\mathrm{D^b}(X) \to \mathcal{A}_X$ is given by $\mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}$. So (1) follows from Lemma 6.2.2. When X is special, this is by Lemma 6.2.13. Note that since $I_C \notin \mathcal{A}_X$, we have $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) \neq 0$. The non-trivial map $\mathcal{Q}^\vee \to I_C$ is surjective by the arguments in Lemma 6.2.2 and 6.2.13. Note that by the stability of $\mathcal{Q}^\vee$, the kernel of $\mathcal{Q}^\vee \twoheadrightarrow I_C$ is μ-stable with the same Chern character as $\mathcal{E}$, hence we have $\ker(\mathcal{Q}^\vee \twoheadrightarrow I_C) \cong \mathcal{E}$ by [DIM12, Proposition 4.1]. □

Proposition 6.3.2 ([JLLZ21, Proposition 7.2]). *Let X be a Gushel–Mukai threefold and $C \subset X$ a conic on X. If $I_C \notin \mathcal{A}_X$, then we have the exact triangle*

$$\mathcal{E}[1] \to \mathrm{pr}(I_C) \to \mathcal{Q}^\vee.$$

Proof. By Proposition 6.3.1, I_C fits into the short exact sequence $0 \to \mathcal{E} \to \mathcal{Q}^\vee \to I_C \to 0$. Applying the projection functor to this short exact sequence, we get a triangle $\mathrm{pr}(\mathcal{E}) \to \mathrm{pr}(\mathcal{Q}^\vee) \to \mathrm{pr}(I_C)$, where $\mathrm{pr} = \mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}$. Note that applying the functor pr to the exact sequence $0 \to \mathcal{Q}^\vee \to \mathcal{O}_X^{\oplus 5} \to \mathcal{E}^\vee \to 0$ gives $\mathrm{pr}(\mathcal{Q}^\vee) = 0$. Thus $\mathrm{pr}(I_C) \cong \mathrm{pr}(\mathcal{E})[1]$. Now we compute the projection $\mathrm{pr}(\mathcal{E}) = \mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}\mathcal{E}$. We have the triangle

$$\mathrm{Hom}^\bullet(\mathcal{E}^\vee, \mathcal{E}) \otimes \mathcal{E}^\vee \to \mathcal{E} \to \mathbf{L}_{\mathcal{E}^\vee}\mathcal{E}.$$

Since $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, \mathcal{E}) \cong \mathbf{C}[-3]$, we get $\mathcal{E}^\vee[-3] \to \mathcal{E} \to \mathbf{L}_{\mathcal{E}^\vee}\mathcal{E}$. Now applying $\mathbf{L}_{\mathcal{O}_X}$ to this triangle, we get $\mathbf{L}_{\mathcal{O}_X}\mathcal{E}^\vee[-3] \to \mathcal{E} \to \mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}\mathcal{E} = \mathrm{pr}(\mathcal{E})$, which is equivalently $\mathcal{Q}^\vee[-2] \to \mathcal{E} \to \mathrm{pr}(\mathcal{E})$. Therefore we obtain the triangle

$$\mathcal{E}[1] \to \mathrm{pr}(\mathcal{E})[1] \to \mathcal{Q}^\vee$$

and the desired result follows. □

6.3.1 Involutions acting on conics

By [KP18, Proposition 2.6], there is a natural involutive autoequivalence functor of $\mathcal{A}_X$, denoted by $\tau_{\mathcal{A}}$. When X is special, it is induced by the natural involution

τ on X, which comes from the double cover $X \to Y$. In this case it is easy to see that

$$\tau_{\mathcal{A}}(\mathrm{pr}(I_C)) \cong \mathrm{pr}(I_{\tau(C)}).$$

When X is ordinary, the situation is more subtle. In the following, we describe the action of $\tau_{\mathcal{A}}$ on the projection into $\mathcal{A}_X$ of an ideal sheaf of a conic $\mathrm{pr}(I_C)$ in this case.

Proposition 6.3.3 ([JLLZ21, Proposition 7.3]). *Let X be an ordinary Gushel–Mukai threefold and C a conic on X.*

1. *If $I_C \in \mathcal{A}_X$, then $\tau_{\mathcal{A}}(I_C)$ is either*
   - *$I_{C'}$ such that $C \cup C' = Z(s)$ for $s \in H^0(\mathcal{E}^\vee)$, where $Z(s)$ is the zero locus of the section s;*
   - *or $i^!(\mathcal{Q}^\vee)$ (cf. Section 6.1), where $i^!(\mathcal{Q}^\vee)$ is given by the triangle*

   $$\mathcal{E}[1] \to i^!(\mathcal{Q}^\vee) \to \mathcal{Q}^\vee.$$

2. *If $I_C \notin \mathcal{A}_X$, then $\tau_{\mathcal{A}}(\mathrm{pr}(I_C)) \cong I_{C''}$ for a conic $C'' \subset X$.*

Remark 6.3.4. Once we have proved Theorem 6.3.13, this will imply that the involution induced by $\tau_{\mathcal{A}}$ on $\mathcal{C}_m(X)$ is the same as ι in Lemma 6.2.5, described in [DIM12, Section 5.2].

We first state some technical lemmas which we require for the proof of Proposition 6.3.3 above.

Lemma 6.3.5 ([JLLZ21, Lemma 7.5]). *Let X be a Gushel–Mukai threefold and E a μ-semistable sheaf on X with truncated Chern character $\mathrm{ch}_{\leq 2}(E) = (2, -H, aL)$. Assume that $a \geq 1$ and $c_3(E) \geq 0$. Then we have $E \cong \mathcal{E}$.*

Proof. By Lemma 4.3.5 (Stronger BG II), we have $a \leq 1$ which means $a = 1$ by our assumption. Thus $c_1(E) = -1$ and $c_2(E) = 4$. Note that $\mathrm{ch}_{\leq 2}(E(H)) = (2, H, L)$, and $c_3(E(H)) \geq 0$ by assumption. Thus, by [BF14, Proposition 3.5(i)] we have $\chi(\mathcal{O}_X, E) = 0$. But by formula (3.4.1),

$$\chi(\mathcal{O}_X, E) = \chi_0(2 - H + L + \frac{1}{3}P + \frac{1}{2}c_3(E)) = \frac{1}{2}c_3(E)$$

which implies that $c_3(E) = 0$. Moreover, E is a globally generated bundle by [BF14, Proposition 3.4(ii)]. Thus $E \cong \mathcal{E}$ by [DIM12, Proposition 4.1]. $\square$

Lemma 6.3.6 ([JLLZ21, Lemma 7.6])**.** *Let X be a Gushel–Mukai threefold and E a μ-semistable sheaf on X with $\mathrm{ch}(E) = \mathrm{ch}(\mathcal{Q})$. Then we have $E \cong \mathcal{Q}$.*

Proof. First we show that $h^2(E) = 0$. Indeed, if $h^2(E) \neq 0$, then we have $\mathrm{Hom}(E, \mathcal{O}_X(-H)[1]) \neq 0$ by Serre duality. Therefore, we have a non-trivial extension

$$0 \to \mathcal{O}_X(-H) \to F \to E \to 0.$$

If F is not μ-semistable, then the minimal destabilising quotient sheaf F' of F has $\mathrm{ch}_{\leq 1}(F') = (1, -H)$ by the μ-stability of $\mathcal{O}_X(-H)$ and E. Thus $F'^{\vee\vee} \cong \mathcal{O}_X(-H)$. Now apply $\mathrm{Hom}(-, \mathcal{O}_X(-H)) = 0$ to the short exact sequence above. We obtain $\mathrm{Hom}(F, \mathcal{O}_X(-H)) = 0$ because $\mathrm{Ext}^1(E, \mathcal{O}_X(-H)) \neq 0$. But this is a contradiction to $F'^{\vee\vee} \cong \mathcal{O}_X(-H)$, thus F is μ-semistable. Note also that $\mathrm{ch}_{\leq 2}(F) = (4, 0, 4L)$, so $\Delta(F) < 0$ which contradicts μ-semistability of F. Thus $h^2(E) = 0$.

Since $\chi(\mathcal{O}_X, E) = 5$, it follows that $h^0(E) \geq 5$. Now take five linearly independent sections of E and consider the map $t \colon \mathcal{O}_X^{\oplus 5} \to E$. Because $\mathcal{O}_X^{\oplus 5}$ and E are μ-semistable, we have $0 \leq \mu(\mathrm{im}(t)) \leq 1/3$, i.e. either $\mu(\mathrm{im}(t)) = 0$ or $1/3$. But the first case cannot happen, because then $\mathrm{im}(t)$ would be some direct sum of a number of copies of $\mathcal{O}_X$, and this would contradict the fact that we took linearly independent sections. Thus $\mu(\mathrm{im}(t)) = 1/3$ and $\mathrm{ch}_{\leq 1}(\mathrm{im}(t)) = (3, H)$. Note also that $\mathrm{ch}_{\leq 2}(\ker(t)) = (2, -H, xL)$ where $x \geq 1$. The sheaf $\ker(t)$ is reflexive and has rank two, so $c_3(\ker(t)) \geq 0$. Then by μ-stability of $\mathcal{O}_X$ and the fact that $\mathrm{Hom}(\mathcal{O}_X, \ker(t)) = 0$, we have that $\ker(t)$ is μ-semistable. Thus by Lemma 6.3.5 we have $\ker(t) \cong \mathcal{E}$. Therefore $\mathrm{ch}(\mathrm{im}(t)) = \mathrm{ch}(E)$ and thus t is surjective.

Now apply $\mathrm{Hom}(\mathcal{Q}, 0)$ to the short exact sequence $0 \to \mathcal{E} \to \mathcal{O}_X^{\oplus 5} \to E \to 0$. Since $\mathrm{Hom}^\bullet(\mathcal{Q}, \mathcal{O}_X) = 0$ and $\mathrm{Ext}^1(\mathcal{Q}, \mathcal{E}) = \mathbf{C}$, we get $\mathrm{Hom}(\mathcal{Q}, E) = \mathbf{C}$. Now since $\mathcal{Q}$ and E are both μ-stable and have the same Chern character, we get $E \cong \mathcal{Q}$. □

Lemma 6.3.7 ([JLLZ21, Lemma 7.7])**.** *Let X be an ordinary Gushel–Mukai threefold and C a ρ-conic on X. Then the natural morphism $s' \colon \mathcal{E}^{\oplus 2} \to I_C$ is surjective and there is a short exact sequence*

$$0 \to \mathcal{Q}(-H) \to \mathcal{E}^{\oplus 2} \to I_C \to 0.$$

Proof. By Lemma 6.2.2, we have $\mathrm{Hom}(\mathcal{E}, I_C) = \mathbf{C}^2$. Thus, taking two linearly independent elements in $\mathrm{Hom}(\mathcal{E}, I_C)$, we have a map $s' \colon \mathcal{E}^{\oplus 2} \to I_C$. Moreover, since $\langle C \rangle = \mathrm{Gr}(2, 3)$ and $\langle C \rangle \cap X = C$, we know that s' is surjective. Let $K :=$

$\ker(s')$. Then one can check that $\mathrm{ch}(K) = \mathrm{ch}(\mathcal{Q}(-H))$. Note that $\mathrm{Hom}(\mathcal{E}, K) = 0$ and K is reflexive.

We claim that K is μ-semistable. Indeed, suppose K is not μ-semistable and let K' be its maximal destabilising subsheaf. Then K' is also reflexive. Since $\mathrm{Hom}(\mathcal{E}, K) = 0$, we have $K' \neq \mathcal{E}$. Since $\mathcal{E}^{\oplus 2}$ is μ-semistable and K' is maximal, we have $\mu(K') = -1/2$. Since $\mathrm{Hom}(K', \mathcal{E}) \neq 0$, by the μ-stability of $\mathcal{E}$ and K' we have $K' \subset \mathcal{E}$. Now since $\mathrm{ch}_{\leq 1}(K') = \mathrm{ch}_{\leq 1}(\mathcal{E})$ we have that $\mathcal{E}/K'$ is supported in codimension ≥ 2. But this is a contradiction since $\mathcal{E}$ and K' are both reflexive. So K is μ-semistable.

Now the result follows from Lemma 6.3.6 because $K(H)$ is μ-semistable with $\mathrm{ch}(K(H)) = \mathrm{ch}(\mathcal{Q})$. $\square$

We define the *derived dual* of an object $E \in \mathrm{D^b}(X)$ to be

$$\mathbb{D}(E) \coloneqq \mathrm{R}\mathcal{H}om(E, \mathcal{O}_X).$$

Lemma 6.3.8 ([JLLZ21, Lemma 7.8])**.** *Let X be an ordinary Gushel–Mukai threefold. Consider the semiorthogonal decomposition $\mathrm{D^b}(X) = \langle \mathsf{Ku}(X), \mathcal{E}, \mathcal{O}_X \rangle$. Let C be a conic on X. Then*

$$\mathbf{L}_{\mathcal{E}}(I_C) = \begin{cases} \mathbb{D}(I_{C'}) \otimes \mathcal{O}_X(-H)[1], & \mathrm{Hom}^\bullet(\mathcal{E}, I_C) = \mathbf{C} \\ i^!(\mathcal{E}), & \mathrm{Hom}^\bullet(\mathcal{E}, I_C) = \mathbf{C}^2 \oplus \mathbf{C}[-1] \end{cases}$$

where C' is the involutive conic of C.

Proof. By Lemma 6.2.2, we have that $\mathrm{Hom}^\bullet(\mathcal{E}, I_C)$ is either $\mathbf{C}$ or $\mathbf{C}^2 \oplus \mathbf{C}[-1]$. If $\mathrm{Hom}^\bullet(\mathcal{E}, I_C) = \mathbf{C}$, then we have the triangle $\mathcal{E} \to I_C \to \mathbf{L}_{\mathcal{E}}(I_C)$. Taking cohomology of this triangle with respect to the standard heart we get

$$0 \to \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C)) \to \mathcal{E} \xrightarrow{s} I_C \to \mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C)) \to 0.$$

The image of the map s is the ideal sheaf of an elliptic quartic D, thus we have following two short exact sequences: $0 \to \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C)) \to \mathcal{E} \to I_D \to 0$ and $0 \to I_D \to I_C \to \mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C) \to 0$. Thus $\mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C))$ is a torsion-free sheaf of rank 1 with the same Chern character as $\mathcal{O}_X(-H)$. So $\mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C)) \cong \mathcal{O}_X(-H)$. On the other hand $\mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C))$ is supported on the residual curve C' of C in D and $\mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C)) \cong \mathcal{O}_{C'}(-H)$. Thus we have the triangle

$$\mathcal{O}_X(-H)[1] \to \mathbf{L}_{\mathcal{E}}(I_C) \to \mathcal{O}_{C'}(-H)$$

and we observe that $\mathbf{L}_{\mathcal{E}}(I_C)$ is exactly the twisted derived dual of the ideal sheaf $I_{C'}$ of a conic $C' \subset X$, i.e. $\mathbf{L}_{\mathcal{E}}(I_C) \cong \mathbb{D}(I_{C'}) \otimes \mathcal{O}_X(-H)[1]$.

