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Essays on Human Capital, Sorting, and Wages

Stuart Alexander Breslin



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Abstract

This thesis is composed of three chapters that study vital linkages in the labour market; the accumulation of human capital on the job, the sorting between workers and firms and the wages that arise through these processes.

In *Chapter 1*, I develop a dynamic model of sorting between workers and firms in which it is possible to endogenously invest in the worker's human capital. The capability of a worker-firm pair to produce both tradeable output and further human capital may depend non-parametrically on both the worker's current human capital and firm type. Supermodularity of the technology with respect to the types does not suffice for strict positive assortative matching (PAM) in the competitive equilibrium; much stronger assumptions must be imposed. If high-productivity firms are better at training and production is concave in human capital, then PAM is not guaranteed even if production is supermodular in the types. In particular, with enough concavity, the importance of getting low-skilled workers paired with the best firms may outweigh the effects of supermodularity. With simple examples, it is shown that randomisation in the matching process may be an endogenous outcome, even in the absence of search or informational frictions. I prove that under weaker conditions, it is sometimes possible to determine whether the correlation between worker and firm type will be positive or negative, without any knowledge of the distribution of types. Furthermore, I prove that under some conditions, workers sort in a manner such that the highest skilled workers see their wages increase at the fastest rate, giving firms a highly active role in the dispersion of wages and inequality over the life cycle.

Chapter 2 builds off the ideas of the first chapter but uses a data-driven, quantitative approach. This is done by adding some more realistic features i.e. search frictions and firm-specific human capital, and taking this model to the data. I document employer-provided training for full-time employed workers in the UK using an "effective training" measure that weighs off different types of self-reported work-related training. This form of training tends to be higher in already highly educated workers and is provided in greater amounts at larger firms. Moreover, occupations and industries that tend to pay higher wages also tend to provide more training; this is consistent with the idea that training enhances the productivity of the worker and that workers with high earning ability may sort into high training environments. In conjunction with these findings, I develop a search model of the labour market that includes heterogeneity in both workers and firms. Workers vary in their level of human capital and firms vary in productivity. Worker-firm pairs can increase the worker's human capital at the cost of losing output. I show that this framework can replicate key facts from the data; namely, higher educated workers receive more training throughout their lifetime and earn more, and that the firms that pay higher wages also provide more training. Finally, the model features inefficiently low human capital investment due to the social returns not being fully internalised under random search; a policy of subsidising

low-skilled young workers covered by income taxation is shown to improve aggregate welfare and social mobility in the model.

In *Chapter 3*, which is a co-authored project with Andy Snell, Heiko Stüber, and Jonathan Thomas, we document distinctive empirical features of wage pass-through in Germany that are consistent with a Thomas-Worrall wage contracting framework in the presence of both idiosyncratic and nonstationary aggregate productivity components. These empirical features are hard to reconcile with the predictions of search models based on period-by-period Nash bargaining over match surplus and with the predictions of financial models where risk-neutral firms may costlessly shield risk-averse workers from idiosyncratic shocks (Guiso, Pistaferri et al. 2005).

Lay Summary

This thesis aims to understand better the relationship between training, the jobs workers do, and the wages that they earn. Labour markets have imperfections, and it is important for economists to understand these imperfections.

In *Chapter 1*, I want to understand the link between two phenomena. One is sorting i.e. which workers will match up with which firms. The other is the development of skills that workers gain through practice and training on the job. I want to know, under what conditions, the “best” workers will go to the “best” firms. Usually, when economists think about this problem, they consider how much output would be produced in the different matches at any given time. My approach goes beyond this by recognising the fact that, while in a job, training and skill development may occur. Variation between firms is important - some may be better at training their workers than others. My main contribution is to treat the level of training provided as something that can be chosen by the worker and firm, while simultaneously considering the problem of which worker goes to which firm when both workers and firms vary in their abilities. This analysis raises interesting dilemmas - for example, low-skilled workers may be hopeless in their performance at the most elite firms, but benefit greatly from the training that they are able to provide. Should such a job go to a low or high-skilled worker?

In *Chapter 2*, I analyse data from the UK to study the interplay between the matching of workers to firms and the training they receive and document some interesting facts. I find that, when comparing different workers, the best educated tend to get more training throughout their career relative to the least educated. The industries and occupations that tend to pay the most also train the most. Higher-educated workers not only start out with higher wages but experience a faster growth rate. To understand these issues, I take the model developed in the first chapter and modify it to make it more realistic. I am then able to show that this model is consistent with a number of facts shown in the data. I then use this model to simulate what would happen if income tax was used to heavily subsidise the training of young low-skilled workers. I find this is able to have a dramatic effect on the overall productive ability of the economy.

In *Chapter 3*, we use German data to study how the wages of workers move in response to different events. These events include booms and busts in the overall economy, as well as successes and failures at the level of the individual firm. What we find is that at the individual firm level, wages fall when the firm does badly, but they don't rise when the firm is successful. Wages do rise, however, when the overall economy booms. We find that this pattern suggests that wages try to stay as still as possible (a bit like a form of insurance) but only move once they reach unacceptable levels (i.e. either the worker would walk away or the firm would go broke). We develop a model of the economy that mimics these features and argue that this model of the labour market is potentially an improvement over other approaches.

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My friends and family have been vital in supporting me, from ensuring that my morale stays high, providing me with shelter, and making the earlier investments in my human capital that got me to this point. In particular, I'd like to thank my parents Sandra and Joseph, my sister Laura, and my grandfather Alexander, for always looking out for me.

Finally, I thank Leyan, for her patience, trust, love and companionship.

Declaration

Chapters 1 and 2 were written by me, and me alone. Chapter 3 is a co-authored work. I, and each of my co-authors, have had a substantial contribution to Chapter 3; my contribution was primarily in the numerical part of solving, simulating, and calibrating the model. None of the works in this thesis have been submitted for any other degree or professional qualification.

Stuart Breslin

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Assortative Matching and Human Capital Investment

Stuart Alexander Breslin *

2023

Abstract

I develop a dynamic model of sorting between workers and firms in which it is possible to endogenously invest in the worker's human capital. The capability of a worker-firm pair to produce both tradeable output and further human capital may depend non-parametrically on both the worker's current human capital and firm type. Supermodularity of the technology with respect to the types does not suffice for strict positive assortative matching (PAM) in the competitive equilibrium; much stronger assumptions must be imposed. If high-productivity firms are better at training and production is concave in human capital, then PAM is not guaranteed even if production is supermodular in the types. In particular, with enough concavity, the importance of getting low-skilled workers paired with the best firms may outweigh the effects of supermodularity. With simple examples, it is shown that randomisation in the matching process may be an endogenous outcome, even in the absence of search or informational frictions. I prove that under weaker conditions, it is sometimes possible to determine whether the correlation between worker and firm type will be positive or negative, without any knowledge of the distribution of types. Furthermore, I prove that under some conditions, workers sort in a manner such that the highest skilled workers see their wages increase at the fastest rate, giving firms a highly active role in the dispersion of wages and inequality over the life cycle.

1 Introduction

The workplace is not just an environment for a worker to produce tradeable commodities but also functions as an opportunity for the worker to gain new skills. While some skill development may be in the form of "learning by doing" which is a by-product of creating the commodity, the worker and/or firm may deliberately try to improve the skills of the worker through on-the-job training. This could be formal (like that of an

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apprenticeship or study for qualification) or informal (such as the worker being allocated to less productive but more insightful tasks or spending time reflecting and evaluating performance). Other than the case in which there is pure learning-by-doing, there will generally be a trade-off between producing the commodity and training the worker. One such way of thinking of this problem is in the model of Ben-Porath (1967), where the worker may spend some time working and the rest of the time training. The training causes the worker to accumulate human capital and increases the worker's capacity to create the commodity in the future, but comes at the cost of lower current net output.

Ben-Porath's model does not consider the way heterogeneous firms might matter. Firm heterogeneity is relevant to on-the-job training for two reasons. The first is that if firms vary in their ability to train the worker, then we may expect to endogenously have different levels of training across different firms even if the same worker type is employed at each. The second reason concerns the worker's aspirations. The return to human capital investment may depend on the worker's future employer; all else equal, we may expect to have more training if the worker anticipates a future employer for which human capital makes a larger marginal gain to production. This suggests that sorting will affect the equilibrium investments. However, the investments alter the distribution of human capital amongst the workers. Under some parametrisations of the model, it is possible that the distribution of types will affect the intensity, and possibly the sign, of sorting. This allows for the possibility of feedback between the sorting process and the investment process. Sorting also acts as a function to create inequality; workers who start with very similar human capital may receive different investments due to their initial match. In a competitive environment, this produces heterogeneous wage slopes; some workers earn a low wage in early life and a high wage in later life, while others have a moderate wage throughout.

This paper makes three primary contributions to the existing literature on the theory of assortative matching. The first is to characterise a set of sufficient conditions for positive and negative assortative matching in a dynamic setting, where there are endogenous investments made by matches that alter the type of one of the agents. This is done for two-period, arbitrary finite-period, and steady-state infinite horizon settings. Importantly, these are sufficient conditions on the technology alone i.e. they require nothing on the distribution of types (save being non-degenerate). A second, and somewhat more challenging contribution, is to show that under weaker conditions it is sometimes possible to sign the Kendall rank correlation (see Kendall (1938)) between the worker and firm types. The third is to show conditions for which the rate of wage increase is a monotonic function of the firm type. The first two serve as building blocks in the theory of dynamic sorting with endogenous investments, while the third serves to directly link this theory to outcomes over time. A fourth, novel, contribution is to show by example that the presence of firm heterogeneity could prevent a worker's human capital from converging where it would otherwise converge if the firms were homogenous. This suggests that the combination of firm heterogeneity and endogenous investment can cause non-convergent behaviour, even where there is no uncertainty.

The rest of the paper continues as follows. Section 2 discusses related literature. Section 3 sets up and analyses a two-period, quadratic, model with normally-distributed types as a social planner's problem that has many analytical properties and is useful for understanding the central concepts of the paper. Section 4 generalises the model to non-parametric functional forms, a general number of periods (including infinite-horizon steady states), and generic distributions of the types. While there is no analytical solution, sufficient conditions for different sorting patterns are provided. Section 5 discusses the competitive equilibrium, its efficiency properties and the characterisation of wages. Section 6 provides various examples of the model, and section 7 concludes.

2 Literature

The role of sorting in the context of transferable utility has been studied by Becker (1974), who identified supermodularity (submodularity) of the production function with respect to the two types as a determinant of positive (negative) assortative matching (from hereon PAM and NAM) under transferable-utility. Put briefly, if types on each side of the market are denoted by $a \in \mathbb{R}$, $b \in \mathbb{R}$ respectively, and $f(a, b)$ is a production function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, then:

$$\forall a_1 > a_{-1}, \quad b_1 > b_{-1} :$$

$$f(a_{-1}, b_{-1}) + f(a_1, b_1) > f(a_{-1}, b_1) + f(a_1, b_{-1}),$$

is sufficient to ensure that in a competitive equilibrium, the high a types will be assigned to the high b types. Attempts have been made to understand the robustness of this relationship when different assumptions and variations to the model are introduced. Shimer & Smith (2000) introduce search frictions in the style of Diamond (1982), Mortensen (1982), Pissarides (1990). In this environment, the outside option of the agents depends on their type which has implications for the sharing of surplus (in Becker's model, outside options can be thought of as zero). This means that in addition to the supermodularity of the production function the supermodularity of both:

$$\begin{aligned} & \frac{\partial}{\partial a} \log(f(a, b)), \\ \text{and } & \frac{\partial^2}{\partial a \partial b} \log(f(a, b)) \end{aligned}$$

are necessary. If directed search is instead used rather than random search - as in Eeckhout & Kircher (2010) - then the conditions for PAM weaken (an "n-th root" rather than a "log" condition is sufficient). This is extended further in Cai et al. (2021), who allow for multiple workers to apply for the same job; they find that the sufficient condition for PAM becomes stronger as the firm can screen more of its applicants. An overview of the relationship between sorting and search frictions is provided by Chade et al. (2017).

In another direction, Eeckhout & Weng (2022) consider how *learning* in the Bayesian sense may affect sorting. In their model, there is heterogeneity on both sides as in Becker's model, but the worker's type is unknown, but in the process of producing output, workers gradually learn their type, as in Jovanovic (1979). Their results are rather robust; even with this learning, supermodularity in the production function is still sufficient for PAM. The role of private information in sorting outcomes is considered in the recent draft by Shimer & Wu (2023). In their model, there is also a two-sided market which wishes to pair up but their types are hidden. A new type of agent, "platforms" can set the terms of access. While they are unable to screen agents, this can sometimes as a mechanism for which assortative matching can be achieved, with the equilibrium resembling a directed search solution. Probably closest to this paper is that by Anderson (2015). In his model, not only does the output of a match depend on the types of agents, but the evolution of one's type may depend on their own and their partner's. In particular, if $\Gamma(a'|a, b)$ is the probability that next period, my type is less than a' , given that I am currently type a and my partner is type b , then he demonstrates that one way to ensure that PAM is the equilibrium is:

$$\begin{aligned} f(a, b) & \text{ Supermodular in } a, b, \\ & \text{ Convex in both } a, b \text{ individually} \\ \Gamma(a'|a, b) & \text{ Submodular in } a, b, \forall a', \\ & \text{ and Concave in } a, b \text{ individually.} \end{aligned}$$

Submodularity of $\Gamma()$ can be interpreted as the types complimenting one another on climbing the ranks of the distribution. His model is very general in the sense that the evolution of types is stochastic, and the distribution of the future types can depend arbitrarily on the current match. It does not however allow for choice in the investment process, and the analysis is limited to steady states. If we think of our agents as workers and firms, then achieving a large increase in human capital over time may come at an expense; time and resources may need to be expended in training the worker. This is particularly relevant in life-cycle issues, where the incentive to invest may be stronger in the early career than in the late. Allowing for endogenous investment also allows for interesting counterfactuals such as a change to the cost of investment. A key insight made in Anderson's paper is that the curvature of $f()$ and $\Gamma()$ with respect to the types *individually* also matters in addition to static complementarities. The findings of my paper replicate this in the context of endogenous investment; the curvature of human capital in producing output can affect the equilibrium sorting pattern in the absence of any changes to complementarities in the types. Anderson also provides conditions for which within-cohort wage inequality rises with age; similarly, I derive sufficient conditions for which wages disperse over the life cycle. Unlike Anderson's paper, only one of the matches (the worker) may evolve in type; this is for the sake of simplicity.

There are various papers which consider sorting over more than two dimensions. Lindenlaub (2017) has a model in which workers have both a manual, and cognitive component to human capital, and firms also have a manual and cognitive type. The concept of sorting between workers and firms therefore must be generalised to involve

two dimensions; for there to be positive assortative matching, the workers with higher manual (cognitive) skills should tend to go to firms of the higher manual (cognitive) type. This is analysed in the context of structural change in which complementarities in the cognitive dimension strengthen over time relative to complementarities in the manual skills. Her paper includes an analysis of the sorting patterns under quadratic technology and Gaussian distributed types, which also serves as an extremely helpful case in my model. Lindenlaub & Postel-Vinay (2023) extend the notion of multidimensional sorting to an environment in which there are search frictions. Using simulated data, they show that using a one-dimensional model as an approximation, when the truth is two-dimensional, can lead to large errors in the conclusions (e.g. there may be strong sorting in truth, but a one-dimensional approximation misses this). Gola (2021) also models workers and firms as having multidimensional types, while including two sectors (manufacturing/services) in the style of a “Roy model” Roy (1951), and uses the model to assess the impact of changes in technology. While the outcome of sorting is not the focus of the paper, the within-sector and between-sector sorting that workers engage in means that the similarity to which the two sectors rank the workers is an important determinant of the wage distribution. In comparison to these models, which all include one-to-one matching but with each agent being multidimensional, the primary model of this paper analyses uni-variate agents who match into groups of three. This is because lifetime discounted production depends on the current worker, the current firm *and* the future firm, and the problem is not separable. Thus the model of this paper shares similarities to those that involve matching over two dimensions, but for the reason of having more than two agents, not multidimensional types.

There is a literature that considers why firms train their workers in the first place, and who bears the cost. Becker (1962) demonstrates that in competitive environments, the cost of *general* human capital will be borne by the workers, whereas *firm-specific* human capital will be borne by the firms. The intuition behind this result is that since general human capital can be taken by the worker to other firms, the returns of the investment are in effect collected by the worker. Under firm-specific human capital, the firm can capture the returns and hence is willing to cover the cost. It also means that workers of differing abilities may earn different amounts because some are investing more in general human capital than others. Variation in earnings conditional on current human capital is also a feature of my model. Acemoglu & Pischke (1998) take a friction discussed by Waldman (1984), namely that the current employer of a worker may have better information about that worker’s ability compared to other potential employers, and what this means for general human capital training. The informational advantage that the current firm has over other firms allows it to capture some of the gains from general training, hence providing a motive for the firm to cover some of the costs.

Another, related area of research considers investments before a matching problem and the inefficiencies that can arise. Hopkins (2012) takes a two-period “matching tournament” structure; in the first period the worker invests, and in the second period they match to a firm. When the worker’s type is unknown to the firm, the investment may perform a signalling role, leading to overinvestment relative to the social

optima, however, the extent of this inefficiency depends largely on the distribution of agents. Mailath et al. (2017) consider the possibility of inefficient investment in a variety of contexts including asymmetric information, the agent's "remuneration values" (the division of surplus in the absence of transfers) and whether one or both sides of the market make investments. The direction and sizes of the inefficiencies depend largely on these market features. In comparison, the models in this paper are all socially optimal and involve matching *before* investment as well as after.

Evidence for sorting in the labour market is not just theoretical - Bonhomme et al. (2019) establish econometric techniques for identifying sorting patterns in matched employer-employee data. They find wages to be log-additive in the worker and firm types, and that there are strong positive assortative matching patterns. This builds on a research area that has sought to measure the effects of workers and firms on wages, with Abowd et al. (1999) providing an important contribution to the econometric methods. The empirical work on matching extends to other areas such as the marriage process, with Chiappori et al. (2022) and Goñi (2022) being recent examples of work that supports (broadly) positive assortative matching by income and other characteristics.

A useful framework for thinking about human capital investment is given by Ben-Porath (1967). In his model, human capital evolves according to a law of motion that includes depreciation, and an investment which is created using the worker's time and other inputs. There is thus an opportunity cost to spending time training; that time could have been used for creating output instead, as well as an explicit cost of buying training resources. This paper uses a reduced-form version of Ben-Porath's model while generalising it to allow for heterogeneous firms, that may vary in their ability to train the worker. In section 4, I show that the generalised model in this paper can embed this technology structure.

3 Two Period, Quadratic Technology Model

3.1 Technology

There is an equal measure of atomistic workers and firms of size 1, and there are two periods of time. In the first period, each worker takes a draw of (scalar) human capital from the distribution $\mathcal{N}(\mu_h, \sigma_h^2)$ and each firm takes a draw of (scalar) productivity type from the distribution $\mathcal{N}(\mu_a, \sigma_a^2)$. All draws are independently distributed. In the first period, each firm may employ one worker from which some output and human capital may be produced, and the worker is paid a wage. Consider a match that consists of a worker with human capital h and firm type a . The *potential* output of such a match is described by function $y()$:

$$y(h, a) = h(1 + \frac{\rho}{2}h + \lambda a) + a$$

The parameter $\rho \in \mathbb{R}$ allows for non-constant marginal productivity of human capital. This is relevant when considering investments; if $\rho > 0$ then there will tend to be

higher benefits to investing in already high-skilled workers relative to their low-skilled counterparts, whereas if $\rho < 0$ then the opposite will tend to be true. Like Anderson (2015), this curvature will be of important consequence. Including a quadratic term in a would affect the equilibrium outputs, but would be of no consequence for the assignments. $\lambda \in \mathbb{R}$ captures complementarities between the worker and firm types. In the special case $\lambda = 0$, production is separable in the types, whereas $\lambda > 0$ introduces supermodularity and $\lambda < 0$ submodularity. The cost of investing in the worker (such that the next period they have human capital x) is described by the function $\kappa()$:

$$\kappa(h, a, x) = \frac{c}{2} \left(x - ((1 - \delta)h + \gamma a) \right)^2.$$

$c \in \mathbb{R}^2$ scales the size of the costs, while $0 \leq \delta \leq 1$ can be interpreted as a depreciation parameter. $\gamma \in \mathbb{R}$ captures the extent to which higher productive firms are better at training workers. With $\gamma = 0$, investment costs are independent of the firm type, whereas $\gamma > 0$ will mean more productive firms help out with human capital investment. $\gamma < 0$ may also be considered; more productive firms may be worse at training (e.g. if the opportunity cost of time devoted to training is higher). The *net output* of a match with worker type h , firm type a with investments leading to the worker finishing period 1 with x is described with $f()$:

$$f(h, a, x) = h(1 + \frac{\rho}{2}h + \lambda a) + a - \frac{c}{2} \left(x - ((1 - \delta)h + \gamma a) \right)^2.$$

In period 2, the distribution of human capital will have changed due to the investments in the worker. The workers and firms again form matches (switching to a different partner from the previous period is costless). A match with worker type h and firm type a will not engage in any further investment, and produce $y(h, a)$. All agents are treated as risk neutral (firms with regards to profits and workers with regards to wages) and discount at rate β . Risk neutrality simplifies the analysis, as it means the model can be analysed as a transferable utility problem; in the competitive equilibrium, wages can arbitrarily transfer utility between the firm and worker. The social optima is therefore the allocation that maximises aggregate output.

The combination of normally distributed types and a quadratic form for $f()$ allows for a highly analytical approach to the problem, not dissimilar to Lindenlaub (2017).

3.2 The Social Planner's Problem

A social planner interested in maximising aggregate discounted output faces various decisions. In period 1, how should workers be assigned to firms, and what should the investments be? In period 2, what should the assignments be? These decisions are not independent of one another; in particular, the choice of investment in the worker may depend on both the first-period firm they are employed by and the second-period firm due to the presence of non-zero λ and γ . The second-period distribution of human capital is affected by the endogenous investments which in turn affects which worker should go to which firm in period 2. A parsimonious way to solve the model is to treat it as a three-agent matching problem. A worker with initial human capital h must be

matched with a current firm a and a future firm \hat{a} . With the three known, the investment problem is trivial. This means there is an implicit payoff from having the triplet $\{h, a, \hat{a}\}$ that is described by function $V()$:

$$V(h, a, \hat{a}) = \max_x \left\{ f(h, a, x) + \beta y(x, \hat{a}) \right\}$$

i.e. given that a worker h currently works for firm a and goes to \hat{a} in the next period, V defines the maximised discounted output attainable. Since the technology is quadratic, the optimal investment (denoted $x^*(h, a, \hat{a})$) has the linear solution:

$$x^*(h, a, \hat{a}) = \frac{c((1-\delta)h + \gamma a) + \beta(1 + \lambda \hat{a})}{c - \beta\rho} \quad (1)$$

$c - \beta\rho > 0$ is assumed for the investment problem to be well-behaved. With the payoff function $V()$ defined, now consider the matching problem. The social planner cannot control the univariate distributions of initial human capital and firm types, but they can control the joint distribution across the matches. Let G represent this joint distribution. Then the social planner's matching problem may be written as:

$$G = \operatorname{argmax}_{G_0} \left\{ \mathbb{E}_{\{(h,a,\hat{a}) \sim G_0\}} \left(V(h, a, \hat{a}) \right) \right\}$$

$$\begin{aligned} \text{s.t. } h &\sim \mathcal{N}(\mu_h, \sigma_h^2), \\ a &\sim \mathcal{N}(\mu_a, \sigma_a^2), \\ \hat{a} &\sim \mathcal{N}(\mu_a, \sigma_a^2), \end{aligned}$$

Without restrictions on the types of joint distribution the social planner could choose from, this problem could be very complicated. However, the quadratic structure to $V()$ means that it may be written as a function of its means and covariance. To see this, write $V(h, a, \hat{a})$ as an exact second-order Taylor expansion around $V(\mu_h, \mu_a, \mu_a)$:

$$\begin{aligned} V(h, a, \hat{a}) &= V + (h - \mu_h)V_1 + (a - \mu_a)V_2 + (\hat{a} - \mu_a)V_3 \\ &\quad + \frac{1}{2} \left((h - \mu_h)^2 V_{11} + (a - \mu_a)^2 V_{22} + (\hat{a} - \mu_a)^2 V_{33} \right) \\ &\quad + (h - \mu_h)(a - \mu_a)V_{12} + (h - \mu_h)(\hat{a} - \mu_a)V_{13} + (a - \mu_a)(\hat{a} - \mu_a)V_{23} \end{aligned}$$

(Derivatives evaluated at $V = V(\mu_h, \mu_a, \mu_a)$).

These cross derivatives take on the following values:

$$\begin{aligned} V_{12} &= \lambda + (1-\delta)\gamma\rho \frac{c\beta}{c - \beta\rho}, \\ V_{13} &= (1-\delta)\lambda \frac{c\beta}{c - \beta\rho}, \\ V_{23} &= \lambda\gamma \frac{c\beta}{c - \beta\rho}. \end{aligned}$$

V_{12} , which can be interpreted as capturing complementarities between worker type and the first-period firm depends on several parameters. There is a one-to-one relationship with λ , which directly captures complementarities in production. However, a second term captures an indirect complementarity, which depends on the sign of $\gamma \times \rho$. In the absence of any firm heterogeneity, the workers who benefit more in their productivity from training depend on the sign of ρ e.g. if ρ is negative then the lowest workers tend to gain the most from training. The firms that are best at training are given by γ e.g. a positive γ means the highest firms are best at training. There is thus an additional benefit from matching the best “coaches” to the workers who benefit most from training. These terms could work against one another, so it is possible for example that production is supermodular ($\lambda > 0$) but the lowest skill workers benefit the most from training, while the highest firms are best at training $\rho\gamma < 0$.

V_{13} captures complementarities in the second period which is easier to interpret since there is no investment decision - its sign simply depends on λ . Since there is no investment in the second period, it is production complementarities that matter.

V_{23} is less straightforward, the sign depending on $\lambda \times \gamma$. Although the first and second-period firms do not meet, it is helpful to think about their interaction via the fact they both interact with the same worker. It captures the benefit that comes from the best “coaches” in the first period having their workers go to the firms that make the best use of that productivity. For example, if $\gamma > 0$, then the highest firms are the best at training, and if $\lambda > 0$ then in the second period the best firms will gain the most from human capital; there would therefore tend to be a benefit here from having the best firms training workers that again go to the best firms in the future. Note that all of the parameters in V_{12} also appear in each of V_{12} , V_{13} and V_{23} , so it is not straightforward to apply monotone comparative statics to the problem. An increase in λ would *seem* like it should increase r_{ha} , but consider that it also increases production complementarities in the second period. This increases the aggregate payoff from investing heavily in the high-skilled workers relative to the low-skilled workers which means both ρ and γ matter.

Taking expectations (i.e. aggregating over the economy) reveals a closed-form expression for the aggregate output, when the social planner chooses a multivariate normal distribution over h, a, \hat{a} :

$$E(V(h, a, \hat{a})) = V + \frac{1}{2} \left(\sigma_h^2 V_{11} + \sigma_a^2 V_{22} + \sigma_{\hat{a}}^2 V_{33} \right) + r_{ha} \sigma_h \sigma_a V_{12} + r_{h\hat{a}} \sigma_h \sigma_{\hat{a}} V_{13} + r_{a\hat{a}} \sigma_a^2 V_{23},$$

Where $r_{ij} \equiv \text{correlation}(i, j)$.

Due to the quadratic structure of $V()$, the social planner’s problem reduces to a choice of the correlations between h , a , and \hat{a} . If V were not quadratic, this simplification would only be locally approximate (under twice differentiability). Furthermore, the assumption of normality on h and a means that any choice of correlations may be achieved using a joint normal distribution, so long as the correlations matrix is positive

V_{12}	V_{13}	V_{23}	r_{ha}	$r_{h\hat{a}}$	$r_{a\hat{a}}$
+	+	+	+1	+1	+1
+	-	-	+1	-1	-1
-	+	-	-1	+1	-1
-	-	+	-1	-1	+1
-	-	-	?	?	?
-	+	+	?	?	?
+	-	+	?	?	?
+	+	-	?	?	?

Table 1: Corner Solutions and the Sign of Cross Derivatives of $V()$

semi-definite. This implies the constraints:

$$\begin{aligned}
r_{ha}^2 &\leq 1, \\
r_{h\hat{a}}^2 &\leq 1, \\
r_{a\hat{a}}^2 &\leq 1, \\
r_{ha}^2 + r_{h\hat{a}}^2 + r_{a\hat{a}}^2 &\leq 1 + 2r_{ha}r_{h\hat{a}}r_{a\hat{a}}
\end{aligned}$$

These constraints would be less trivial without the assumption of normally distributed types. If, for example, h had a uniform distribution and a an exponential distribution, a correlation of 1 between h and a would not be feasible. When the types *are* normally distributed, the social planner *could* choose a joint distribution that is not a multivariate normal (e.g. a Gumbel copula), but this would only further constrain the space of correlations the planner could choose from, so is endogenously ruled out. All correlation matrices must be positive definite, but multivariate Gaussians have the special feature of not imposing any further restrictions. The optimal correlations $[r_{ha}^*, r_{h\hat{a}}^*, r_{a\hat{a}}^*]$ may be summarised by the maximisation problem:

$$[r_{ha}^*, r_{h\hat{a}}^*, r_{a\hat{a}}^*] = \underset{[r_{ha}, r_{h\hat{a}}, r_{a\hat{a}}]}{\operatorname{argmax}} \left\{ r_{ha}\sigma_h\sigma_a V_{12} + r_{h\hat{a}}\sigma_h\sigma_a V_{13} + r_{a\hat{a}}\sigma_a^2 V_{23} \right\},$$

subject to:

$$\begin{aligned}
r_{ha}^2 &\leq 1, \\
r_{h\hat{a}}^2 &\leq 1, \\
r_{a\hat{a}}^2 &\leq 1, \\
r_{ha}^2 + r_{h\hat{a}}^2 + r_{a\hat{a}}^2 &\leq 1 + 2r_{ha}r_{h\hat{a}}r_{a\hat{a}}.
\end{aligned}$$

Corner solutions (where all three correlations are absolute size 1) exist for some combinations of signs of the cross derivatives of $V()$. This is shown in Table 1, with question marks denoting cases in which there is not an obvious corner solution.

If, for example, all three cross derivatives are negative, then the social planner would *like* for the correlation between the three agents to be -1 , but this is not possible; this

violates the final constraint ($r_{ha}^2 + r_{h\hat{a}}^2 + r_{a\hat{a}}^2 \leq 1 + 2r_{ha}r_{h\hat{a}}r_{a\hat{a}}$). A similar problem occurs if two of the cross derivatives are positive and the third is negative. How is sorting determined in these cases, when $V_{12}V_{13}V_{23} < 0$? First, introduce three coefficients:

$$\begin{aligned} C_{ha} &\equiv \lambda + (1 - \delta)\gamma\rho \frac{c\beta}{c - \beta\rho} \\ C_{h\hat{a}} &\equiv (1 - \delta)\lambda \frac{c\beta}{c - \beta\rho} \\ C_{a\hat{a}} &\equiv \lambda\gamma \frac{c\beta}{c - \beta\rho} \frac{\sigma_a}{\sigma_h}. \end{aligned}$$

Now introduce the “determinant”:

$$D = \frac{1}{C_{ha}^4} + \frac{1}{C_{h\hat{a}}^4} + \frac{1}{C_{a\hat{a}}^4} - 2\left(\frac{1}{C_{ha}^2 C_{h\hat{a}}^2} + \frac{1}{C_{ha}^2 C_{a\hat{a}}^2} + \frac{1}{C_{h\hat{a}}^2 C_{a\hat{a}}^2}\right)$$

Proposition 1 *Let $C_{ha}C_{h\hat{a}}C_{a\hat{a}} < 0$ and $D > 0$. Then the solution to the optimal correlation problem is a corner solution in which the sign of the weakest term is sacrificed.*

Mathematically, let

$$i = \underset{i \in \{(ha), (h\hat{a}), (a, \hat{a})\}}{\operatorname{argmin}} \{|C_i|\},$$

then:

$$\begin{aligned} r_i^* &= -\operatorname{sign}(C_i), \\ r_j^* &= \operatorname{sign}(C_j), \quad \forall j \neq i \end{aligned}$$

The $\operatorname{sign}()$ function takes on value 1 if the argument is positive, -1 if negative.

Proposition 2 *Let $C_{ha}C_{h\hat{a}}C_{a\hat{a}} < 0$ and $D \leq 0$. Then the solution to the optimal correlation problem is:*

$$r_m^* = \frac{1}{2}C_n C_o \left(\frac{1}{C_m^2} - \frac{1}{C_n^2} - \frac{1}{C_o^2} \right), \quad \forall \{m, n, o\} = \{(h, a), (h, \hat{a}), (a, \hat{a})\}.$$

Proofs of both propositions 1 and 2 can be found in appendix A.

Now that there is a solution to the correlations, it is worth thinking about what sorting looks like in each period. The second period by itself is a special case of Becker (1974); since there is no investment, the modularity $\gamma()$ fully determines sorting

Lemma 1 (Becker 1974):

$\lambda > 0 \implies$ PAM in the second period,
 $\lambda < 0 \implies$ NAM in the second period,

It is important to stress that $r_{h\hat{a}}$ *does not* explain second-period sorting. $r_{h\hat{a}}$ is the correlation between the workers' initial human capital and the second-period firm. "PAM in the second period" refers to the positive relationship between the worker's *second period* human capital and the second-period firm type. To define PAM or NAM in any other manner would be confusing. Of special interest is r_{ha} , which reveals the sorting in the first period. Notice however that the solution to the correlations can depend on $\frac{\sigma_a}{\sigma_h}$. This seems problematic for interpretation; two economies with the same underlying technology, but different type distributions may undergo different patterns of sorting. To come up with a more robust interpretation, I, therefore, aim to find conditions which are sufficient to ensure PAM/NAM *under any non-degenerate normal distribution of the types*.

The ratio $\frac{\sigma_a}{\sigma_h}$ can be scaled arbitrarily; this means the coefficient $C_{a\hat{a}}$ can be shifted in relative size to C_{ha} and $C_{h\hat{a}}$, but its sign cannot be changed. This means that in the cases for which $V_{12}V_{13}V_{23} \geq 0$, it is sufficient to look at the sign of V_{12} to determine sorting:

Proposition 3 (*Pure Sorting*)

$$\begin{aligned} \left(\lambda + (1 - \delta)\gamma\rho \frac{c\beta}{c - \beta\rho} > 0, \quad \text{and} \quad \gamma \geq 0 \right) &\implies \text{PAM in the first period,} \\ \left(\lambda + (1 - \delta)\gamma\rho \frac{c\beta}{c - \beta\rho} < 0, \quad \text{and} \quad \gamma \leq 0 \right) &\implies \text{NAM in the first period,} \end{aligned}$$

Proposition 3 works by setting C_{ha} to positive for PAM and negative for NAM, and then the sign of γ ensures that $C_{h\hat{a}} \times C_{a\hat{a}}$ matches the sign of C_{ha} , producing a clear-cut corner solution. Now let's suppose that $C_{ha}C_{h\hat{a}}C_{a\hat{a}} < 0$. In one extreme, as $\frac{\sigma_a}{\sigma_h}$ goes to zero, the term $C_{a\hat{a}}$ will become arbitrarily small, hence r_{ha} can match the sign of C_{ha} , and $r_{h\hat{a}}$ match the sign of $C_{h\hat{a}}$. In the other extreme, as $\frac{\sigma_a}{\sigma_h}$ goes to infinity, $C_{a\hat{a}}$ becomes infinitely large in comparison to C_{ha} and $C_{h\hat{a}}$. This creates a situation in which the determinant is non-negative:

$$\lim_{C_{a\hat{a}} \rightarrow \infty} (D) = \left(\frac{1}{C_{ha}^2} - \frac{1}{C_{h\hat{a}}^2} \right)^2$$

In the special case in which the absolute size of C_{ha} and $C_{h\hat{a}}$ are exactly equal, there are two solutions (either one of the correlation signs can be sacrificed). In the more general case, where they are unequal, the sign of the correlation of the weaker of C_{ha} and $C_{h\hat{a}}$ has its correlation sacrificed. If $|C_{ha}| \leq |C_{h\hat{a}}|$, then first period sorting could be the opposite sign of C_{ha} .

Proposition 4 (*Ambiguous Sorting*)

$$\begin{aligned} \text{If: } & \lambda + (1 - \delta)\gamma\rho \frac{c\beta}{c - \beta\rho} > 0, \quad \gamma < 0, \quad \text{and} \quad |\lambda + (1 - \delta)\gamma\rho \frac{c\beta}{c - \beta\rho}| \leq |(1 - \delta)\lambda \frac{c\beta}{c - \beta\rho}|, \\ \text{or: } & \lambda + (1 - \delta)\gamma\rho \frac{c\beta}{c - \beta\rho} < 0, \quad \gamma > 0, \quad \text{and} \quad |\lambda + (1 - \delta)\gamma\rho \frac{c\beta}{c - \beta\rho}| \leq |(1 - \delta)\lambda \frac{c\beta}{c - \beta\rho}|, \end{aligned}$$

then the correlation between worker and firm type in the first period could be as low as -1, or as high as 1, depending on the distribution of types i.e. no useful distribution-free statement about sorting is possible.

Now consider the cases in which $C_{ha}C_{h\hat{a}}C_{a\hat{a}} < 0$, but $|C_{ha}| > |C_{h\hat{a}}|$. In such cases, at the extremes of the distributions, where $\frac{\sigma_a}{\sigma_h}$ is zero or infinite, the correlation of h, a is 1 in absolute size and matches the sign of C_{ha} , but the correlation r_{ha}^* can be non-monotone in $C_{a\hat{a}}$. However, in these cases, it is possible to *sign* the correlation. Consider solving the interior solution for $r_{ha} = 0$:

$$\begin{aligned} & \frac{1}{2}C_{h\hat{a}}C_{a\hat{a}}\left(\frac{1}{C_{ha}^2} - \frac{1}{C_{h\hat{a}}^2} - \frac{1}{C_{a\hat{a}}^2}\right) = 0 \\ \Rightarrow & \frac{1}{C_{a\hat{a}}^2} = \frac{1}{C_{ha}^2} - \frac{1}{C_{h\hat{a}}^2}. \end{aligned}$$

Since $|C_{ha}| > |C_{h\hat{a}}|$, the right hand side is negative. This means that there is no real solution for $C_{a\hat{a}}$. There is therefore no distribution of the types for which $r_{ha}^* = 0$. As C_{ha} is varied from 0 to ∞ , at no point does the correlation cross zero, and since the sign of r_{ha}^* is the same at both extremes, it must be the same sign everywhere. These logical steps require r_{ha}^* to be continuous in $C_{a\hat{a}}$, which is shown in appendix B. This leads to proposition 5

Proposition 5 (*Weak Sorting*)

$$\begin{aligned} & \left(\lambda + (1 - \delta)\gamma\rho \frac{c\beta}{c - \beta\rho} > 0, \quad \gamma < 0, \quad \text{and} \quad |\lambda + (1 - \delta)\gamma\rho \frac{c\beta}{c - \beta\rho}| > |(1 - \delta)\lambda \frac{c\beta}{c - \beta\rho}| \right) \Rightarrow r_{ha}^* > 0, \\ & \left(\lambda + (1 - \delta)\gamma\rho \frac{c\beta}{c - \beta\rho} < 0, \quad \gamma > 0, \quad \text{and} \quad |\lambda + (1 - \delta)\gamma\rho \frac{c\beta}{c - \beta\rho}| > |(1 - \delta)\lambda \frac{c\beta}{c - \beta\rho}| \right) \Rightarrow r_{ha}^* < 0 \end{aligned}$$

Although the size of the correlation between worker and firm may depend on the distributions of the types, proposition 5 demonstrates that under certain conditions, the correlation can be signed.

The value of ρ plays a key role in the outcome of sorting. In the special case for which $\rho = 0$, two of the inequalities in the conditions for $r_{ha}^* > 0$ simplify:

for $\rho = 0$:

$$\lambda + (1 - \delta)\gamma\rho \frac{c\beta}{c - \beta\rho} > 0$$

$$\rightarrow \lambda > 0,$$

$$|\lambda + (1 - \delta)\gamma\rho \frac{c\beta}{c - \beta\rho}| > |(1 - \delta)\lambda \frac{c\beta}{c - \beta\rho}|$$

$$\rightarrow |\lambda| > |(1 - \delta)\beta\lambda|$$

The second of these conditions is always true for $\lambda \neq 0$ and $0 < \delta < 1$, $0 < \beta < 1$. Thus, in the absence of any curvature on the potential output function with respect to human capital, the sign of sorting in the first period can always be determined by λ alone.

3.3 Example

To understand the various mechanics involved in the outcome of sorting, it is helpful to work through an example. Let's fix some of the parameters of the model:

$$\lambda = 1$$

$$\delta = 1/4$$

$$\beta = 5/6$$

$$c = 1.$$

Since there are complementarities in production ($\lambda > 0$), the one-period model would favour PAM, but in the two-period model, the first period's sorting pattern will also depend on γ , ρ , and $\frac{\sigma_a}{\sigma_h}$. The regularity condition requires $\rho < \frac{c}{\beta}$ - in this example $\rho < \frac{6}{5}$. Given the results on sorting of this section, regardless of the distributions, the sorting can be characterised in some regions; this is shown in figure 1. Figures 2 and 3 show what the correlation would be, if the ratio $\frac{\sigma_a}{\sigma_h}$ were set to 1/2 and 2 respectively. The intuition behind the general pattern here is best understood by the relationship between γ and ρ . If $\rho > 0$, then it tends to be the high workers who benefit the most from training, whereas $\rho < 0$ suggests lower workers benefit more from training. The sign of γ determines whether the high firms are better at training. Thus, far enough into the northwest and southeast regions of the ρ, γ space, negative assortative matching tends to dominate, even in the presence of complementarities in production.

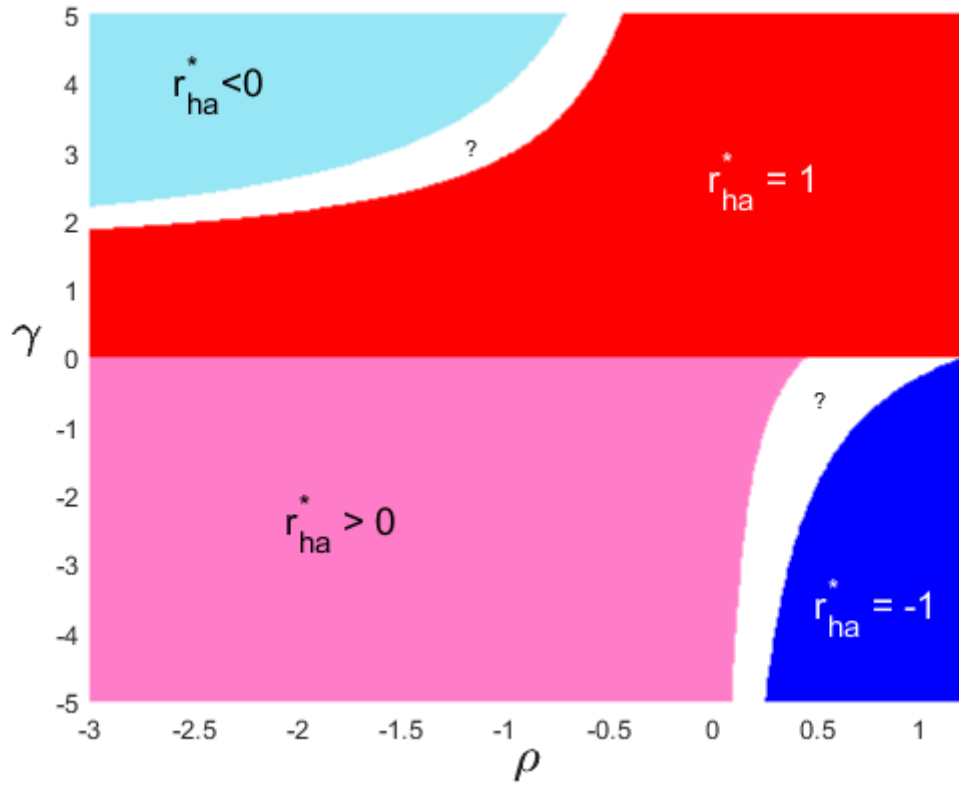


Figure 1: An example of different regions over ρ and γ that characterise sorting in the first period, independent of the distribution. The red region will feature PAM, the blue region NAM, the pink region positive correlation in sorting, and the turquoise negative correlation in sorting. The white regions are ambiguous.

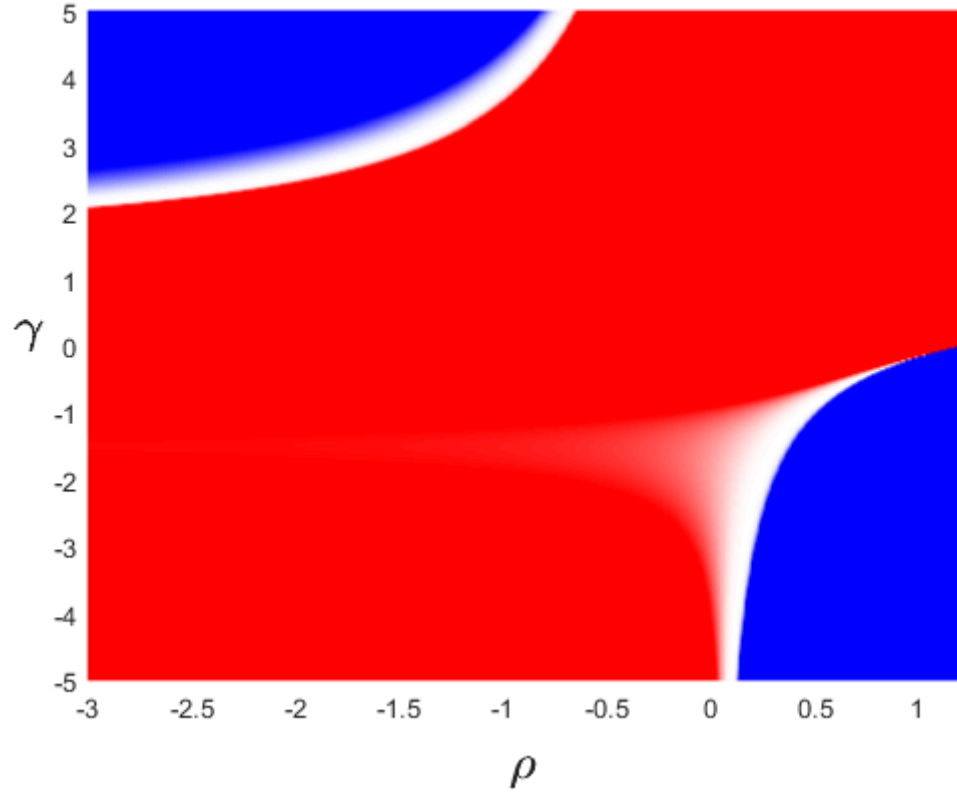


Figure 2: An example of first period sorting correlation, r_{ha}^* over values of ρ and γ . Here, $\frac{\sigma_a}{\sigma_h} = 1/2$. The red central region denotes PAM and the blue regions to the north-west and south-east denote NAM. White areas have weaker sorting, with the whiter the region, the weaker the correlation.

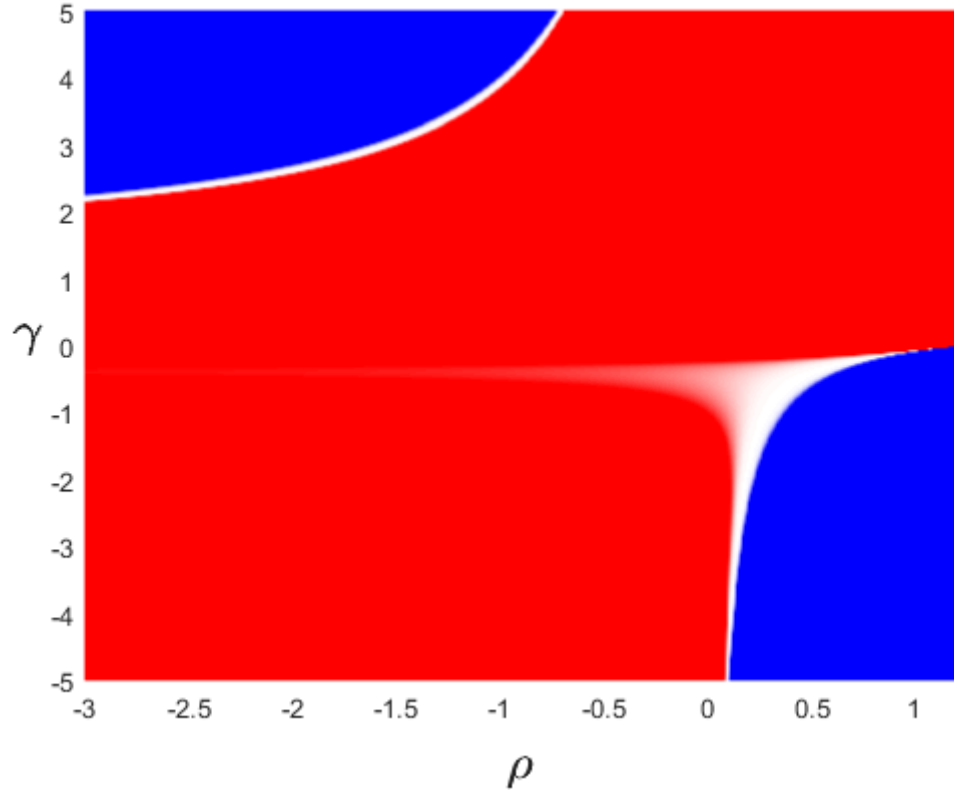


Figure 3: An example of first period sorting correlation, r_{ha}^* over values of ρ and γ . Here, $\frac{\sigma_a}{\sigma_h} = 2$. The red central region denotes PAM and the blue regions to the north-west and south-east denote NAM. White areas have weaker sorting, with the whiter the region, the weaker the correlation.

3.4 Comparison to Learning-By-Doing

Since the endogeneity of investment choices might seem to be a complication to the model, it is worth exploring what new this adds to the framework. To make the model as comparable as possible to the one with endogenous investment, suppose all matches are forced to invest such that the marginal cost of investment reaches some level, m^* . Then the level of investment for each match is:

$$x(h, a) = (1 - \delta)h + \gamma a + \frac{m^*}{c}.$$

Again, γ has an intuitive meaning; a higher γ means more human capital is gained by matching to the high-type firms. The implied output of a match in the first period is therefore:

$$h(1 + \frac{\rho}{2}h + \lambda a) + a - \frac{m^{*2}}{2c}.$$

Let $V()$ represent the payoff of a triple (h, a, \hat{a}) :

$$V^{\text{ex}}(h, a, \hat{a}) = h(1 + \frac{\rho}{2}h + \lambda a) + a - \frac{m^{*2}}{2c} + \beta \left(x(1 + \frac{\rho}{2}x + \lambda \hat{a}) + a - \frac{m^{*2}}{2c} \right),$$

where $x = (1 - \delta)h + \gamma a + \frac{m^*}{c}$

The cross derivatives of such a function are:

$$\begin{aligned} V_{12}^{\text{ex}}(h, a, \hat{a}) &= \lambda + \beta(1 - \delta)\gamma\rho, \\ V_{13}^{\text{ex}}(h, a, \hat{a}) &= \beta(1 - \delta)\lambda, \\ V_{23}^{\text{ex}}(h, a, \hat{a}) &= \beta\gamma\lambda, \end{aligned}$$

So far, the model is very similar, albeit with β where $\frac{c\beta}{c-\beta\rho}$ would appear. Importantly, $0 < \beta < 1$, whereas $0 < \frac{c\beta}{c-\beta\rho} < \infty$. The sign of $\frac{c\beta}{c-\beta\rho} - \beta$ matches the sign of ρ . Define C_{ha} etc. in the same manner as before:

$$\begin{aligned} C_{ha}^{\text{ex}} &= \lambda + (1 - \delta)\gamma\rho\beta \\ C_{h\hat{a}}^{\text{ex}} &= (1 - \delta)\lambda\beta \\ C_{a\hat{a}}^{\text{ex}} &= \lambda\gamma\beta \frac{\sigma_a}{\sigma_h} \end{aligned}$$

The solution to the optimal correlations only depends on these coefficients relative to one another:

$$\begin{aligned} \tilde{C}_{ha}^{\text{ex}} &= \frac{\lambda}{\beta} + (1 - \delta)\gamma\rho \\ \tilde{C}_{h\hat{a}}^{\text{ex}} &= (1 - \delta)\lambda \\ \tilde{C}_{a\hat{a}}^{\text{ex}} &= \lambda\gamma \frac{\sigma_a}{\sigma_h} \end{aligned}$$

Now the coefficients have been $C_{h\hat{a}}$ and $C_{a\hat{a}}$ have been re-scaled. If this same rescaling were done in the endogenous model, the coefficients would be:

$$\begin{aligned}\tilde{C}_{ha} &= \lambda \frac{c - \beta\rho}{c\beta} + (1 - \delta)\gamma\rho \\ \tilde{C}_{h\hat{a}} &= (1 - \delta)\lambda \\ \tilde{C}_{a\hat{a}} &= \lambda\gamma \frac{\sigma_a}{\sigma_h}.\end{aligned}$$

Now the endogenous and exogenous models are comparable, with \tilde{C}_{ha} being the only coefficient that changes. Given that ρ only enters $\tilde{C}_{ha}^{\text{ex}}$ and \tilde{C}_{ha} , it can be adjusted without affecting any of the other coefficients. When switching from the endogenous to exogenous model, there is therefore a switch of ρ that can be performed, which will leave all three coefficients the same. This is found by solving:

$$\frac{\lambda}{\beta} + (1 - \delta)\gamma\rho^{\text{ex}} = \lambda \frac{c - \beta\rho}{c\beta} + (1 - \delta)\gamma\rho,$$

and has the solution:

$$\rho^{\text{ex}} = \left(1 - \frac{\lambda}{(1 - \delta)c\gamma}\right)\rho \quad (2)$$

This means that one could consider two models, one with endogenous investment and the other with exogenous investment. All conclusions regarding sorting will be equivalent across those two models, provided the curvature parameter, ρ , is adjusted using equation 2. This effectively means that the endogenous model can embed the structure of the exogenous model while not introducing any new phenomena regarding the sorting analysis. The endogenous model can be made “more exogenous” by arbitrarily increasing the value of c , effectively fixing the level of investment each match can achieve. However, the models are not identical just by switching up the value of ρ . The two models have different implications for dynamics; the return to investing in human capital changes over time, which is something an exogenous investment model cannot capture. Suppose, for example, we observe identical matches in periods 1 and 2, both (h, a) . Under endogenous investment, we would expect the net output of the second-period match to be higher than that of the first period since the second-period match will not actively invest in human capital, but the first will. In the exogenous investment model, these are forced to be equal. The endogenous model therefore has the advantage of generalisation and realism, while not overly-complicating the implications of sorting.

4 Generalisation

The quadratic model is useful for demonstrating the basic ideas but its structure is rather restrictive. Firstly, it is natural to consider technology that is non-quadratic, and secondly, it is interesting to consider more than two periods. To address the first point, let technology be represented with:

$$f(h, a, x) = y(h, a) - \kappa(x - (1 - \delta)h, a). \quad (3)$$

$y(h, a)$ can be interpreted as a “potential output” function, and $\kappa(x - (1 - \delta)h, a)$ a “cost of investment” function. Some regularity conditions are introduced:

$$\begin{aligned} y(\cdot), \kappa(\cdot) & \text{ twice differentiable,} \\ y_1(\cdot) & > 0, \\ y_2(\cdot) - \kappa_2(\cdot) & > 0, \\ \kappa(0, \cdot) & = 0 \\ \kappa_1(\cdot) & > 0 \\ \kappa_{11}(\cdot) & > 0 \end{aligned}$$

The second issue will be dealt with by considering various possibilities; a two-period setting, a $T > 2$ finite setting, and an infinite horizon steady-state. In the two-period and steady-state cases, a representation of the problem by a three-agent matching problem in which two of the types are identical in distribution is possible, which keeps the problem tractable. Since in principle the pattern of sorting could change from period to period, and workers could switch ranks, it is important to have clear definitions of sorting.

Definition 1 (*Strong PAM and NAM*)

A strong PAM (NAM) allocation is one in which there is perfect positive (negative) assortative matching in every period. PAM (NAM) in period t is defined using the period t human capital ranks.

Definition 2 (*Weak PAM and NAM*)

A weak PAM (NAM) allocation is one in which there is a non-negative (non-positive) Kendall rank correlation between the worker and their assigned firm in every period. The worker ranks in period t are based on the period t human capital ranks.

Definition 2 makes use of the rank correlation measure from Kendall (1938). This is distinct from the classical correlation coefficient used in the quadratic model. While in the Gaussian distributions, the signs of these two correlation measures will always be equal, the same is not true more generally. A relationship which has a positive Kendall rank correlation coefficient could have a negative classical correlation coefficient. However, the former is robust to arbitrary monotone transformations of the types, while the latter is not, which makes the rank-based correlation a much more powerful analytical tool.