If $\mathrm{Hom}^\bullet(\mathcal{E}, I_C) = \mathbf{C}^2 \oplus \mathbf{C}[-1]$, then we have the triangle $\mathcal{E}^2 \oplus \mathcal{E}[-1] \to I_C \to \mathbf{L}_{\mathcal{E}}(I_C)$. Taking the long exact sequence in cohomology with respect to the standard heart, we get

$$0 \to \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C)) \to \mathcal{E}^{\oplus 2} \xrightarrow{s'} I_C \to \mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C)) \to \mathcal{E} \to 0.$$

Now by Lemma 6.3.7, s' is surjective and the cohomology objects are given by $\mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C)) \cong \mathcal{Q}(-H)$ and $\mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C)) \cong \mathcal{E}$, which implies that $\mathbf{L}_{\mathcal{E}}(I_C) \cong i^!(\mathcal{E})$ (see Lemma 6.1.1). $\square$

Proof of Proposition 6.3.3. Since $S_{\mathcal{A}_X} = \tau_{\mathcal{A}}[2]$, by Proposition 3.2.6 we have that $\tau_{\mathcal{A}}^{-1} = \mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}(- \otimes \mathcal{O}_X(H))[-1]$. Thus $\tau_{\mathcal{A}}(I_C) = \mathbf{L}_{\mathcal{O}_X}(\mathbf{L}_{\mathcal{E}}(I_C) \otimes \mathcal{O}_X(H))[-1]$. Recall that there are two cases for $\mathbf{L}_{\mathcal{E}}(I_C)$ by Lemma 6.3.8; either $\mathrm{Hom}^\bullet(\mathcal{E}, I_C) = \mathbf{C}$ or $\mathrm{Hom}^\bullet(\mathcal{E}, I_C) = \mathbf{C}^2 \oplus \mathbf{C}[-1]$.

If $\mathrm{Hom}^\bullet(\mathcal{E}, I_C) = \mathbf{C}$ then $\tau_{\mathcal{A}}(I_C) \cong \mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))$. Associated to this left mutation we have the triangle

$$\mathrm{Hom}^\bullet(\mathcal{O}_X, \mathbb{D}(I_{C'})) \otimes \mathcal{O}_X \to \mathbb{D}(I_{C'}) \to \mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'})).$$

Note that[2] $\mathrm{Hom}^\bullet(\mathcal{O}_X, \mathbb{D}(I_{C'})) \cong \mathrm{Hom}^\bullet(I_{C'}, \mathcal{O}_X) = \mathbf{C} \oplus \mathbf{C}[-1]$. Then we have the triangle

$$\mathcal{O}_X \oplus \mathcal{O}_X[-1] \to \mathbb{D}(I_{C'}) \to \mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'})). \tag{6.3.1}$$

The derived dual $\mathbb{D}(I_{C'})$ is given by the triangle $\mathcal{O}_X \to \mathbb{D}(I_{C'}) \to \mathcal{O}_{C'}[-1]$. Then taking cohomology with respect to the standard heart of triangle (6.3.1) we have the long exact sequence

$$\begin{array}{llll} 0 \longrightarrow & 0 = \mathcal{H}^{-1}(\mathbb{D}(I_{C'})) \longrightarrow & \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) \longrightarrow & \\ \mathcal{O}_X \longrightarrow & \mathcal{O}_X \longrightarrow & \mathcal{H}^0(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) \longrightarrow & \\ \mathcal{O}_X \longrightarrow & \mathcal{O}_{C'} \longrightarrow & \mathcal{H}^1(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) \longrightarrow 0. & \end{array}$$

Therefore we have $\mathcal{H}^{-1}(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) = 0$, $\mathcal{H}^1(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) = 0$ and also $\mathcal{H}^0(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) \cong I_{C'}$. Hence $\tau_{\mathcal{A}}(I_C) \cong \mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'})) \cong I_{C'}$.

[2] For these cohomology computations, it is useful to recall that for conics C we have $\mathcal{O}_X(H)|_C \cong \mathcal{O}_C(2)$

If we are in the second case and $\mathrm{Hom}^\bullet(\mathcal{E}, I_C) = \mathbf{C}^2 \oplus \mathbf{C}[-1]$, then $\tau_{\mathcal{A}}(I_C) \cong \mathbf{L}_{\mathcal{O}_X}(i^!(\mathcal{E}) \otimes \mathcal{O}_X(H)[-1])$. Then using Lemma 3.4.6, we have $\tau_{\mathcal{A}}(I_C) \cong i^!(\mathcal{Q}^\vee)$.

So we have proved both cases of Proposition 6.3.3(1). To see part (2) of the proposition, if $I_C \notin \mathcal{A}_X$ then we have the triangle from Proposition 6.3.2, and in particular we are in the second bullet point of Proposition 6.3.3(1), i.e. $\tau_{\mathcal{A}}(I_{C''}) \cong \mathrm{pr}(I_C)$ for some conic C''. But then $\tau_{\mathcal{A}}^{-1} = \tau_{\mathcal{A}}$ gives the result. □

6.3.2 Stability of objects associated to conics

Lemma 6.3.9 ([JLLZ21, Lemma 7.9])**.** *Let X be a Gushel–Mukai threefold. If $C \subset X$ is a conic such that $I_C \notin \mathcal{A}_X$, then*

1. $\mathrm{Hom}^\bullet(\mathrm{pr}(I_C), \mathrm{pr}(I_C)) = \mathbf{C} \oplus \mathbf{C}^2[-1]$ *when X is ordinary.*
2. $\mathrm{Hom}^\bullet(\mathrm{pr}(I_C), \mathrm{pr}(I_C)) = \mathbf{C} \oplus \mathbf{C}^3[-1] \oplus \mathbf{C}[-2]$ *when X is special.*

Proof. Recall that by Remark 6.1.6 we have $\Xi(i^!(\mathcal{E})) \cong i^!(\mathcal{Q}^\vee)[1]$. Furthermore by Proposition 6.3.2 we have $i^!(\mathcal{Q}^\vee) \cong \mathrm{pr}(I_C)$ for $C \subset X$ such that $I_C \notin \mathcal{A}_X$. Using these two facts and Lemma 6.1.4 gives the required result. □

Lemma 6.3.10 ([JLLZ21, Lemma 7.10])**.** *Let X be a Gushel–Mukai threefold. If $I_C \notin \mathcal{A}_X$, the projection $\mathrm{pr}(I_C)[1]$ is stable with respect to every Serre-invariant stability condition on $\mathcal{A}_X$.*

Proof. Since the dimensions of the Hom spaces are the same as in Lemma 6.1.4, the same argument gives the result. □

When $I_C \in \mathcal{A}_X$, we cannot use Proposition 4.5.4 to prove the Bridgeland stability of I_C, since $\mathcal{C}(X)$ may be singular and $\mathrm{Ext}^1(I_C, I_C)$ may have large dimension. Instead, we use a wall-crossing argument and the uniqueness of Serre-invariant stability conditions (see Theorem 4.5.14).

Lemma 6.3.11 ([JLLZ21, Lemma 7.11])**.** *Let X be a Gushel–Mukai threefold. Let F be an object with $\mathrm{ch}_{\leq 2}(F) = (1, 0, -2L)$. Then there are no walls for F in the range $-\frac{1}{2} \leq \beta < 0$ and $\alpha > 0$.*

Proof. Firstly, $\beta = 0$ is the unique vertical wall of F. Any other wall is a semicircle centered along the β-axis, and its apex lies on the hyperbola $\mu_{\alpha,\beta}(F) = 0$. Moreover, no two walls intersect. These facts are all by e.g. [BBF+22, Theorem 4.12].

Note that when $\mu_{\alpha,\beta}(F) = 0$ holds, we have $\beta < -\sqrt{\frac{2}{5}} < -\frac{1}{2}$, thus we know that there is no semicircular wall centered in the interval $-\frac{1}{2} \leq \beta < 0$. Therefore, any semicircular wall passing through the range $-\frac{1}{2} \leq \beta < 0$ will intersect the line $\beta = -\frac{1}{2}$. Therefore, to prove the statement of the lemma, we only need to show that there are no walls when $\beta = -\frac{1}{2}$. But this follows from the fact that $\mathrm{ch}_1^{-\frac{1}{2}}(F) = \frac{1}{2}$ is minimal. $\square$

Proposition 6.3.12 ([JLLZ21, Proposition 7.12])**.** *Let $C \subset X$ be a conic on a Gushel–Mukai threefold X such that $I_C \in \mathcal{A}_X$. Then $I_C[1] \in \mathcal{A}_X$ is stable with respect to every Serre-invariant stability condition on $\mathcal{A}_X$.*

Proof. By Lemma 4.3.3 and Lemma 6.3.11, I_C is $\sigma_{\alpha,\beta}$-stable for all $(\alpha, \beta) \in V$. Since I_C is torsion-free, we know that $I_C[1] \in \mathrm{Coh}^0_{\alpha,\beta}(X)$ is $\sigma^0_{\alpha,\beta}$-semistable. Thus $I_C[1] \in \mathcal{A}_{\mathrm{alt}}(\alpha, \beta)$ is $\sigma_{\mathrm{alt}}(\alpha, \beta)$-semistable.

The stability with respect to every Serre-invariant stability condition follows from Theorem 4.5.2 and Theorem 4.5.14. $\square$

6.3.3 Contraction of Fano surface of conics as a Bridgeland moduli space

We are now ready to realise the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ as a contraction of the Fano surface of conics $\mathcal{C}(X)$.

Theorem 6.3.13 ([JLLZ21, Theorem 7.13])**.** *Let X be a Gushel–Mukai threefold and σ a Serre-invariant stability condition on $\mathcal{A}_X$. The projection functor $\mathrm{pr}\colon \mathrm{D}^{\mathrm{b}}(X) \to \mathcal{A}_X$ induces an isomorphism $\mathcal{S} := p(\mathcal{C}(X)) \cong \mathcal{M}_\sigma(\mathcal{A}_X, -x)$, where $p\colon \mathcal{C}(X) \to \mathcal{S}$ is*

1. *a blow-down morphism to a smooth point when X is ordinary;*
2. *a contraction of the component $\mathbf{P}^2$ to a singular point when X is special.*

In particular, when X is general and ordinary, $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ is isomorphic to the minimal model $\mathcal{C}_m(X)$ of the Fano surface of conics on X. When X is general and special, the moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ has only one singular point.

Proof. Suppose that X is ordinary. By Proposition 6.3.1, it is known that the family of σ-conics[3] $C \subset X$ with the property that $I_C \notin \mathcal{A}_X$ is parametrized by

[3] Note σ here is not the stability condition.

the line L_σ. By Lemma 6.3.10, $\mathrm{pr}(I_C)[1]$ is σ-stable when $I_C \notin \mathcal{A}_X$. The ideal sheaves I_C for all the conics C in the complement of L_σ in the Fano surface $\mathcal{C}(X)$ of conics are contained in $\mathcal{A}_X$ by Proposition 6.3.1(1). Then $\mathrm{pr}(I_C[1]) = I_C[1] \in \mathcal{A}_X$, and they are σ-stable by Proposition 6.3.12.

Using the universal family of conics on $X \times \mathcal{C}(X)$, the functor pr induces a morphism $p\colon \mathcal{C}(X) \to \mathcal{M}_\sigma(\mathcal{A}_X, -x)$ factoring through one of the irreducible components $\mathcal{S}$ of $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ as in [LZ22, Lemma 4.3]. The complement of L_σ in $\mathcal{C}(X)$ is a dense open subset U_1 of $\mathcal{C}(X)$ since $\mathcal{C}(X)$ is irreducible. The morphism $p|_{U_1}$ is injective and étale, so $p(U_1) \subset \mathcal{S}$ is also a dense open subset of $\mathcal{S}$. But p is proper, so $p(\mathcal{C}(X)) = \mathcal{S}$. In particular, L_σ is contracted by p to a smooth point by Lemma 6.3.9. By Proposition 6.5.5, we have $\mathcal{S} = \mathcal{M}_\sigma(\mathcal{A}_X, -x)$. Thus part (1) follows. Part (2) follows by a similar argument.

When X is general and ordinary, the Fano surface $\mathcal{C}(X)$ is smooth (Theorem 6.2.3). Thus $\mathcal{S}$ is a smooth surface obtained by blowing down a smooth rational curve L_σ on the smooth irreducible projective surface $\mathcal{C}(X)$. This implies that $\mathcal{S}$ is also a smooth projective surface. Also, it is known that there is a unique rational curve $L_\sigma \subset \mathcal{C}(X)$ and it is the unique exceptional curve by Lemma 6.2.4. Thus $\mathcal{S}$ is the minimal model $\mathcal{C}_m(X)$ of Fano surface of conics on X.

When X is general and special, the last statement follows by a similar argument to the preceding paragraph, by Theorem 6.2.12 and Lemma 6.3.9. □

6.4 Another moduli space

In this section we investigate the moduli space of rank 2 Gieseker-semistable torsion-free sheaves on a Gushel–Mukai threefold X with Chern classes $c_1 = 1$ and $c_2 = 5$, denoted $M_G^X(2,1,5)$. We drop X from the notation when it is clear from context on which threefold we work. Note that if $F \in M_G(2,1,5)$, then

$$\mathrm{ch}(F) = (2, H, 0, -\frac{5}{6}P).$$

Recall the following theorem [DIM12, Section 8]:

Theorem 6.4.1. *Let X be a Gushel–Mukai threefold and $F \in M_G^X(2,1,5)$. Then F is either a*

1. *globally generated bundle which fits into a short exact sequence*

$$0 \to \mathcal{O}_X \to F \to I_Z(H) \to 0$$

where Z is a projective normal smooth elliptic quintic curve;

2. *non-locally free sheaf with a short exact sequence*

$$0 \to F \to \mathcal{E}^\vee \to \mathcal{O}_L \to 0$$

where L is a line on X. Moreover, F is uniquely determined by L;

3. *non-globally generated vector bundle which fits into the exact sequence*

$$0 \to \mathcal{E} \to H^0(X, F) \otimes \mathcal{O}_X \to F \to \mathcal{O}_L(-1) \to 0.$$

Moreover, F is uniquely determined by L.

Furthermore, in all of the cases above we have $\mathrm{Hom}^\bullet(\mathcal{O}_X, F) = \mathbf{C}^4$ *and the vanishing* $\mathrm{Hom}^\bullet(\mathcal{O}_X, F(-H)) = 0$.

Proof. The proofs for statements in this theorem can be found in [DIM12, Section 8]. The result also follows from [BF14, Proposition 3.5]. □

A natural question to ask is what Bridgeland moduli space we get after projecting an object from $M_G(2,1,5)$ into the Kuznetsov component. Since it is easier in this setting, we will work with the alternative Kuznetsov component $\mathcal{A}_X$ in this section (like we did when investigating $\mathcal{C}(X)$ in Section 6.3). Our analysis of the projections of objects in $M_G(2,1,5)$ is based on the three cases listed in Theorem 6.4.1. We begin with a Hom-vanishing result.

Proposition 6.4.2 ([JLLZ21, Proposotion 8.2])**.** *Let X be a Gushel–Mukai threefold and $F \in M_G^X(2,1,5)$. Then we have* $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, F) = 0$.

Proof. We have $\mathrm{Ext}^3(\mathcal{E}^\vee, F) \cong \mathrm{Hom}(F, \mathcal{E})^\vee = 0$ by Serre duality, $1/2 = \mu(F) > \mu(\mathcal{E}) = -1/2$ and the μ-stability of F and $\mathcal{E}$. We also have $\mathrm{Hom}(\mathcal{E}^\vee, F) = 0$ because $\mu(\mathcal{E}^\vee) = \mu(F)$, $\mathcal{E}^\vee$ and F are μ-stable, and $\mathcal{E}^\vee \ncong F$. Since $\chi(\mathcal{E}^\vee, F) = 0$, we only need to show that $\mathrm{Ext}^1(\mathcal{E}^\vee, F) = 0$ or $\mathrm{Ext}^2(\mathcal{E}^\vee, F) = 0$.

1. First, let F be globally generated. Applying $\mathrm{Hom}(\mathcal{E}^\vee, -)$ to the sequence in Theorem 6.4.1(1), from $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, \mathcal{O}_X) = 0$ we obtain that $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, I_Z(H)) \cong \mathrm{Hom}^\bullet(\mathcal{E}^\vee, F)$.