The technology in equation 3 is capable of embedding the model of Ben-Porath (1967) that is generalised to include heterogeneous firms while abstracting away from time and training resources. To see how this works, suppose that the *potential* output of a match is represented with $y^P(h, a)$, and the match spends time s training and uses quantity D of “training inputs” at price p . The realised output is:

$$y^R(h, a) = (1 - s)y^P(h, a) - pD.$$

Human capital production comes from the effective time use: $sy^P(h, a)$, the training inputs D , and the firm type:

$$x = (1 - \delta)h + G\left((sy^P(h, a))^\phi D^{1-\phi}, a\right).$$

This maintains the Cobb-Douglas structure between effective time and training inputs in Ben-Porath's model while allowing the heterogeneous firms to play a general role. The constraint on training intensity $0 \leq s \leq 1$ is ignored for simplicity. As long as G is invertible in its first argument (which is reasonable), there is an implied amount of the composite input $(sy^P(h, a))^\phi D^{1-\phi}$ that is required to produce the desired investment, conditional on the firm i.e.

$$(sy^P(h, a))^\phi D^{1-\phi} = \tilde{G}^{-1}\left(x - (1 - \delta)h, a\right),$$

where $\tilde{G}^{-1}(G(n, a), a) = n$. Now consider minimising the cost of investment subject to a desired level. The Cobb-Douglas structure means that the share of output lost from effective time versus that spent on the training inputs must follow:

$$\frac{sy^P(h, a)}{pD} = \frac{\phi}{1 - \phi},$$

and then there will be some implied solution that can be written in the form:

$$\begin{aligned} sy^P(h, a) &= \phi v^*(x - (1 - \delta)h, a), \\ pD &= (1 - \phi) v^*(x - (1 - \delta)h, a), \end{aligned}$$

$$\text{where } v^*(x - (1 - \delta)h, a) = \frac{1}{\phi^\phi (1 - \phi)^{1-\phi}} \tilde{G}^{-1}\left(x - (1 - \delta)h, a\right)$$

implying that if x is the desired next period human capital, then the output that is realised at the optimum can be represented with:

$$y^{*R}(h, a; x) = y^P(h, a) - v^*(x - (1 - \delta)h, a),$$

which is the same underlying structure as used for $f(h, a, x)$. The technology used can therefore capture the same underlying idea as a Ben-Porath model while generalising to have heterogeneous firms and leaving the training intensity and input choices as implicit in the background.

4.1 Two Periods

Use $V(h, a, \hat{a})$ similarly to the quadratic model i.e.

$$V(h, a, \hat{a}) = \max_x \left\{ y(h, a) - \kappa(x - (1 - \delta)h, a) + \beta y(x, \hat{a}) \right\}.$$

The first order condition for x is:

$$\kappa_1(x - (1 - \delta)h, a) = \beta y_1(x, \hat{a}),$$

and this implies marginal relationships between the investment and the types:

$$\begin{aligned}\frac{\partial x}{\partial h} &= \frac{(1-\delta)\kappa_{11}(x - (1-\delta)h, a)}{\kappa_{11}(x - (1-\delta)h, a) - \beta y_{11}(x, \hat{a})}, \\ \frac{\partial x}{\partial a} &= \frac{-\kappa_{12}(x - (1-\delta)h, a)}{\kappa_{11}(x - (1-\delta)h, a) - \beta y_{11}(x, \hat{a})}, \\ \frac{\partial x}{\partial \hat{a}} &= \frac{\beta y_{12}(x, \hat{a})}{\kappa_{11}(x - (1-\delta)h, a) - \beta y_{11}(x, \hat{a})}.\end{aligned}$$

This first order characterisation implicitly assumes that $y()$ is not too convex relative to the concavity of $\kappa()$, the formal assumption is:

$$\inf_{h,a,x,\hat{a}} \left(\kappa_{11}(x - (1-\delta)h, a) - \beta y_{11}(x, \hat{a}) \right) > 0.$$

The cross derivatives of V (using the envelope condition once and condensing the notation) are:

$$\begin{aligned}V_{12} &= y_{12}(h, a) - (1-\delta)\beta \frac{\kappa_{12}(x - (1-\delta)h, a)}{\kappa_{11}(x - (1-\delta)h, a) - \beta y_{11}(x, \hat{a})}, \\ V_{13} &= (1-\delta)\beta \frac{\kappa_{11}(x - (1-\delta)h, a)}{\kappa_{11}(x - (1-\delta)h, a) - \beta y_{11}(x, \hat{a})}, \\ V_{23} &= \beta \frac{-\kappa_{12}(x - (1-\delta)h, a)}{\kappa_{11}(x - (1-\delta)h, a) - \beta y_{11}(x, \hat{a})}.\end{aligned}$$

When do both periods exhibit perfect sorting? For strong PAM, we need $y_{12}() > 0$ to get PAM in the last period, and then $V_{12} > 0$, $V_{13}V_{23} \geq 0$ are sufficient for PAM in $T-1$. And for strong NAM, $y_{12} < 0$, $V_{12} < 0$ and $V_{13}V_{23} \leq 0$ are sufficient. These can in turn be determined in some cases depending on the signs of y_{12} , κ_{12} and y_{11} in combination. This allows for a straightforward characterisation of the sorting patterns under some combinations:

Proposition 6 (*Two Period Strong Sorting: Sufficient Conditions*)

$$\begin{aligned}y_{12} > 0, \quad \kappa_{12} \leq 0, \quad y_{11} \geq 0 &\implies \text{Strong PAM,} \\ y_{12} < 0, \quad \kappa_{12} \geq 0, \quad y_{11} \geq 0 &\implies \text{Strong NAM,}\end{aligned}$$

$$\begin{aligned}\text{or } y_{12} > 0, \quad \kappa_{12} \leq 0, \quad |y_{12}| \geq |\kappa_{12}| &\implies \text{Strong PAM,} \\ y_{12} < 0, \quad \kappa_{12} \geq 0, \quad |y_{12}| \geq |\kappa_{12}| &\implies \text{Strong NAM,}\end{aligned}$$

Some shortcuts in notation are used here; $y_{12}() > 0$ is used to mean $\inf_{h,a} y(h, a) > 0$, and $\kappa_{12} \leq 0$ used for $\sup_{h,a,x} (\kappa(x - (1-\delta)h, a)) \leq 0$ etc.

Proposition 6 does not contain the weakest conditions that could be constructed but are straightforward to understand. y_{12} has the same interpretation as λ ; it captures complementarities in production, whereas κ_{12} has the same interpretation as γ , albeit

with the sign flipped. A positive value of κ_{12} means that higher firms tend to inflate the marginal cost of the investment; so κ_{12} can be thought of as negatively related to investment complementarities. In the quadratic, Gaussian model, it was found that the correlation could be bounded to be positive in cases where $V_{12} > 0$, $|V_{12}| > |V_{13}|$. Similarly, if the strong conditions fail, it may sometimes be possible to bound the rank correlation:

Proposition 7 (*Two Period Weak Sorting: Sufficient Conditions*)

$$\begin{aligned} y_{12} > 0, \quad \kappa_{12} > 0, \quad y_{11} \leq 0, \quad \left| \frac{\kappa_{12}}{\kappa_{11}} \right| \geq \left| \frac{y_{12}}{y_{11}} \right| &\implies \text{Weak PAM,} \\ y_{12} < 0, \quad \kappa_{12} < 0, \quad y_{11} \geq 0, \quad \left| \frac{\kappa_{12}}{\kappa_{11}} \right| \geq \left| \frac{y_{12}}{y_{11}} \right| &\implies \text{Weak NAM,} \end{aligned}$$

As an example; some intuition can be built for the weak PAM case. There are complementarities in production ($y_{12} > 0$) but there are negative complementarities in investment ($\kappa_{12} > 0$). However, since the output is weakly concave in human capital ($y_{11} \leq 0$), the low-skilled workers will tend to benefit from more investment (not accounting for the second-period sorting). The final condition, $\left| \frac{\kappa_{12}}{\kappa_{11}} \right| \geq \left| \frac{y_{12}}{y_{11}} \right|$ ensures that $|V_{12}| > |V_{13}|$, which in simpler terms means that ensuring a currently good worker going to a good firm now is more important than a currently good worker going to a good firm in the future. Proofs for both propositions 6 and 7 are given in appendix C.

4.2 T>2 periods

It can be shown that under similar conditions to the 2 period case, strong PAM and strong NAM allocations remain socially optimal in a longer period setting.

Proposition 8 (*T period strong sorting: sufficient conditions*)

$$\begin{aligned} y_{12} > 0, \quad \kappa_{12} \leq 0, \quad y_{11} \geq 0 &\implies \text{Strong PAM} \\ y_{12} < 0, \quad \kappa_{12} \geq 0, \quad y_{11} \geq 0 &\implies \text{Strong NAM} \end{aligned}$$

The trick to the proof of proposition 8 is to use a recursive approach, that accounts for the fact that the combination of $y_{11} > 0$ and that there is sorting means that the economy tends to benefit from spread out human capital distributions. The problem can be recursively using the *distribution* of human capital and the period as the state space. Let $Y^t(H)$ denote the aggregate discounted output that can be generated from

a human capital distribution (CDF) H at time t , with the recursive formulation:

$$Y^t(H) = \max_{\mathcal{P}(h,a,x),X} \left\{ \int \left((y(h,a) - \kappa(x - (1-\delta)h, a)) \mathcal{P}(h,a,x) \right) dh da dx + \beta Y^{t+1}(X) \right\}$$

subject to:

$$\begin{aligned} \int 1(h \leq h') \mathcal{P}(h,a,x) dh da dx &= H(h'), \\ \int 1(a \leq a') \mathcal{P}(h,a,x) dh da dx &= F_A(a'), \\ \int 1(x \leq x') \mathcal{P}(h,a,x) dh da dx &= X(x'). \end{aligned}$$

$$Y^T(H) = \max_{\mathcal{P}(h,a)} \left\{ \int \left(y(h,a) \mathcal{P}(h,a) \right) dh da \right\}$$

subject to:

$$\begin{aligned} \int 1(h \leq h') \mathcal{P}(h,a,x) dh da dx &= H(h'), \\ \int 1(a \leq a') \mathcal{P}(h,a,x) dh da dx &= F_A(a'). \end{aligned}$$

(4)

Equation 4 effectively says that in period t , the social planner can choose any joint distribution across h, a, x , but takes the individual distributions of current human capital and firm type as given. The future human capital distribution, X , is a choice variable, which enters the continuation value - $Y^{t+1}(X)$. Since period T is the final period, this does not feature here. The proof for the strong PAM case, which is included in appendix D relies on the following logical steps:

- Period T features PAM
- $Y^T(\hat{H}) > Y^T(H)$ if \hat{H} first-order stochastically dominates H , or \hat{H} is a mean preserving spread of H
- This implies PAM in period $T - 1$
- This implies $Y^{T-1}(\hat{H}) > Y^{T-1}(H)$ if \hat{H} first-order stochastically dominates H , or \hat{H} is a mean preserving spread of H
- ...

Also noteworthy is the fact that under the T period setting with strong wages, it is possible to make claims about how the wage distribution should evolve under a competitive equilibrium; this is discussed further in section 6.1.

4.3 Infinite Horizon Steady State

Now suppose that there are an infinite number of periods and that there exists some steady state human capital distribution. Taking such a steady state for granted, under what conditions would it exhibit particular sorting patterns? This problem can be represented as a three-agent matching problem, in which the current worker type is matched to the current firm type and *future* worker type. The payoff from such a triple is represented with the function $M()$:

$$M(h, a, x) = y(h, a) - \kappa(x - (1 - \delta)h, a).$$

The cross derivatives of M are:

$$M_{12}(h, a, x) = y_{12} + (1 - \delta)\kappa_{12},$$

$$M_{13}(h, a, x) = (1 - \delta)\kappa_{11},$$

$$M_{23}(h, a, x) = -\kappa_{12}.$$

If all three are positive, or exactly one of these is positive, the sorting pattern is trivial since $M_{12}M_{13}M_{23} \geq 0$, and will depend on the sign of $y_{12} + (1 - \delta)\kappa_{12}$.

Proposition 9 (*Infinite Horizon Steady State: Sufficient Conditions for Strong Sorting*)

$$y_{12} > 0, \quad \kappa_{12} \leq 0, \quad |y_{12}| > |\kappa_{12}| \implies \text{Strong PAM},$$

$$y_{12} < 0, \quad \kappa_{12} \geq 0, \quad |y_{12}| > |\kappa_{12}| \implies \text{Strong NAM},$$

This is very similar to the proof is given in appendix C, but for $M()$ rather than $V()$. Proposition 10 gives analogous conditions for weak sorting:

Proposition 10 (*Infinite Horizon Steady State: Sufficient Conditions for Weak Sorting*):

$$y_{12} > 0, \quad \kappa_{12} > 0, \quad |y_{12}| > 2|\kappa_{12}| \implies \text{Weak PAM},$$

$$y_{12} < 0, \quad \kappa_{12} < 0, \quad |y_{12}| > 2|\kappa_{12}| \implies \text{Weak NAM},$$

The proof of the weak case is similar to the weak case in appendix C i.e. it uses the fact that there is a three-agent matching problem in which two of the “types” are identically distributed but is provided in E for further reading. Furthermore, this section only provides general conditions for which different patterns of sorting will occur *conditional on the existence of such a steady state*. While the steady states for weak sorting are complex, a proof of concept is given in section 6.2, which shows a weak PAM steady state along with the wages that would exist under a competitive equilibrium.

5 Competitive Equilibrium

So far there has only been attention to the social optimum, but not a market equilibrium. This section extends the allocations and analysis given by the social planner's perspective, to an economy with a competitive equilibrium. The competitive equilibrium is efficient (as shown in section 5.1), so attention is not given to the allocation, but rather the wages that would support an efficient allocation. In a two-period setting, the competitive equilibrium is defined as an allocation $\mathcal{P}(h, a, \hat{a}, x)$ and wage schedules $w^1(h, x)$, $w^2(x)$, such that:

- Firms choose their first-period worker, the investment provided, and the second-period workers to maximise profit, taking wage schedules as given
- Workers choose their investments in the first period to maximise discounted wages, taking wage schedules as given
- All markets clear

Markets in this setting are defined over the period, the human capital type, and (for the first period) the investments. This inclusion of investment is important; even if two workers have the same human capital, if they receive different investments there would be no reason for these jobs to pay the same wage. $w^1(h, x)$, therefore, describes the wage paid to a worker in period 1 who initially has human capital h and receives investments such that their second period human capital is x . In the second period, however, there are no such investments; $w^2(x)$ describes the wage paid to a worker of type x . Nowhere should the firm type appear in the wage schedule; the worker does not intrinsically care about the firm type, only the investments received, hence the law of one price arises. The allocation $\mathcal{P}(h, a, \hat{a}, x)$ describes the density of workers who match with a in period 1, enter period two with human capital x , and match with firm \hat{a} in period 2. This keeps the allocation comparable to the three agent matching in the social planner's problem, but it is important to include investment in the definition of the allocation. Optimising behaviour for a firm of type a for any given wage schedule \mathbf{w} implies indirect profit function π :

$$\begin{aligned}\pi^1(a; \mathbf{w}) &= \max_{h, x} \left\{ f(h, a, x) - w^1(h, x) \right\}, \\ \pi^2(\hat{a}; \mathbf{w}) &= \max_x \left\{ f(x, \hat{a}) - w^2(x) \right\},\end{aligned}$$

and optimising behaviour for a worker of type h implies indirect utility function u :

$$u(h; \mathbf{w}) = \max_x \left\{ w^1(h, x) + \beta w^2(x) \right\}.$$

In general, the solution to these problems may be non-unique i.e. the same types may supply or demand over various markets in the same period. Let $d^1(a \rightarrow (h, x))$ denote the demand from firms of type a for market (h, x) and $s^1(h \rightarrow (h, x))$ the equivalent supply function. Similarly, $d^2(a \rightarrow x)$ and $s^2(x \rightarrow x)$ describe the second-period counterparts. These are non-negative functions. The profit and utility maximising behaviour means that markets which are dominated by an alternative choice available to

that agent will not receive any demand (from the firm's perspective) or supply (from the worker's perspective):

$$\begin{aligned} f(h, a, x) - w^1(h, x) < \pi^1(a; \mathbf{w}) &\implies d^1(a \rightarrow (h, x)) = 0, \\ y(x, \hat{a}) - w^2(x) < \pi^2(\hat{a}; \mathbf{w}) &\implies d^2(\hat{a} \rightarrow x) = 0, \\ w^1(h, x) + \beta w^2(x) < u(h; \mathbf{w}) &\implies s^1(h \rightarrow (h, x)) = 0 \end{aligned}$$

Then total demand and supply each type exerts must match their density (which is taken as exogenous for the first period) i.e.

$$\begin{aligned} \int_{h,x} d^1(a \rightarrow (h, x)) dh dx &= \mathcal{F}_A(a) \\ \int_x s^1(h \rightarrow (h, x)) dx &= \mathcal{F}_H(h), \\ \int_x d^2(\hat{a} \rightarrow x) dx &= \mathcal{F}_A(\hat{a}) \\ s^2(x) &= \int_h s^1(h \rightarrow (h, x)) dx, \end{aligned}$$

Notice that the human capital supply in each market is endogenous in the second period. Market clearing requires the demand and supply cast onto each market to match i.e.

$$\begin{aligned} \int_a d^1(a \rightarrow (h, x)) da &= \int_h s^1(h \rightarrow (h, x)) dh \quad \forall (h, x) \\ \int_a d^2(a \rightarrow x) da &= s^2(x) \quad \forall x. \end{aligned}$$

Given a wage schedule that clears all markets, the allocation can then be described with:

$$\mathcal{P}(h, a, \hat{a}, x) = s^1(h \rightarrow (h, x)) \times \frac{d^1(a \rightarrow (h, x))}{\int_{a'} d^1(a' \rightarrow (h, x))} \times \frac{d^2(\hat{a} \rightarrow x)}{\int_{a'} d^2(a' \rightarrow x)}$$

i.e. the first term is the measure of workers in market (h, x) , the second term describes the proportion of firms in market (h, x) which are of type a , and the third term is the proportion of firms of type \hat{a} which are employing worker x in the second period.

5.1 Efficiency

Aggregate welfare in this economy is given by the sum of discounted profits and wages. Since output in each match is transferable between the worker and firm, it suffices to use total discounted output as a measure of welfare. Much abuse of the integral notation is used in this section to save space; hence $\int_x f(x)$ is used in place of $\int_{x \in \mathbf{X}} f(x) dx$. For any allocation $\mathcal{P}()$, let $S(\mathcal{P})$ denote the social welfare, which is given by:

$$S(\mathcal{P}) = \int_{h,a,x,\hat{a}} \left(f(h, a, x) + \beta y(x, \hat{a}) \right) \times \mathcal{P}(h, a, x, \hat{a})$$

i.e. given the output of each (h, a, \hat{a}, x) combination, the total discounted output can be measured by integrating over these combinations, weighed by their density implied by the allocation. Now a proof by contradiction shows the efficiency properties of the equilibrium. Suppose there were an alternative feasible allocation, $\hat{\mathcal{P}}$ such that $S(\hat{\mathcal{P}}) > S(\mathcal{P})$.

$$\int_{h,a,x,\hat{a}} \left(f(h, a, x) + \beta y(x, \hat{a}) \right) \times \hat{\mathcal{P}}(h, a, x, \hat{a}) > \int_{h,a,x,\hat{a}} \left(f(h, a, x) + \beta y(x, \hat{a}) \right) \times \mathcal{P}(h, a, x, \hat{a})$$

The RHS integral may be adjusted to reflect the equilibrium payoffs:

$$\begin{aligned} & \int_{h,a,x,\hat{a}} \left(f(h, a, x) + \beta y(x, \hat{a}) \right) \times \hat{\mathcal{P}}(h, a, x, \hat{a}) \\ & > \int_{h,a,x,\hat{a}} \left(\left(\pi^1(a; \mathbf{w}) + w^1(h, x; \mathbf{w}) + \beta \pi^2(\hat{a}; \mathbf{w}) + \beta w^2(x, \mathbf{w}) \right) \times \mathcal{P}(h, a, x, \hat{a}) \right) \end{aligned}$$

The reason this expression is equivalent is that in every non-empty market, the agents must be earning the maximal values. So the fact that this increases the values in empty markets does not increase the size of the integral. Now consider one (h, a, x, \hat{a}) combination out of the integral. Compare $f(h, a, x) + \beta y(x, \hat{a})$ to $\pi^1(a; \mathbf{w}) + w^1(h, x; \mathbf{w}) + \beta \pi^2(\hat{a}; \mathbf{w}) + \beta w^2(x, \mathbf{w})$.

$$\text{if } f(h, a, x) + \beta y(x, \hat{a}) > \pi^1(a; \mathbf{w}) + w^1(h, x; \mathbf{w}) + \beta \pi^2(\hat{a}; \mathbf{w}) + \beta w^2(x, \mathbf{w}),$$

then at least one of the following is true:

$$\begin{aligned} f(h, a, x) - w^1(h, x; \mathbf{w}) &> \pi^1(a; \mathbf{w}) \\ f(\hat{a}, x) - w^2(x; \mathbf{w}) &> \pi^2(\hat{a}; \mathbf{w}) \end{aligned}$$

If this equality was true it would violate profit maximisation under the original allocation. Hence, each term within the integral on the LHS must be weakly smaller than the RHS. This puts an upper bound on the LHS. Thus for $\hat{\mathcal{P}}$ to beat \mathcal{P} , it is *necessary* that the following holds:

$$\begin{aligned} & \int_{h,a,x,\hat{a}} \left(\pi^1(a; \mathbf{w}) + w^1(h, x; \mathbf{w}) + \beta \pi^2(\hat{a}; \mathbf{w}) + \beta w^2(x, \mathbf{w}) \right) \times \hat{\mathcal{P}}(h, a, x, \hat{a}) \\ & > \int_{h,a,x,\hat{a}} \left(\pi^1(a; \mathbf{w}) + w^1(h, x; \mathbf{w}) + \beta \pi^2(\hat{a}; \mathbf{w}) + \beta w^2(x, \mathbf{w}) \right) \times \mathcal{P}(h, a, x, \hat{a}) \end{aligned}$$

Separability in the terms means that many of these functions can be removed from the integral. For example, the profit function $\pi^1(a; \mathbf{w})$ is only a function of the firm type,

and the marginal density with respect to firm type is exogenous:

$$\begin{aligned} & \left(\int_a \pi^1(a; \mathbf{w}) \times \mathcal{F}_A(a) \right) + \beta \left(\int_{\hat{a}} \pi^2(\hat{a}; \mathbf{w}) \times \mathcal{F}_A(\hat{a}) \right) + \left(\int_{h,x} \left(w^1(h, x; \mathbf{w}) + \beta w^2(x, \mathbf{w}) \right) \times \hat{\mathcal{P}}(h, ., x, .) \right) \\ & > \left(\int_a \pi^1(a; \mathbf{w}) \times \mathcal{F}_A(a) \right) + \beta \left(\int_{\hat{a}} \pi^2(\hat{a}; \mathbf{w}) \times \mathcal{F}_A(\hat{a}) \right) + \left(\int_{h,x} \left(w^1(h, x; \mathbf{w}) + \beta w^2(x, \mathbf{w}) \right) \times \mathcal{P}(h, ., x, .) \right) \end{aligned}$$

Here $\mathcal{P}(h, ., x, .) = \int_{a, \hat{a}} \mathcal{P}(h, a, x, \hat{a})$ i.e. it is the density of agents at market (h, x) . Since the firm terms cancel, all that is left is to compare the wage part i.e.

$$\left(\int_{h,x} \left(w^1(h, x; \mathbf{w}) + \beta w^2(x, \mathbf{w}) \right) \times \hat{\mathcal{P}}(h, ., x, .) \right) > \left(\int_{h,x} \left(w^1(h, x; \mathbf{w}) + \beta w^2(x, \mathbf{w}) \right) \times \mathcal{P}(h, ., x, .) \right)$$

The terms in the RHS integral can be substituted for $u(h; \mathbf{w})$ which is the value worker type h gets amongst all non-empty markets; the density will be zero at all empty markets so the value of the integral is unaffected. Similarly, on the LHS, each $w^1(h, x; \mathbf{w}) + \beta w^2(x, \mathbf{w})$ must be strictly less than $u(h; \mathbf{w})$, hence the following is necessary:

$$\left(\int_{h,x} u(h; \mathbf{w}) \times \hat{\mathcal{P}}(h, ., x, .) \right) > \left(\int_{h,x} u(h; \mathbf{w}) \times \mathcal{P}(h, ., x, .) \right)$$

Since these are not functions of x , they can be integrated out:

$$\left(\int_h u(h; \mathbf{w}) \times \mathcal{F}_H(h) \right) > \left(\int_h u(h; \mathbf{w}) \times \mathcal{F}_H(h) \right),$$

which is a contradiction, hence there is not another feasible $\hat{\mathcal{P}}$ for which $S(\hat{\mathcal{P}}) > S(\mathcal{P})$.

5.2 Local Behaviour of Wages

The equilibrium wages, in general, will be indeterminate; a level shift in all of the first-period schedules, or the second, will maintain a competitive equilibrium. Furthermore, empty markets may have a wide range of possible wages, such that they do not disturb the equilibrium, and if there are discontinuities in the distribution of types, the equilibrium may be described as a set of bounds. A more intuitive case to study is that in which there are a continuum of types, and have a characterisation of marginal wages for non-empty markets. When there is a continuum of types and investment choices, the first-order conditions are necessary for non-zero $\mathcal{P}(h, a, x, \hat{a})$:

$$\begin{aligned} \frac{\partial}{\partial h} w^1(h, x) &= \frac{\partial}{\partial h} f(h, a, x), \\ \frac{\partial}{\partial x} w^1(h, x) &= \frac{\partial}{\partial x} f(h, a, x), \\ \frac{\partial}{\partial x} w^2(x) &= \frac{\partial}{\partial x} y(x, \hat{a}), \\ \frac{\partial}{\partial x} w^1(h, x) + \beta \frac{\partial}{\partial x} w^2(x) &= 0, \end{aligned}$$

In the second period, the assignment of workers to firms should be strict provided $y^2()$ is either supermodular or submodular. Let $g^2(x)$ represent this assignment; i.e. $g^2(x)$ is the firm type a worker who enters the second period with human capital x will match with. The second-period wages have the following characterisation:

Proposition 11 (*Characterisation of Second Period Wages*)

$$w^2(x) = C_2 + \int_0^x y_1(x', g^2(\phi)) dx'$$

i.e. the marginal wage reflects the marginal productivity of the worker at the assigned firm. The first period is more complicated. In the strict cases, use $g^1()$ to denote the first-period assignment function, and $q()$ is the equilibrium investment associated with the worker type. Then, with some manipulation of the FOCs:

$$\frac{\partial}{\partial h} w^1(h, q(h)) \equiv w_1^1(h, q(h)) + q'(h) w_2^1(h, q(h))$$

(Using FOC for first period human capital and investment):

$$\frac{\partial}{\partial h} w^1(h, q(h)) = f_1(h, g^1(h), q(h)) + q'(h) f_3(h, g^1(h), q(h))$$

$$w^1(h, q(h)) = C_1 + \int_0^h f_1(v, g^1(v), q(v)) + q'(v) f_3(v, g^1(v), q(v)) dv$$

First-period wages reflect both the marginal productivity but also a compensating term. Since firms lose output from training the workers, the workers end up implicitly paying for the training out of their wages. It is worth considering the worker's discounted earnings in the equilibrium, which is perhaps more intuitive. Since the investment is being chosen by the worker to optimise this, the envelope theorem can be applied:

$$\begin{aligned} \frac{\partial}{\partial h} (w^1(h, q(h)) + \beta w^2(q(h))) &= w_1^1(h, q(h)) \\ \Rightarrow \frac{\partial}{\partial h} (w^1(h, q(h)) + \beta w^2(q(h))) &= f_1(h, g^1(h), q(h)) \end{aligned}$$

This means discounted earnings have a surprisingly simple representation; they are the marginal productivity in the first period, holding second-period human capital constant:

$$w^1(h, q(h)) + \beta w^2(q(h)) = C_1 + \beta C_2 + \int_0^h f_1(h, g^1(h), q(h)) dh.$$

This may seem counterintuitive since the second-period technology does not appear, but remember that $f_1()$ is defined by the marginal productivity of human capital,

holding the next period constant. This is a rather difficult interpretation since it's unnatural to imagine all workers reaching the same human capital level in the second period. To make the result clearer, let's explicitly break f down i.e.

$$f(h, a, x) = y(h, a) - \kappa(x - (1 - \delta)h, a),$$

then the result would take on the form:

$$w^1(h, q(h)) + \beta w^2(q(h)) = C_1 + \beta C_2 + \int_0^h y_1(h, g^1(h)) + (1 - \delta)\kappa_1(q(h) - (1 - \delta)h, g^1(h))dh,$$

and in equilibrium, the marginal cost of investment would always match the second period's discounted marginal output - this is implied from using the first-order conditions:

$$\begin{aligned} \kappa_1(x - (1 - \delta)h, a) &\equiv f_3(h, a, x) = w_2^1(h, q(h)), \\ w_1^1(h, q(h)) + \beta w_1^2(q(h)) &= 0, \\ w_1^2(q(h)) &= y_1(q(h), g^2(q(h))), \\ \implies \kappa_1(x - (1 - \delta)h, a) &= \beta y_1(g^2(q(h)), q(h)) \end{aligned}$$

This implies:

Proposition 12 (*Characterisation of lifetime discounted earnings*)

$$w^1(h, q(h)) + \beta w^2(q(h)) = C_1 + \beta C_2 + \int_0^h y_1(h, g^1(h)) + (1 - \delta)\beta y_1(q(h), g^2(q(h)))dh,$$

This gives a more intuitive representation of equilibrium wages: the marginal discounted earnings of human capital represents its marginal contribution to discounted output, after accounting for depreciation. Equilibrium profits also take on integral representations (use $k()$ to denote assignments from firm to worker and $r()$ to denote the investment associated with the firm). With the envelope theorem, deriving by a :

$$\begin{aligned} \frac{\partial}{\partial a}(\pi^2(a)) &= \frac{\partial}{\partial a}(y(k^2(a), a) - w^2(k^2(a))) \\ &= y_2(k(a), a) \\ \pi^2(a) &= D_2 + \int_0^a y_2(k(\tilde{a}), \tilde{a})d\tilde{a} \end{aligned}$$

In the first period:

$$\begin{aligned} \frac{\partial}{\partial a}(\pi^1(a)) &= \frac{\partial}{\partial a}(f(k(a), a, r(a)) - w^1(k(a), r(a))) \\ &= f_2(k(a), a, r(a)) \\ \pi^1(a) &= D_1 + \int_0^a f_2(k(\tilde{a}), \tilde{a}, r(\tilde{a}))d\tilde{a} \end{aligned}$$

If an additive structure for the firm's involvement were to be imposed, e.g.

$$f(h, a, x) = y(h, a) - \kappa(x - ((1 - \delta)h + \gamma a)),$$

then the firm's contribution by helping out with human capital investment i.e. the γ term shows up neatly:

$$\pi^1(a) = D_1 + \int_0^a y_2(k^1(\tilde{a}), \tilde{a}, r(\tilde{a})) + \gamma \beta y_1(r(\tilde{a}), g^2(r(\tilde{a}))) d\tilde{a}$$

In the second period, marginal profits with respect to firm type are equal to their marginal productivity. In the first period, the marginal profits reflect not just the current marginal productivity with respect to potential output, but also their contribution to the worker's second-period output via their training ability (even if the worker is at another firm).

6 Examples

6.1 Strong Sorting and Inequality

Given that workers in this economy sort into different firms, and receive different levels of investment, it is interesting to consider how the distribution of wages may change over time. Two examples are considered, which exhibit sufficient conditions for strong PAM and NAM respectively. In the first example, technology is given by:

$$\begin{aligned} y(h, a) &= \frac{1}{1.25} h^{1.25} a, \\ \kappa(I, a) &= 40 \times I^2 a^{-1}, \\ \delta &= 0.03, \\ \beta &= 0.9. \end{aligned}$$

This technology exhibits sufficient conditions for strong PAM because production is supermodular, the cost of investment function is submodular, and production is convex up in human capital. Now consider a finite number of periods, e.g. $T = 50$, a starting point in which all workers start with $h = 1$, and a discrete firm distribution that takes support at $a = \{0.9, 0.95, 1, 1.05, 1.1\}$ with equal measure. the paths of human capital, net output, and wages are shown for each worker in figures 4 to 6.

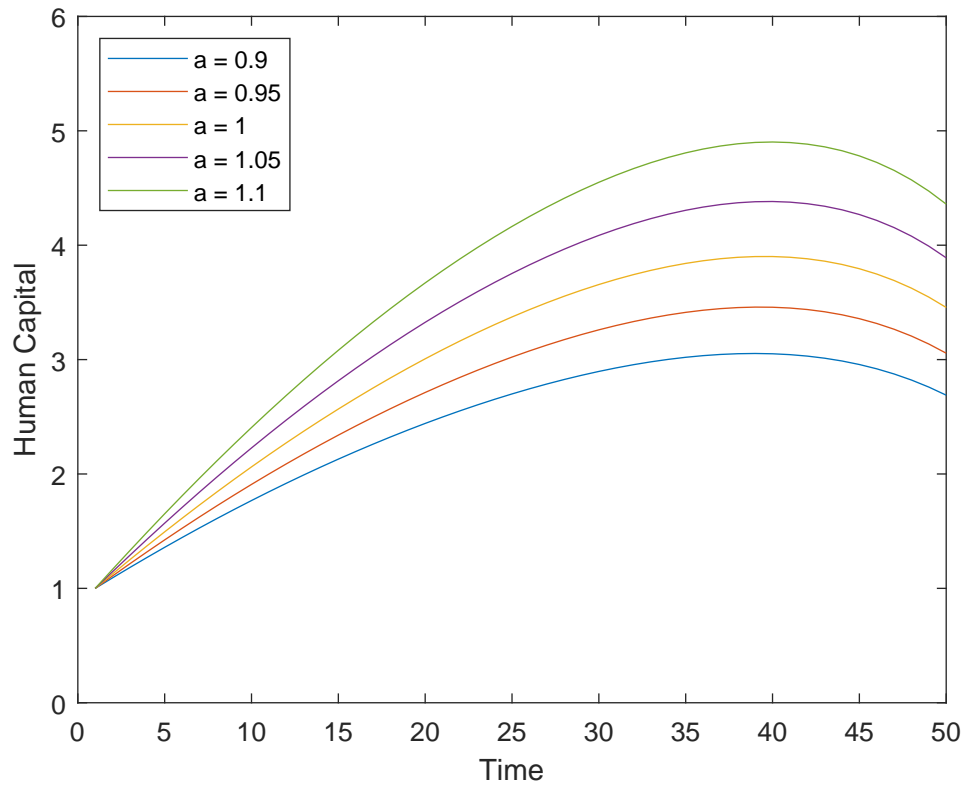


Figure 4: The path of human capital experienced by workers at different firm types over the life-cycle.

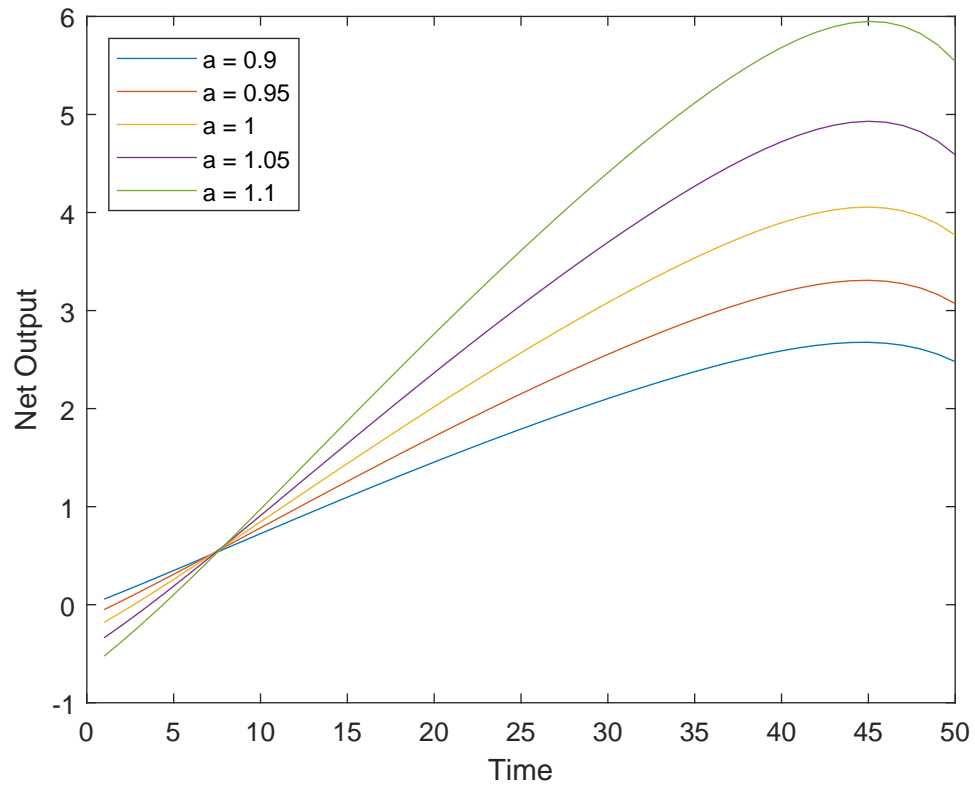


Figure 5: The path of net output experienced by workers at different firm types over the life-cycle. All workers start with the same initial human capital.

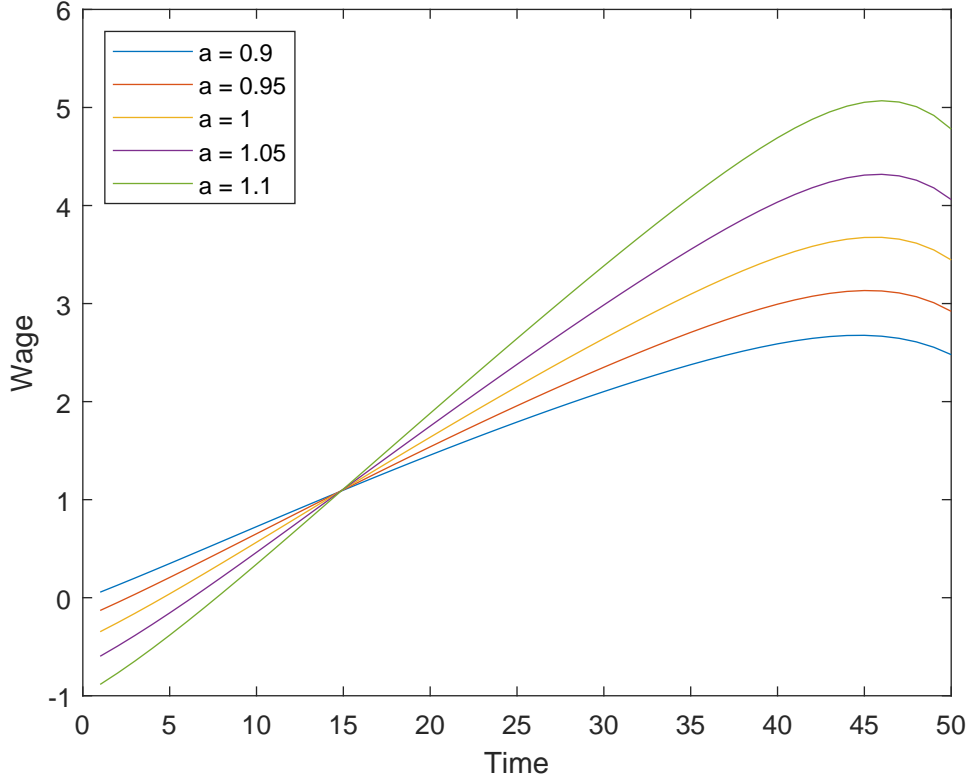


Figure 6: The path of wages experienced by workers at different firm types over the life-cycle. All workers start with the same initial human capital.

Since all workers start with the same human capital, they must receive the same present discounted value of all the wage flows. This means that the workers at the best firm receive a lower wage to begin which trades off with higher levels of human capital investment, and hence higher wages in the future. Notice that in this example, the wages overlap, then spread out, and compress towards the end of the life cycle. The compression towards the end is a result of letting the human capital depreciate. When there are few periods left, the incentive to invest reduces, and the depreciation reduces the human capital of the high-skilled workers more. In the special case in which $\delta = 0$, a theoretical result is possible on the behaviour of wages:

Proposition 13 *Let $\delta = 0$, consider a finite time setting ($1 < T < \infty$), and let the sufficient conditions for period- T strong PAM (NAM) hold i.e. $y_{12} > 0, \kappa_{12} \leq 0, y_{11} \geq 0$ (for NAM: $y_{12} < 0, \kappa_{12} \geq 0, y_{11} \geq 0$). The absolute rate at which wages grow must be increasing (decreasing) in the firm type.*

The proof of 13 is given in appendix F.

This is a useful result since it tells us that fanning out of wages over the life cycle is possible as a result of sorting, even when workers are initially identical, there are no costs to switching firms and investment is endogenous. Intuition comes from the fact that sorting results in different workers being exposed to different investment technologies, and the anticipation of different future workplaces means the marginal value of these investments will not be equalised across the workers.

6.2 A Weak PAM steady state

Sufficient conditions for a steady state to be a weak PAM were given, but this implicitly assumes that such a steady state might exist; perhaps the human capital distribution could degenerate in the limit or always converge to some form of strong or weak PAM. An example is given which demonstrates such steady states can exist. Since weak PAM is much more computationally demanding, the example is very simple and is meant to act as a proof of concept. Let there be three possible human capital types; low, medium, and high, and three firm types; low, medium, and high. There is an equal measure of each of the three firm types. Let $\beta = 0.9$, and the net output function, $f(h, a, x)$ is given by 6.2:

		$x = \text{Low}$		
		a		
		Low	Medium	High
h	Low	3	4.75	5.75
	Medium	3.5	4.75	8
	High	5	6.5	8

		$x = \text{Medium}$		
		a		
		Low	Medium	High
h	Low	2	3	3.75
	Medium	2.75	4.25	6
	High	4.25	6.25	6.25

		$x = \text{High}$		
		a		
		Low	Medium	High
h	Low	1.25	1.5	1.75
	Medium	2.5	2.75	4.5
	High	4.25	5	5

Table 2: Realised Output for different (h, a, x) combinations

$f()$ is well behaved, satisfying:

- Weakly increasing in h , a and weakly decreasing in x (otherwise the ordering of types is unclear)
- Weakly supermodular in (h, x) (otherwise investment costs cannot be considered convex)

A competitive, steady-state equilibrium exists with the following properties:

- The measure of workers of each type is of equal size
- Low firms always employ low workers and invest in the workers to reach high human capital

- Medium firms always employ high workers and let the human capital depreciate to medium
- High firms always employ medium workers and let the human capital depreciate to low

Wages (defined over (h, x)) that support such an equilibrium are given by table 6.2:

		x		
		Low	Medium	High
h	Low	4	2.25	1.25
	Medium	6	4.25	2.75
	High	7	5.5	4.25

Table 3: Competitive Equilibrium Wages in the Weak PAM steady-state example.

The Kendall rank correlation coefficient is $\frac{1}{3}$ in this example; the matches are shown in 7 which shows double the amount of concordant pairwise comparisons to discordant.

The equilibrium periodic profits are 0, 0.75, and 2 for the low, medium, and high-type firms respectively, and the present discounted values for the workers are 40.8, 42.7 and 43.96 for the low, medium, and high worker types respectively. The worker's wage will cycle between 1.25, 6.25 and 8 repeatedly, as their human capital cycles between low, high and medium, and their employer cycles between low, medium, and high. Any deviations from the equilibrium allocation leaves the worker worse off (the choice of round numbers for the outputs and wages means the firms have some deviations in which they are indifferent, for example, the low firm can choose $(h, x) = (\text{low}, \text{high})$ or $(\text{high}, \text{high})$, and the medium firm could choose $(h, x) = (\text{low}, \text{low})$, $(\text{low}, \text{medium})$, $(\text{high}, \text{medium})$ or $(\text{high}, \text{high})$).

In this economy, the average periodic profit is $\frac{15.5}{3}$. It is useful to consider a few other alternatives and show that they are inefficient. For example, why doesn't the human capital distribution simply degenerate in a steady state? The steady-state output associated with a degenerate human capital distribution would be $\{\frac{13.5}{3}, \frac{13}{3}, \frac{14.25}{3}\}$ for the low, medium, and high human capital types respectively; which are all strictly worse than the weak PAM allocation. Another form of allocation to consider is strong PAM and strong NAM. If low always matches to low, medium to medium, high to high, the best investment policy that can be achieved from such an arrangement is for the workers to flip rank each period i.e. this strong PAM allocation could achieve $\frac{13.5}{3}$. Under strong NAM, the best that can be achieved is $\frac{14.25}{3}$, which involves the workers staying with the same firm permanently. It is worth pointing out that this cycle could not occur if the firms were homogenous. If all firms were the low type, human capital would always converge to the high level, if they are the medium type, then they will stay at low if already there, but converge to high otherwise, and if the firms are high types, the human capital always converges to medium. The cyclical behaviour is therefore dependent on firm heterogeneity; its presence encourages the human capital distribution not to collapse into a degenerate distribution.

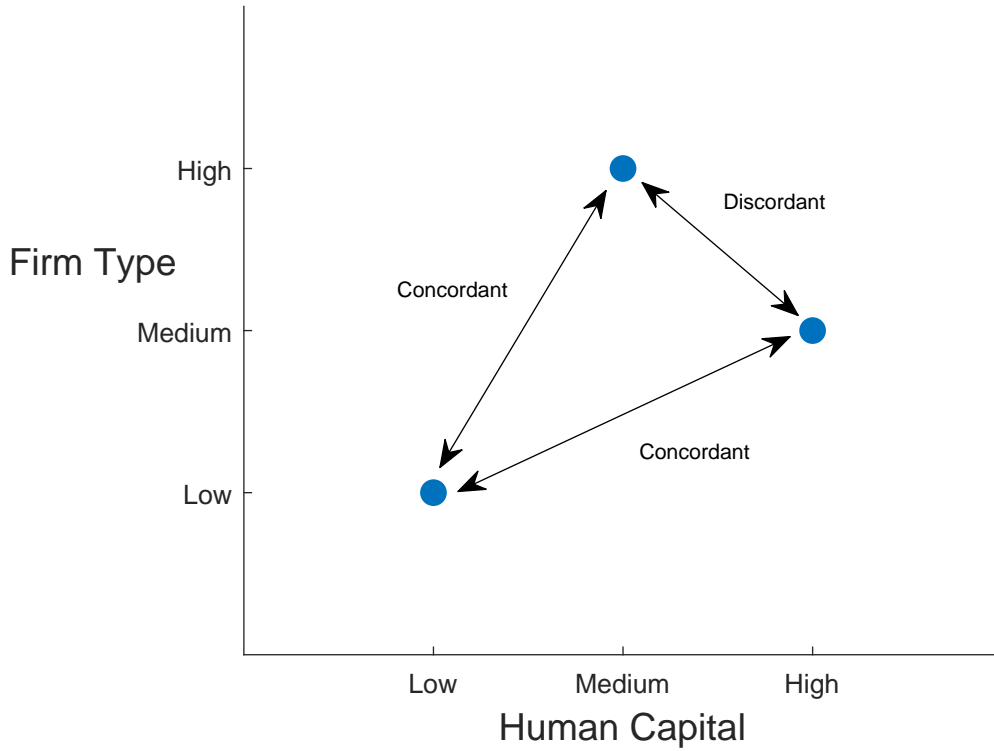


Figure 7: Assignment of workers to firms. There are more concordant pairwise comparisons than discordant, meaning the Kendall rank correlation coefficient is positive.

7 Conclusion

This paper considers the effects of endogenous investment in a dynamic assignment model. In a single period setting, whether or not the static production function is supermodular or submodular will ultimately determine the sign of sorting. However, when there are multiple periods, and agents may invest; the conditions under which different sorting patterns arise are substantially more complicated. As the quadratic, Gaussian model shows, while sorting is complicated, it is still possible to make distribution-free statements about the socially optimal sorting pattern based on the underlying technology. This also shows that imperfect sorting can be an endogenous outcome. This paper establishes various sufficient conditions under a wide variety of scenarios, ranging from a two-period setting, a $T > 2$ finite setting, and an infinite horizon steady state. Under fairly strong assumptions, it can be guaranteed that PAM or NAM occurs in every period, whereas under looser assumptions, it is possible to sign the Kendall rank correlation of sorting in each period, based on the features of the technology alone. The model can be recast not as a social planner's problem, but as a competitive equilibrium, and the marginal behaviour of wages is intuitive. Furthermore, in the finite setting, under strong sorting patterns, it is possible to establish dynamic results on the behaviour of wages; in the absence of any discounting, they will tend to disperse over time. Finally, an example of a steady state with imperfect sorting is given. This equilibrium is both novel and artificial in the sense that human capital cycles, but demonstrates the existence of such equilibria.

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Appendices

A Solution to the optimal correlation problem

The values of the cross derivatives for P in the quadratic model are:

$$V_{12} = \lambda + (1 - \delta)\gamma\rho \frac{c\beta}{c - \beta\rho},$$

$$V_{13} = (1 - \delta)\lambda \frac{c\beta}{c - \beta\rho},$$

$$V_{23} = \lambda\gamma \frac{c\beta}{c - \beta\rho},$$

and the objective function is:

$$r_{ha}\sigma_h\sigma_a V_{12} + r_{h\hat{a}}\sigma_h\sigma_a V_{13} + r_{a\hat{a}}\sigma_a^2 V_{23}.$$

The solution to the correlations will not change if the three coefficients ($\sigma_h\sigma_a V_{12}$), $\sigma_h\sigma_a V_{13}$ and $\sigma_a^2 V_{23}$ are all scaled by an arbitrary constant, hence it is neater to write the objective as:

$$r_{ha}C_{ha} + r_{h\hat{a}}C_{h\hat{a}} + r_{a\hat{a}}C_{a\hat{a}},$$

where:

$$C_{ha} = \lambda + (1 - \delta)\gamma\rho \frac{c\beta}{c - \beta\rho}$$

$$C_{h\hat{a}} = (1 - \delta)\lambda \frac{c\beta}{c - \beta\rho}$$

$$C_{a\hat{a}} = \lambda\gamma \frac{c\beta}{c - \beta\rho} \frac{\sigma_a}{\sigma_h}.$$

For simplicity, cases in which $C_{12} \neq 0$, $C_{13} \neq 0$ and $C_{23} \neq 0$ are considered. Furthermore, $|C_{12}| \neq |C_{13}|$, $|C_{12}| \neq |C_{23}|$, $|C_{13}| \neq |C_{23}|$ is imposed. This is simply to rule out indeterminate and multiple solutions to the problem, but such scenarios are considered in brief at the end of this section.

Lemma 2 *The constraint*

$$r_{ha}^2 + r_{h\hat{a}}^2 + r_{a\hat{a}}^2 \leq 1 + 2r_{ha}r_{h\hat{a}}r_{a\hat{a}}$$

should always bind with equality.

To see why lemma 2 holds, consider any combination such that:

$$r_{ha}^2 + r_{h\hat{a}}^2 + r_{a\hat{a}}^2 < 1 + 2r_{ha}r_{h\hat{a}}r_{a\hat{a}}.$$

If this is true then a local improvement can always be made in the objective. At least one of the correlations must not be equal to 1 in absolute size (if they all were, then the constraint is either binding or has been violated). This correlation can therefore be adjusted in the direction of the associate coefficient by a small amount without breaking the constraints (and the coefficients are assumed to be non-zero). Any optimum must therefore have the constraint binding. Since there is symmetry in the problem, let's relabel the indices in the following manner, with ordering by absolute size:

ordering: $|C_m| > |C_n| > |C_o|$, (where $\{m, n, o\} = \{(ha), (h\hat{a}), (a\hat{a})\}$)

objective: $r_m C_m + r_n C_n + r_o C_o$.

Given the binding constraint, it is always possible to write one of the correlations as a function of the others:

$$r_o = r_m r_n + \text{sign}(C_o) \sqrt{(1 - r_m^2)(1 - r_n^2)}$$

Note that a plus or minus could be used in front of the square root, but it will always be best to place r_o as large as possible if $C_o > 0$ and as small as possible if $C_o < 0$. The problem may be re-written in terms of r_m and r_n , while substituting the constraint for r_o , and label the objective for any general r_m, r_n $G()$:

$$\begin{aligned} & \max_{r_m, r_n} \left\{ r_m C_m + r_n C_n + r_m r_n C_o + \sqrt{(1 - r_m^2)(1 - r_n^2)} |C_o| \right\}, \\ \text{s.t. } & r_m^2 \leq 1, \\ & r_n^2 \leq 1. \end{aligned}$$

$$G(r_m, r_n) \equiv r_m C_m + r_n C_n + r_m r_n C_o + \sqrt{(1 - r_m^2)(1 - r_n^2)} |C_o|$$

Consider the derivative of G with respect to r_n as r_n approaches either 1 or -1 :

$$\begin{aligned} \lim_{r_n \rightarrow 1} G_2(r_m, r_n) &= \lim_{r_n \rightarrow 1} \left(C_n + r_m C_o - r_n \sqrt{\frac{1 - r_m^2}{1 - r_n^2}} |C_o| \right) \rightarrow -\infty \\ \lim_{r_n \rightarrow -1} G_2(r_m, r_n) &= \lim_{r_n \rightarrow -1} \left(C_n + r_m C_o - r_n \sqrt{\frac{1 - r_m^2}{1 - r_n^2}} |C_o| \right) \rightarrow +\infty \end{aligned}$$

G becomes infinitely downwards sloping in r_n as r_n tends to 1 and infinitely upwards sloping in r_n as r_n tends to -1 which rules out these extremes. However, this relies on $|r_m| \neq 1$ i.e. $|r_m| \neq 1 \implies |r_n| = 1$. This means

$$|r_n| = 1 \implies |r_m| = 1 \implies |r_o| = 1.$$

The second implication come from the fact that if both $|r_n|$ and $|r_m|$ are 1, this forces $|r_o|$ to be 1 also. Since there is symmetry in the analysis, this effectively means only two classes of solutions could exist:

- All three correlations are 1 in absolute size
- None of the correlations are 1 in absolute size

Let's introduce the function $\tilde{G}(r_n)$ which will give the objective when r_m and r_o are both optimised conditional on r_n i.e:

$$\tilde{G}(r_n) = \max_{r_m} \left\{ r_m C_m + r_n C_n + r_m r_n C_o + \sqrt{(1 - r_m^2)(1 - r_n^2)} |C_o| \right\},$$

s.t. $r_m^2 \leq 1$.

The solution for r_m is:

$$r_m^* = \operatorname{argmax}_{r_m} \left\{ r_m C_m + r_n C_n + r_m r_n C_o + \sqrt{(1 - r_m^2)(1 - r_n^2)} |C_o| \right\} = \frac{C_m + r_n C_o}{\sqrt{(1 - r_n^2) C_o^2 + (C_m + r_n C_o)^2}},$$

which can be found with first-order conditions (this is concave in r_m). This can be substituted back into \tilde{G} , and after some manipulation yields the expression:

$$\tilde{G}(r_n) = r_n C_n + \sqrt{C_m^2 + C_o^2 + 2r_n(C_m C_o)}$$

This is concave in r_n due to the fact that $\operatorname{sign}(C_n) \neq \operatorname{sign}(C_m C_o)$. If the solution is interior, then:

$$r_n = \frac{1}{2} C_m C_o \left(\frac{1}{C_n^2} - \frac{1}{C_m^2} - \frac{1}{C_o^2} \right).$$

Given the symmetry of the problem, the solution for all three (conditional on being interior) is given by:

$$\begin{aligned} r_m^{\text{interior}} &= \frac{1}{2} C_n C_o \left(\frac{1}{C_m^2} - \frac{1}{C_n^2} - \frac{1}{C_o^2} \right), \\ r_n^{\text{interior}} &= \frac{1}{2} C_m C_o \left(\frac{1}{C_n^2} - \frac{1}{C_m^2} - \frac{1}{C_o^2} \right), \\ r_o^{\text{interior}} &= \frac{1}{2} C_m C_n \left(\frac{1}{C_o^2} - \frac{1}{C_m^2} - \frac{1}{C_n^2} \right). \end{aligned}$$

These satisfy the constraint $r_m^2 + r_n^2 + r_o^2 = 1 + 2r_m r_n C$. But what about whether they are less than 1 in absolute value? A corner solution will occur if $\tilde{G}'(1) > 0$ or $\tilde{G}'(-1) < 0$, which is equivalent to the condition:

$$\frac{1}{C_m^4} + \frac{1}{C_n^4} + \frac{1}{C_o^4} \geq 2 \left(\frac{1}{C_m^2 C_n^2} + \frac{1}{C_n^2 C_o^2} + \frac{1}{C_m^2 C_o^2} \right).$$

This condition (rearranged to be greater than or equal to 0) may define the “determinant” of the problem i.e. whether the solution type will be a corner or interior. If the solution is a corner one, 4 different combinations can be chosen. The best corner takes the largest two in absolute value of C_m, C_n, C_o and matches the sign of their relevant correlations, and sacrifices the sign of the third (this is because $C_m C_n C_o < 0$, but $r_m r_n r_o = 1$). With the ordering that has been chosen, this means:

$$\begin{aligned} r_m &= \text{sign}(C_m), \\ r_n &= \text{sign}(C_n), \\ r_o &= -\text{sign}(C_o), \end{aligned}$$

and the objective is:

$$|C_m| + |C_n| - |C_o|$$

No other corner can beat this; it is easy to check the other three are inferior:

$$\begin{aligned} &|C_m| - |C_n| + |C_o|, \\ &-|C_m| + |C_n| + |C_o|, \\ &-|C_m| - |C_n| - |C_o|. \end{aligned}$$

Cases in which one of the coefficients is zero are rather trivial; the other two can have the associated correlations set to 1 or -1 to match the sign. If two of the coefficients were zero, then the remaining non-zero coefficient can have its correlation set to 1 or -1 to match its sign and the other two are indeterminate. If the determinant is D is positive (implying a corner solution), but the “weakest” coefficient is tied either between two, or all three, then any of the smallest in absolute value can be chosen to have its sign of correlation sacrificed, hence there may be multiple solutions to such a knife edge scenario.