   Now we turn to $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, I_Z(H)) \cong \mathrm{Hom}^\bullet(\mathcal{E}, I_Z)$. We have $\mathrm{Hom}(\mathcal{E}, I_Z) = \mathrm{Ext}^3(\mathcal{E}, I_Z) = 0$ from Serre duality and stability (by a similar argument as at the beginning of this proof). Since $\chi(\mathcal{E}, I_Z) = 0$, we only need to show that $\mathrm{Ext}^2(\mathcal{E}, I_Z) = 0$. To this end, we apply $\mathrm{Hom}(\mathcal{E}, -)$ to the ideal sheaf sequence $0 \to I_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$. Since $\operatorname{Ext}^i(\mathcal{E}, \mathcal{O}_X) = 0$ for $i \neq 0$, we only need to show that $\operatorname{Ext}^1(\mathcal{E}, \mathcal{O}_Z) = 0$. We claim that

$$\operatorname{Hom}^\bullet(\mathcal{E}, \mathcal{O}_Z) = \operatorname{Hom}^\bullet(\mathcal{O}_Z, \mathcal{E}^\vee|_Z) = \mathbf{C}^5.$$

Indeed, by Atiyah's classification of vector bundles on elliptic curves [Ati57] and the case described in e.g. [IM07, Section 5.2], we have that $\mathcal{E}^\vee|_Z$ can only split as the direct sum of line bundles with degrees $(2, 3)$ or $(0, 5)$. Note that the second case in [IM07, Section 5.2] is not possible because $\mathcal{E}^\vee|_Z$ has odd degree. But as shown in *loc. cit.*, $\mathcal{E}^\vee|_Z$ cannot split as the sum of line bundles with degrees $(0, 5)$, otherwise Z would not be projectively normal as explained in [IM07, Section 5.2], which is a contradiction. So $\mathcal{E}^\vee|_Z \cong \mathcal{O}_Z(2p) \oplus \mathcal{O}_Z(3p)$ where $p \in Z$ is a point. Then a cohomology computation shows that $H^0(Z, \mathcal{O}_Z(2p) \oplus \mathcal{O}_Z(3p)) = \operatorname{Hom}(\mathcal{O}_X, \mathcal{E}^\vee|_Z) = \mathbf{C}^5$. Finally, an Euler characteristic computation shows that

$$\chi(\mathcal{E}, \mathcal{O}_Z) = 5 = \hom(\mathcal{E}, \mathcal{O}_Z) - \operatorname{ext}^1(\mathcal{E}, \mathcal{O}_Z),$$

as required for the claim. Hence it follows that $\operatorname{Hom}^\bullet(\mathcal{E}^\vee, F) = 0$ as required.

2. Now let F be non-locally free. Apply $\operatorname{Hom}(\mathcal{E}^\vee, -)$ to the sequence from Theorem 6.4.1(2). Because $\operatorname{Hom}^\bullet(\mathcal{E}^\vee, \mathcal{E}^\vee) = \mathbf{C}$ by exceptionality of $\mathcal{E}$, and $\operatorname{Hom}^\bullet(\mathcal{E}^\vee, \mathcal{O}_L) = \mathbf{C}$ by a cohomology calculation ($\mathcal{E}|_L$ splits as $\mathcal{O}_L \oplus \mathcal{O}_L(-1)$), we get the exact sequence

$$0 \to \operatorname{Hom}(\mathcal{E}^\vee, F) \to \mathbf{C} \to \mathbf{C} \to \operatorname{Ext}^1(\mathcal{E}^\vee, F) \to 0.$$

Hence by $\operatorname{Hom}(\mathcal{E}^\vee, F) = 0$, we obtain $\operatorname{Ext}^1(\mathcal{E}^\vee, F) = 0$ and $\operatorname{Hom}^\bullet(\mathcal{E}^\vee, F) = 0$ as required.

3. Now let F be a non-globally generated vector bundle. Recall from Theorem 6.4.1(3) the exact sequence

$$0 \to \mathcal{E} \to H^0(X, F) \otimes \mathcal{O}_X \to F \to \mathcal{O}_L(-1) \to 0.$$

Let $G := \operatorname{im}(H^0(X, F) \otimes \mathcal{O}_X \to F)$. Then the exact sequence above can be split up into the short exact sequences

$$0 \to \mathcal{E} \to H^0(X, F) \otimes \mathcal{O}_X \to G \to 0 \tag{6.4.1}$$

and

$$0 \to G \to F \to \mathcal{O}_L(-1) \to 0. \tag{6.4.2}$$

Applying $\mathrm{Hom}(\mathcal{E}^\vee, -)$ to sequence (6.4.1), we have the long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(\mathcal{E}^\vee, \mathcal{E}) \to \mathrm{Hom}(\mathcal{E}^\vee, \mathcal{O}_X^{\oplus m}) \to \mathrm{Hom}(\mathcal{E}^\vee, G) \to \\ \to \mathrm{Ext}^1(\mathcal{E}^\vee, \mathcal{E}) \to \ \cdots \end{aligned}$$

where $m := h^0(X, F)$. Firstly, we know that $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, \mathcal{O}_X) = 0$. Next we find $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, \mathcal{E})$. By Serre duality, $\mathrm{Ext}^i(\mathcal{E}^\vee, \mathcal{E}) \cong \mathrm{Ext}^{3-i}(\mathcal{E}, \mathcal{E})$ which is $\mathbf{C}$ for $i = 3$ and 0 else by exceptionality of $\mathcal{E}$. Therefore we have $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, G) = \mathbf{C}[-2]$.

Next we apply $\mathrm{Hom}(\mathcal{E}^\vee, -)$ to the sequence (6.4.2). We get the long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(\mathcal{E}^\vee, G) \to \mathrm{Hom}(\mathcal{E}^\vee, F) \to \mathrm{Hom}(\mathcal{E}^\vee, \mathcal{O}_L(-1)) \to \\ \to \mathrm{Ext}^1(\mathcal{E}^\vee, G) \to \ \cdots . \end{aligned}$$

Since $\mathcal{E}|_L(-1)$ splits as $\mathcal{O}_L(-1) \oplus \mathcal{O}_L(-2)$, cohomology computations show that $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, \mathcal{O}_L(-1)) = \mathbf{C}[-1]$, so the resulting long exact sequence and the paragraph above gives that $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, F) = 0$.

□

6.4.1 Involutions on $M_G(2, 1, 5)$

In this subsection, we briefly recall the involutions which exist on $M_G(2, 1, 5)$. We follow [DIM12]. Let F be a globally generated vector bundle, and consider the short exact sequence

$$0 \to \ker(\mathrm{ev}) \to H^0(X, F) \otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}} F \to 0. \tag{6.4.3}$$

Note that $\ker(\mathrm{ev})$ is a rank 2 vector bundle with $c_1 = -1$ and $c_2 = 5$ and no global sections, hence $\ker(\mathrm{ev})^\vee \in M_G(2, 1, 5)$. Define $\iota F := \ker(\mathrm{ev})^\vee$. The bundle ιF is globally generated, and we have $H^0(X, \iota F) \cong H^0(X, F)^\vee$ (see [DIM12, p. 29]). If F is a non-locally free sheaf, then the same construction gives a non-globally generated bundle $\iota F = \ker(\mathrm{ev})^\vee$ ([DIM12, p. 32]).

Remark 6.4.3. To summarise, under the involution ι, globally generated vector bundles get sent to globally generated vector bundles, and non-globally generated vector bundles and non-locally free sheaves get exchanged.

Note that for a special Gushel–Mukai threefold, there is another involution on $M_G(2,1,5)$ induced by the involution τ on X,

$$\tau^*\colon M_G(2,1,5) \to M_G(2,1,5), \qquad F \mapsto \tau^* F,$$

which is different from the one we just defined, since if F is not a bundle, then ιF is a bundle but $\tau^* F$ is not.

6.4.2 An explicit description of $\mathrm{pr}(F)$

We are now ready to give an explicit description of $\mathrm{pr}(F)$, for all objects $F \in M_G(2,1,5)$.

Proposition 6.4.4 ([JLLZ21, Lemma 8.3]). *Let X be a Gushel–Mukai threefold and $F \in M_G(2,1,5)$. Then we have*

$$\mathrm{pr}(F) = \begin{cases} (\iota F)^\vee[1] \cong \ker(\mathrm{ev})[1], & F \text{ globally generated} \\ & \text{or non-locally free} \\ \mathcal{E}[1] \to \mathrm{pr}(F) \to \mathcal{O}_L(-1), & F \text{ non-globally generated} \end{cases}$$

where ι is the involution on $M_G(2,1,5)$, defined in Section 6.4.1.

Proof. As a result of Proposition 6.4.2, $\mathbf{L}_{\mathcal{E}^\vee} F = F$, so $\mathrm{pr}(F) = \mathbf{L}_{\mathcal{O}_X} F$. By Theorem 6.4.1 we have $\mathrm{Hom}^\bullet(\mathcal{O}_X, F) = \mathbf{C}^4$, and the triangle defining the left mutation is

$$\mathcal{O}_X^{\oplus 4} \xrightarrow{\mathrm{ev}} F \to \mathrm{pr}(F). \tag{6.4.4}$$

In the cases where F is globally generated or non-locally free, the evaluation map ev is surjective, so $\mathrm{pr}(F) = \ker(\mathrm{ev})[1]$. Section 6.4.1 relates $\ker(\mathrm{ev})$ to ιF as required.

If F is non-globally generated, ev is not surjective. So we take the long exact sequence in cohomology with respect to the standard heart of the triangle (6.4.4). This gives an exact sequence

$$0 \to \mathcal{H}^{-1}(\mathrm{pr}(F)) \to \mathcal{O}_X^{\oplus 4} \to F \to \mathcal{H}^0(\mathrm{pr}(F)) \to 0. \tag{6.4.5}$$

Comparing the sequence (6.4.5) with the sequence in Theorem 6.4.1(3) gives that

$$\mathcal{H}^i(\mathrm{pr}(F)) = \begin{cases} \mathcal{E}, & i = -1 \\ \mathcal{O}_L(-1), & i = 0 \\ 0, & \text{else.} \end{cases}$$

Thus $\mathrm{pr}(F)$ in this case fits into the triangle $\mathcal{E}[1] \to \mathrm{pr}(F) \to \mathcal{O}_L(-1)$ as required. □

6.4.3 Compatibility of categorical and classical involutions for ordinary Gushel–Mukai threefolds

Let X be an ordinary Gushel–Mukai threefold, $\tau_{\mathcal{A}}$ be the involution of $\mathcal{A}_X$, and ι be the geometric involution of $M_G(2,1,5)$ defined in Section 6.4.1. Then $\tau_{\mathcal{A}}$ induces involutions of the Bridgeland moduli spaces of σ-stable objects $\mathcal{M}_\sigma(\mathcal{A}_X,-x)$ and $\mathcal{M}_\sigma(\mathcal{A}_X,y-2x)$. In Proposition 6.3.3, we already showed that the action of $\tau_{\mathcal{A}}$ on $\mathcal{M}_\sigma(\mathcal{A}_X,-x)$ induces a geometric involution on $\mathcal{C}_m(X)$. In this section, we show that the involution induced by $\tau_{\mathcal{A}}$ is also compatible with ι on $M_G(2,1,5)$.

Proposition 6.4.5 ([JLLZ21, Proposition 8.4])**.** *Let X be an ordinary Gushel–Mukai threefold and $F\in M_G^X(2,1,5)$. Then $\tau_{\mathcal{A}}\mathrm{pr}(F)\cong \mathrm{pr}(\iota F)$.*

Proof.

1. Let F be globally generated. Recall that F fits into the short exact sequence (6.4.3) $0\to \ker(\mathrm{ev})\to H^0(X,F)\otimes\mathcal{O}_X\xrightarrow{\mathrm{ev}}F\to 0$. Dualising this sequence and applying pr, we get the triangle

$$\mathrm{pr}(F^\vee)\to\mathrm{pr}(\mathcal{O}_X^{\oplus 4})\to\mathrm{pr}(\ker(\mathrm{ev})^\vee)\cong\mathrm{pr}(\iota F).$$

   Note that $\mathrm{pr}(\mathcal{O}_X)=0$. Also, $F^\vee\in\mathcal{A}_X$ since $\mathrm{Hom}^\bullet(\mathcal{O}_X,F)=0$ by the last part of Theorem 6.4.1, and $\mathrm{Hom}^\bullet(\mathcal{E}^\vee,F)=0$ by Proposition 6.4.2. Therefore we get $\mathrm{pr}(\iota F)\cong F^\vee[1]$.

   Since $F\in M_G(2,1,5)$ is a globally generated vector bundle, we have $F\cong\iota E$ for some[4] globally generated vector bundle E. Then $\mathrm{pr}(F)=\mathrm{pr}(\iota E)\cong E^\vee[1]\cong E\otimes\mathcal{O}_X(-H)[1]$, hence

$$\tau_{\mathcal{A}}(\mathrm{pr}(F))\cong\tau_{\mathcal{A}}(E\otimes\mathcal{O}_X(-H))[1]\cong\mathrm{pr}(E)\cong\mathrm{pr}(\iota F).$$

2. Let F be non-globally generated. Then by Corollary 6.4.4 we have the triangle $\mathcal{E}[1]\to\mathrm{pr}(F)\to\mathcal{O}_L(-1)$. Then $\tau_{\mathcal{A}}(\mathrm{pr}(F))$ is given by the triangle

$$\mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}(\mathcal{E}^\vee)\to\tau_{\mathcal{A}}(\mathrm{pr}(F))\to\mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}(\mathcal{O}_L)[-1].$$

   Note that $\mathbf{L}_{\mathcal{E}^\vee}(\mathcal{E}^\vee)=0$, hence $\tau_{\mathcal{A}}(\mathrm{pr}(F))\cong\mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}(\mathcal{O}_L)[-1]$. Since $\mathcal{E}^\vee|_L\cong\mathcal{O}_L\oplus\mathcal{O}_L(1)$, we have $\mathrm{Hom}^\bullet(\mathcal{E}^\vee,\mathcal{O}_L)=\mathbf{C}$, therefore we have

[4] By Section 6.4.1 E is $\ker(\mathrm{ev})^\vee$.

the triangle $\mathcal{E}^\vee \to \mathcal{O}_L \to \mathbf{L}_{\mathcal{E}^\vee}\mathcal{O}_L$. Also, since $\mathcal{E}^\vee \to \mathcal{O}_L$ is surjective, we have $\mathbf{L}_{\mathcal{E}^\vee}\mathcal{O}_L \cong \ker(\mathcal{E}^\vee \to \mathcal{O}_L)[1]$, where $F' := \ker(\mathcal{E}^\vee \to \mathcal{O}_L)$ is a non-locally free sheaf in $M_G(2,1,5)$ by Theorem 6.4.1. Thus $\tau_{\mathcal{A}}(\mathrm{pr}(F)) \cong \mathbf{L}_{\mathcal{O}_X}F' \cong \ker(\mathrm{ev})[1]$ where $\mathrm{ev}\colon H^0(X,F') \otimes \mathcal{O}_X \to F'$. But note that $F' = \ker(\mathcal{E}^\vee \to \mathcal{O}_L) \cong \iota F$ since F' and F are associated with the same line L (recall that the line L from Theorem 6.4.1(3) determines the bundle uniquely).

Thus $\tau_{\mathcal{A}}(\mathrm{pr}(F)) \cong \mathbf{L}_{\mathcal{O}_X}(\iota F)$. Note that ιF is a non-locally free sheaf and $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, \iota F) = 0$ by Proposition 6.4.2. Thus we have $\iota F \cong \mathbf{L}_{\mathcal{E}^\vee}\iota F$. Then $\tau_{\mathcal{A}}(\mathrm{pr}(F)) \cong \mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}\iota F \cong \mathrm{pr}(\iota F)$ as required.