B Continuity of r_{ha}^* in $C_{a\hat{a}}$

The interior solution:

$$r_{ha}^* = \frac{1}{2} C_{h\hat{a}} C_{a\hat{a}} \left(\frac{1}{C_{ha}^2} - \frac{1}{C_{h\hat{a}}^2} - \frac{1}{C_{a\hat{a}}^2} \right), \quad (5)$$

$$(6)$$

is continuous in $C_{a\hat{a}}$ ($\forall 0 < C_{a\hat{a}} < \infty$). This means that as long as the determinant is non-negative, there is continuity of r_{ha}^* in $C_{a\hat{a}}$. The determinant itself:

$$D = \frac{1}{C_{ha}^4} + \frac{1}{C_{h\hat{a}}^4} + \frac{1}{C_{a\hat{a}}^4} - 2 \left(\frac{1}{C_{ha}^2 C_{h\hat{a}}^2} + \frac{1}{C_{ha}^2 C_{a\hat{a}}^2} + \frac{1}{C_{h\hat{a}}^2 C_{a\hat{a}}^2} \right)$$

is also continuous in $C_{a\hat{a}}$. This is important as it means *within* the positive regions of the determinant, $r_{ha}^* = \text{sign}(C_{ha})$. This allows for the claim that *within* both the negative and positive regions of the determinant, r_{ha}^* is continuous in $C_{a\hat{a}}$. The “regime”

will switch at points in which $D = 0$. Since D is a quadratic in $C_{a\hat{a}}^{-2}$, this could occur at two points:

$$C_{a\hat{a}}^{-2} = C_{ha}^{-2} + C_{h\hat{a}}^{-2} \pm 2\sqrt{C_{ha}^{-2}C_{h\hat{a}}^{-2}}. \quad (7)$$

Now check at what points the interior solution delivers a correlation of $r_{ha}^* = \text{sign}(C_{ha})$. This reduces to a quadratic in $C_{a\hat{a}}^{-1}$ with solutions at the points:

$$C_{a\hat{a}}^{-1} = -|C_{ha}^{-1}| \pm |C_{h\hat{a}}^{-1}| \quad (8)$$

Under the assumption of $|C_{ha}| > |C_{h\hat{a}}|$, this is well behaved.

The solution implied by equation 8 can be squared, which will lead to the same expression as equation 7. This shows that the points at which $C_{a\hat{a}}$ delivers $r_{ha}^* = \text{sign}(C_{ha})$ are also the points at which $D = 0$. This means that there is no jump in the value of r_{ha}^* when the regime switches, and hence r_{ha}^* is continuous in $C_{a\hat{a}} \forall 0 < C_{a\hat{a}} < \infty$.

C Proof of Strong and Weak Sorting in the Two Period Model

C.1 Strong Sorting

The sufficient conditions in 6 ensure that both $V_{12}V_{13}V_{23} \geq 0$, and that V_{12} is positive for the PAM case and negative for the NAM case. To see why work through the PAM case, and note that:

$$V_{12} = y_{12}(h, a) - (1 - \delta)\beta \frac{\kappa_{12}y_{11}(x, \hat{a})}{\kappa_{11} - \beta y_{11}(x, \hat{a})},$$

so if both $y_{12} > 0$, $\kappa_{12} \leq 0$ and $y_{11} \geq 0$, then $V_{12} > 0$ since the first term is positive and the second term non-negative. The alternative condition, $y_{12} > 0$, $\kappa_{12} \leq 0$ and $|y_{12}| \geq |\kappa_{12}|$ works because the term $(1 - \delta)\beta \frac{y_{11}(x, \hat{a})}{\kappa_{11} - \beta y_{11}(x, \hat{a})}$ is strictly smaller than 1 in absolute size when $y_{11} < 0$, hence it is possible to drop the $y_{11} \geq 0$ condition and replace it with $|y_{12}| > |\kappa_{12}|$, which ensures the first term in V_{12} always dominates the second. The conditions $\kappa_{12} \leq 0$ ensures that V_{13} and V_{23} are the same sign in the PAM case, meaning $V_{12}V_{13}V_{23} \geq 0$ is guaranteed. The NAM conditions work in the same manner but ensure $V_{12} < 0$ and $V_{12}V_{13}V_{23} \geq 0$.

For any two trios observed in an equilibrium (indexed by i and j), it must not be the case that swapping the worker could increase output i.e.

$$\begin{aligned} V(h_i, a_i, \hat{a}_i) + V(h_j, a_j, \hat{a}_j) - V(h_j, a_i, \hat{a}_i) - V(h_i, a_j, \hat{a}_j) &\geq 0 \\ (h_i - h_j) \left(V_1(\tilde{h}, a_i, \hat{a}_i) + V_1(\tilde{h}, a_j, \hat{a}_j) \right) &\geq 0 \\ (h_i - h_j) \left(V(\tilde{h}, a_i, \hat{a}_i) - V(\tilde{h}, a_i, \hat{a}_j) + V(\tilde{h}, a_i, \hat{a}_j) + V(\tilde{h}, a_j, \hat{a}_j) \right) &\geq 0 \\ (h_i - h_j) \left((a_i - a_j)V_{12}(\tilde{h}, \bar{a}, \hat{a}_j) + (\hat{a}_i - \hat{a}_j)V_{13}(\tilde{h}, a_i, \hat{a}) \right) &\geq 0 \end{aligned}$$

This uses the mean value theorem, where there must exist a \tilde{h} in-between h_i and h_j , a \bar{a} in-between a_i and a_j , and a \hat{a} in-between \hat{a}_i and \hat{a}_j , which makes this equivalence

work. The same argument could be repeated, but instead of swapping the worker, the current firm, or the future firm could be swapped instead. There are therefore three restrictions that are implied:

$$\begin{aligned}(h_i - h_j) \left((a_i - a_j) V_{12} + (\hat{a}_i - \hat{a}_j) V_{13} \right) &\geq 0 \\ (a_i - a_j) \left((h_i - h_j) V_{12}^* + (\hat{a}_i - \hat{a}_j) V_{23}^* \right) &\geq 0 \\ (\hat{a}_i - \hat{a}_j) \left((h_i - h_j) V_{13}^{**} + (a_i - a_j) V_{23}^{**} \right) &\geq 0\end{aligned}$$

The asterisks are used to distinguish that different values of these derivatives may be used while employing the mean value theorem. Let's suppose, in the case where $V_{12} > 0$ and $V_{12} V_{13} V_{23} \geq 0$, we observe that there is negative alignment between two worker firm pairs i.e. $h_i < h_j$, $a_i > a_j$. This implies some signs that can be placed into these three inequalities:

$$\begin{aligned}\text{negative} \times V_{12} + \text{negative} \times (\hat{a}_i - \hat{a}_j) V_{13} &\geq 0 \\ \text{negative} \times V_{12}^* + \text{positive} \times (\hat{a}_i - \hat{a}_j) V_{23}^* &\geq 0 \\ (\hat{a}_i - \hat{a}_j) \left(\text{negative} \times V_{13}^{**} + \text{positive} \times V_{23}^{**} \right) &\geq 0\end{aligned}$$

Since $V_{12} > 0$, the first inequality implies that $(\hat{a}_i - \hat{a}_j) V_{13} < 0$. Similarly, the second inequality implies $(\hat{a}_i - \hat{a}_j) V_{23}^* > 0$. This however leads to a contradiction; it implies that V_{13} and V_{23}^* must be opposite signs, which goes against $V_{12} V_{13} V_{23} > 0$. This means that any allocation which involves negative alignment between the worker and firm ranks in the first period cannot be socially optimal; this ensures that first-period sorting is PAM. The steps are the same for NAM, which are not shown here.

C.2 Weak Sorting

Under the sufficient conditions for weak PAM and NAM, both feature the assumption $|\frac{\kappa_{12}}{\kappa_{11}}| > |\frac{y_{12}}{y_{11}}|$. In the weak PAM case, the combination of $y_{12} > 0$, $\kappa_{12} > 0$, $y_{11} \leq 0$ ensures that V_{12} is positive. To see why $|V_{12}| > |V_{13}|$: compare:

$$\begin{aligned}V_{12} &= y_{12}(h, a) - (1 - \delta)\beta \frac{\kappa_{12}() y_{11}(x, \hat{a})}{\kappa_{11}() - \beta y_{11}(x, \hat{a})} \\ V_{13} &= (1 - \delta)\beta \frac{\kappa_{11}() y_{12}(x, \hat{a})}{\kappa_{11}() - \beta y_{11}(x, \hat{a})}\end{aligned}$$

The second term of V_{12} is larger in absolute size than V_{13} when $|\kappa_{12} y_{11}| > |\kappa_{11} y_{12}|$, and the fact that both terms in V_{12} are the same sign means $|V_{12}| > |V_{13}|$ must hold. Like section C.1, let's consider the restriction that in a socially optimal assignment, it must not be possible to improve output by swapping the worker between triplets (h_i, a_i, \hat{a}_i) and (h_j, a_j, \hat{a}_j) . This condition can be decomposed:

$$(h_i - h_j) \left((a_i - a_j) V_{12} + (\hat{a}_i - \hat{a}_j) V_{13} \right) \geq 0$$

$$\implies (h_i - h_j) \left((a_i - a_j) (V_{12} - V_{13}) + ((a_i + \hat{a}_i) - (a_j + \hat{a}_j)) V_{13} \right) \geq 0$$

Notice that in the weak PAM conditions, the terms $V_{12} - V_{13}$, and V_{13} , are positive. This carries an important implication:

$$(h_i - h_j) \left((a_i - a_j) \times \text{positive} + ((a_i + \hat{a}_i) - (a_j + \hat{a}_j)) \times \text{positive} \right) \geq 0$$

This implies:

$$(h_i - h_j)(a_i - a_j) < 0 \implies (h_i - h_j) \left((a_i + \hat{a}_i) - (a_j + \hat{a}_j) \right) > 0,$$

$$(h_i - h_j) \left((a_i + \hat{a}_i) - (a_j + \hat{a}_j) \right) < 0 \implies (h_i - h_j)(a_i - a_j) > 0$$

This means that if a worker is negatively aligned with the first-period firm, it must be positively aligned with the sum of the current and future firm types. This at first appears a rather strange object since there are no reasons to think that *adding* firm types together makes sense, but it does allow bounds to be made on the Kendall rank correlation. To see how this helps, consider all pairwise comparisons between workers in period 1. We can compare whether worker j has more human capital than worker i , whether the firm a_j is a higher type than a_i , and which of the second-period firms associated with the workers, \hat{a}_i and \hat{a}_j is higher. The following table demonstrates all the possible combinations.

h	a	\hat{a}	$a + \hat{a}$	Measure of Instances
+	+	+	+	P
+	+	-	+	Q
			-	R
+	-	+	+	S
			-	0
+	-	-	-	0
-	+	+	+	0
-	+	-	+	0
			-	S
-	-	+	+	R
			-	Q
-	-	-	-	P

Table 4: Comparing two triplets: Change in h , a , \hat{a} , $a + \hat{a}$ and the measure of instances for which this happens.

Notice some of these are empty. In particular, if h increases, a decreases, then \hat{a} should not decrease, and likewise for the reverse. If h increases, and $a + \hat{a}$ decreases, then a cannot decrease (and similarly for the reverse). Exact ties have been removed for simplicity, but the analysis will still hold if only pairwise comparisons in which $h_i \neq h_j$ and $a_i \neq a_j$ are considered. The Kendall Tau correlation for h and a is:

$$\tau_{ha} = \frac{P + Q + R - S}{P + Q + R + S}$$

It is not immediately clear that this is positive, but compare it to the correlation of a and $a + \hat{a}$:

$$\tau_{a,a+\hat{a}} = \frac{P + Q - R - S}{P + Q + R + S}$$

Clearly, $\tau_{ha} \geq \tau_{a,a+\hat{a}}$. Now I make the claim that $\tau_{a,a+\hat{a}} \geq 0$. A lower bound on $\tau_{a,a+\hat{a}}$ can be constructed by taking the case where a and \hat{a} are perfectly negatively correlated. Suppose a discrete set is considered, with $a_1 < a_2 < \dots < a_N$. Then the alignment of a and $a + \hat{a}$ is shown below:

a	$a + \hat{a}$
a_1	$a_1 + a_N$
a_2	$a_2 + a_{N-1}$
\dots	\dots
a_N	$a_N + a_1$

Table 5: Perfect negative rank correlation between a and \hat{a} ; implied values for $a + \hat{a}$.

For every concordant pair, there will be another discordant pair. To see how this works, take any two K, L with $K < L$. Then inspect the four pairings, as shown in Table 6:

a	$a + \hat{a}$
a_K	$a_K + a_{N-K+1}$
a_L	$a_L + a_{N-L+1}$
a_{N-K+1}	$a_{N-K+1} + a_K$
a_{N-L+1}	$a_{N-L+1} + a_L$

Table 6: Comparison of two firms in the perfectly negative rank correlation case

Or mathematically:

$$\text{sign}(a_L - a_K) \left(a_L + a_{N-L+1} - a_K - a_{N-K+1} \right) \neq \text{sign}(a_{N-L+1} - a_{N-K+1}) \left(a_L + a_{N-L+1} - a_K - a_{N-K+1} \right)$$

This bounds the correlation $\tau_{a,a+\hat{a}} \geq 0$, in turn implying $\tau_{ha} \geq 0$. The weak NAM two-period case follows the same steps but is not included for brevity.

D Proof of the T Period Strong Sorting Sufficient Conditions

The PAM case is shown, with assumptions $y_{12} > 0$, $\kappa_{12} \leq 0$, $y_{11} \geq 0$. The first step of the proof is to show that $Y^t(\hat{H}) > Y^t(H)$ if \hat{H} first-order stochastically dominates H . This is straightforward; take any trio from the original allocation (h, a, x) , and maintain the rank of h that is associated with a, x i.e. the new allocation has trio $(\hat{H}^{-1}(H(h)), a, x)$. This will increase output while maintaining the distribution of human capital in the next period;

$$\begin{aligned} & \hat{H}^{-1}(H(h)) > h \\ \implies & \left(y(\hat{H}^{-1}(H(h)), a) - \kappa(x - (1 - \delta)\hat{H}^{-1}(H(h)), a) \right) > \left(y(h, a) - \kappa(x - (1 - \delta)h, a) \right). \end{aligned}$$

(Since y is increasing in the human capital input and κ decreasing). Since $\hat{H}^{-1}(H(h)) \geq h$ everywhere and the equality is strict in some quantiles, it is possible to increase total current output while maintaining the same distribution for the next period under a new human capital distribution which first-order stochastically dominates the original. This reasoning applies to the final period, applying the inequality to $y()$ alone, and ignoring $\kappa()$. The next step is to show that if \hat{H} is a mean preserving spread of H , then $Y(\hat{H}) > Y(H)$. This is easy to show for the final period; using the same transformation:

$$\begin{aligned} y(\hat{H}^{-1}(H(h)), a) - y(h, a) & \equiv y(h + \varepsilon_h, a) - y(h, a), \\ \text{where } \varepsilon_h & \equiv \hat{H}^{-1}(H(h)) - h \end{aligned}$$

Let a_h be the assigned firm to h (and since there is PAM in the final period, $\frac{\partial a_h}{\partial h} \geq 0$). The gain in output associated with a particular worker when switching distribution is:

$$\begin{aligned} & \exists \tilde{h} \text{ inbetween } h, h + \varepsilon_h : \\ & y(h + \varepsilon_h, a_h) - y(h, a_h) = \varepsilon_h y_1(\tilde{h}, a_h) \end{aligned}$$

Since both $y_{11} > 0$, the term $y_1(\tilde{h}, a_h) - y_1(h, a_h)$ is the same sign as ε_h i.e.

$$y(h + \varepsilon_h, a_h) - y(h, a_h) \geq \varepsilon_h y_1(h, a_h).$$

Furthermore, since h and a_h are increasing in ε_h (because this is a mean preserving spread and there is PAM), and $y_{11} > 0$, $y_{12} > 0$, then $y_1(h, a_h)$ is also increasing in ε_h . This means that ε_h and $y_1(h, a_h)$ must be positively correlated:

$$\begin{aligned} E_h(\varepsilon_h y_1(h, a_h)) &> E_h(\varepsilon_h) E(y_1(h, a_h)), \\ \implies E_h(\varepsilon_h y_1(h, a_h)) &> 0, \\ \implies E_h(y(h + \varepsilon_h, a_h) - y(h, a_h)) &> 0, \end{aligned}$$

hence there is a gain to aggregate output from the mean preserving spread. Now a recursive approach can be taken. Suppose we know that in period $t + 1$, $Y^t()$ is increasing in mean preserving spreads and first-order stochastic dominance transformations. Now consider two worker firm pairs who are matched negatively and have investments: (h_0, a_1, x_q) and (h_1, a_0, x_r) , with $h_0 < h_1$ and $a_0 < a_1$. It is unclear whether x_r or x_q is bigger since the better firm is better at investing. Two main cases can be considered, the first is:

$$x_r - (1 - \delta)h_1 \geq x_q - (1 - \delta)h_0.$$

In these cases, it is possible to improve current output with a positive alignment while keeping the next period distribution the same. Consider the allocation (h_0, a_0, x_q) and (h_1, a_1, x_r) . Now compare the difference in output:

$$\begin{aligned} &\left(y(h_0, a_0) + y(h_1, a_1) - \kappa(x_q - (1 - \delta)h_0, a_0) - \kappa(x_r - (1 - \delta)h_1, a_1) \right) \\ &- \left(y(h_0, a_1) + y(h_1, a_0) - \kappa(x_q - (1 - \delta)h_0, a_1) - \kappa(x_r - (1 - \delta)h_1, a_0) \right) \end{aligned}$$

This difference is positive due to $y_{12} > 0$ and $\kappa_{12} \leq 0$ i.e. the supermodularity in production is taken advantage of, as is the submodularity in investment costs (the higher firm is lined up with the higher $x - (1 - \delta)h$). The other case is

$$x_r - (1 - \delta)h_1 < x_q - (1 - \delta)h_0.$$

In these cases, a different approach is needed (otherwise there will be a tradeoff between y_{12} and κ_{12}). Instead of keeping the investment levels the same, allocate the higher investment to the higher worker, and the lower investment to the lower worker i.e. $(h_0, a_0, (1 - \delta)h_0 + (x_r - (1 - \delta)h_1))$ and $(h_1, a_1, (1 - \delta)h_1 + (x_q - (1 - \delta)h_0))$. Compare the outputs:

$$\begin{aligned} &\left(y(h_0, a_0) + y(h_1, a_1) - \kappa(x_r - (1 - \delta)h_1, a_0) - \kappa(x_q - (1 - \delta)h_0, a_1) \right) \\ &- \left(y(h_0, a_1) + y(h_1, a_0) - \kappa(x_q - (1 - \delta)h_0, a_1) - \kappa(x_r - (1 - \delta)h_1, a_0) \right) \end{aligned}$$

Here the investment costs are identical, but because of the supermodularity in output, the total production will be higher. However, the next period distribution has changed. Under the original, negative alignment, the next period human capitals are x_q, x_r , but under the new arrangement, now they are $(1 - \delta)h_0 + (x_r - (1 - \delta)h_1)$ and $(1 - \delta)h_1 + (x_q - (1 - \delta)h_0)$. It is easy to show that the new allocation is a mean preserving spread of the old one i.e.

$$\begin{aligned} \left((1 - \delta)h_1 + (x_q - (1 - \delta)h_0) \right) - \left((1 - \delta)h_0 + (x_r - (1 - \delta)h_1) \right) &= 2(1 - \delta)(h_1 - h_0) + (x_q - x_r) \\ &> x_q - x_r. \end{aligned}$$

Thus in either case, for any two negatively aligned worker firm pairs, it is always possible to come up with a set of investments such that a positive alignment can beat it. This relied on the next period benefitting from mean preserving spreads, hence:

$$Y^{t+1} \text{ benefits from mean preserving spreads} \implies \text{PAM in period } t.$$

Now it needs to be shown that period t also benefits from mean preserving spreads in the human capital distribution. To do this, take any two worker firm pairs (h_0, a_0, x_0) and (h_1, a_1, x_1) , and spread out the worker types by an arbitrarily small amount, $\varepsilon > 0$, such that they are $(h_0 - \varepsilon, a_0, x_0 - (1 - \delta)\varepsilon)$ and $(h_1 + \varepsilon, a_1, x_1 - (1 - \delta)\varepsilon)$. The gain in current output is:

$$\begin{aligned} &\left(y(h_0 - \varepsilon, a_0) + y(h_1 + \varepsilon, a_1) - \kappa(x_0 - (1 - \delta)h_0) - \kappa(x_1 - (1 - \delta)h_1) \right) \\ &- \left(y(h_0, a_0) + y(h_1, a_1) - \kappa(x_0 - (1 - \delta)h_0) - \kappa(x_1 - (1 - \delta)h_1) \right) \\ &= y(h_0 - \varepsilon, a_0) + y(h_1 + \varepsilon, a_1) - (y(h_0, a_0) + y(h_1, a_1)) \\ &\quad \text{(First order Taylor Expansion)} \\ &\approx \varepsilon(y_1(h_1, a_1) - y_1(h_0, a_0)) \\ &= \varepsilon(y_1(h_1, a_1) - y_1(h_0, a_1) + y_1(h_0, a_1) - y_1(h_0, a_0)) \\ &\quad \text{(Mean Value Theorem)} \\ &= \varepsilon \left((h_1 - h_0)y_{11}(\tilde{h}, a_1) + (a_1 - a_0)y_{12}(h_0, \tilde{a}) \right) \end{aligned}$$

Since $y_{11} \geq 0$ and $y_{12} > 0$, the spreads in human capital are beneficial to current production. The future human capital is also spread out, which is beneficial to the future payoff. This means that backward induction can be used to complete the proof.

$$y_{12} > 0, \quad \kappa_{12} \leq 0, \quad y_{11} \geq 0:$$

- Period T features PAM and Y^T is improving in mean preserving spreads and FOSD of the human capital distribution
- If period $t + 1$ benefits from mean preserving spreads and FOSD, then period t features PAM
- If period t features PAM, then period t also benefits from mean preserving spreads and FOSD

- By backwards induction, all periods feature PAM

This means that if the production function is supermodular in human capital and firm type, the cost of investment function is submodular in the investment quantity and firm type, and production is convex up in human capital, then regardless of the distribution or number of periods, there will be positive assortative matching in every period. Similarly, following the same steps, $y_{12} < 0$, $\kappa_{12} \geq 0$ and $y_{11} \geq 0$ will ensure NAM in every period for an arbitrary T period model.

E Proof of the Sufficient Conditions for Weak Sorting (Infinite Horizon Steady State)

This proof is similar to that of weak sorting in the two-period case. Only the weak PAM case is shown. With the weak PAM assumptions $y_{12} > 0$, $\kappa_{12} > 0$, $|y_{12}| > 2|\kappa_{12}|$, notice that (for any $0 < \delta < 1$), we have $M_{12} > 0$, $M_{13} > 0$, $M_{23} < 0$, and crucially $|M_{12}| > |M_{13}|$.

Start with the notion that if any allocation is optimal, there cannot be a gain in output by swapping the *firm* between two triplets (h_i, a_i, x_i) , (h_j, a_j, x_j) . Notice that, unlike the two-period case, here the firm is being switched. This is because, in the two-period case, two firms were being matched to one worker (albeit at different times), whereas now there are two “workers” being matched to one firm (albeit the same worker, but two human capital values at different times). The condition to ensure swapping the firm doesn’t lead to an output gain is (using the mean value theorem as in appendix C.

$$(a_i - a_j) \left((h_i - h_j) M_{12} + (x_i - x_j) M_{23} \right) \geq 0$$

$$\implies (a_i - a_j) \left((h_i - h_j) (M_{12} + M_{23}) + ((x_i - h_i) - (x_j - h_j)) M_{23} \right) \geq 0$$

Since both $M_{12} + M_{23}$ is positive and $-M_{23}$ negative, some bounds can be made:

$$\begin{aligned} (h_i - h_j)(a_i - a_j) < 0 &\implies ((x_i - h_i) - (x_j - h_j))(a_i - a_j) < 0, \\ ((x_i - h_i) - (x_j - h_j))(a_i - a_j) > 0 &\implies (h_i - h_j)(a_i - a_j) > 0. \end{aligned}$$

This rules out some patterns from appearing in the optimal allocation, as shown in table E

The correlation τ_{ha} is given by:

$$\tau_{ha} = \frac{P + Q + R - S}{P + Q + R + S}.$$

For this correlation to be negative, it must be the case that $S > P + Q + R$. This in turn would imply:

$$\tau_{h,x-h} = \frac{P - Q - R + S}{P + Q + R + S} > 0$$

h	a	x	$x - h$	Measure of Instances
+	+	+	+	P
			-	Q
+	+	-	-	R
+	-	+	+	S
			-	0
+	-	-	-	0
-	+	+	-	0
-	+	-	+	0
			-	S
-	-	+	+	R
-	-	-	+	Q
			-	P

Table 7: Measure of Instances: Assigning worker to firm to future human capital in the infinite horizon steady state (with weak PAM assumptions)

This is not possible. To see why, the most correlated h and $x - h$ could be would be if h and x were perfectly negatively correlated (in terms of ranks), then table E would show the relationship between h and $x - h$ (when ordered by rank $h_1 < h_2 < \dots < h_N$):

h	$x - h$
h_1	$x_N - h_1$
h_2	$x_{N-1} - h_2$
...	...
h_N	$x_1 - h_N$

Table 8: Perfect negative rank correlation between h and x ; implied values for $x - h$.

For every concordant pair, there will be another discordant pair. To see how this works, take any two K, L with $K < L$. Then inspect the four pairings, as shown in E:

h	$x - h$
h_K	$h_{N-K+1} - h_K$
h_L	$h_{N-L+1} - h_L$
h_{N-K+1}	$h_K - h_{N-K+1}$
h_{N-L+1}	$h_L - h_{N-L+1}$

Table 9: Comparison of two workers in the perfect negatively rank correlated case

Mathematically:

$$\begin{aligned} & \text{sign}\left((h_K - h_L)((h_{N-K+1} - h_K) - (h_{N-L+1} - h_L))\right) \\ &= \text{sign}\left((h_{N-L+1} - h_{N-K+1})((h_L - h_{N-L+1}) - (h_K - h_{N-K+1}))\right). \end{aligned}$$

The greatest that the correlation $\tau_{h,x-h}$ can be is zero, which therefore rules out negative τ_{ha} i.e. $\tau_{ha} \geq 0$; the steady state has weak PAM.

F Without discounting, the wages at the higher firm grow faster (Under Strong PAM): A proof

The absolute growth rate of human capital must be increasing in the firm type. This can be proven by contradiction; let $h_0 < h_1$ be human capital levels, $a_0 < a_1$ be firm types, and let x_0, x_1 be the next period human capital analogues of h_0, h_1 . Suppose the investment by the low match is greater i.e. $h_0 - x_0 > h_1 - x_1$. Swapping the investment levels must increase the current net output, the gain is given by the following term:

$$\begin{aligned} & \left(y(h_0, a_0) - \kappa((x_1 - h_1), a_0) + y(h_1, a_1) - \kappa((x_0 - h_0), a_1) \right) \\ & - \left(y(h_0, a_0) - \kappa(x_0 - h_0, a_0) + y(h_1, a_1) - \kappa(x_1 - h_1, a_1) \right) \\ & = \left(\kappa(x_0 - h_0, a_0) + \kappa(x_1 - h_1, a_1) \right) - \left(\kappa((x_1 - h_1), a_0) + \kappa((x_0 - h_0), a_1) \right) \end{aligned}$$

which is positive due to $x_0 - h_0 > x_1 - h_1$, $a_0 < a_1$, and submodularity of $\kappa()$. Furthermore, as a result of swapping the investments around, the spread of human capital in the next period increases:

$$\begin{aligned} & h_1 > h_0, \quad (x_0 - h_0) > (x_1 - h_1) \\ \implies & |(h_1 + (x_0 - h_0)) - (h_0 + (x_1 - h_1))| > |x_1 - x_0| \end{aligned}$$

Because mean-preserving spreads are beneficial in the strong PAM environment, it is unambiguously better for the stronger match to invest more. This means between any two periods, $t, t+1$, the growth in human capital must be greater at the higher firm type. Furthermore, since there is no depreciation, and a larger investment is allocated to workers with the most human capital, the human capital must spread out over time. To get from human capital to wages, the marginal contributions of human capital to output must be involved. The envelope theorem can be used to show that the marginal value of human capital takes the following form:

$$\begin{aligned} W^t(h, a) &= \max_I \left\{ y(h, a) - \kappa(I, a) + \beta W^{t+1}(h + I, a) \right\} \\ \frac{\partial}{\partial h} W^t(h, a) &= y_1(h, a) + \beta \frac{\partial}{\partial h} W^{t+1}(h + I, a), \\ \implies \frac{\partial}{\partial h} W^t(h^t, a) &= \sum_{s=t}^T \beta^{s-t} y_1(h^s, a), \end{aligned}$$

where h^t, h^{t+1}, \dots, h^T is the path of human capital the worker at firm a takes in equilibrium. The marginal wage can then be backed out with $w^t = W^t - \beta W^{t+1}$:

$$\begin{aligned} w_1^{*t}(h) &= \sum_{s=t}^T \beta^{s-t} y_1(h^s, a_h) - \beta \sum_{s=t+1}^T \beta^{s-(t+1)} y_1(h^s, a_h) \\ \implies w_1^{*t}(h) &= y_1(h, a_h), \end{aligned}$$

where $w^{*t}(h)$ is the equilibrium wage earned by a worker in period t . This simplification is only possible because $\delta = 0$, otherwise the future periods also come into play. With this simple characterisation, compare the wages of workers $h_0^t < h_1^t$ at firms $a_0 < a_1$:

$$w^{*t}(h_1) = w^{*t}(h_0) + \int_{h_0}^{h_1} y_1(h, a^t(h)) dh.$$

$a^t(h)$ is used simply to denote the firm assignment at time t . Due to PAM, this is given by:

$$a^t(h) = F_A^{-1}(F_{Ht}(h)).$$

The difference between the worker's wages is given by the integral term, which can also be expressed (using a change of variable) as:

$$w^{*t}(h_1) - w^{*t}(h_0) = \int_{a_0}^{a_1} y_1(h^t(a), a) \frac{f_A(a)}{f_{Ht}(h^t(a))} da,$$

(where $h^t(a)$ is used as the inverse of $a^t(h)$). The gap between the wages in the next period is:

$$w^{*t}(h_1 + I_1) - w^{*t}(h_0 + I_0) = \int_{a_0}^{a_1} y_1(h^t(a) + I^t(a), a) \frac{f_A(a)}{f_{H,t+1}(h^{t+1}(a))} da,$$

where $I^t(a)$ is the investment associated with firm a . To show that the gap has become more positive, it suffices to show both of the following of the following inequalities are true:

$$y_1(h^t(a) + I^t(a), a) - y_1(h^t(a), a) > 0,$$

$$f_{H,t+1}(h^{t+1}(a)) < f_{Ht}(h^t(a)).$$

The first of these is true from $y_{11} > 0$; since investments are strictly positive, $y_1(h, a)$ is increasing in h . The second of these represents the distribution of workers spreading out, and hence the density of the human capital distribution tending to thin. This is easy to show since investments are increasing in the firm type:

$$F_{t+1,H}(h^t(a) + I^t(a)) = F_{tH}(h^t(a)),$$

$$\implies (h_1^t(a) + I_1^t(a)) \times f_{t+1,H}(h^{t+1}(a)) = h_1^t(a) \times f_{tH}(h^t(a))$$

Since $I_1^t(a) > 0$:

$$\implies f_{t+1,H}(h^{t+1}(a)) < f_{tH}(h^t(a))$$

The intuition here is best understood from the firm's perspective. Since the higher firms invest more, the human capital distribution has to spread out. This, along with

the convex return to human capital strengthens the returns to human capital, and hence the “slope” of wages over the rank of workers. This strong PAM setting is therefore consistent with patterns of wages over the life cycle in which wages tend to disperse. However, a very similar pattern of wages will be observed under strong NAM. In a strong NAM setting, the highest workers go to the lowest firms and will receive the largest investments.