3. Let F be non-locally free. Then $F \cong \iota E$ for some non-globally generated vector bundle E by Section 6.4.1. Thus we only need to check $\tau_{\mathcal{A}}(\mathrm{pr}(\iota E)) \cong \mathrm{pr}(\iota \circ \iota(E)) \cong \mathrm{pr}(E)$, but this is true by part (2) of the proof.

□

6.4.4 A Bridgeland moduli space interpretation of $M_G(2,1,5)$

We arrive at the first of the main results of Section 6.4.

Theorem 6.4.6 ([JLLZ21, Theorem 8.5])**.** *Let X be a Gushel–Mukai threefold and σ a Serre-invariant stability condition on $\mathcal{A}_X$. Then the projection functor $\mathrm{pr}\colon \mathrm{D}^{\mathrm{b}}(X) \to \mathcal{A}_X$ induces an isomorphism $M_G(2,1,5) \cong \mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$.*

We split the proof of this theorem into a series of lemmas and propositions.

Proposition 6.4.7 ([JLLZ21, Proposition 8.6])**.** *The functor $\mathrm{pr}\colon \mathrm{D}^{\mathrm{b}}(X) \to \mathcal{A}_X$ is injective on all objects in $M_G(2,1,5)$, i.e. if $\mathrm{pr}(F_1) \cong \mathrm{pr}(F_2)$, then $F_1 \cong F_2$.*

Proof. For the case of globally generated vector bundles or non-locally free sheaves, by Corollary 6.4.4, $\mathrm{pr}(F_1) \cong \mathrm{pr}(F_2)$ implies that

$$(\iota F_1)^\vee \cong (\iota F_2)^\vee. \tag{6.4.6}$$

Note that $(\iota F_i)^\vee \cong \iota F_i \otimes \mathcal{O}_X(-H)$ for $i = 1,2$. Then we get $\iota F_1 \cong \iota F_2$. Finally, we apply ι to both sides. Since it is an involution $\iota^2 = \mathrm{id}$, so $F_1 \cong F_2$ as required.

For the case of non-globally generated vector bundles, we have that

$$\mathrm{Ext}^1(\mathcal{O}_L(-1), \mathcal{E}[1]) = \mathbf{C}$$

because of the splitting of $\mathcal{E}|_L$. Thus the extension $\mathcal{E}[1] \to \mathrm{pr}(F) \to \mathcal{O}_L(-1)$ is unique, and L uniquely determines F, so $\mathrm{pr}(F)$ is uniquely determined by L too. Thus $\mathrm{pr}(F_1) \cong \mathrm{pr}(F_2)$ implies $F_1 \cong F_2$, as required. □

Proposition 6.4.8 ([JLLZ21, Proposition 8.7])**.** *The functor* $\mathrm{pr}\colon \mathrm{D^b}(X) \to \mathcal{A}_X$ *induces isomorphisms between* $\mathrm{Ext}^k(\mathrm{pr}(F_1), \mathrm{pr}(F_2))$ *and* $\mathrm{Ext}^k(F_1, F_2)$ *for all* k *and for all* $F_1, F_2 \in M_G(2,1,5)$.

Proof. We apply $\mathrm{Hom}(F_1, -)$ to the exact triangle $\mathcal{O}_X^{\oplus 4} \to F_2 \to \mathrm{pr}(F_2)$. By adjunction of pr and the inclusion $\mathcal{A}_X \hookrightarrow \mathrm{D^b}(X)$, we have $\mathrm{Ext}^k(F_1, \mathrm{pr}(F_2)) \cong \mathrm{Ext}^k(\mathrm{pr}(F_1), \mathrm{pr}(F_2))$ for all k. Thus we get a long exact sequence

$$\begin{aligned} \cdots \to \mathrm{Ext}^k(F_1, \mathcal{O}_X^{\oplus 4}) \to \mathrm{Ext}^k(F_1, F_2) \to \mathrm{Ext}^k(\mathrm{pr}(F_1), \mathrm{pr}(F_2)) \\ \to \mathrm{Ext}^{k+1}(F_1, \mathcal{O}_X^{\oplus 4}) \to \cdots . \end{aligned}$$

Note that $\mathrm{Ext}^k(F_1, \mathcal{O}_X) \cong \mathrm{Ext}^{3-k}(\mathcal{O}_X, F_1(-H))^\vee = 0$ for all k by Serre duality and [BF14, Proposition 3.5]. Thus the desired result follows. □

In what follows, we show the stability of $\mathrm{pr}(F)$ in $\mathcal{A}_X$.

Proposition 6.4.9 ([JLLZ21, Proposition 8.8])**.** *Let* X *be a Gushel–Mukai threefold and* $F \in M_G^X(2,1,5)$. *Then we have*

1. $\mathrm{Hom}^\bullet(F,F) = \mathbf{C} \oplus \mathbf{C}^2[-1]$ *when* X *is ordinary;*
2. $\mathrm{Hom}^\bullet(F,F) = \mathbf{C} \oplus \mathbf{C}^2[-1]$ *or* $\mathrm{Hom}^\bullet(F,F) = \mathbf{C} \oplus \mathbf{C}^3[-1] \oplus \mathbf{C}[-2]$ *when* X *is special.*

Proof. Fist we assume that X is ordinary. By [DIM12, Theorem 8.2], we have $\mathrm{ext}^1(F,F) = 2$. Now $\hom(F,F) = 1$ and $\mathrm{ext}^3(F,F) = 0$ by Serre duality and the Gieseker stability of F. Note that $\chi(F,F) = -1$, so $\mathrm{ext}^2(F,F) = 0$.

Now we assume that X is special. Then by Proposition 6.4.8 and Serre duality in $\mathsf{Ku}(X)$, we have

$$\begin{aligned} \mathrm{Ext}^2(F,F) &\cong \mathrm{Ext}^2(\mathrm{pr}(F), \mathrm{pr}(F)) \\ &\cong \mathrm{Hom}(\mathrm{pr}(F), \tau_{\mathcal{A}}(\mathrm{pr}(F)))^\vee \\ &\cong \mathrm{Hom}(\mathrm{pr}(F), \mathrm{pr}(\tau^* F))^\vee \\ &\cong \mathrm{Hom}(F, \tau^* F)^\vee \end{aligned}$$

where τ is the involution on X induced by the double cover. Thus when $F \cong \tau^* F$, we have $\mathrm{Ext}^2(F,F) = \mathbf{C}$, and $\mathrm{Ext}^2(F,F) = 0$ otherwise. Since $\mathrm{Ext}^3(F,F) = 0$ and $\mathrm{Hom}(F,F) = \mathbf{C}$ by Serre duality and Gieseker stability of F as before, the result follows from $\chi(F,F) = -1$. □

Lemma 6.4.10 ([JLLZ21, Lemma 8.9]). *For every $F \in M_G(2,1,5)$, the object $\mathrm{pr}(F)$ is stable with respect to every Serre-invariant stability condition on $\mathcal{A}_X$.*

Proof. This follows from Proposition 6.4.8, Proposition 6.4.9, and Proposition 4.5.4. □

We are now ready to prove Theorem 6.4.6.

Proof of Theorem 6.4.6. First note that $M_G(2,1,5)$ is a fine moduli space. This is a consequence of [HL10, Theorem 4.6.5]. Using Lemma 6.4.10, by the same argument as in [Zha21, Theorem 8.10], the projection functor pr induces a morphism

$$p\colon M_G(2,1,5) \to \mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$$

which is bijective on points by Proposition 6.4.7 and Theorem 6.5.2, and bijective on tangent spaces by Proposition 6.4.8. Hence it is an isomorphism. □

6.5 Irreducibility of certain Bridgeland moduli spaces

In this section we prove that the Bridgeland moduli spaces of numerical class $y-2x$ and $-x$ are irreducible.

We first fix some notation. Let $\alpha > 0$ and $\beta < 0$. For an object $E \in \mathrm{D^b}(X)$, the *limit central charge* $Z^0_{0,0}(E)$ is defined as the limit of $Z^0_{\alpha,\beta}(E)$ when $(\alpha,\beta) \to (0,0)$. Note that $Z^0_{\alpha,\beta}(E)$ is given by $\mathbf{Q}$-linear combinations of $\alpha, \beta, \alpha^2, \beta^2$, thus the limit $Z^0_{0,0}(E)$ always exists. For $Z^0_{0,0}(E) \neq 0$, we can also define the *limit slope* $\mu^0_{0,0}(E)$ as follows:

$$\mu^0_{0,0}(E) := \begin{cases} -\frac{\Re(Z^0_{0,0}(E))}{\Im(Z^0_{0,0}(E))}, & \Im(Z^0_{0,0}(E)) \neq 0 \\ -\infty, & \Im(Z^0_{0,0}(E)) = 0 \text{ and } \Re(Z^0_{0,0}(E)) > 0 \\ +\infty & \Im(Z^0_{0,0}(E)) = 0 \text{ and } \Re(Z^0_{0,0}(E)) < 0 \end{cases}$$

We will use the following two facts repeatedly throughout this section.

Remark 6.5.1.

1. Note that $Z^0_{0,0}(E) = 0$ if and only if $\mathrm{ch}_{\leq 2}(E)$ is a multiple of $\mathrm{ch}_{\leq 2}(\mathcal{O}_X)$.

2. Let $E \in \mathrm{Coh}^0_{\alpha,\beta}(X)$. By continuity, we can find a neighborhood U_E of the origin such that for any $(\alpha, \beta) \in U_E$, the slopes $\mu^0_{\alpha,\beta}(E)$ and $\mu^0_{0,0}(E)$ are both positive or negative. Let $F \in \mathrm{Coh}^0_{\alpha,\beta}(X)$ be another object such that E, F are both $\sigma^0_{\alpha,\beta}$-semistable in a neighborhood $U_{E,F}$ of the origin. If $\mu^0_{0,0}(E) > \mu^0_{0,0}(F)$, then by continuity, we can find a smaller neighborhood $U'_{E,F}$ such that $\mu^0_{\alpha,\beta}(E) > \mu^0_{\alpha,\beta}(F)$ holds for every $(\alpha, \beta) \in U'_{E,F}$. Thus we have $\mathrm{Hom}(E, F) = 0$.

6.5.1 The moduli space of class $y - 2x$

In this section, we show that the moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, y - 2x)$ is irreducible, i.e. that $\mathcal{M}_\sigma(\mathcal{A}_X, y - 2x) \cong M_G(2, 1, 5)$.

Theorem 6.5.2 ([JLLZ21, Theorem 9.1]). *Let $F \in \mathcal{A}_{\mathrm{alt}}(\alpha, \beta)$ be a $\sigma_{\mathrm{alt}}(\alpha, \beta)$-stable object with numerical class $y - 2x$ for every $(\alpha, \beta) \in V$. Then $F = \mathrm{pr}(E)$ for some $E \in M_G(2, 1, 5)$.*

Proof. We argue as in [PY22, Proposition 4.6]. When $(\alpha_0, \beta_0) = (0, 0)$, we have $\mu^0_{\alpha_0,\beta_0}(F) = -\infty$. Since there are no walls intersecting with $\beta = 0$ as in [PY22, Proposition 4.6], we know that F is $\sigma^0_{\alpha,0}$-semistable for all $\alpha > 0$. By the definition of the double tilted heart, we have a triangle

$$A[1] \to F \to B$$

such that A (respectively B) is in $\mathrm{Coh}^0(X)$ with its $\sigma_{\alpha,0}$-semistable factors having slope $\mu_{\alpha,0} \leq 0$ (respectively $\mu_{\alpha,0} > 0$). Since F is $\sigma^0_{\alpha,0}$-semistable and $\mu^0_{\alpha,0}(F) < 0$, we have that $A[1] = 0$ and $B \cong F$. Since $\mathrm{ch}^0_1(F)$ is minimal, there are no walls intersecting $\beta = 0$, and we know that F is $\sigma_{\alpha,0}$-semistable for every $\alpha > 0$. Thus by Lemma 4.3.3, $\mathcal{H}^{-1}(F)$ is a μ-semistable reflexive sheaf and $\mathcal{H}^0(F)$ is 0 or supported in dimension ≤ 1.

If $\mathcal{H}^0(F)$ is supported in dimension 0, then $\mathrm{ch}(\mathcal{H}^0(F)) = bP$ for $b \geq 1$. But this is impossible since then $c_3(\mathcal{H}^{-1}(F)) > 0$ and by [BF14, Proposition 3.4(i)] we have $\chi(\mathcal{H}^{-1}(F)) = 0$, which implies that $b = 0$.

If $\mathcal{H}^0(F)$ is supported in dimension 1, we can assume that $\mathrm{ch}(\mathcal{H}^0(F)) = aL + \frac{b}{2}P$ where $a \geq 1$ and b are integers. Thus $\mathrm{ch}(\mathcal{H}^{-1}(F)) = 2 - H + aL + (\frac{5}{6} + \frac{b}{2})P$. Now from Lemma 6.3.5, we know $\mathcal{H}^{-1}(F) \cong \mathcal{E}$ and $\mathrm{ch}(\mathcal{H}^0(F)) = L - \frac{P}{2}$. Thus $\mathcal{H}^0(F) \cong \mathcal{O}_L(-1)$ for some line L on X. Therefore we have a triangle

$$\mathcal{E}[1] \to F \to \mathcal{O}_L(-1).$$

In this case we have

$$\begin{aligned}\operatorname{Hom}(\mathcal{O}_L(-1), \mathcal{E}[2]) &= \operatorname{Hom}(\mathcal{E}^\vee(H), \mathcal{O}_L[1]) \\ &= H^1(L, \mathcal{E}(-H)|_L) \\ &= H^1(L, \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-2)) = \mathbf{C}.\end{aligned}$$

Hence by Lemma 6.4.4, $F \cong \operatorname{pr}(E)$ for some $E \in M_G(2,1,5)$ such that E is locally free but not globally generated.

If $\mathcal{H}^0(F) = 0$, we have $F[-1] \cong \mathcal{H}^{-1}(F)$. Then $F[-1]$ is a μ-semistable sheaf. Since $F[-1]$ is reflexive and $c_3(F[-1]) = 0$, $F[-1] \in M_G(2,-1,5)$ is a stable vector bundle. Thus by Lemma 6.4.4, we know $F[-1] = \operatorname{pr}(E)$ for some $E \in M_G(2,1,5)$ such that E is a globally generated vector bundle or non-locally free sheaf. □

6.5.2 The moduli space of class $-x$

In this subsection, we show that $\mathcal{C}_m(X) \cong \mathcal{M}_\sigma(\mathcal{A}_X, -x)$.

Lemma 6.5.3 ([JLLZ21, Lemma 9.2]). *If $F \in \mathcal{A}_{\text{alt}}(\alpha,\beta)$ is $\sigma_{\text{alt}}(\alpha,\beta)$-stable such that $[F] = -x$ and F is $\sigma^0_{\alpha,\beta}$-semistable for some $(\alpha,\beta) \in V$, then $F \cong I_C[1]$ for some conic C on X.*

Proof. Since F is $\sigma^0_{\alpha,\beta}$-semistable and $\mu^0_{\alpha,\beta}(F) > 0$, as in [PY22, Proposition 4.6] there is a triangle

$$F_1[1] \to F \to F_2$$

where $F_1 \in \operatorname{Coh}^\beta(X)$ with $\mu^+_{\alpha,\beta}(F_1) < 0$ and F_2 is supported on points. Thus $\operatorname{ch}(F_1) = (1, 0, -2L, mP)$, where m is the length of F_2. By Lemmas 6.3.11 and 4.3.3, F_1 is a rank 1 torsion free sheaf, hence it is the ideal sheaf of a closed subscheme. Thus by [San14, Corollary 1.38] (see the Case (2) in the proof), we have $m \leq 0$, which means $F_2 = 0$ and $F_1 \cong F[-1]$. Thus by Lemmas 6.3.11 and 4.3.3 again, $F[-1]$ is a μ-semistable torsion free sheaf, which is of the form $F[-1] \cong I_C$ for some conic C on X since $\operatorname{Pic}(X) = \mathbf{Z} \cdot H$. □

When F is not $\sigma^0_{\alpha,\beta}$-semistable for $(\alpha,\beta) \in V$, the argument is slightly more complicated.