The steps to prove this are identical to the strong PAM case. A worker with higher human capital (who is at a lower firm) will always receive more investment, otherwise, the investments can be swapped which improves current net output and creates a mean preserving spread in the human capital distribution of the next period which is beneficial. Since $\delta = 0$, marginal wages have the simple characterisation that they are equal to the marginal productivity of human capital. Since the human capital spreads out over time, and $y(h, a)$ is convex-up in h , it will again be true that the difference in wages of the high vs. low worker will spread out between periods. Since there is NAM, this means the higher the firm, the lower the rate of wage growth.

Employer-Provided Training in the UK

A Search-Theoretic Model

Stuart Alexander Breslin *

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Abstract

I document employer-provided training for full-time employed workers in the UK using an “effective training” measure that weighs off different types of self-reported work-related training. This form of training tends to be higher in already highly educated workers and is provided in greater amounts at larger firms. Moreover, occupations and industries that tend to pay higher wages also tend to provide more training; this is consistent with the idea that training enhances the productivity of the worker and that workers with high earning ability may sort into high training environments. In conjunction with these findings, I develop a search model of the labour market that includes heterogeneity in both workers and firms and endogenous human capital distribution. Workers vary in their level of human capital and firms vary in productivity. Worker-firm pairs can increase the worker’s human capital at the cost of losing output. I show that this framework can replicate key facts from the data; namely, higher educated workers receive more training throughout their lifetime and earn more, and that the firms that pay higher wages also provide more training. Finally, the model features inefficiently low human capital investment due to the social returns not being fully internalised under random search; a policy of subsidising low-skilled young workers covered by income taxation is shown to improve aggregate welfare and social mobility in the model.

1 Introduction

Do some firms invest more in their workers’ skills than others? Do some workers tend to receive more of these investments? If so, why? How does this contribute to differences in wages and human capital, and what does it mean for social mobility and economic efficiency? To gain some insight into these questions, I develop a search model of the labour market where jobs are characterised both by a firm’s productivity as well

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as the human capital of the worker. Investments in the worker's human capital may be chosen by the pair, which comes at the cost of lost output in the job. In doing so, I link the works of Becker's assortative matching (Becker (1974)) and worker training provision (Becker (1962)) with Ben-Porath's (Ben-Porath (1967)) ideas of human capital accumulation over the life cycle. In the real world, this form of training corresponds most closely to employer-provided training for full-time workers. This contrasts with the likes of formal degrees or apprenticeships, which are heavily regulated and are predominantly aimed at younger people. The training that workers receive once they are heavily embedded in the labour market is the focus of this paper.

In analysing UK survey data that includes employment and training information, I find various stylised facts that closely link employer-provided training and wage outcomes. In particular, higher-educated workers tend to receive more training throughout their careers. Larger firms also tend to provide more training. The industries and occupations that tend to pay more also tend to train more, on average. This means that however workers are grouped, whether it be by education level, the size of the firm they work for, the occupation they are in, or the industry their employer is in, the higher-paid groups tend to also have higher training. This is coupled with a life-cycle pattern that has the higher-educated workers see their wages grow at a faster rate.

The model of this paper builds upon the model of Breslin (2023) by adding two features which add some realism. The first of these is search frictions a la Shimer & Smith (2000). The second of these is firm-specific human capital. Both these features make it costly for the match to break up and create uncertainty in the matching process. A binary, Poisson, ageing process is also used which allows for a computationally simple way to distinguish in-model between young and old agents while avoiding the conceptual limitations of having infinitely lived agents. Search frictions also introduce inefficiencies which were not present in the competitive framework, which leaves some space for a social planner to make efficiency improvements.

Markets with two-sided heterogeneity may also give rise to assortative matching, in which the matching between types is correlated. In the model of this paper, there is positive assortative matching, whereby workers with higher human capital tend to work for more productive firms. This, along with a functional form that has strong convexity in the relationship between human capital and production encourages a positive feedback effect, by which the highest-skilled workers tend to get the fastest gains in human capital. When considering social mobility, this might not be desirable, and policies to help low-skilled workers may be seen as a way to reduce these effects. In this model, there is both an efficiency reason and an equity reason to encourage more investment in low-skilled workers.

The paper continues as follows. Section 2 outlines related literature. Section 3 describes the data and establishes some key empirical facts about employer-provided training in the UK. Section 4 presents the model. Section 5 calibrates and demonstrates the model and the impact of training subsidies. Section 6 concludes.

2 Related Literature

An interesting feature of human capital investment is that should the match separate, the worker retains the benefits of the training but the firm does not. However, workers may be able to implicitly pay for their training by taking a lower wage during the period of training. In addition to this, some training may develop firm-specific human capital, which the worker cannot take elsewhere. As pointed out by Becker (1962), firms will be willing to pay towards this. Kiyotaki & Zhang (2018) for example consider an overlapping generation model with inefficient investment that arises from lack of commitment, particularly when skill is less specific to particular firms. This is discussed in the context of low economic growth in Japan. Bagger & Lentz (2014) have a similar quirk to my model; where I have human capital investment as a choice that matches must be made, they have the on-the-job search intensity which is chosen. While they note that an inability to commit will lead to the search intensity being inefficiently high, they instead focus on a bilaterally efficient choice of search intensity, with bargaining only important for the resulting split of wage and profit. Analogous to their model, I will take as a benchmark the assumption that the investment choice is bilaterally efficient between the worker-firm pair and bargaining determines the wage/profit split. I show that several different mechanisms can deliver the same outcome, and also provide a counterexample in which there is under-investment relative to the bilaterally efficient level. The model will feature under-investment relative to the socially optimal levels, however. This is because *future employers* on average will benefit from the training of workers they have not yet met by being able to extract additional surplus from better-trained workers, but cannot contract on this.

The inefficiency that arises in this model is the same as in Acemoglu (1997), which also considers a random search model with training decisions. This result does however hinge on aspects of the search structure of the model. Moen & Rosén (2004) demonstrate that in a directed search framework with bilaterally efficient training and investment decisions, the investment levels in the equilibrium are efficient, and training subsidies would reduce aggregate welfare. They do, however, conclude that the combination of both training subsidies and policies to reduce turnover may be welfare improving. Bilateral efficiency in their model is, perhaps, less plausible than in my framework as there is an on-the-job search, meaning long-term contracts must be able to account for the possibility of a poaching firm meeting the worker. Evidence regarding the effect of non-compete clauses in the US - see Starr (2019) - suggests that allowing for such contracts does tend to increase employer-sponsored training, but depresses wages. In directed search frameworks that feature heterogeneity, there are however other distortions. Galenianos et al. (2011) for instance consider homogenous workers and heterogenous firms with directed search, and find that the market power that firms hold results in too many workers matched to the low productivity firms, relative to the optimum. In my model, random search is favoured for its simplicity, and there is no poaching as job search only occurs while unemployed - in doing this I bypass the potential complications involved with bilateral efficiency under the ability to be poached by competing firms.

The evidence on whether training is under (or over) provided relative to the efficient benchmark does not have a consensus. In an extensive review of workplace

training in Europe, Bassanini et al. (2005) do not find compelling evidence that firm-provided training is under-provided, which is repeated by Brunello & De Paola (2009). There are however measurement issues, particularly with the measurement of training costs. However, more recently, Martins (2021) used a quasi-experimental approach to assess the impact of European Social Fund grants (for workplace training) on the performance of firms that received these funds. Firms that received the funds were found to have improved substantially in sales, value-added, employment, productivity, and exports. The size of the positive impact on the firms relative to the size of the grants is evidence in favour of market under-provision in training. A randomised control trial - see Adhvaryu et al. (2023) - which provided soft skills training to Indian garment workers found the program had substantial (13.5%) increases on the treatment group's productivity, but little effect on the wages or worker turnover; an empirical finding consistent with the theory of Acemoglu & Pischke (1998).

Other studies do find evidence of wage returns to training. A meta-analysis by Haelermans & Borghans (2012) that attempts to control for publication bias estimates that on-the-job training programmes typically increase wages by 2.6%. Using employer-employee matched data, Almeida & Faria (2014) estimates the wage returns from receiving on-the-job training using a propensity-score-matching approach that controls for selection into training by the worker and firm. They find a positive effect of 7.7% on the wages of Malaysian workers who have received on-the-job training and an effect of 4.5% when Thailand is considered. Interestingly, they find heterogeneity in the return to training, with higher returns to the better educated.

Ljungqvist & Sargent (1998) consider exogenous human capital dynamics. Their framework focuses on the skill accumulation of workers in employment vs. unemployment in the context of high European unemployment in the 1980s; employed workers tend to accumulate human capital while unemployed workers tend to depreciate, and search effort while unemployed is endogenous. This allows for an analysis that focuses on the effects of unemployment insurance on the labour market. Related to this idea of state-dependent human capital gains is Gregory (2020), which incorporates a rich heterogeneity structure on both firms and workers. In her paper, accumulation in human capital is partially exogenous, modelled as a direct function of the learning ability of the worker, the worker's age, and the "learning environment" of the employer. In a different direction, Lise & Postel-Vinay (2020) uses a model of multidimensional skills (manual vs. cognitive vs. interpersonal), which develop over time depending on the worker's type and that of the employer. This model can address issues such as skill mismatch; in their framework, the social planner may be able to make improvements by matching the agents differently than the market equilibrium. While models that have human capital development as a function of the worker's environment are very useful for considering some policy and welfare questions, the homogeneity of the investments limits these approaches in some areas. In particular, policies addressed at increasing the level of training cannot easily be analysed in these frameworks.

Fu (2011) is closer to my model in respect of training being a deliberate process. Her paper allows for homogenous firms to post take-it or leave-it piece-rate/training menus, in the style of a Burdett & Mortensen (1998) model. While the model can capture heterogeneity in wages and training, the process by which human capital is accumulated is somewhat for analytical convenience; human capital is accumulated expo-

nentially in training time and this is constant across the worker's entire tenure at the firm. I instead opt for a human capital investment process that can be adapted as time goes on. Under this specification, the worker-firm pair bargains over the rate at which human capital can be improved, but the value of such an improvement is endogenous.

A recent job market paper that takes an endogenous training approach in a Burdett and Mortensen style model is Ma et al. (2022). Here the focus is on cross-country differences and the role of the informal vs. modern sector under imperfect contract enforceability. Their model differs greatly in that workers just live for two periods.

Intrinsic heterogeneity in both workers and firms with search frictions is examined closely by Shimer & Smith (2000), where the productivity of a match is determined by worker type and firm type, although in their model the worker's type is permanent. This builds mathematically upon the framework of Becker (1974). Their model shuts down features such as endogenous entry of firms and consideration of the matching function to make it easy to study the sorting patterns that occur in the model; I will follow this minimalistic approach to the search environment while generalising to allow for investment in general and firm-specific human capital.

The model is also related to work on income dynamics. For example, Guvenen (2009) documents evidence of "heterogenous wage profiles"; this corresponds to this paper in the sense that the agents in my model systematically diverge in the logarithm of their wages; workers who start with higher human capital are not just scaled up versions of their lower skilled counterparts plus noise. Bardhi et al. (2020) develop a model which features discrimination, which in the early career can lead to large persistent effects across the lifetime. While my model does not feature discrimination and has perfect information, the same pattern is generated via a different mechanism - namely that the higher-skilled workers early on in their careers gain better training opportunities.

Finally, a very closely related paper to mine is that of Blundell et al. (2021). They use the same data as I, but instead focus on the 1991 - 2008 period, to measure the impact of training on women, with a particular focus on those returning to the labour force after maternity leave. Their empirical approach treats policy changes as exogenous. Like my paper, they find much heterogeneity in the provision of training, with the higher educated generally receiving more. The earlier, pre-2009 data has the advantage of a longer period but has the drawbacks of a smaller number of individuals. Their model is richer on the supply side, with the state space of the worker allowing to account for childbirth and the wage of the spouse, but the demand side is not explicitly modelled. In their model, they find small training subsidies to be an effective policy to help redevelop skills; similarly, in my paper, I find that subsidies aimed at young low-skill workers are effective (albeit with a very different methodology).

3 Data

The data is taken from "Understanding Society" (University Of Essex (2023)) which is a UK longitudinal household study. Households are surveyed annually regarding a wide range of issues including health, beliefs, lifestyle patterns, family life, and economic situation. This data includes detailed information about employment, earnings, and training. Since this paper focuses on employer-provided training, it is therefore suit-

able for making inferences about the training provided to workers by their employers and how their wages change in response. Since major changes were made in 2009, I focus only on the stretch from 2009 - 2020 which avoids problems merging some of the variables and changes to the way training data was surveyed. I restrict the sample to observations with the following criteria:

- The individual was between the ages of 25 and 60 at the time of the survey
- They were in paid employment, working at least 30 hours in a typical week with one job only, and reported at least £300 in gross monthly labour income
- Their gender, ethnic group, occupation, industry, and firm size (by number of employees) were all reported

Wages are imputed using the income and working hours reported and are adjusted for the consumer price index so that values are at 2015 levels. These restrictions leave 25,085 unique individuals in the sample. When individuals are surveyed, and were also surveyed in the previous year, they are asked about forms of education and training they have undertaken within the space of the interviews. Of interest to this paper are reports of training provided by the employer. Workers are asked to recall how many spells of training they have received (which could still be in progress). If there are more than three spells, the individual is asked to consider only the three most significant spells. For each spell, the worker is asked who provided the training (with the main options being the employer, a government agency, or a college/university), the number of days of training involved in the spell, the average number of hours spent per day in the training, and its purpose. With regards to the "purpose" of each spell, 7 options are provided, which are ordered by the most common responses from first spells (with the percentage of responses with this category for those who did report employer-provided training given):

- "To improve your skills in your current job" (55%)
- "To maintain professional status and/or meet occupational standards" (40%)
- "To prepare you for a job you might do in the future" (24%)
- "Health and Safety Training" (19%)
- "For hobbies and leisure" (10%)
- "To help you get started in your job" (9.6%)
- "To help you get a promotion" (7.6%)

In cases for which more than one type is mentioned for the spell, the spell is categorized as the least common type. For example, if the individual reported that the training spell was to improve their skills in their current job, *and* it was for health and safety training, then the spell is categorised as health and safety. This will allow for a count of the number of hours the worker spends on each type of training while avoiding double-counting. The rule of applying the rarest category has the advantage of the

fact that relatively specific responses may be paired with generic responses, in which case the relatively specific response gives more of an idea of the flavour of the training involved. The categories “to help you get started in a job” and “to help you get a promotion” are very close together, but it is unusual to put down both - within those who said “for hobbies and leisure”, only five per cent also included “to help you get started in your job” (within first spell responses). Since the number of days of training as well as the hours of training per day is recorded, it is possible to measure the amount of hours the worker spends in different training categories. This means, for example, that a worker could report training 50 hours in maintaining professional status... and 26 hours in health and safety. Some workers report more than three spells of training. Unfortunately, it is not possible to observe what form of training or hours were involved in these spells. As a robustness check, I check two options. The baseline approach will be to multiply each set of training hours by $\frac{n}{3}$ if the individual reports $n > 3$ spells. The alternative approach is simply to ignore the additional spells. The training data by hours has many zeros. The percentage of observations that have zero hours of training reported are 98.5, 89.6, 90.1, 97.2, 96.9, 92.7, and 99.7 for the seven different respective categories. In addition to this, the non-zero segments of the distribution are heavy-tailed. I investigate two methods to deal with this. In the first, I opt to treat training as a binary variable i.e. either the worker received a particular type of training or they did not. To get rid of very small values of hours, I introduce a minimum of 12 hours for the worker to be considered trained; this cutoff typically needs just over two work days to complete. This cutoff of 12 is a bit arbitrary, but robustness checks are used to test different thresholds. In the baseline, the frequency of different training types over all observations, and their means and medians are summarised in Table 1.

Type	Frequency (%)	Median (hrs)	Mean (hrs)
job induction	1.19	48	126
skill improvement in current job	7.47	29	49
maintain status/standards	7.14	29	53
prepare for future job	2.20	35	65
help get a promotion	2.69	47	92
health and safety	4.48	25	46
hobbies/leisure	0.19	38	67

Table 1: Summary statistics of training types over the restricted sample.

In the second method, I add one hour to the number of hours trained in a particular category and then take the logarithm of this value. This transforms the non-zero-hour section of the distribution to be closer to normal. The different categories of training may not have equal importance in generating productivity gains for workers. Training for hobbies and leisure, for example, is by definition not primarily designed with work in mind. Similarly, it is not obvious that health and safety training should be thought of as productivity-enhancing, or a regulatory minimum that firms must cover, and this may also be true of training to help get workers started with the job. These judgements are rather subjective however; a more objective way to determine the relative importance of training purposes would be to examine how such training sessions translate

into wage rises. Let w_{it} denote the logarithm of hourly wages that individual i earns in year t , then $\Delta w_{it} \equiv w_{it} - w_{i,t-1}$. The training indicators are defined in the following manner:

$$\tau_{i,t-1,k} = \begin{cases} 1, & \text{if individual } i \text{ received at least 12 hours of training type } k \\ & \text{between year } t-1 \text{ and } t \\ 0, & \text{otherwise.} \end{cases}$$

$\tau_{it} \equiv [\tau_{it1}, \dots, \tau_{it7}]$ denotes the vector of training indicators for a particular individual-year observation. Let X_{it} represent a vector of control variables for the observation. These include the occupation and industry of the worker (defined by the 9-category Standard Occupational Classification 2000, and 21-category 2007 Standard Industrial Classification of Economic Activities), firm size (dummies for different bins by number of employees), age dummies (by year), gender, ethnicity (whether white British or not), their highest qualification level, and dummies that denote whether the worker switched job in the previous year, or the year before that. Let α_i denote an individual fixed "growth" effect and ε_{it} the error term on each observation. Then the primary regression of interest is:

$$\Delta w_{it} = \tau_{i,t-1} \lambda + X_{t-1} \beta + \alpha_i + \varepsilon_{it}.$$

The use of an individual fixed effect, α_i allows for some unobserved heterogeneity amongst individuals in their wage growth rate. In the baseline specification, a fixed effects approach is applied to α_i , but as a robustness check, random effects is also considered. The vector λ denotes the predicted wage rises that workers with various training types are expected to receive. Table 2 shows the results of the baseline regression and the various robustness checks. In the baseline specification, only three of the coefficients are significant at the 95% level - "To improve your skills in your current job", "To maintain professional status and/or meet occupational standards", and "To help you get a promotion". This is a rather robust finding across the different variations. When controls are not included in the regression, "training to help you get started in your job" is statistically significant. This is perhaps unsurprising since it does not control for whether workers have recently started the job. Interestingly, "training to prepare you for a job you might do in the future" does not appear to be predictive of wage rises either in terms of statistical significance or the practical significance of the estimated effect. Since only three categories appear to predict wage rises, and this is fairly robust across a wide range of specifications, I propose an "effective training measure", which weighs these three categories proportionately by their respective coefficients:

$$\begin{aligned} \text{Effective Training} = & 1.44 \times 1 \left(\text{Received skill improvement in current job training} \right) \\ & + 1.40 \times 1 \left(\text{Received training to maintain standards} \right) \\ & + 2.12 \times 1 \left(\text{Received training to help get a promotion} \right) \end{aligned}$$

With training being collapsed into a single dimension which captures the most important categories, it is helpful to investigate how such training is distributed throughout the economy. To do this, I consider simply the raw averages of effective training

Training Coefficient	Baseline Regression	Random Effects	No Adjustment for >3 Spells	6 Hour Cutoff	24 Hour Cutoff	No Controls	$\tau = \log(\text{hours} + 1)$ Transformation
job induction	0.0139936 (0.0171424)	0.0172821 (0.0132493)	0.0162692 (0.0172235)	0.0210641 (0.0166871)	0.0111441 (0.0195431)	0.027887** (0.0118515)	0.0037635 (0.004155)
skill improvement in current job	0.0143787** (0.0057607)	0.0157975*** (0.0044787)	0.0134724** (0.0058402)	0.0120007** (0.053273)	0.022192*** (0.0069729)	0.0087481** (0.0043075)	0.0043111*** (0.0015599)
maintain status/standards	0.0140474** (0.0057649)	0.0136583*** (0.0045659)	0.0153348*** (0.0057913)	0.0120283** (0.0053085)	0.021887*** (0.00636)	0.0082787* (0.004247)	0.0047755*** (0.0015398)
prepare for future job	-0.0072271 (0.0096344)	-0.0043716 (0.0081149)	-0.0073543 (0.0097852)	-0.0023147 (0.0094171)	-0.0045693 (0.0111047)	0.0055752 (0.0069827)	-0.0011118 (0.0024713)
help get a promotion	0.0212456** (0.0085033)	0.0183913** (0.0078025)	0.0234608*** (0.0085808)	0.0189171** (0.0079532)	0.0253268*** (0.0091795)	0.0116058** (0.0058535)	0.0049157** (0.0020667)
health/ safety	0.0101121 (0.0064976)	0.0073067 (0.0057007)	0.0069045 (0.006751)	0.001614 (0.0057562)	0.0077851 (0.0081286)	0.0016588 (0.0052587)	0.0023448 (0.0017489)
hobbies/leisure	-0.007089 (0.0286248)	0.0021443 (0.0285695)	-0.0098884 (0.0033486)	0.0033486 (0.0254906)	0.0025077 (0.0382002)	0.0046498 (0.0219299)	0.0010284 (0.0075727)

Table 2: *p<0.1, **p<0.05, ***p<0.01, Standard errors are clustered at the individual level, with the exception of Random Effects which uses maximum likelihood estimation. The Baseline regression uses a 12 hour cutoff to binarize the training indicators, includes the controls, adjusts for >3 training spells, and uses a Fixed Effects estimation approach. Each of the respective robustness checks changes one key aspect of the baseline approach.

across different groups - namely - by education level, firm size, occupation and industry. Figures 1 through 4 demonstrate the variation in training amongst different groups.

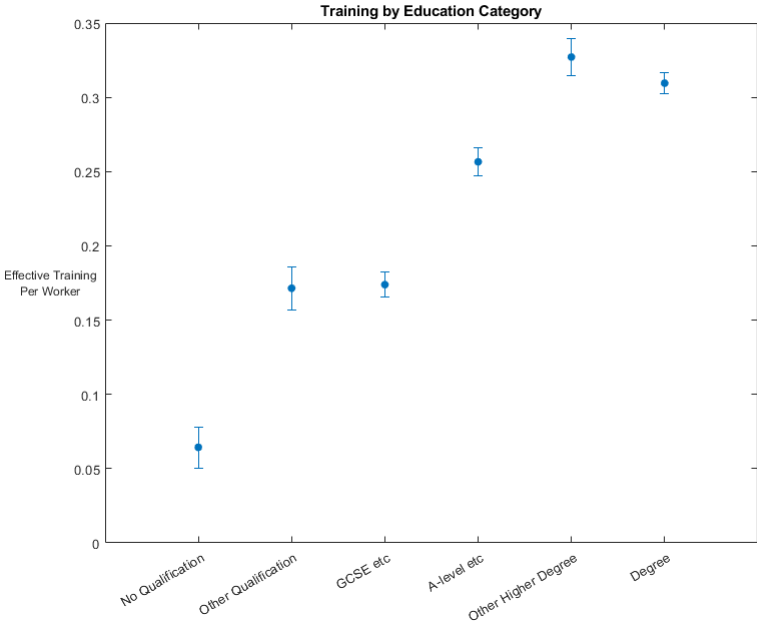


Figure 1: Average Effective Training Per Worker (Annually) By Education Level (standard errors clustered at the individual level)

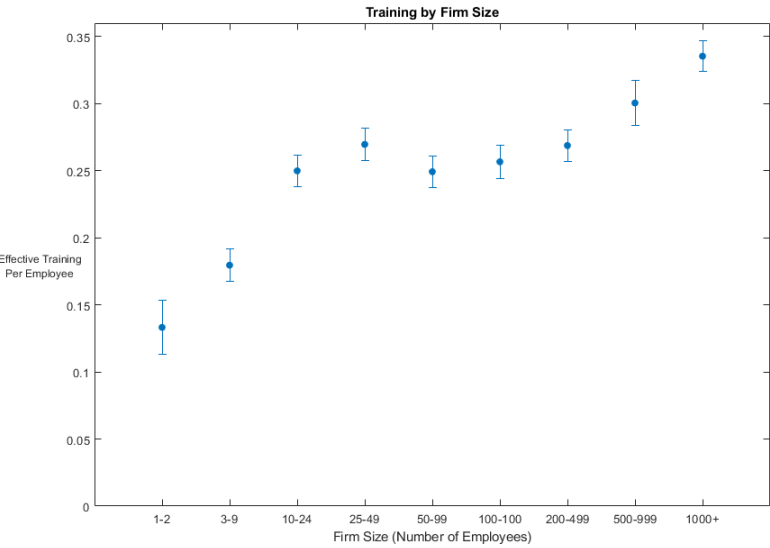


Figure 2: Average Effective Training Per Worker (Annually) By Firm Size (standard errors clustered at the individual level)

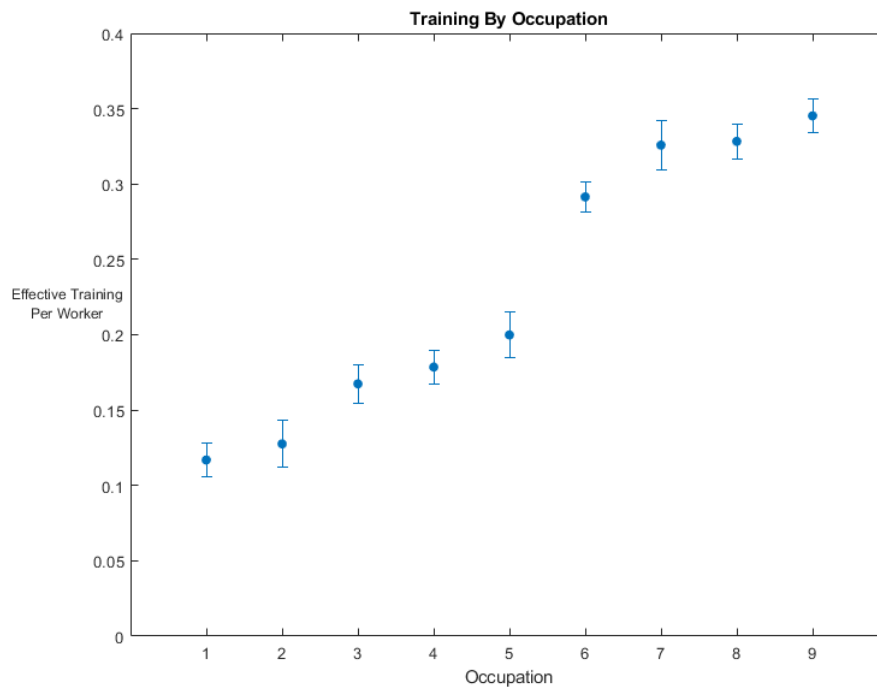


Figure 3: Average Effective Training Per Worker (Annually) By Occupation (Ordered from lowest to highest). Intervals denote 95% confidence intervals (standard errors clustered at the individual level)

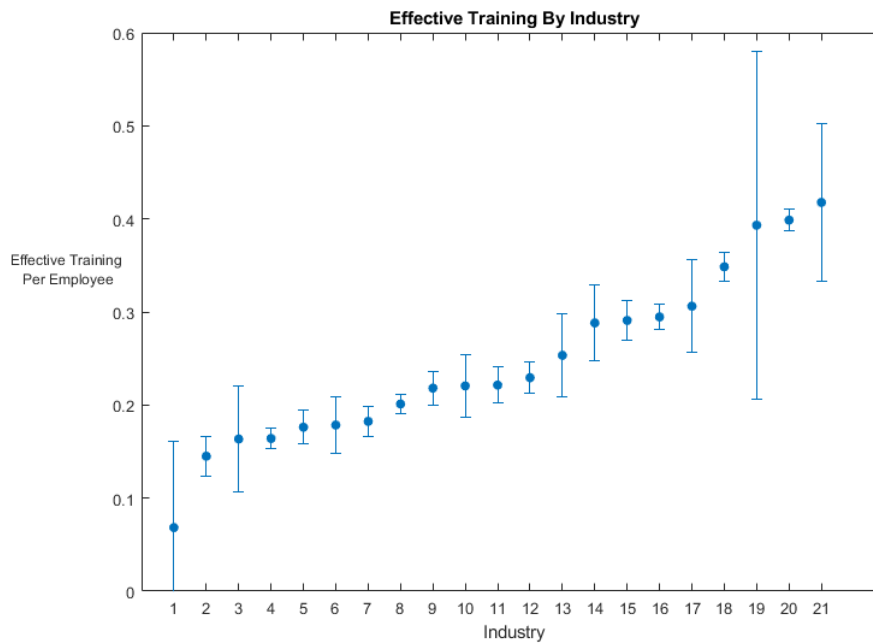


Figure 4: Average Effective Training Per Worker (Annually) By Industry (standard errors clustered at the individual level)

While not perfectly monotone, the better-educated workers tend to receive more training, and larger firms tend to provide more. The variation in training by education type is quite striking, with degree holders receiving nearly double the level of those with A-levels only. When it comes to comparisons across occupations and industries, unsurprisingly there is much variation between groups. The list of occupations and industries is given in Table 3

Occupation List	
1	Elementary occupations
2	Sales and customer service occupations
3	Process, plant and machine operatives
4	Administrative and secretarial occupations
5	Skilled trades occupations
6	Managers and senior officials
7	Personal service occupations
8	Professional occupations
9	Associate professional and technical occupations
Industry List	
1	Activities Of Households As Employers; Undifferentiated Goods- And Services-Producing Activities Of Households For Own Use
2	Accommodation And Food Service Activities
3	Agriculture, Forestry And Fishing
4	Wholesale And Retail Trade; Repair Of Motor Vehicles And Motorcycles
5	Administrative And Support Service Activities
6	Arts, Entertainment And Recreation
7	Transportation And Storage
8	Manufacturing
9	Information And Communication
10	Other Service Activities
11	Construction
12	Professional, Scientific And Technical Activities
13	Water Supply; Sewerage, Waste Management And Remediation Activities
14	Real Estate Activities
15	Financial And Insurance Activities
16	Education
17	Electricity, Gas, Steam And Air Conditioning Supply
18	Public Administration And Defence; Compulsory Social Security
19	Activities Of Extraterritorial Organisations And Bodies
20	Human Health And Social Work Activities
21	Mining And Quarrying

Table 3: List of Occupations and Industries

Unlike education and firm size, occupations and industries are unordered groups. However, one way of thinking about ordering is through the average wages that the groups tend to pay. Figures 5 through 8 demonstrate the robust relationship between groups that tend to earn/pay more and groups that tend to train more.