Lemma 6.5.4 ([JLLZ21, Lemma 9.3]). *If $F \in \mathcal{A}_{\text{alt}}(\alpha,\beta)$ is $\sigma_{\text{alt}}(\alpha,\beta)$-stable such that $[F] = -x$ and F is not $\sigma^0_{\alpha,\beta}$-semistable for every $(\alpha,\beta) \in V$, then F fits into a triangle*

$$\mathcal{E}[2] \to F \to \mathcal{Q}^\vee[1].$$

Proof. Since there are no walls for F tangent to the wall $\beta = 0$, by the local finiteness of walls and [BMT13, Proposition 2.2.2] we can find an open neighborhood U' of the origin such that the Harder–Narasimhan filtration with respect to $\sigma^0_{\alpha,\beta}$ is constant for every $(\alpha, \beta) \in U := U' \cap V$. In the following we will only consider $\sigma^0_{\alpha,\beta}$ for $(\alpha, \beta) \in U$.

Let B be the minimal destabilizing quotient object of F and $0 \to A \to F \to B \to 0$ be the destabilizing short exact sequence of F in $\mathrm{Coh}^0_{\alpha,\beta}(X)$. Hence we know that $A, B \in \mathrm{Coh}^0_{\alpha,\beta}(X)$ and B is $\sigma^0_{\alpha,\beta}$-semistable with $\mu^{0,-}_{\alpha,\beta}(A) > \mu^0_{\alpha,\beta}(F) > \mu^0_{\alpha,\beta}(B)$ for all $(\alpha, \beta) \in U$. By [BLMS23, Remark 5.12], we have $\mu^0_{\alpha,\beta}(B) \geq \min\{\mu^0_{\alpha,\beta}(F), \mu^0_{\alpha,\beta}(\mathcal{O}_X), \mu^0_{\alpha,\beta}(\mathcal{E}^\vee)\}$. Hence the following relations hold for all $(\alpha, \beta) \in U$:

a. $\mu^0_{\alpha,\beta}(A) > \mu^0_{\alpha,\beta}(F) > \mu^0_{\alpha,\beta}(B)$,

b. $\mathrm{Im}(Z^0_{\alpha,\beta}(A)) \geq 0$, $\mathrm{Im}(Z^0_{\alpha,\beta}(B)) > 0$,

c. $\mu^0_{\alpha,\beta}(B) \geq \min\{\mu^0_{\alpha,\beta}(F), \mu^0_{\alpha,\beta}(\mathcal{O}_X), \mu^0_{\alpha,\beta}(\mathcal{E}^\vee)\}$,

d. $\Delta(B) \geq 0$.

By continuity we have this list which we call (★):

1. $\mu^0_{0,0}(A) \geq \mu^0_{0,0}(F) = 0 \geq \mu^0_{0,0}(B)$,
2. $\mathrm{Im}(Z^0_{0,0}(A)) \geq 0$, $\mathrm{Im}(Z^0_{0,0}(B)) \geq 0$,
3. $\mu^0_{0,0}(B) \geq \min\{\mu^0_{0,0}(F), \mu^0_{0,0}(\mathcal{O}_X), \mu^0_{0,0}(\mathcal{E}^\vee)\}$,
4. $\Delta(B) \geq 0$.

Assume $[A] = a[\mathcal{O}_X] + b[\mathcal{O}_H] + c[\mathcal{O}_L] + d[\mathcal{O}_P]$. Then $[B] = (-1-a)[\mathcal{O}_X] - b[\mathcal{O}_H] + (2-c)[\mathcal{O}_L] - (1+d)[\mathcal{O}_P]$. Then $\mathrm{ch}(A) = (a, bH, \frac{c-5b}{10}H^2, \frac{\frac{5}{3}b+\frac{c}{2}+d}{10}H^3)$ and $Z^0_{0,0}(A) = bH^3 + (\frac{c-5b}{10}H^3)\cdot i$, $Z^0_{0,0}(B) = -bH^3 + (\frac{2-c+5b}{10}H^3)\cdot i$ and $\mu^0_{0,0}(A) = \frac{10b}{5b-c}$, $\mu^0_{0,0}(B) = \frac{-10b}{c-5b-2}$. Note that $[F] = -[\mathcal{O}_X] + 2[\mathcal{O}_L] - [\mathcal{O}_P]$. From (2) we know $c - 5b = 0$, 1 or 2. But when $c - 5b = 2$, it is not hard to see that (c) fails near the origin. Thus $c - 5b = 0$ or 1.

We begin with two claims.

Claim 1: $\mathrm{Hom}^\bullet(\mathcal{O}_X, B) = \mathrm{Hom}(\mathcal{O}_X, B)$ and $\mathrm{Hom}^\bullet(\mathcal{O}_X, A) = \mathrm{Ext}^1(\mathcal{O}_X, A)[-1]$.

Since $F \in \mathcal{A}_X$, we only need to prove that $\mathrm{Ext}^i(\mathcal{O}_X, A) = 0$ for $i \neq 1$. Indeed, since $\mathcal{O}_X \in \mathrm{Coh}^0_{\alpha,\beta}(X)$ and $F \in \mathcal{A}_X$, we have $\mathrm{Ext}^i(\mathcal{O}_X, A) = 0$ for all

$i \leq 0$. Also, by Serre duality we have $\mathrm{Ext}^i(\mathcal{O}_X, A) = \mathrm{Hom}(A, \mathcal{O}_X(-H)[3-i])$. Thus from $\mathcal{O}_X(-H) \in \mathrm{Coh}^0_{\alpha,\beta}(X)$, we obtain $\mathrm{Hom}(A, \mathcal{O}_X(-H)[3-i]) = 0$ for $i \geq 2$. Therefore we have $\mathrm{Ext}^i(\mathcal{O}_X, A) = 0$ for $i \neq 1$.

Claim 2: $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, B) = \mathrm{Hom}(\mathcal{E}^\vee, B)$ and $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, A) = \mathrm{Ext}^1(\mathcal{E}^\vee, A)[-1]$.

Since $\mathcal{E}^\vee$ and $\mathcal{E}[2] \in \mathrm{Coh}^0_{\alpha,\beta}(X)$, the argument is the same as Claim 1.

Now we deal with the cases $c - 5b = 0$ and $c - 5b = 1$ separately.

Case 1 ($c - 5b = 0$):

First we assume that $c - 5b = 0$. By (★), we have:

1. $-2 \leq b \leq 0$,
2. $b^2 + \frac{2a+2}{5} \geq 0$.

Case 1.1 ($b = 0$): If $b = 0$, then $c = 0$ and $a \geq -1$. In this case we have $\mathrm{ch}_{\leq 2}(B) = (-1-a, 0, 2L)$. If $a = -1$, then $\mathrm{ch}_{\leq 2}(B) = (0, 0, 2L)$, which is impossible since $B \in \mathrm{Coh}^0_{\alpha,\beta}(X)$. Thus $a \geq 0$, but then we have $\mu^0_{\alpha,\beta}(F) \geq \mu^0_{\alpha,\beta}(B)$ when $(\alpha, \beta) \in U$ is sufficiently close to the origin. This contradicts our assumption on B.

Case 1.2 ($b = -1$): If $b = -1$, we have $c = -5$. In this case $\mathrm{ch}_{\leq 2}(A) = (a, -H, 0)$. Since $A \in \mathrm{Coh}^0_{\alpha,\beta}(X)$, we have $\mathrm{Im}(Z^0_{\alpha,\beta}(A)) \geq 0$ for every $(\alpha, \beta) \in U$. Note that $\mathrm{Im}(Z^0_{\alpha,\beta}(A)) = (\beta + \frac{a(\beta^2-\alpha^2)}{2})H^3$ and $\alpha < -\beta$, and we have $a \geq \frac{-2\beta}{\beta^2-\alpha^2}$. But note that when $\alpha = \frac{-\beta}{2}$ and $\beta \to -0$, we have $\frac{-2\beta}{\beta^2-\alpha^2} \to +\infty$, thus we get a contradiction since a is a finite number.

Case 1.3 ($b = -2$): If $b = -2$, we have $c = -10$. In this case we have $\mathrm{ch}_{\leq 2}(A) = (a, -2H, 0)$. Similarly to case 1.2, we have $\mathrm{Im}(Z^0_{\alpha,\beta}(A)) \geq 0$ for every $(\alpha, \beta) \in U$. Note that $\mathrm{Im}(Z^0_{\alpha,\beta}(A)) = (2\beta + \frac{a(\beta^2-\alpha^2)}{2})H^3$ and $\alpha < -\beta$, and we have $a \geq \frac{-4\beta}{\beta^2-\alpha^2}$. Then as in Case 1.2, we get a contradiction.

Case 2 ($c - 5b = 1$): Now we assume that $c - 5b = 1$. Then by (★), we have:

1. $-1 \leq b \leq 0$,
2. $b^2 + \frac{a+1}{5} \geq 0$.

Case 2.1 ($b = 0$): If $b = 0$, then $c = 1$. Therefore $-1 \leq a$. If $a = -1$, since B is $\sigma^0_{\alpha,\beta}$-semistable, we know $\mathcal{H}^0_{\mathrm{Coh}^\beta(X)}(B)$ is either 0 or supported on points. Thus $\mathrm{ch}_{\leq 2}(\mathcal{H}^{-1}_{\mathrm{Coh}^\beta(X)}(B)) = (0, 0, -L)$. But $\mathrm{Re}(Z_{\alpha,\beta}(\mathcal{H}^{-1}_{\mathrm{Coh}^\beta(X)}(B))) > 0$ which is impossible since $\mathcal{H}^{-1}_{\mathrm{Coh}^\beta(X)}(B) \in \mathrm{Coh}^\beta(X)$ with $\mathrm{Im}(Z_{\alpha,\beta}(\mathcal{H}^{-1}_{\mathrm{Coh}^\beta(X)}(B))) = 0$.

Therefore we have $a \geq 0$. Hence $\mathrm{ch}_{\leq 2}(B) = -(a+1, 0, -L)$, where $a+1 \geq 1$. This is also impossible since when $(\alpha, \beta) \in U$ is sufficiently close to the origin, we have $\mu^0_{\alpha,\beta}(B) > \mu^0_{\alpha,\beta}(F)$.

Case 2.2 ($b = -1$): We have $b = -1$ and $c = -4$. Hence $-6 \leq a$. In this case $\mathrm{ch}_{\leq 2}(B) = (-1-a, H, L)$ and we have $\mu^0_{\alpha,\beta}(B) < 0$ for when $(\alpha, \beta) \in U$ is sufficiently close to the origin. Therefore, $B \in \mathrm{Coh}^\beta(X)$ is $\sigma_{\alpha,\beta}$-semistable. Applying Lemma 4.3.4 to B, we have $a \geq -3$.

We first prove a claim.

Claim 3: In the situation of Case 2.2, we have A is $\sigma^0_{\alpha,\beta}$-semistable. Hence $\mathrm{Hom}^\bullet(\mathcal{O}_X, A) = 0$, $\mathrm{ch}(A) = (a, -H, L, (\frac{7}{3} - a)P)$ and $\chi(\mathcal{E}^\vee, A) = 3 - 2a$.

Assume A is not $\sigma^0_{\alpha,\beta}$-semistable for some $(\alpha, \beta) \in U$. Then we can take a neighborhood U'_A of the origin such that A has constant Harder–Narasimhan factors for any $(\alpha, \beta) \in U_A := U \cap U'_A \cap V$. Let C be the minimal destabilizing quotient object of A with respect to $\sigma^0_{\alpha,\beta}$ for $(\alpha, \beta) \in U_A$. In this case we have $\mathrm{ch}_{\leq 2}(A) = (a, -H, L)$. Since $\mathrm{Im}(Z^0_{0,0}(A)) = \frac{1}{10}H^3$, we know that $\mathrm{Im}(Z^0_{0,0}(C)) = 0$ or $\frac{1}{10}H^3$. If $\mathrm{Im}(Z^0_{0,0}(C)) = 0$, then $\mu^0_{0,0}(C) = +\infty$ or $-\infty$. But the previous case contradicts $\mu^0_{\alpha,\beta}(A) > \mu^0_{\alpha,\beta}(C)$ and the latter case contradicts $\mu^0_{\alpha,\beta}(C) > \mu^0_{\alpha,\beta}(F)$. Therefore we have $\mathrm{Im}(Z^0_{0,0}(C)) = \frac{1}{10}H^3$ and we can assume that $\mathrm{ch}_{\leq 2}(C) = (e, fH, L)$ where $e, f \in \mathbb{Z}$. Since $\mu^0_{0,0}(A) \geq \mu^0_{0,0}(C) \geq \mu^0_{0,0}(F) = 0$, we have $10 \geq -10f \geq 0$. If $f = 0$, then $\mathrm{ch}_{\leq 2}(C) = (e, 0, L)$ and $\mathrm{ch}_{\leq 2}(D) = (a-e, -H, 0)$, where $D = \mathrm{cone}(A \to C)[-1]$. Then $\mu^{0-}_{\alpha,\beta}(D) > \mu^0_{\alpha,\beta}(A)$ for any $(\alpha, \beta) \in U_A$. Hence $\mu^0_{\alpha,\beta}(D) = \frac{1+(a-e)\beta}{\beta + \frac{a-e}{2}(\beta^2 - \alpha^2)}$. But note that if we take $\alpha = -\frac{\beta}{2}$ and $|\beta| < |\frac{1}{a-e}|$, when $(\alpha, \beta) \in U_A$ and $|\beta|$ is sufficiently small we get $1 + (a-e)\beta > 0$ and $\beta + \frac{a-e}{2}(\beta^2 - \alpha^2) < 0$. This implies $\mu^0_{\alpha,\beta}(D) < 0$ for such (α, β), which gives a contradiction since $\mu^0_{\alpha,\beta}(D) > \mu^0_{\alpha_0,\beta_0}(F)$ holds for any $(\alpha, \beta) \in U_A$.

Therefore the only possible case is $f = -1$, and hence $\mu^0_{0,0}(C) = 10$. Since $\mu^0_{\alpha,\beta}(A) > \mu^0_{\alpha,\beta}(C)$ for $(\alpha, \beta) \in U_A$, we have $\mathrm{rk} C > a$. But this is impossible since $D, \mathcal{O}_X \in \mathrm{Coh}^0_{\alpha,\beta}(X)$ but $\mathrm{ch}_{\leq 2}(D) = (s, 0, 0) = s \cdot \mathrm{ch}_{\leq 2}(\mathcal{O}_X)$ where $s = a - \mathrm{rk} C < 0$. Now for the last statement, note that $\mathcal{O}_X(-H)[2] \in \mathrm{Coh}^0_{\alpha,\beta}(X)$ is $\sigma^0_{\alpha,\beta}$-semistable with $\mu^0_{0,0}(\mathcal{O}_X(-H)[2]) = 2$, hence we have $\mathrm{Hom}(A, \mathcal{O}_X(-H)[2]) = \mathrm{Hom}(\mathcal{O}_X, A[1]) = 0$. Now combining with Claim 1, this proves our claim.

Now we deal with the three cases $a = -3$, $-2 \leq a \leq 1$ and $a \geq 2$ separately.

When $a = -3$, we have $\mathrm{ch}_{\leq 2}(B) = \mathrm{ch}_{\leq 2}(\mathcal{E}^\vee)$. Then since $\mathrm{ch}_{\leq 2}(B)$ is on the boundary of Lemma 4.3.4, by a standard argument we know that B is $\sigma_{\alpha,\beta}$-semistable for every $\alpha > 0$ and $\beta < 0$, as explained in [PR23, Proposition

3.2]. Thus by Lemma 4.3.3, B is a μ-semistable sheaf. From Claim 3 we have $\chi(\mathcal{O}_X, B) = 0$, hence $\mathrm{ch}(B) = \mathrm{ch}(\mathcal{E}^\vee)$ and by Lemma 6.3.5 we have $B \cong \mathcal{E}^\vee$. But this implies $\mathrm{Hom}(\mathcal{O}_X, A[1]) = \mathbf{C}^5$ since $F \in \mathcal{A}_X$, which contradicts Claim 3.

When $-2 \leq a \leq 1$, we have $\mu^0_{\alpha,\beta}(A) > \mu^0_{\alpha,\beta}(\mathcal{E}[2])$. Since A is $\sigma^0_{\alpha,\beta}$-semistable, we have $\mathrm{Hom}(A, \mathcal{E}[2]) = \mathrm{Hom}(\mathcal{E}^\vee, A[1]) = 0$. Thus $\mathrm{Hom}^\bullet(\mathcal{E}^\vee, A) = 0$ by Claim 2. But this contradicts Claim 3 since $\chi(\mathcal{E}^\vee, A) = 3 - 2a$.

When $a \geq 2$, applying Lemma 4.3.5 to B, we have $a = 2$. Thus $\mathrm{ch}_{\leq 2}(B) = \mathrm{ch}_{\leq 2}(\mathcal{Q}^\vee[1])$. By Claim 3, we know that $\mathrm{Hom}^\bullet(\mathcal{O}_X, B) = 0$ and we get $\mathrm{ch}(B) = \mathrm{ch}(\mathcal{Q}^\vee[1])$. Thus $\chi(\mathcal{E}^\vee, B) = \hom(\mathcal{E}^\vee, B) > 0$. Therefore, if we apply $\mathrm{Hom}(-, B)$ to the exact sequence $0 \to \mathcal{Q}^\vee \to \mathcal{O}_X^{\oplus 5} \to \mathcal{E}^\vee \to 0$, we obtain $\hom(\mathcal{Q}^\vee[1], B) > 0$. Now by stability, we have $B \cong \mathcal{Q}^\vee[1]$. Now $\mathrm{ch}(A) = \mathrm{ch}(\mathcal{E}[2])$. By Claims 2 and 3, we have $\mathrm{ext}^1(\mathcal{E}^\vee, A) = \hom(A, \mathcal{E}[2]) = 1$. Since A is $\sigma^0_{\alpha,\beta}$-semistable and $\mathcal{E}[2]$ is $\sigma^0_{\alpha,\beta}$-stable, we have $A \cong \mathcal{E}[2]$. □

Theorem 6.5.5 ([JLLZ21, Theorem 9.4]). *Let X be a Gushel–Mukai threefold. Then the the irreducible component $\mathcal{S}$ in Theorem 6.3.13 is the whole moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$.*

Proof. Note that $\hom(\mathcal{Q}^\vee[1], \mathcal{E}[2]) = 1$. Then the result follows from Lemma 6.5.3 and Lemma 6.5.4. □

6.6 Categorical Torelli theorems for Gushel–Mukai threefolds

In this section, we will prove several refined/birational categorical Torelli theorems for Gushel–Mukai threefolds, using results from the previous sections.

We first show that the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x) \cong \mathcal{C}_m(X)$ admits a universal family, which in turn implies that it is a fine moduli space. Note that $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x) \cong M_G(2,1,5)$ immediately admits a universal family, since it is isomorphic to a Gieseker moduli space.

6.6.1 The universal family for $\mathcal{C}_m(X)$

Let $\mathcal{I}$ be the universal ideal sheaf of conics on $X \times \mathcal{C}(X)$, i.e. for every $x \in \mathcal{C}(X)$, $\mathcal{I}|_{X\times x}$ is an ideal sheaf of a conic on X. Let $\mathcal{I}_{L_\sigma}$ be the universal ideal sheaf of conics restricted to $X \times L_\sigma$. Let $q\colon X \times \mathcal{C}(X) \to X$ and $\pi\colon X \times \mathcal{C}(X) \to \mathcal{C}(X)$ be the projection maps on the first and second factors, respectively.

Let $\mathcal{G}' := \mathrm{pr}(\mathcal{I}_{L_\sigma})$ be the projected family in $\mathcal{A}_{X\times L_\sigma}$. Let $t \in L_\sigma \cong \mathbf{P}^1$ be any point. Then $j_t^*\mathrm{pr}(\mathcal{I}_{L_\sigma}) \cong A$, where $j_t : X_t \to X_t \times L_\sigma$ and $A \in \mathcal{A}_X$ is $A \cong \mathrm{pr}(I_C)$ for $I_C \notin \mathcal{A}_X$ by Proposition 6.3.2. Then $\mathcal{G}' \cong q^*(A) \otimes \pi^*\mathcal{O}_{L_\sigma}(k)$ for some $k \in \mathbf{Z}$. Now let $\mathcal{G} := \mathrm{pr}(\mathcal{I}) \otimes \pi^*\mathcal{O}_{\mathcal{C}(X)}(kE)$, where $E \cong L_\sigma \cong \mathbf{P}^1$ is the unique exceptional curve on $\mathcal{C}(X)$.

Proposition 6.6.1 ([JLLZ21, Proposition 10.1])**.** *The object $(p_X)_*\mathcal{G}$ is the universal family of $\mathcal{C}_m(X)$, where $p_X = \mathrm{id}_X \times p\colon X \times \mathcal{C}(X) \to X \times \mathcal{C}_m(X)$.*

Proof. We first fix some notation via the commutative diagrams below which summarise the maps in the proof:

$$\begin{tikzcd}
X_s \arrow[r, "\cong"] \arrow[d, hook, "j_s"] & X_s \arrow[r, "\pi_s"] \arrow[d, hook, "i_s"] & \{s\} \arrow[d] \\
X \times \mathcal{C}(X) \arrow[r, "p_X"'] & X \times \mathcal{C}_m(X) \arrow[r] & \mathcal{C}_m(X)
\end{tikzcd}$$

$$\begin{tikzcd}
X_s \arrow[r, "j_s"] \arrow[d, "\pi_s"] & X \times \mathcal{C}(X) \arrow[d, "\pi"] \\
\{s\} \arrow[r, "f_s"] & \mathcal{C}(X)
\end{tikzcd}$$

1. If $s = [A] = \pi \in \mathcal{C}_m(X)$, s is contracted from the unique rational curve $L_\sigma \cong \mathbf{P}^1 \subset \mathcal{C}(X)$. Note that in this case $p_X|_{L_\sigma} = q$. Then

$$\begin{aligned}
i_s^*(p_X)_*\mathcal{G} &\cong i_s^*(p_X)_*(\mathcal{G}' \otimes \pi^*\mathcal{O}_{\mathcal{C}(X)}(kE)) \\
&\cong i_s^*q_*(q^*(A) \otimes \pi^*\mathcal{O}_{L_\sigma}(k) \otimes \pi^*\mathcal{O}_{\mathcal{C}(X)}(kE)) \\
&\cong i_s^*q_*(q^*(A) \otimes (\pi^*\mathcal{O}_{L_\sigma}(k) \otimes \mathcal{O}_{L_\sigma}(kE))) \\
&\cong i_s^*q_*(q^*(A) \otimes \pi^*(\mathcal{O}_{L_\sigma}(k) \otimes \mathcal{O}_{L_\sigma}(-k))) \\
&\cong i_s^*q_*(q^*(A)) \cong i_s^*(A) \cong A.
\end{aligned}$$

2. If $s = [I_C]$, then $\mathcal{C}_m(X)$ and $\mathcal{C}(X)$ are isomorphic outside L_σ. Note that p restricts to id on $\mathcal{C}(X) \smallsetminus L_\sigma$. Then

$$\begin{aligned}
i_s^*(p_X)_*\mathcal{G} &\cong i_s^*(p_X)_*(\mathrm{pr}(\mathcal{I}) \otimes \pi^*\mathcal{O}_{\mathcal{C}(X)}(kE)) \\
&\cong j_s^*(\mathrm{pr}(\mathcal{I})) \otimes j_s^*\pi^*\mathcal{O}_{\mathcal{C}(X)}(kE) \\
&\cong I_C \otimes (\pi \circ j_s)^*\mathcal{O}_{\mathcal{C}(X)}(kE) \\
&\cong I_C \otimes (f_s \circ \pi_s)^*\mathcal{O}_{\mathcal{C}(X)}(kE) \cong I_C.
\end{aligned}$$

$\square$

6.6.2 Refined categorical Torelli for Gushel–Mukai threefolds

We now prove a refined categorical Torelli theorem for ordinary Gushel–Mukai threefolds.

Theorem 6.6.2 ([JLLZ21, Theorem 10.2])**.** *Let X and X' be general ordinary Gushel–Mukai threefolds such that $\Phi\colon \mathsf{Ku}(X) \simeq \mathsf{Ku}(X')$ is an equivalence and $\Phi(i^!(\mathcal{E})) \cong (i')^!(\mathcal{E}')$. Then $X \cong X'$.*

Proof. Note that $\Xi(i^!(\mathcal{E})) \cong \mathrm{pr}(I_C)[1] \cong i^!(\mathcal{Q}^\vee)[1]$, where $I_C \notin \mathcal{A}_X$. Then the equivalence Φ induces an equivalence $\Psi := \Xi \circ \Phi \circ \Xi^{-1} : \mathcal{A}_X \simeq \mathcal{A}_{X'}$ such that $\Psi(\pi) = \pi'$, where $\pi := i^!(\mathcal{Q}^\vee)[1] \cong \mathrm{pr}(I_C)[1] \in \mathcal{A}_X$ and $\pi' := (i')^!(\mathcal{Q}'^\vee)[1] \cong \mathrm{pr}'(I_{C'})[1] \in \mathcal{A}_{X'}$. The existence of the universal family on $\mathcal{C}_m(X)$ guarantees a projective dominant morphism from $\mathcal{C}_m(X)$ to $\mathcal{C}_m(X')$, denoted by ψ, which is induced by Ψ (for more details on the construction of the morphism ψ, see [BMMS12, APR22]). Since Ψ is an equivalence, ψ is bijective on closed points by Theorem 6.3.13 and Theorem 4.5.14. It also identifies the tangent spaces of each point on $\mathcal{C}_m(X)$ and $\mathcal{C}_m(X')$, hence it is an isomorphism. On the other hand, we have $\psi(\pi) = \pi'$. Then ψ induces an isomorphism $\phi\colon \mathcal{C}(X) \cong \mathcal{C}(X')$ by blowing up $\pi \in \mathcal{C}_m(X)$ and $\pi' \in \mathcal{C}_m(X')$, respectively. Then we have $X \cong X'$ by Logachev's Reconstruction Theorem 6.2.7. □

6.6.3 Birational categorical Torelli for Gushel–Mukai threefolds

In this subsection, we show a birational categorical Torelli theorem for ordinary Gushel–Mukai threefolds, i.e. assuming the Kuznetsov components are equivalent leads to a birational equivalence of the ordinary Gushel–Mukai threefolds.

Theorem 6.6.3 ([JLLZ21, Theorem 10.3])**.** *Let X and X' be general ordinary Gushel–Mukai threefolds such that $\mathcal{A}_X \simeq \mathcal{A}_{X'}$. Then X' is a conic transform, or a conic transform of a line transform of X. In particular, we have $X \simeq X'$.*

Proof. Assume that $\Phi\colon \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_{X'}$, and fix a (-1)-class $-x$ in $\mathcal{N}(\mathcal{A}_X)$. The equivalence Φ sends $-x$ to either itself or $y - 2x$ in $\mathcal{N}(\mathcal{A}_{X'})$ up to sign. By the same argument as in [BMMS12, APR22], we thus get two possible induced morphisms between Bridgeland moduli spaces

$$\begin{tikzcd} M_\sigma(\mathcal{A}_X, -x) \arrow[r, "\gamma"] \arrow[dr, "\gamma'"] & M_\sigma(\mathcal{A}_{X'}, -x) \\ & M_\sigma(\mathcal{A}_{X'}, y-2x) \end{tikzcd}$$

As we have seen in Theorems 6.3.13 and 6.4.6, $\mathcal{M}_\sigma(\mathcal{A}_X, -x) \cong \mathcal{C}_m(X)$ and $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x) \cong M_G(2,1,5)$. So we have two cases: either $\mathcal{C}_m(X) \cong \mathcal{C}_m(X')$ or $\mathcal{C}_m(X) \cong M_G^{X'}(2,1,5)$.

For the first case, blow up $\mathcal{C}_m(X)$ at the distinguished point $\pi := \Xi(i^!(\mathcal{E}))$, and blow up $\mathcal{C}_m(X')$ at the point $c := \Phi(\pi)$. We have $\mathcal{C}(X) \cong \mathrm{Bl}_\pi\mathcal{C}_m(X)$ and we have $\mathrm{Bl}_c\mathcal{C}_m(X') \cong \mathcal{C}(X'_c)$ by Theorem 6.2.9, so $\mathcal{C}(X) \cong \mathcal{C}(X'_c)$. Therefore by Logachev's Reconstruction Theorem 6.2.7 we have $X \cong X'_c$. But X'_c is birational to X', so X and X' are birational.

For the second case, we get $\mathcal{C}_m(X) \cong M_G^{X'}(2,1,5)$ but we have a birational equivalence $M_G^{X'}(2,1,5) \simeq \mathcal{C}(X'_L)$ of surfaces by [DIM12, Proposition 8.1]. Thus $\mathcal{C}_m(X)$ is birationally equivalent to $\mathcal{C}(X'_L)$. Let $\mathcal{C}_m(X'_L)$ be the minimal surface of $\mathcal{C}(X'_L)$. Note that the surfaces here are all smooth surfaces of general type. By the uniqueness of minimal models of surfaces of general type, we get $\mathcal{C}_m(X) \cong \mathcal{C}_m(X'_L)$, which implies $X \cong (X'_L)_c \simeq X'$ as in the first case. $\square$

As a corollary, we obtain a stronger result than what is proved in [DIM12], which claims that $\mathcal{C}_m(X_L)$ is birational to $M_G^X(2,1,5)$.

Corollary 6.6.4 ([JLLZ21, Corollary 10.5]). *Let X be a general ordinary Gushel–Mukai threefold, and X_L be a line transform of X. Then we have $\mathcal{C}_m(X_L) \cong M_G^X(2,1,5)$. Moreover, this isomorphism commutes with involutions ι and ι' on both sides, thus giving an isomorphism $\mathcal{C}_m(X_L)/\iota \cong M_G^X(2,1,5)/\iota'$.*

Proof. By the same argument as in the proof of Theorem 6.6.3, we have $\mathcal{C}_m(X_L) \cong \mathcal{C}_m(X)$ or $\mathcal{C}_m(X_L) \cong M_G^X(2,1,5)$. Note that $\mathcal{C}_m(X_L) \cong \mathcal{C}_m(X)$ implies that $X_L \cong X_c$ for some conic $c \subset X$ as in Theorem 6.6.3. But this is impossible by [DIM12, Remark 7.3]. Thus we always have $\mathcal{C}_m(X_L) \cong M_G^X(2,1,5)$. The last statement follows from the fact that any equivalence between Kuznetsov components commutes with Serre functors, and the involutions on $\mathcal{C}_m(X_L)$ and $M_G^X(2,1,5)$ can be induced by Serre functors up to shift by Propositions 6.3.3 and 6.4.5. $\square$

Since the intermediate Jacobian $J(X)$ is invariant under conic and line transforms, we have the following corollary.