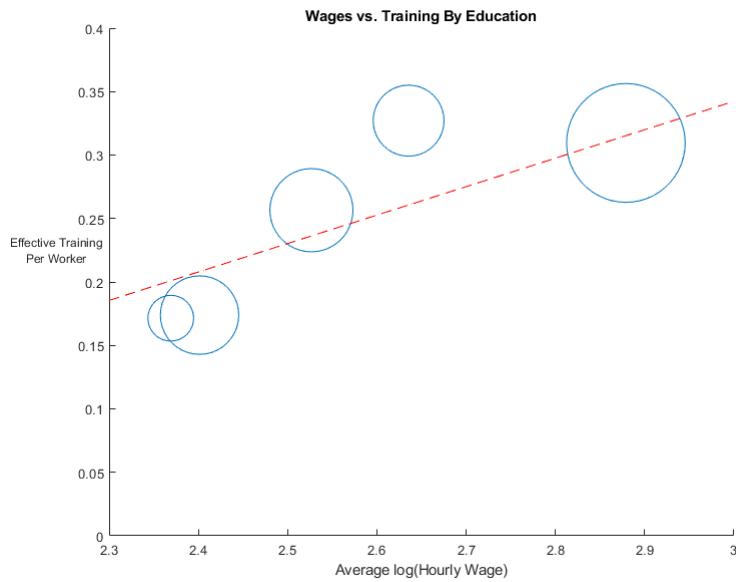


Figure 5: Scatterplot of log-wages and effective training, with workers grouped by education type

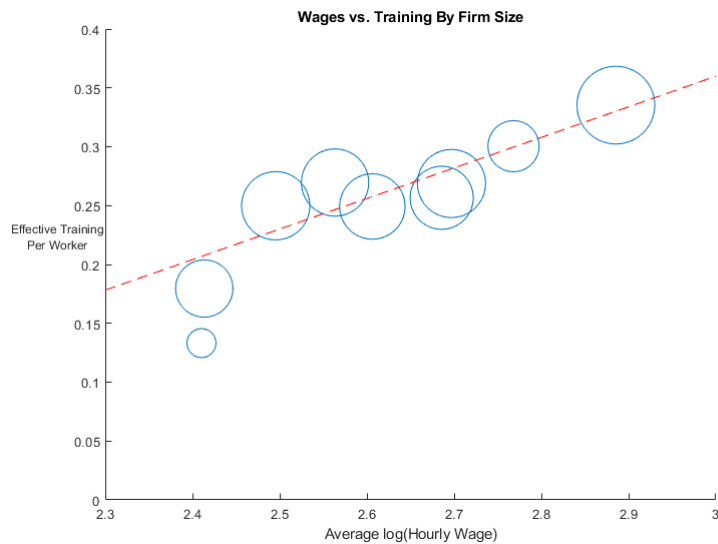


Figure 6: Scatterplot of log-wages and effective training, with workers grouped by the firm size of their employer

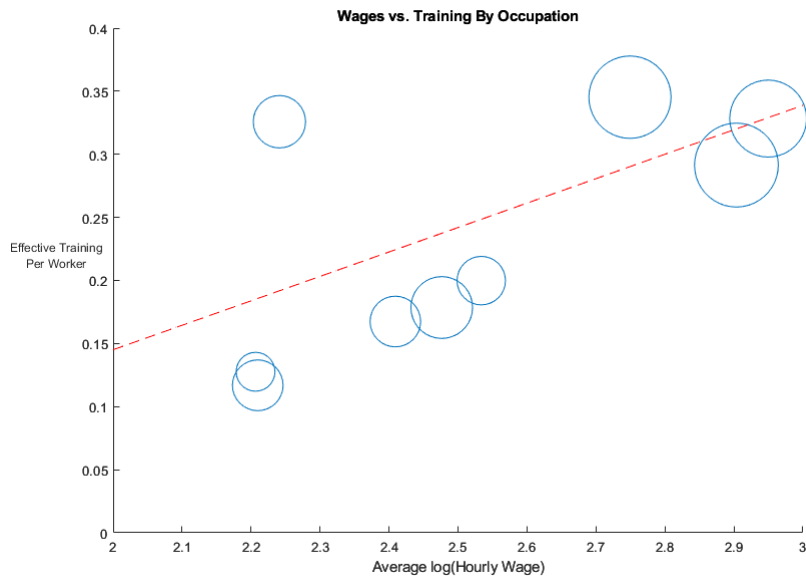


Figure 7: Scatterplot of log-wages and effective training, with workers grouped by the firm size of their occupation

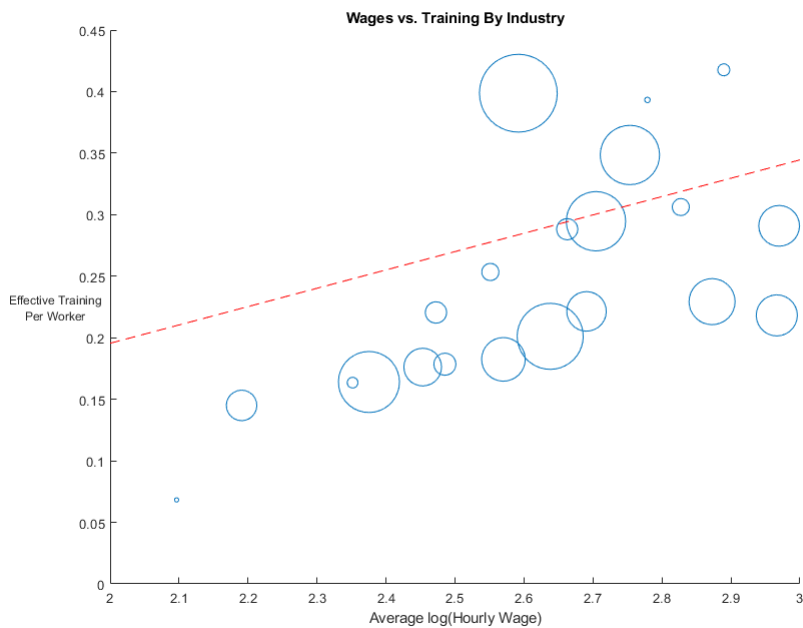


Figure 8: Scatterplot of log-wages and effective training, with workers grouped by the firm size of the industry of their employer

The occupation group to the northwest of Figure 7 is "Personal Service Occupations" which tend to involve a relatively large amount of training despite the relatively low wages. Nonetheless, there is a positive relationship between training and wages that is robust over different groupings of workers. This general relationship is consistent both with the idea that increased training may lead to higher wages and that work-

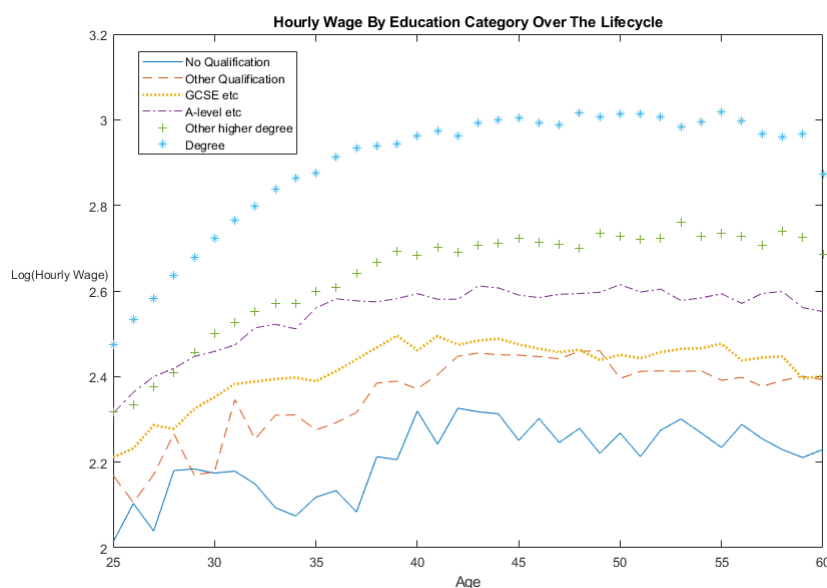


Figure 9: Average Log-wage by education level and age, controlling for time fixed effects.

ers with high earning ability may tend to be more likely to receive training, whether that be specifically aimed at them or via sorting into occupations/industries or firms that have higher levels of training. This paper does not attempt to pick apart these various mechanisms, but instead provides a simple model that can replicate the facts. The fact that higher educated workers tend to get more training over their lifetime could be a mechanism which causes dispersion over the lifecycle. Indeed, average wages across education groups tend to spread out, even in logarithmic terms, as workers age; not only do degree holders have higher wages than their A-level counterparts at age 25, but they experience a faster rate of growth through much of their career (this is shown in Figure 9). The fact that workers with higher wages tend to receive more training is interesting from a welfare perspective; one would think that training those who have already accumulated much human capital may be problematic from the perspective of contributing to inequality. There are reasons why this is desirable however; the jobs that high-skilled workers end up in may benefit much more greatly than those that their low-skilled counterparts do for example. Nevertheless, a concern that arises with investment is that in the presence of sufficiently high market power by firms, there may be inefficiently low investment. If firms have substantial bargaining power over the worker, they may be able to extract some of the value of a worker's human capital. This is particularly true if the worker's outside options are weak, which may be the case with low-skilled workers. With these ideas in mind, I proceed to develop a model which has the ingredients necessary to address some of these issues.

4 Model

Time is continuous. There exists a continuum of workers and firms, with the measure of workers normalised to 1. Workers (firms) aim to maximise the lifetime sum of their wages (profits), which is subject to discount rate ρ_{dis} , which applies to all workers and firms. At any given time, there may be flows in and out of the labour force, and in and out of employment. A “match” is a worker-firm pair that can produce a combination of output and human capital investment. The state that a match with worker i and firm j finds themselves in can be characterised among several dimensions. The first is the general human capital of the worker, h_{it} . General human capital has a linearly-spaced discrete support, with R possible values:

$$h \in \{h_0, h_1, \dots, h_R\}.$$

The second dimension is firm-specific human capital f_{it} , which is binary i.e. $f_{it} \in \{0, 1\}$. The third is their age; age_{it} , which is either young or old i.e. $age_{it} \in \{\text{young}, \text{old}\}$. Workers enter the model as young, transition to being old at rate ρ_{old} , and once old they retire at rate ρ_{exit} . The last dimension is the type of their employer; a_j which has a discrete support with G levels:

$$a \in \{a_0, a_1, \dots, a_G\}.$$

Such a match has a potential output (flow) of:

$$y_{ijt}^{potential} = a_j e^{h_{it} \theta f_{it}}$$

The multiplicative effect of a_j means that the firm type can be thought of as a productivity type. The fact that this function is supermodular in the firm type and general human capital will tend to encourage positive assortative matching between these two measures. $\theta > 1$ is the ratio of potential outputs between two equivalent matches, with one having firm-specific capital, and the other not. The inclusion of firm-specific human capital in addition to general capital is done to add some realism to the model; it will encourage more training when workers start a new job, and allow matches to develop a “love”. Notice that the exponentiation of h_{it} means that each climb of the human capital ladder has a multiplicative effect on production; this convexity will generally encourage more investment in workers with higher human capital. The investment process works in the following manner. The match may choose the rate of human capital investment, $\rho_{invest} \geq 0$. A successful investment means that the worker moves up one unit of the human capital distribution (unless they are already at the top level), and gains firm-specific human capital (if not already acquired). This investment is costly; if the match chooses this rate ρ_{invest} , their realised output is:

$$y_{ijt}^{realised} = a_j e^{h_{it} \theta f_{it}} - \frac{c}{2a_j} \rho_{invest}^2$$

$c > 0$ is a cost of investment parameter. The quadratic functional form is chosen for its simplicity and the fact that the costs are inversely proportional to a_j means that the firm productivity scales both output and investment capability in a Hicks-Neutral manner. This will help tie together the idea that higher-paying firms also tend to train

their workers more. Matches are also subject to exogenous job separation which occurs at rate $\rho_{separate}$. The match may choose to split up at any time, in which case the firm becomes a vacancy (characterised by its type) and the worker becomes unemployed (characterised by their general human capital and age only). This can occur in the equilibrium; the gain of human capital can lead to a state in which the match opts to split up. The notation for the value of vacancies and unemployed workers are $V(a)$ and $W(h, \text{age})$ respectively; which will be given more attention later. The value of a match is denoted by $J(h, f, \text{age}, a)$. If the worker is young, the value function is given by:

$$\begin{aligned} \rho_{dis}J(h_r, f, \text{young}, a) = \max_{\rho_{invest}} \bigg\{ & ae^{(h_r)}\theta^f - \frac{c}{2a}\rho_{invest}^2 \\ & + \rho_{invest} \left(\hat{J}(h_{\min\{R, r+1\}}, 1, \text{young}, a) - J(h_r, f, \text{young}, a) \right) \\ & + \rho_{age} \left(\hat{J}(h_{\min\{R, r\}}, f, \text{old}, a) - J(h_r, f, \text{young}, a) \right) \\ & + \rho_{dep} \left(\hat{J}(h_{\max\{1, r-1\}}, 0, \text{young}, a) - J(h_r, f, \text{young}, a) \right) \\ & + \rho_{sep} \left(W(h, \text{young}) + V(a) - J(h_r, f, \text{young}, a) \right) \\ & \bigg\} \end{aligned}$$

The notation \hat{J} is used to represent the maximum of the joint value by continuing to stay together, or separation i.e.

$$\hat{J}(h^r, f, \text{young}, a) = \max\{W(h, \text{young}) + V(a), J(h_r, f, \text{young}, a)\}.$$

Similarly, if the worker is old, then the value function is much the same, but instead of facing the ageing shock, there is the exit shock, which results in the worker receiving a value of zero and the firm becoming a vacancy:

$$\begin{aligned} \rho_{dis}J(h^r, f, \text{old}, a) = \max_{\rho_{invest}} \bigg\{ & ae^{(h_r)}\theta^f - \frac{c}{2a}\rho_{invest}^2 \\ & + \rho_{invest} \left(\hat{J}(h_{\min\{R, r+1\}}, 1, \text{old}, a) - J(h_r, f, \text{old}, a) \right) \\ & + \rho_{exit} \left(V(a) - J(h_r, f, \text{old}, a) \right) \\ & + \rho_{dep} \left(\hat{J}(h_{\max\{1, r-1\}}, 0, \text{old}, a) - J(h_r, f, \text{old}, a) \right) \\ & + \rho_{sep} \left(W(h, \text{old}) + V(a) - J(h_r, f, \text{old}, a) \right) \\ & \bigg\} \end{aligned}$$

Notice that the assumption that investment is chosen to maximise the joint value of the match implicitly relies on the idea that the worker and firm can contract on future levels of training, wages, and separation decisions. If this level of commitment were not possible, then there may for example be hold-up inefficiencies that could arise. This will serve as a benchmark, but other potential mechanisms could be considered, for example:

1. The worker chooses the level of investment, then Nash bargaining over the wage occurs (with unemployment & vacancy as the outside options)

2. The firm chooses the level of investment, then Nash bargaining over the wage occurs (with unemployment & vacancy as the outside options)
3. The level of investment and the wage are simultaneously decided via Nash bargaining (with unemployment & vacancy as the outside options)

I show in appendix B that all three schemes are equivalent to the bilaterally-efficient investment decision that is used as the benchmark, and discuss variations to the model where bilateral efficiency would fail. In unemployment, a worker with human capital h receives flow value $ba_1 e^h$ (where $b < 1$ ensures that the worker does not obtain more than they would if they took all of the output working at the worst firm type), meets a firm at rate ρ_{meet} , and is still subject to age/exit shocks. Matching is random, so conditional on meeting a vacancy, the probability that the vacancy is of type “ a ” is given by the proportion of vacancies that are type a . $E_{a'}[\dots]$ is used crudely to denote the expected value given that a' follows the distribution of vacant firm types. The value functions for young and old unemployed workers are respectively:

$$\begin{aligned}\rho_{dis}W(h_r, \text{young}) &= ba_1 e^{(h_r)} \\ &\quad + \rho_{meet} E_{a'} \left[\eta (\hat{J}(h_r, f, \text{old}, a') - W(h_r, \text{young}) - V(a)) \right] \\ &\quad + \rho_{age} (W(h_r, \text{old}) - W(h_r, \text{young}))\end{aligned}$$

$$\begin{aligned}\rho_{dis}W(h_r, \text{old}) &= ba_1 e^{(h_r)} \\ &\quad + \hat{\rho}_{meet} E_{a'} \left[\eta (\hat{J}(h_r, f, \text{old}, a') - W(h_r, \text{old}) - V(a')) \right] \\ &\quad + \rho_{exit} (0 - W(h_r, \text{old}))\end{aligned}$$

The parameter η ($0 \leq \eta \leq 1$) denotes the share of the surplus of the match that the worker takes. The value function for vacancies is given by:

$$\rho_{dis}V(a) = -\nu + \hat{\rho}_{meet} E_{h', \text{age}'} \left[(1 - \eta) (\hat{J}(h', f, \text{age}', a) - W(h', \text{age}') - V(a)) \right].$$

$\nu > 0$ is a vacancy flow cost, and note that the firm’s share of the surplus, $1 - \eta$, is accounted for. Notice that the rate at which workers and firms meet potential matches is treated as exogenous, and the stock of firms is treated as fixed. This could be modelled explicitly with a free entry condition and matching function, but this is left out for simplicity; there are no business cycle effects at play. Instead, the model is calibrated such that the ratio of vacancies to unemployed workers is $\frac{\rho}{\hat{\rho}}$, which provides stock-flow consistency. The model is treated as a steady state, meaning the endogenous distribution of agents in the various possible states stays fixed over time. This is elaborated upon in the computational appendix.

5 Calibration And Results

Some of the parameters regarding the search structure of the model are calibrated following Hagedorn & Manovskii (2008). Their calibration implies a low bargaining

weight for the worker, but a fairly high outside flow value for the worker, which can replicate fluctuations in the DMP model over business cycles. The general human capital grid is set to match the range from the first to 99th percentiles of the log-wage distribution in the understanding society data. Although human capital and wages are not the same, this provides sufficient space to deal with a wide range of skill levels. This is split into 100 parts, meaning moving up one level corresponds to a 2.35% increase in the potential output that the worker is capable of (holding firm type and specific capital fixed). The firm-specific component is set to correspond to one of these rungs - this makes the initial period of training rather important as it can enhance the worker's productivity by $\sim 4.7\%$. The rate at which agents transition from young to old, and old to exiting is set to $\frac{1}{17.5}$, to correspond to two chunks of the 25-60 age range being studied. The bargaining power is taken straight from Hagedorn & Manovskii (2008), as is the value of outside work relative to in-work productivity (in my model this is set relative to the worst firm type). The remaining parameters are calibrated to bring in line moments in the model to that of the data. The separation rate is used to target unemployment in the model, with 5% being a broad average over the period of the survey in the UK. The vacancy cost is targeted relative to the average wage following Hagedorn & Manovskii (2008), as is the relative job filling to job finding rate. In my model, since some matches will reject one another upon meeting, meaning that the meet rate needs to be calibrated to ensure that the match rate hits its target. The cost of investment parameter, c is chosen to ensure that the average rate of wage growth in the data corresponds to that in the model. In the data, this is recovered by extracting the effect of age, controlling for time-fixed effects:

$$\log(\text{wage}_{it}) = \alpha_0 + \alpha_1 \text{age}_{it} + \delta_t + \varepsilon_{it},$$

where α_1 is the target. This removes drifts in long-run productivity movements which are not a part of the model. The distribution of firm types is normalised to be equally spaced around 1, with 15 types. The spread of these types is then calibrated to target the between-variation of log-wages by firm-size in the data. Formally, let \bar{w}_j be the average log-wage of employees at firms of size category j . In the data, the standard deviation (across the pool of workers) of \bar{w}_j is 0.149. Similarly, in the model, let \hat{w}_j be the average log-wage of workers at firm type j . Then the spread of firms is chosen to try and bring the standard deviation of \hat{w}_j close to 0.149. While the distribution of workers is endogenous, the entry distribution is not. This is set up to try and replicate different workers by education category in the data. To do this, the average log-wages of each education group from ages 25-30 are computed in the data, and compared relative to the no-qualification group. Similarly, in the model, the rungs at which workers enter are chosen to try and mimic the relative log-wages for workers 25-30. Since the early wages of A-level holders and other-higher-degree holders are similar, they are bunched into one group for the model. The proportion of workers entering the market by these different types is chosen to match that in the data directly. To recover some of the moments from the model, a simulation is run, which is further described in the computational appendix. The calibration is summarised in table 4 and the fit of moments in table 5

The model can replicate various features of training and wages as seen in the UK data. Like the data, the workers who start with more education (in the model, this

Externally Calibrated Parameters	
General Human Capital Grid	Ranges from 1st to 99th percentile of log-wage distribution, with 100 levels. This means each rung enhance productivity by 2.35%
θ	1.0235 (comparable to one rung on the general grid)
ρ_{old}	1/17.5
ρ_{exit}	1/17.5
ρ_{dis}	0.04 (Hagedorn & Manovskii (2008))
η	0.052 (Hagedorn & Manovskii (2008))
b	0.955 (Hagedorn & Manovskii (2008))
Internally Calibrated Parameters	
ρ_{sep}	0.19
v	6.5
c	17.45
Distribution of a	Uniformly spaced with 15 types. Minimum at 1-0.073, maximum at 1+0.073
ρ_{meet}	1.08×26
$\hat{\rho}_{meet}$	$\rho_{meet} \times (1/0.634)$ (Hagedorn & Manovskii (2008))
Human Capital Entry Distribution	5 Entry levels (rungs 25, 27, 32, 37, 47) with frequencies to match education groups

Table 4: Calibration summary

Relevant Parameter	Moment	Data	Model
ρ_{sep}	Rate of Unemployment	0.05	0.0484
v	Ratio of vacancy cost to average wage (Hagedorn & Manovskii (2008))	0.584	0.581
c	Average log-growth in wages by age, time effects removed	0.0106	0.0102
Distribution of a	Between log-wage variance by firm type (data: by firm size)	0.149	0.143
ρ_{meet}	Job match rate (since some may reject) (Hagedorn & Manovskii (2008))	0.257×26	0.257×26
Entry Rungs	Average log-wage age 25-30, relative to No Qualification Group:		
	No Qualification	0	0
	Other Qualification	0.0438	0.0474
	GCSE etc.	0.157	0.162
	A-level or other Higher Degree	0.280	0.279
	Degree	0.501	0.513

Table 5: The fit of moments to data and the relevant parameters

is simply a higher starting level of general human capital) tend to also have higher wages and a faster growth rate; this is shown in figure 11. The shape of the wage profiles is not humped like the data, however; this is a limitation of the binary ageing process and lack of depreciation (which does not appear in the model out of parsimony). The general human capital of the different groups also follows a similar pattern, with the higher educated workers accumulating general human capital at a faster rate (figure 12). The idea that investing more in the already high-skilled workers may have a higher payoff is baked into the model with the exponentiation of human capital in the production function. Another mechanism at play is sorting. On average, the higher firm types tend to employ high-skilled workers; this is demonstrated in figure 10. This means that high skilled workers tend to work at firms that are better at training. A pattern in the data that was rather robust was that groups of workers with higher wages also tend to receive higher training, whether they are grouped by education/firm-size/occupation/industry. Although the model does not have the same richness over occupations, industries, and firm size, it does have a notion of both education and *firm type*. With that in mind, like with the data, the wage-training patterns across these groups in the model exhibit a positive relationship, shown in figures 13 and 14.

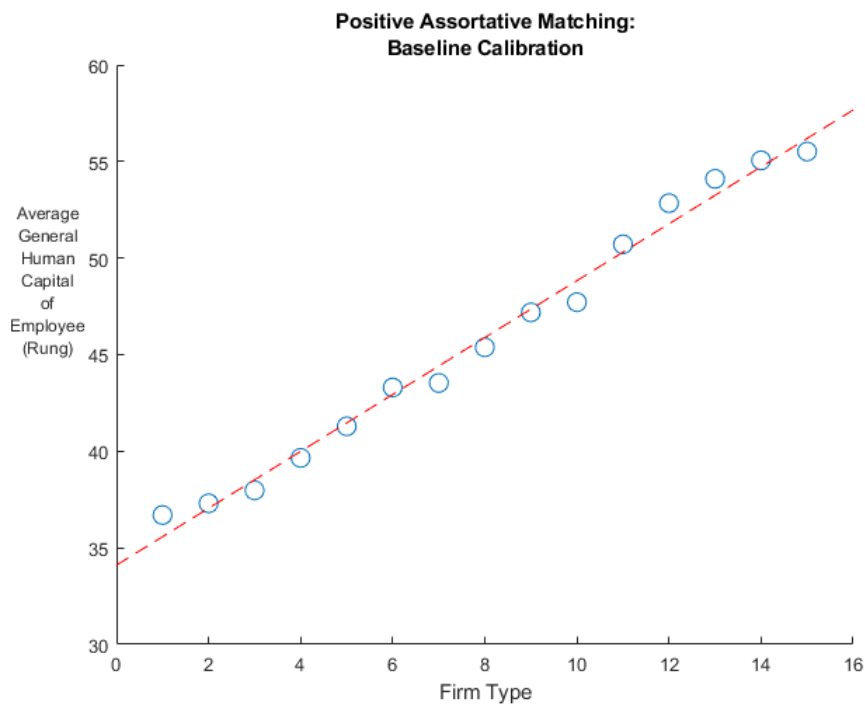


Figure 10: Bubbles show the average human capital of workers (in terms of the rung) employed at each firm type. The dotted line is the OLS regression line.