Corollary 6.6.5 ([JLLZ21, Corollary 10.6])**.** *Let X and X' be general ordinary Gushel–Mukai threefolds. If $\mathsf{Ku}(X) \simeq \mathsf{Ku}(X')$, then we have $J(X) \cong J(X')$.*

Note the corollary above also follows from Perry's construction of the intermediate Jacobian of a category [Per22].

6.7 A categorical Torelli theorem for special Gushel–Mukai threefolds

In this section, we show that the Kuznetsov component of a general special Gushel–Mukai threefold X determines the isomorphism class of X.

Recall from Section 2 that every special Gushel–Mukai threefold X is a double cover of a degree 5 index 2 prime Fano threefold Y branched over a quadric hypersurface $\mathcal{B}$ in Y. Since X is smooth and general, $(\mathcal{B}, h)$ is a smooth degree $h^2 = 10$ K3 surface with Picard rank 1. There is a natural geometric involution τ on X induced by the double cover. The Serre functor on $\mathsf{Ku}(X)$ is given by $S_{\mathsf{Ku}(X)} = \tau_{\mathcal{A}} \circ [2]$.

Theorem 6.7.1 ([JLLZ21, Theorem 10.9])**.** *Let X and X' be smooth general special Gushel–Mukai threefolds such that there is an equivalence $\Phi\colon \mathsf{Ku}(X) \simeq \mathsf{Ku}(X')$. Then $X \cong X'$.*

Proof. By [KP17, Theorem 1.1, Section 8.2], the equivariant triangulated category $\mathsf{Ku}(X)^{\mu_2}$ is equivalent to $\mathrm{D}^{\mathrm{b}}(\mathcal{B})$, where μ_2 is the group of square roots of 1 generated by the involution $\tau_{\mathcal{A}}$ acting on $\mathsf{Ku}(X)$[5]. Assume there is an equivalence $\Phi\colon \mathsf{Ku}(X) \simeq \mathsf{Ku}(X')$.

We now check that Φ descends to an equivalence of equivariant Kuznetsov components. The argument will be analogous to those in [DJR23, Lemmas 6.2 and 6.3]. We first check that Φ preserves 1-categorical actions (all actions we discuss in this proof will be understood to be $\mathbf{Z}/2 \cong \mu_2$-actions). Let $E \in \mathsf{Ku}(X)$ and consider the arrow $E \to \tau_{\mathcal{A}}(E)$ in $\mathsf{Ku}(X)$. The equivalence Φ sends this arrow to the arrow $\Phi(E) \to \Phi(\tau_{\mathcal{A}}(E)) \cong \tau'_{\mathcal{A}}(\Phi(E))$ in $\mathsf{Ku}(X')$, where we have used the fact that $\tau_{\mathcal{A}} = S_{\mathsf{Ku}(X)}[-2]$ and $\tau'_{\mathcal{A}} = S_{\mathsf{Ku}(X')}[-2]$, and that Serre functors

[5]By abuse of notation, we denote the involution on $\mathsf{Ku}(X)$ as $\tau_{\mathcal{A}}$, which is also the involution on $\mathcal{A}_X$.

commute with equivalences of categories. The above holds for all objects $E \in \mathsf{Ku}(X)$, so Φ takes the 1-categorical action $\tau_{\mathcal{A}} \in \mathrm{Aut}(\mathsf{Ku}(X))$ to the 1-categorical action $\tau'_{\mathcal{A}} \in \mathrm{Aut}(\mathsf{Ku}(X'))$.

Next, we check that Φ respects 2-categorical actions. The 1-categorical actions from the previous paragraph lift to 2-categorical actions, because the functors $\tau_{\mathcal{A}} \colon \mathsf{Ku}(X) \to \mathsf{Ku}(X)$ and $\tau'_{\mathcal{A}} \colon \mathsf{Ku}(X') \to \mathsf{Ku}(X')$ are given by pulling back the geometric involutions $\tau \colon X \to X$ and $\tau' \colon X' \to X'$. Since pullbacks are functorial, the 1-categorical actions τ and τ' lift to 2-categorical actions.

Finally, these lifts are unique because $H^2(B\mathbf{Z}/2, \mathbf{C}^\times) = 0$ (see [BP23, Corollary 3.4] for the lifting criterion, and [BP23, Example 3.12] for the vanishing). Thus Φ sends the 2-categorical action $\tau_{\mathcal{A}}$ to the unique 2-categorical action $\tau'_{\mathcal{A}}$. So Φ respects 2-categorical actions, as required.

We thus get an induced equivalence

$$\Psi \colon \mathsf{Ku}(X)^{\mu_2} \simeq \mathsf{Ku}(X')^{\mu'_2}$$

where $\mu_2 = \langle \tau_{\mathcal{A}} \rangle$, $\mu'_2 = \langle \Phi \circ \tau_{\mathcal{A}} \circ \Phi^{-1} = \tau'_{\mathcal{A}} \rangle$ and $\mu_2 \cong \mu'_2$. Thus we have $\Psi \colon \mathrm{D^b}(\mathcal{B}) \simeq \mathrm{D^b}(\mathcal{B}')$. We know that $\mathcal{B}$ and $\mathcal{B}'$ are smooth projective surfaces with polarisations h and h', respectively, so Ψ is a Fourier–Mukai functor by Orlov's Representability Theorem [Orl97, Theorem 2.2]. Moreover, $(\mathcal{B}, h)$ and $(\mathcal{B}', h')$ are both Picard rank 1 smooth projective K3 surfaces of degree $h^2 = h'^2 = 10 = 2 \cdot 5$. Then by [Ogu02, Theorem 1.10] and [HLOY03, Corollary 1.7], there is an isomorphism $\phi \colon \mathcal{B} \cong \mathcal{B}'$. Since they both have Picard rank one, we obtain $\phi^*(h') = h$.

We claim that the polarised isomorphism $\phi \colon \mathcal{B} \cong \mathcal{B}'$ implies that $X \cong X'$. Indeed, ϕ sends the Mukai bundle $\mathcal{E}|_{\mathcal{B}}$ on $\mathcal{B}$ to the Mukai bundle $\mathcal{E}'|_{\mathcal{B}'}$ on $\mathcal{B}'$ because $\mathcal{E}|_{\mathcal{B}}$ is the unique stable vector bundle of its Chern character on $\mathcal{B}$. So ϕ preserves the embeddings of $\mathcal{B}, \mathcal{B}'$ in $\mathrm{Gr}(2, 5)$. But recall that Y_5 is a linear section of $\mathrm{Gr}(2, 5)$, so the embeddings $\mathcal{B} \subset Y_5$ and $\mathcal{B}' \subset Y_5$ are preserved. This proves the claim that $X \cong X'$, and hence the categorical Torelli theorem. □

To make a more general categorical Torelli statement for Gushel–Mukai threefolds, we can relax the assumptions on X by looking at the singularities of Bridgeland moduli spaces:

Theorem 6.7.2. *Let X and X' both be general Gushel–Mukai threefolds, and suppose we have an equivalence $\mathcal{A}_X \simeq \mathcal{A}_{X'}$. Then X and X' are both either ordinary or special. In particular, in the ordinary case they are birationally equivalent, and in the special case they are isomorphic.*

Proof. First we claim that if X and X' are general Gushel–Mukai threefolds such that $\Phi\colon \mathcal{A}_X \simeq \mathcal{A}_{X'}$, then both X and X' are ordinary or special simultaneously. Indeed, we may assume X' is ordinary and X is special. Then the equivalence would identify the moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ of stable objects of class $-x$ in $\mathcal{A}_X$ with either the moduli space $\mathcal{M}_{\sigma'}(\mathcal{A}_{X'}, -x)$ or $\mathcal{M}_{\sigma'}(\mathcal{A}_{X'}, y-2x)$. Then the surface $\mathcal{C}_m(X)$ for a special Gushel–Mukai threefold X would be identified with the minimal Fano surface $\mathcal{C}_m(X')$ or the moduli space $M_G^{X'}(2,1,5)$ for a general ordinary Gushel–Mukai threefold X'. But $\mathcal{C}_m(X)$ has a unique singular point and both $\mathcal{C}_m(X')$ and $M_G^{X'}(2,1,5)$ are smooth for X' general. This means that neither identification is possible, so the claim follows.

Now X and X' are both general ordinary or general special. Hence the birational categorical Torelli result follows from Theorem 6.6.3 if X and X' are both ordinary, and the categorical Torelli result if X and X' are both special follows from 6.7. $\square$

6.8 The Debarre–Iliev–Manivel conjecture

In [DIM12, pp. 3-4], the authors make the following conjecture regarding the general fiber of the period map:

Conjecture 6.8.1 ([DIM12, pp. 3-4])**.** *A general fiber $\mathcal{P}^{-1}(J(X))$ of the period map $\mathcal{P}\colon \mathcal{X}_6 \to \mathcal{A}_{10}$ at an ordinary Gushel–Mukai threefold X is the union of $\mathcal{C}_m(X)/\iota$ and a surface birationally equivalent to $M_G(2,1,5)/\iota'$, where ι, ι' are geometrically meaningful involutions.*

Remark 6.8.2. Note that by Corollary 6.6.4, the surface birationally equivalent to $M_G(2,1,5)/\iota'$ in [DIM12], parametrising conic transforms of a line transform of X, is actually isomorphic to $M_G(2,1,5)/\iota'$. Thus this conjecture predicts that a general fiber $\mathcal{P}^{-1}(J(X))$ is actually the disjoint union of $\mathcal{C}_m(X)/\iota$ and $M_G(2,1,5)/\iota'$.

We will prove a categorical analogue of this conjecture. Consider the *categorical period map*

$$\mathcal{P}_{\mathrm{cat}}\colon \mathcal{X}_6 \to \{\mathcal{A}_X\}/\simeq, \quad X \mapsto \mathcal{A}_X$$

where $\mathcal{X}_6$ is the moduli space of isomorphism classes of Gushel–Mukai threefolds and $\{\mathcal{A}_X\}/\simeq$ is the set of equivalence classes of Kuznetsov components of Gushel–Mukai threefolds. Note that a global description of a "moduli of Kuznetsov components" $\{\mathcal{A}_X\}/\simeq$ is not known, however local deformations are controlled by the second Hochschild cohomology $\mathrm{HH}^2(\mathcal{A}_X)$. The fiber of the categorical period map $\mathcal{P}_{\mathrm{cat}}$ over $\mathcal{A}_X$ for an ordinary Gushel–Mukai threefold is defined as the isomorphism classes of all ordinary Gushel–Mukai threefolds X' such that $\mathcal{A}_{X'} \simeq \mathcal{A}_X$.

Theorem 6.8.3 ([JLLZ21, Theorem 11.3]). *The general fiber $\mathcal{P}_{\mathrm{cat}}^{-1}(\mathcal{A}_X)$ of the categorical period map over the alternative Kuznetsov component of an ordinary Gushel–Mukai threefold X is the union of $\mathcal{C}_m(X)/\iota$ and $M_G^X(2,1,5)/\iota'$ where ι, ι' are geometrically meaningful involutions.*

Proof. The general fiber $\mathcal{P}_{\mathrm{cat}}^{-1}(\mathcal{A}_X)$ of the categorical period map consists of Gushel–Mukai threefolds X' such that there is an equivalence of Kuznetsov components $\mathcal{A}_{X'} \simeq \mathcal{A}_X$. Then by Theorem 6.7.2, X' is also a general ordinary Gushel–Mukai threefold. Thus by Theorem 6.6.3 and Theorem 6.2.11, we know that $\mathcal{A}_{X'} \simeq \mathcal{A}_X$ if and only if X' is a conic transform of X, or a conic transform of a line transform of X. Then the result follows from Proposition 6.2.10 and Corollary 6.6.4. □

Remark 6.8.4. The Kuznetsov components of prime Fano threefolds of index 1 and 2 are often regarded as categorical analogues of the intermediate Jacobians of these threefolds.

Proposition 6.8.5. *Let X and X' be smooth Picard rank 1 Fano threefolds of index 1 or 2 in the same deformation class. Suppose $\mathsf{Ku}(X) \simeq \mathsf{Ku}(X')$ is a Fourier–Mukai equivalence*[6]*. Then $J(X) \cong J(X')$ as principally polarised abelian varieties.*

Proof. By Perry's construction [Per22, Section 5] of the intermediate Jacobian of an admissible subcategory of $\mathrm{D}^{\mathrm{b}}(X)$, we have $J(\mathsf{Ku}(X)) \cong J(X)$. Thus the result follows. □

For the converse in the Gushel–Mukai case, we have the following conjecture.

Conjecture 6.8.6 ([JLLZ21, Conjecture 11.5]). *The intermediate Jacobian $J(X)$ of a Gushel–Mukai threefold X uniquely determines the Kuznetsov component $\mathsf{Ku}(X)$, i.e. if X and X' are Gushel–Mukai threefolds then $J(X) \cong J(X') \implies \mathsf{Ku}(X) \simeq \mathsf{Ku}(X')$.*

[6] We require the Fourier–Mukai assumption on the equivalence, since for the Hodge structure to be preserved in Perry's construction of the intermediate Jacobian of a category, the stable ∞-category structure of the category must be preserved, and this is equivalent to requiring the equivalence to be Fourier–Mukai by [Toë07, Theorem 8.9].

If we replace Gushel–Mukai threefolds by certain other Fano threefolds in Conjecture 6.8.6, then it becomes a theorem:

Theorem 6.8.7 ([JLLZ21, Theorem 11.6])**.** *Let X and X' be both be Fano threefolds in one of the following deformation classes:*

- Y_d, $\quad 2 \leq d \leq 5$
- X_{2g-2}, $\quad g = 5, 7, 8, 9, 10, 12.$

Then we have the following implication: $J(X) \cong J(X') \implies \mathsf{Ku}(X) \simeq \mathsf{Ku}(X')$.

Proof. If X is an index 2 prime Fano threefold Y_d where $2 \leq d \leq 5$, then the statement follows from the Torelli theorems for Y_d. If $X = X_8$, the statement follows from its Torelli theorem. If $X = X_{12}, X_{18}, X_{16}$, their intermediate Jacobians are Jacobians of curves: $J(X_{12}) \cong J(C_7)$, $J(X_{16}) \cong J(C_3)$, and $J(X_{18}) \cong J(C_2)$. But $\mathsf{Ku}(X_{12}) \simeq \mathrm{D^b}(C_7)$, $\mathsf{Ku}(X_{16}) \simeq \mathrm{D^b}(C_3)$ and $\mathsf{Ku}(X_{18}) \simeq \mathrm{D^b}(C_2)$. Thus the statement follows from the classical Torelli theorem for curves. If $X = X_{14}$, the statement follows from [Kuz09, Corollary 4.9] and the Torelli theorem for cubic threefolds [CG72, Tju70]. If $X = X_{22}$, the statement is trivial since $\mathsf{Ku}(X_{22}) \cong \mathsf{Ku}(Y_5)$ ([KPS18]) and Y_5 is rigid, so $\mathsf{Ku}(X) \simeq \mathsf{Ku}(X')$ is always true. □

In the case of general ordinary Gushel–Mukai threefolds X_6, we have the following equivalence of conjectures.

Proposition 6.8.8 ([JLLZ21, Proposition 11.7])**.** *The Debarre–Iliev–Manivel Conjecture 6.8.1 is equivalent to Conjecture 6.8.6.*

Proof. First we assume that Conjecture 6.8.6 holds. Then by Corollary 6.6.5 and Theorem 6.8.3, the Debarre–Iliev–Manivel Conjecture 6.8.1 holds.

On the other hand, we assume the Debarre–Iliev–Manivel Conjecture 6.8.1 holds. Then for any X and X' such that $J(X) \cong J(X')$, X is either a conic transform of X', or X is a conic transform of a line transform of X'. In both cases, we have $\mathsf{Ku}(X) \simeq \mathsf{Ku}(X')$ by Theorem 6.2.11. Thus Conjecture 6.8.6 holds. □

References

[APR22] Matteo Altavilla, Marin Petkovic, and Franco Rota. Moduli spaces on the Kuznetsov component of Fano threefolds of index 2. *Épijournal de Géométrie Algébrique*, 6, 2022.

[Ati57] Michael Francis Atiyah. Vector bundles over an elliptic curve. *Proceedings of the London Mathematical Society*, 3(1):414–452, 1957.

[BBF$^+$22] Arend Bayer, Sjoerd Beentjes, Soheyla Feyzbakhsh, Georg Hein, Diletta Martinelli, Fatemeh Rezaee, and Benjamin Schmidt. The desingularization of the theta divisor of a cubic threefold as a moduli space. *Geometry and Topology*, 2022.

[Bel23] Pieter Belmans. Fanography. *An online database: www.fanography.info*, 2023.

[BF14] Maria Chiara Brambilla and Daniele Faenzi. Vector bundles on Fano threefolds of genus 7 and Brill–Noether loci. *International Journal of Mathematics*, 25(03):1450023, 2014.

[BLMS23] Arend Bayer, Martí Lahoz, Emanuele Macrì, and Paolo Stellari. Stability conditions on Kuznetsov components. *Ann. Sci. Éc. Norm. Supér. (4)*, 56(2):517–570, 2023. With an appendix by Bayer, Lahoz, Macrì, Stellari and X. Zhao.

[BM01] Tom Bridgeland and Antony Maciocia. Complex surfaces with equivalent derived categories. *Mathematische zeitschrift*, 236(4):677–697, 2001.

[BMMS12] Marcello Bernardara, Emanuele Macrì, Sukhendu Mehrotra, and Paolo Stellari. A categorical invariant for cubic threefolds. *Advances in Mathematics*, 229(2):770–803, 2012.

[BMS16] Arend Bayer, Emanuele Macrì, and Paolo Stellari. The space of stability conditions on abelian threefolds, and on some Calabi-Yau threefolds. *Inventiones mathematicae*, 206:869–933, 2016.

[BMT13] Arend Bayer, Emanuele Macrì, and Yukinobu Toda. Bridgeland Stability conditions on threefolds I: Bogomolov-Gieseker type inequalities. *Journal of Algebraic Geometry*, 2013.

[BO95] Alexei Bondal and Dmitri Orlov. Semiorthogonal decomposition for algebraic varieties. *arXiv preprint alg-geom/9506012*, 1995.

[BO01] Alexei Bondal and Dmitri Orlov. Reconstruction of a variety from the derived category and groups of autoequivalences. *Compositio Mathematica*, 125(3):327–344, 2001.

[BP23] Arend Bayer and Alexander Perry. Kuznetsov's Fano threefold conjecture via K3 categories and enhanced group actions. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, (0), 2023.

[Bri07] Tom Bridgeland. Stability conditions on triangulated categories. *Annals of Mathematics*, pages 317–345, 2007.

[BT16] Marcello Bernardara and Gonçalo Tabuada. From semi-orthogonal decompositions to polarized intermediate Jacobians via Jacobians of noncommutative motives. *Moscow Mathematical Journal*, 16(2):205–235, 2016.

[Căl07] Andrei Căldăraru. Non-birational Calabi–Yau threefolds that are derived equivalent. *International Journal of Mathematics*, 18(05):491–504, 2007.

[CG72] C Herbert Clemens and Phillip A Griffiths. The intermediate Jacobian of the cubic threefold. *Annals of Mathematics*, 95(2):281–356, 1972.

[Deb90] Olivier Debarre. Sur le théoreme de Torelli pour les solides doubles quartiques. *Compositio Mathematica*, 73(2):161–187, 1990.

[Del97] Pierre Deligne. Action du groupe des tresses sur une catégorie. *Inventiones mathematicae*, 128(1):159–175, 1997.

[DIM12] Olivier Debarre, Atanas Iliev, and Laurent Manivel. On the period map for prime Fano threefolds of degree 10. *J. Algebraic Geom*, 21(1):21–59, 2012.

[DJR23] Hannah Dell, Augustinas Jacovskis, and Franco Rota. Cyclic covers: Hodge theory and categorical Torelli theorems. *arXiv preprint arXiv:2310.13651*, 2023.

[DK18] Olivier Debarre and Aleksander Kuznetsov. Gushel–Mukai varieties: Classification and birationalities. *Algebraic Geometry*, pages 15–76, 2018.

[Don83] Ron Donagi. Generic Torelli for projective hypersurfaces. *Compositio Mathematica*, 50(2-3):325–353, 1983.

[Ela14] Alexey Elagin. On equivariant triangulated categories. *arXiv preprint arXiv:1403.7027*, 2014.

[Fan29] Gino Fano. Sulle varietá algebriche a tre dimensioni aventi tutti i generi nulli. In *Atti del Congresso Internazionale dei Matematici: Bologna del 3 al 10 de settembre di 1928*, pages 115–122, 1929.

[Fan41] Cino Fano. Su alcune varietá algebriche a tre dimensioni razionali, e aventi curve-sezioni canoniche. *Commentarii Mathematici Helvetici*, 14(1):202–211, 1941.

[FLZ23] Soheyla Feyzbakhsh, Zhiyu Liu, and Shizhuo Zhang. New perspectives on categorical Torelli theorems for del Pezzo threefolds. *arXiv preprint arXiv:2304.01321*, 2023.

[FP23] Soheyla Feyzbakhsh and Laura Pertusi. Serre-invariant stability conditions and Ulrich bundles on cubic threefolds. *Épijournal de Géométrie Algébrique*, 7, 2023.

[Gab62] Pierre Gabriel. Des catégories abéliennes. *Bulletin de la Société Mathématique de France*, 90:323–448, 1962.

[Gus83a] NP Gushel. Fano varieties of genus 8. *Uspekhi Matematicheskikh Nauk*, 38(1):163–164, 1983.

[Gus83b] NP Gushel. On Fano varieties of genus 6. *Izvestiya: Mathematics*, 21(3):445–459, 1983.

[Gus92] NP Gushel. Fano 3-folds of genus 8. *Algebra i Analiz*, 4(1):120–134, 1992.

[HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge University Press, 2010.

[HLOY03] Shinobu Hosono, Bong H Lian, Keiji Oguiso, and Shing-Tung Yau. Fourier-Mukai partners of a K3 surface of Picard number one. *Contemporary Mathematics*, pages 43–55, 2003.

[HLT21] Katrina Honigs, Max Lieblich, and Sofia Tirabassi. Fourier–Mukai partners of Enriques and bielliptic surfaces in positive characteristic. *Mathematical Research Letters*, 28(1):65–91, 2021.

[HR19] Daniel Huybrechts and Jørgen Rennemo. Hochschild cohomology versus the Jacobian ring, and the Torelli theorem for cubic fourfolds. *Algebraic Geometry*, 6(1):76–99, 2019.

[HRS96] Dieter Happel, Idun Reiten, and Sverre O Smalø. *Tilting in abelian categories and quasitilted algebras*, volume 575. American Mathematical Soc., 1996.

[Huy06] Daniel Huybrechts. *Fourier–Mukai transforms in algebraic geometry*. Clarendon Press, 2006.

[Ili94] Atanas Iliev. The Fano surface of the Gushel threefold. *Compositio Mathematica*, 94(1):81–107, 1994.

[IM07] Atanas Iliev and Laurent Manivel. Pfaffian lines and vector bundles on Fano threefolds of genus 8. *Journal of Algebraic Geometry*, 16(3):499–530, 2007.

[Isk99] Vasilii A Iskovskikh. Fano varieties. *Algebraic geometry V*, page 1, 1999.

[JLLZ21] Augustinas Jacovskis, Xun Lin, Zhiyu Liu, and Shizhuo Zhang. Categorical Torelli theorems for Gushel-Mukai threefolds. *arXiv preprint arXiv:2108.02946*, 2021.

[JLZ22] Augustinas Jacovskis, Zhiyu Liu, and Shizhuo Zhang. Brill–Noether theory for Kuznetsov components and refined categorical Torelli theorems for index one Fano threefolds. *arXiv preprint arXiv:2207.01021*, 2022.

[KL15] Alexander Kuznetsov and Valery A Lunts. Categorical resolutions of irrational singularities. *International Mathematics Research Notices*, 2015(13):4536–4625, 2015.

[Kon95] Maxim Kontsevich. Homological algebra of mirror symmetry. In *Proceedings* *of the International Congress of Mathematicians: August 3–11, 1994 Zürich, Switzerland*, pages 120–139. Springer, 1995.

[Kos22] Naoki Koseki. On the Bogomolov-Gieseker inequality for hypersurfaces in the projective spaces. *Mathematical Research Letters*, 2022.

[KP17] Alexander Kuznetsov and Alexander Perry. Derived categories of cyclic covers and their branch divisors. *Selecta Mathematica*, 23(1):389–423, 2017.

[KP18] Alexander Kuznetsov and Alexander Perry. Derived categories of Gushel–Mukai varieties. *Compositio Mathematica*, 154(7):1362–1406, 2018.

[KP23] Alexander Kuznetsov and Alexander Perry. Categorical cones and quadratic homological projective duality. *Annales Scientifiques de l'École Normale Supérieure*, (1):159–175, 2023.

[KPS18] Alexander G Kuznetsov, Yuri G Prokhorov, and Constantin A Shramov. Hilbert schemes of lines and conics and automorphism groups of Fano threefolds. *Japanese Journal of Mathematics*, 13(1):109–185, 2018.

[Kuz04] Alexander Kuznetsov. Derived categories of cubic and V_{14} threefolds. *Proc. V.A.Steklov Inst. Math*, 246:183–207, 2004.

[Kuz06] Alexander Kuznetsov. Hyperplane sections and derived categories. *Izvestiya: Mathematics*, 70(3):447, 2006.

[Kuz07] Alexander Kuznetsov. Homological projective duality. *Publications mathématiques*, 105(1):157–220, 2007.

[Kuz08] Alexander Kuznetsov. Derived categories of quadric fibrations and intersections of quadrics. *Advances in Mathematics*, 218(5):1340–1369, 2008.

[Kuz09] Alexander Kuznetsov. Derived categories of Fano threefolds. *Proceedings of the Steklov Institute of Mathematics*, 264(1):110–122, 2009.

[Kuz10] Alexander Kuznetsov. Derived categories of cubic fourfolds. *Cohomological and Geometric Approaches to Rationality Problems: New Perspectives*, pages 219–243, 2010.

[Kuz19] Alexander Kuznetsov. Calabi–Yau and fractional Calabi–Yau categories. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2019(753):239–267, 2019.

[Li18] Chunyi Li. Stability conditions on Fano threefolds of Picard number 1. *Journal of the European Mathematical Society*, 21(3):709–726, 2018.

[Li23] Chunyi Li. Stronger Bogomolov–Gieseker inequality for Fano varieties of Picard number one. *In preparation*, 2023.

[LNSZ21] Chunyi Li, Howard Nuer, Paolo Stellari, and Xiaolei Zhao. A refined derived Torelli theorem for Enriques surfaces. *Mathematische Annalen*, 379:1475–1505, 2021.

[Log82] Dmitry Logachev. Fano threefolds of genus 6. *arXiv preprint arXiv:0407147*, 1982.

[LPS23] Martí Lahoz, Laura Pertusi, and Paolo Stellari. Categorical Torelli theorems for weighted hypersurfaces. *In preparation*, 2023.

[LSZ22] Chunyi Li, Paolo Stellari, and Xiaolei Zhao. A refined derived Torelli theorem for Enriques surfaces, II: the non-generic case. *Mathematische Zeitschrift*, 300(4):3527–3550, 2022.

[LZ22] Zhiyu Liu and Shizhuo Zhang. A note on Bridgeland moduli spaces and moduli spaces of sheaves on X_{14} and Y_3. *Mathematische Zeitschrift*, 302(2):803–837, 2022.

[LZ23] Xun Lin and Shizhuo Zhang. Serre algebra, matrix factorization and categorical Torelli theorem for hypersurfaces. *arXiv preprint arXiv:2310.09927*, 2023.

[Mac07] Emanuele Macrì. Stability conditions on curves. *Mathematical Research Letters*, 14(4):657–672, 2007.

[MS17] Emanuele Macrì and Benjamin Schmidt. Lectures on Bridgeland stability. *Moduli of Curves: CIMAT Guanajuato, Mexico 2016*, pages 139–211, 2017.

[Muk89] Shigeru Mukai. Biregular classification of Fano 3-folds and Fano manifolds of coindex 3. *Proc. Nat. Acad. Sci. U.S.A.*, 86(9):3000–3002, 1989.

[Muk92] Shigeru Mukai. Fano 3—folds. *Complex Projective Geometry: Selected Papers*, (179):255, 1992.

[Mum62] David Mumford. Projective invariants of projective structures and applications. *Int'l. Cong. Math. Stockholm*, pages 526–530, 1962.

[New68] Peter E Newstead. Stable bundles of rank 2 and odd degree over a curve of genus 2. *Topology*, 7(3):205–215, 1968.

[NR69] Mudumbai S Narasimhan and Sundararaman Ramanan. Moduli of vector bundles on a compact Riemann surface. *Annals of Mathematics*, pages 14–51, 1969.

[Ogu02] K Oguiso. K3 surfaces via almost-primes. *Mathematical Research Letters*, 9(1):47–64, 2002.

[Orl91] Dmitri Olegovich Orlov. Exceptional set of vector bundles on the variety V_5. *Vestnik Moskovskogo Universiteta. Seriya 1. Matematika. Mekhanika*, (5):69–71, 1991.

[Orl97] Dmitri O Orlov. Equivalences of derived categories and K3 surfaces. *Journal of Mathematical Sciences*, 84:1361–1381, 1997.

[Per22] Alexander Perry. The integral Hodge conjecture for two-dimensional Calabi–Yau categories. *Compositio Mathematica*, 158(2):287–333, 2022.

[Pir22] Dmitrii Pirozhkov. Categorical Torelli theorem for hypersurfaces. *arXiv preprint arXiv:2208.13604*, 2022.

[PR23] Laura Pertusi and Ethan Robinett. Stability conditions on Kuznetsov components of Gushel-Mukai threefolds and Serre functor. *Math. Nachr.*, 296(7):2975–3002, 2023.

[PS22] Laura Pertusi and Paolo Stellari. Categorical Torelli theorems: results and open problems. *Rendiconti del Circolo Matematico di Palermo Series 2*, pages 1–63, 2022.

[PY22] Laura Pertusi and Song Yang. Some remarks on Fano three-folds of index two and stability conditions. *International Mathematics Research Notices*, 2022(17):13396–13446, 2022.

[San14] Giangiacomo Sanna. Rational curves and instantons on the Fano threefold Y_5. *arXiv preprint arXiv:1411.7994*, 2014.

[Tak72] Fumio Takemoto. Stable vector bundles on algebraic surfaces. *Nagoya Mathematical Journal*, 47:29–48, 1972.

[Tho00] Richard P Thomas. Derived categories for the working mathematician. *arXiv preprint math/0001045*, 2000.

[Tju70] AN Tjurin. On the Fano surface of a nonsingular cubic in $\mathbf{P}^4$. *Mathematics of the USSR-Izvestiya*, 4(6):1207, 1970.

[Toë07] Bertrand Toën. The homotopy theory of dg-categories and derived Morita theory. *Inventiones mathematicae*, 167(3):615–667, 2007.

[Tor13] R Torelli. Sulle varietà di Jacobi, rendiconti ara d, Lincei. 1913.

[Voi88] Claire Voisin. Sur la Jacobienne intermédiaire du double solide d'indice deux. *Duke Math. J.*, 56(1):629–646, 1988.

[Zha21] Shizhuo Zhang. Bridgeland moduli spaces and Kuznetsov's Fano threefold conjecture. *arXiv preprint arXiv:2012.12193*, 2021.