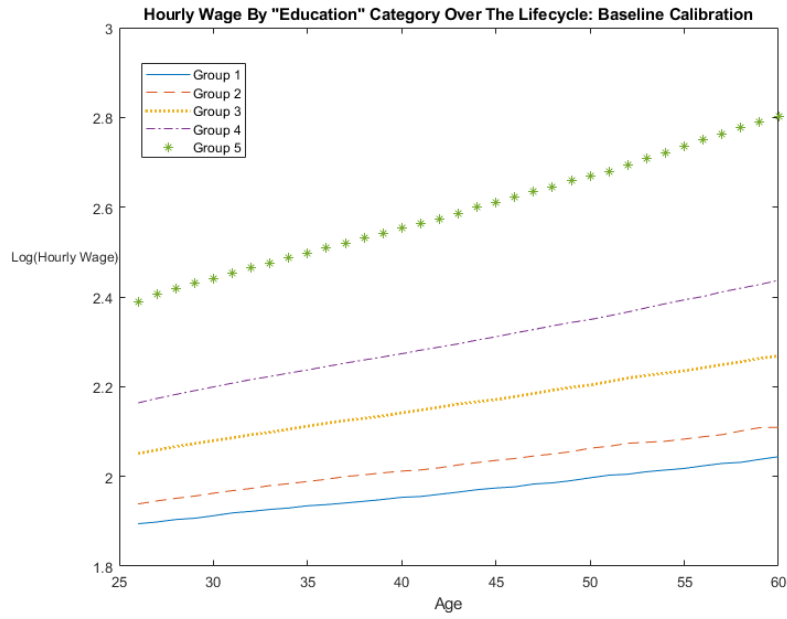


Figure 11: Average log-wage by “education group” in the model by age. Wages can be negative in the model. To deal with these values, values with $w < 1$ are dropped (the average wage is 11.89)

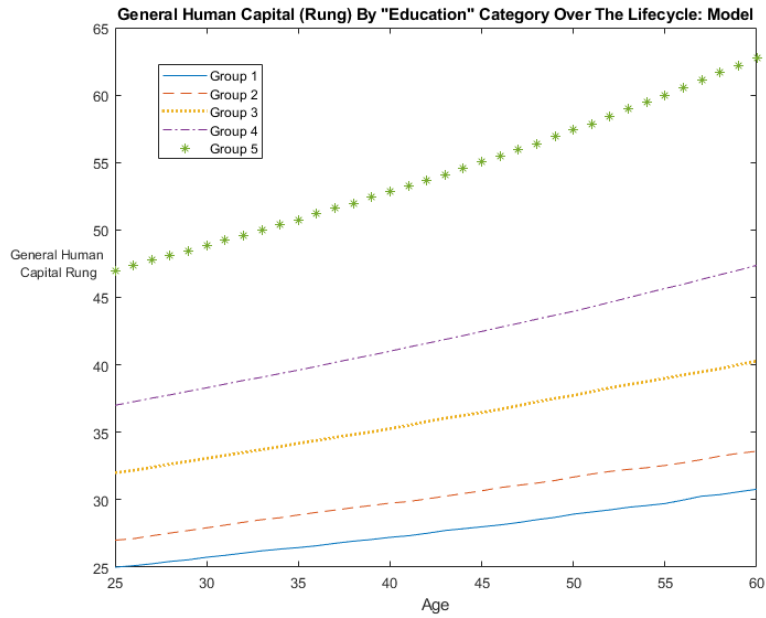


Figure 12: Average human-capital (in terms of which rung the agent is on) by “education group” in the model by age.

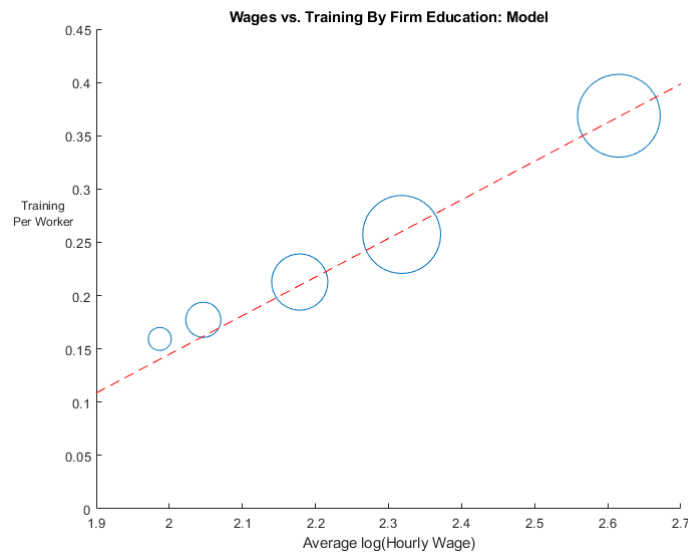


Figure 13: Wages and Training, workers grouped by “education” (model). The dotted line shows the weighted OLS regression line.

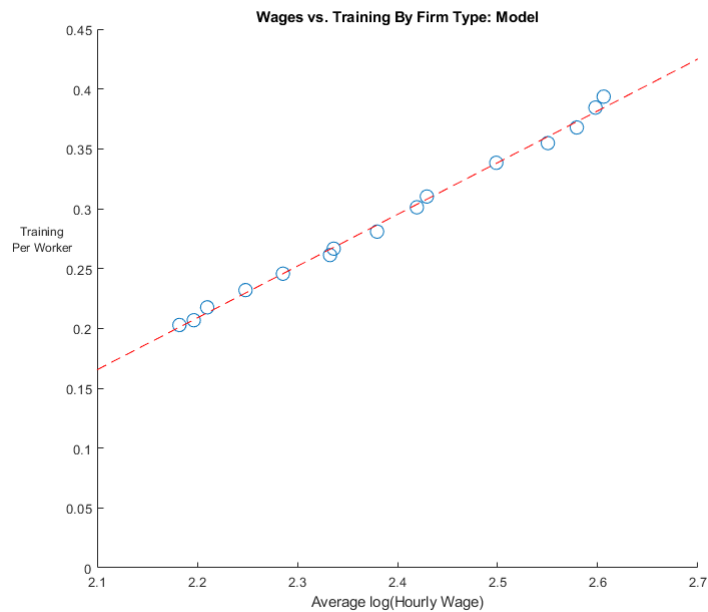


Figure 14: Wages and Training, workers grouped by firm type (model). The dotted line shows the OLS regression line.

Given that low-skilled workers tend to receive less training in this model, there is a welfare question about whether this is desirable from the point of social mobility. There is also a question of efficiency in this model; the bargaining power held by firms means that some of the return to training is extracted from the workers. This inefficiency is discussed in further detail in appendix C. While an “optimal” policy might consist of a set of state-dependent non-linear training subsidies, which get covered

by lump sum taxes, this is far from realistic. A narrower policy set is instead considered - a flat-rate subsidy on the training rate (ρ_{invest}) for young, low-skilled, workers which is funded via a flat-rate income tax. Such a policy deals with the concern about social mobility by focusing on the low-skilled, and the use of an income tax (in the model this is simply a production tax) means the social planner must balance off the gains from subsidising the under-invested workers against the inefficiencies of introducing taxation. Taxation on production also has the potential to reduce the desire to invest in human capital since it decreases its take-home return. Matches eligible for the subsidy effectively have a lower cost to implement training, but must face that some of the future gains from investment will be subject to the tax. I define low-skilled workers as those that have less general human capital than the starting level of general human capital as those in the highest education group. Young agents correspond to approximately those aged 25 to 43 given the binary aging process in the model. With this definition, this applies to 26.6% of the working population in the baseline calibration, which gives a sense of the broadness of the subsidy program. I find within this restricted class of policy that approximately 4% taxation is optimal from a utilitarian standpoint, generating a 6.7% improvement to steady-state aggregate welfare (defined by the average cross-section expected discounted value of all agents in the economy).

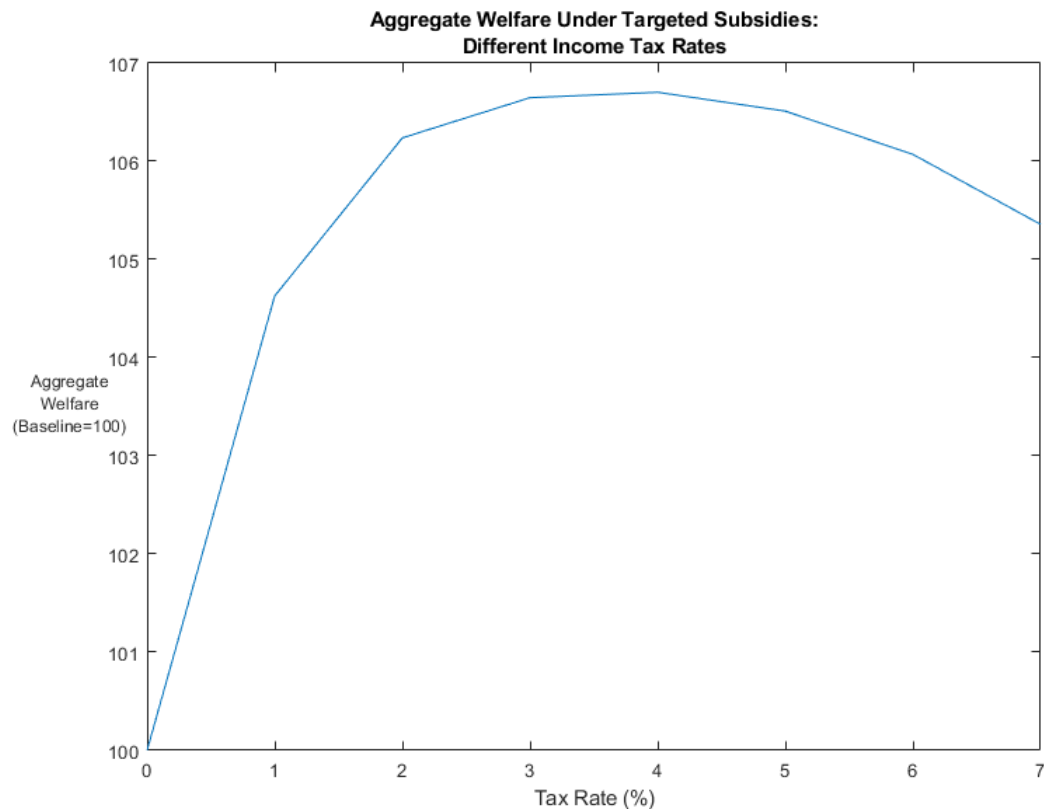


Figure 15: Aggregate welfare under different income tax rates. Income tax is collected at a flat rate on all non-negative production. This is then used to subsidise training, with the size of the subsidy being proportional to ρ_{invest} . The subsidy rate is set such that the policy is revenue-neutral.

Group	Change in Average Welfare in each group due to policy (%)
Low Skilled Young	+30.9
High Skilled Young	-5.3
Low Skilled Old	+1.7
High Skilled Old	-5.7
Firms	-7.1

Table 6: Welfare Effects of the Policy

Group	Proportion of Working Population under Baseline Calibration (%)	Proportion of Working Population under Tax/Subsidy Policy (%)
Low Skilled Young	26.6	18.3
High Skilled Young	23.4	31.7
Low Skilled Old	23.1	14.5
High Skilled Old	26.9	35.5

Table 7: The distributional effects of the policy

There are welfare gains and losses to different agents in this setting as well as large distributional changes associated with this policy, as shown in tables 6 and 7. Unsurprisingly, low-skilled young workers are better off under the policy, with the losses occurring to high-skilled agents and in terms of profits. The low-skilled older workers see a small benefit, which may be in part due to reduced competition from their low-skilled younger counterparts. The policy has a large effect on the distribution of workers due to the ability of many to escape from low levels of human capital; so even though the average low-skilled young worker is substantially better off after the reform, there are fewer of them around due to the increase in those who reach a high skill level.

Under the subsidy, the growth in the general human capital of the lower-educated workers is much higher, giving them a chance to catch up with the most educated group. The sorting between workers and firms is also substantially weaker; the average general human capital across the different firm types is rather similar.

While the policy seems attractive for various reasons - there is an improvement to social mobility in terms of human capital, low-skilled workers get more of an opportunity to work at better firms, and there is an overall improvement to welfare, caution must be exercised with the magnitudes of these effects. Changes to labour market outcomes aged 25-60 will undoubtedly impact the education and early training decisions up to that point; the model treats the distribution of these agents as fixed. Implementation of the policy also assumes the social planner can observe the training rate as well as be able to reliably estimate the human capital of the worker, which has difficulties in practice. Furthermore, the inefficiencies that are associated with raising taxes are limited to possible distortions in the human capital investment process; this does not capture other potential inefficiencies such as investment in physical capital, and labour supply effects.

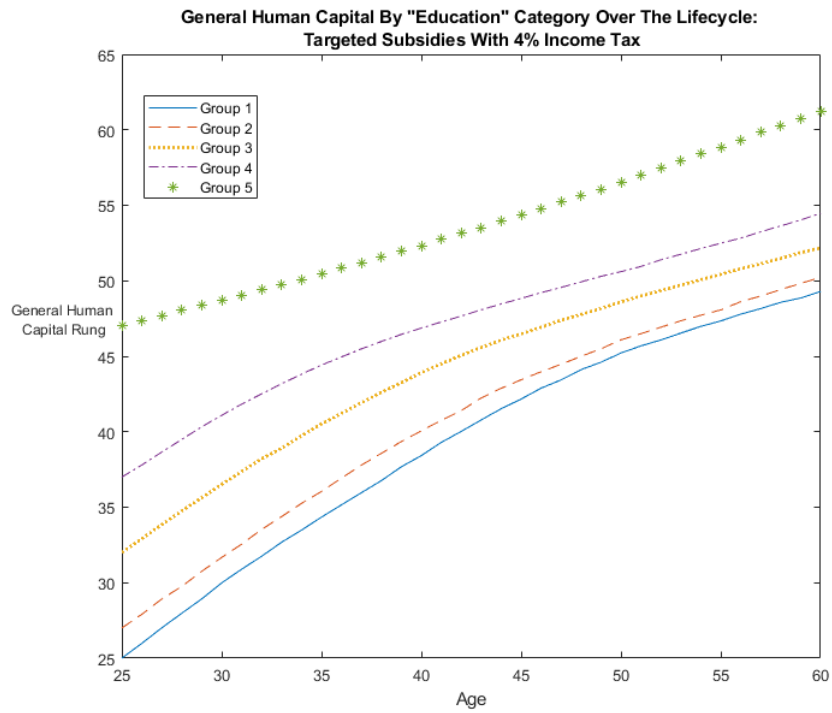


Figure 16: Average general human capital (in terms of the rung) by education group over age, when the counterfactual subsidy is implemented.

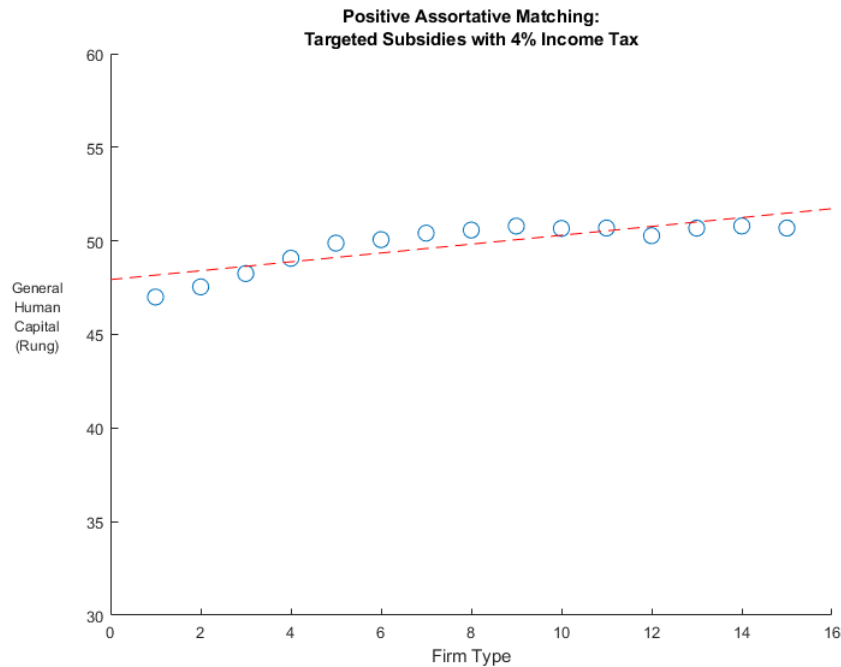


Figure 17: Bubbles show the average general human capital (by rung) employed at each firm type when the counterfactual subsidy is implemented. The dotted line shows the OLS estimate.

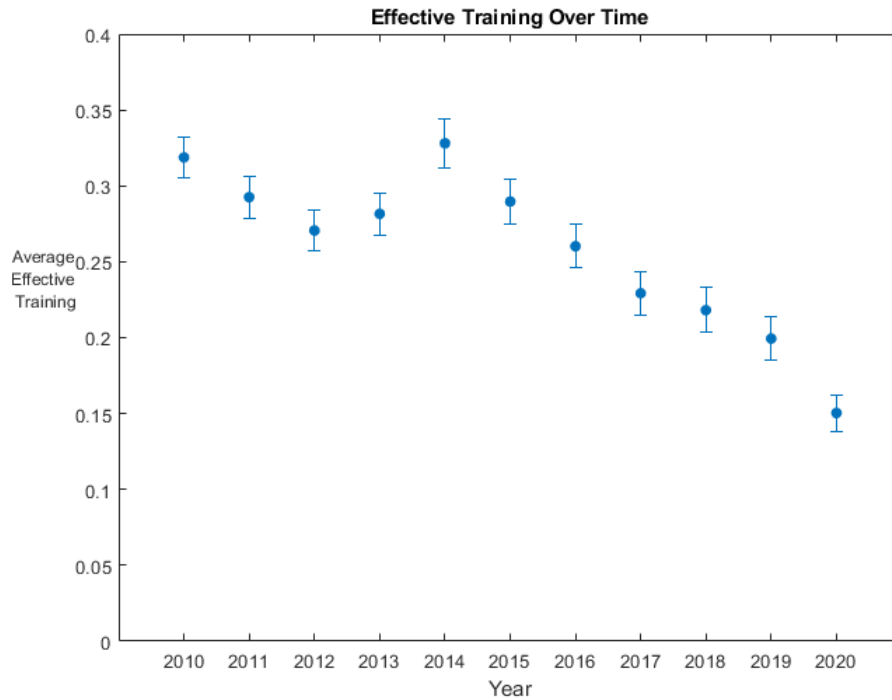


Figure 18: Understanding Society data: effective training by year in the restricted sample. Intervals show 95% confidence intervals, using clustered standard errors at the individual level.

Regardless of these shortcomings, the exercise is designed to provoke; the effects of small worker bargaining power and search frictions are potentially very large with regards to under-investment in human capital, and even less-than-ideal policy can improve both social mobility and aggregate welfare. These issues are, perhaps, of growing importance, since there appears to be a decline in employer training over time according to the effective training measure. While the model is not designed to account for such a shift, the effective training measure shows a stark decline over the period of the sample (see figure 18). Other researchers have also found a decline in training over time in the UK amongst other groups and over longer periods. Green et al. (2016) also find large declines in workplace training between 1997 and 2012 in the UK. While there has been some renewed interest in reforms to education to emphasise routes that are less traditionally academic (e.g. the introduction of T levels in England), the training that workers receive from their employers throughout the rest of their career while more embedded in the labour market may also be a useful channel for policymakers to consider.

6 Conclusion

This paper's main contribution is a two-sided heterogeneous model of the labour market that includes endogenous investment decisions by the matches. This model can replicate key facts; workers who start with higher human capital start with higher wages

and experience faster wage growth, workers who enter with higher human capital tend to have more training throughout their lives, and firm types that pay high wages also tend to provide higher levels of training. This is documented in the Understanding Society data, which shows that the positive relationship between wages and training holds, whether the data is grouped by education, firm size, occupation, or industry. From the perspective of social mobility, this is troubling as it means low-skilled workers may struggle to keep up with their high-skilled counterparts. The model of this paper contains search frictions. In a perfectly competitive environment, the private returns to investing in human capital would be equal to the social returns, but in this model, some of the returns to investments are captured by future employers, who are not involved in the investment decision. Furthermore, the quit and hiring decisions need not be socially optimal under the random search framework. While it is a rather crude and artificial policy, I find that aggregate welfare can be improved by subsidising low-skilled, younger workers, with beneficial effects on social mobility. Further research into this topic could proceed in various directions. First is the use of more extensive data to check the robustness of the empirical findings. The second is to make tweaks to the model, either to add realism or further test the robustness of the findings (there are many avenues here - more dimensions to human capital, depreciation, permanent learning abilities, on-the-job search, directed search, training being bargained separately from wages, firms that vary by both production and training ability, explicit market entry conditions for firms along with a matching function, and a realistic ageing process to name a few). Thirdly in the calibration, we could use different approaches for the likes of the bargaining power and outside options. While these may bring improvements over the approach used in this paper, this one stands to make the point that there are potentially large inefficiencies in the provision of employer-provided training to workers in the presence of plausible market frictions.

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Appendices

A Computational Approach

In order to solve the model, there are in effect two main blocks that must be considered; the value functions and the distributions of the agents. The value space may be represented with three arrays; a (101×2) array, W , which denotes the value to unemployed workers by general human capital and age, (15×1) vector V which denotes the value of being a vacancy for the 15 respective firm types, and $(101 \times 2 \times 2 \times 15)$ array J , which denotes the joint values for each of the possible combinations of match over general human capital, firm specific capital, age, and firm type. Similarly, the distribution block consists of corresponding arrays that describe the measure of agents of each type, with (101×2) array $Work$, (15×1) vector Vac and $(101 \times 2 \times 2 \times 15)$ array $Measure$ which describes the measure of agents of each type who are unemployed, vacant firms, and joint matches respectively. 1 unit of “Measure” becomes 1 unit of “Work” and 1 unit of “Vac” if a match breaks up, so the units in “Measure” describe the number of worker-firm pairs. The value functions are all initialised at zeros, while “Measure” is set to zeros, and “Work” and “Vac” are set to arbitrary non-zero values (this avoids division by zero when using the random matching process). Also recorded in each iteration are the matching indicators, “Mind”, and the investment rates “Optimal” (which are each $(101 \times 2 \times 2 \times 15)$). Then, in each round of iteration, all of these arrays are updated. This is done in the following steps, which provide an overview of the main loop:

1. For each combination of $\{h, f, age, a\}$ $\hat{J}(h, f, age, a) = \min\{J(h, f, age, a), W(h, age) + V(a)\}$
2. Compute the optimal investment rates by first computing the “upgrade benefit”, defined as $\hat{J}(h + 1, 1, age, a) - J(h, 1, age, a)$, and the $Optimal(h, 1, age, a) = \frac{a}{c} \times$ “upgrade benefit”
3. Given the optimal rate, compute the realised output for each match
4. It is now trivial to run through all the different matches and create J_{new} , based on the recursive equations
5. With unemployed workers, the expected gain from a meet must be computed from the weight of the relative vacancy type by the gain conditional on meeting

that type i.e.

$$\text{Expected Gain From Meet } (h, \text{age}) = \eta \sum_{j=1}^{15} \frac{Vac(j)}{Sum(Vac)} (\hat{J}(h, 1, a_j, \text{age}) - W(h, \text{age}) - V(a))$$

6. Once the expected gain from a meeting is recovered, W_{new} is constructed for each h , age, using the recursive equation for W
7. For vacancies, the process is similar to unemployed workers; V_{new} is constructed
8. Now for distributions; a tuning parameter "tdis" (=0.005) is used to simulate small adjustments to the distributions. Initially set $Work_{new} = Work$, $Vac_{new} = Vac$ and $Measure_{new} = Measure$
9. Now run through every possible match, unemployed worker, and vacancy, and simulate the flows of agents between possible states, concerning the optimal investment rates in "Optimal", and the Match indicators in "Mind". Various factors must be kept in mind:
 - Some matches will upgrade their human capital. Depending on the match indicator associated with this new human capital level, they either flow into being a match with this changed human capital, or they flow into unemployment and vacancy respectively, with the worker's human capital updated
 - When workers age from young to old, they may similarly break out of a match
 - When workers exit, they are replaced by young unemployed workers that are in total the same mass as the exiting agents, but distributed according to the exogenous entry distribution. If currently in a match, the firm must be moved into a vacancy
 - When accounting for the movement of unemployed workers and vacancies into matches, this is only done from the worker's side. This is because, if it is done from the firm's side, there will be double-counting
 - Once all the distributions have been updated, there should be a "Worknew", "Vacnew" and "Measurenew" which record the new distribution, as if tdis \times one year has passed
 - The total measure of vacancies, in "Vacnew" should be rescaled such that the ratio: $\frac{sum(Vac_{new})}{sum(Work_{new})} = \frac{\hat{\rho}_{meet}}{\rho_{meet}}$
 - One way to check the distributions have been updated properly is that the total measure of workers should be unchanged i.e.

$$sum(Work_{new}) + sum(Measure_{new}) = sum(Work) + sum(Measure)$$

- The “convergence measure” is defined by measuring the differences in the main objects:

$$\begin{aligned}
\Delta W &= W_{new} - W, \\
\Delta V &= V_{new} - V, \\
\Delta J &= J_{new} - J, \\
\Delta Wk &= Work_{new} - Work, \\
\Delta Vc &= Vac_{new} - Vac, \\
\Delta M &= Measure_{new} - Measure
\end{aligned}$$

$$\begin{aligned}
\text{convergence measure} &= \frac{1}{2} \frac{|\Delta W| + |\Delta V| + |\Delta J|}{|W_{new}| + |V_{new}| + |J_{new}|} \\
&+ \frac{1}{2\text{tdis}} \frac{|\Delta Wk| + |\Delta Vc| + |\Delta M|}{|Work_{new}| + |Vac_{new}| + |Measure_{new}|},
\end{aligned}$$

(where $|x|$ is used to denote the sum of absolute values of all elements in x)

- To stop the value functions overshooting between iterations, a tuning parameter, $tval$ ($=0.005\rho_{dis}$) is introduced, and the value functions are overwritten with:

$$\begin{aligned}
W_{new} &= (1 - tval) \times W + tval \times W_{new}, \\
V_{new} &= (1 - tval) \times V + tval \times V_{new}, \\
J_{new} &= (1 - tval) \times J + tval \times J_{new}.
\end{aligned}$$

- The loop is then repeated until a satisfactory level for the convergence measure is reached (≤ 0.0025)

After the model has been solved (in terms of the value functions, steady-state distribution, optimal investment rates and match indicators), the match rate is computed (since this is a targeted moment) by first computing the probability that a worker of a particular type, h , age, finds a match:

$$\text{match-rate}(h, \text{age}) = \rho_{meet} \times \sum_{j=1}^{15} \text{Mind}(h, 1, \text{age}, a) \frac{Vac(a_j)}{\text{sum}(Vac)}.$$

The overall match rate can then be computed by taking the average of match rates, weighted by the proportion of each unemployed type:

$$\text{match-rate}^* = \sum_{i=1}^R \sum_{\text{age}=1}^2 \text{match-rate}(h, \text{age}) \frac{Work(h_r, \text{age})}{\text{sum}(Work)}.$$

The aggregate welfare may be computed by weighing the different measures of agents by their associated values;

$$\text{aggregate-welfare} = \text{Work} \cdot W + \text{vac} \cdot V + \text{Measure} \cdot J.$$

Similar breakdowns for low-skilled young... etc. can be computed similarly by restricting the summation to the relevant subpopulation. Wages are not required to solve the model but can be inferred from the value functions. To see how this works, first construct W^e , the value of an employed worker using:

$$W^e(h, f, \text{age}, a) = W(h, \text{age}) + \eta \left(\hat{J}(h, f, \text{age}, a) - W(h, \text{age}) - V(a) \right)$$

Notice that this will take on the value of unemployment if the match indicator is zero. Now consider the recursive formulation for a young employed worker as an example:

$$\begin{aligned} \rho_{dis} W^e(h, f, \text{young}, a) = & w + \rho_{invest} \left(W^e(h+1, f, \text{young}, a) - W^e(h, f, \text{young}, a) \right) \\ & + \rho_{old} \left(W^e(h, f, \text{old}, a) - W^e(h, f, \text{old}, a) \right) \\ & + \rho_{sep} \left(W(h, \text{young}) - W^e(h, f, \text{old}, a) \right) \end{aligned}$$

It is therefore possible to infer the wage being paid (call this $w(h, f, \text{young}, a)$):

$$\begin{aligned} w(h, f, \text{young}, a) = & \rho_{dis} W^e(h, f, \text{young}, a) - \rho_{invest} \left(W^e(h+1, f, \text{young}, a) - W^e(h, f, \text{young}, a) \right) \\ & - \rho_{old} \left(W^e(h, f, \text{old}, a) - W^e(h, f, \text{old}, a) \right) \\ & - \rho_{sep} \left(W(h, \text{young}) - W^e(h, f, \text{old}, a) \right), \end{aligned}$$

and a similar formula is used for old agents (albeit with the exit rather than ageing shock).

With the wages now known, a simulation can be run. This is done by constructing a large panel. Time is broken into weekly periods (two weekly would not be fine enough for the job meeting rate) and the life of a worker is simulated. In each entry, 11 variables are tracked:

- An ID number for the worker
- The time
- The worker's age (in years)
- The age type (young vs. old)
- Whether the worker is employed or not
- The wage of the worker (if employed)
- The general human capital

- The firm-specific human capital
- The employer type
- The intensity of training
- Whether the worker started the job in the current period

This is done by rescaling all the “rates” in the model (investment, ageing, separation, job finding) by a factor of $\frac{1}{52}$. In each period, a uniform random draw is taken to decide which type of shock is appropriate to the worker e.g. if they are young and employed they could face a human capital upgrade shock with probability $\frac{\rho_{invest}}{52}$ (where ρ_{invest} is taken from “Optimal”), a job separation shock with probability $\frac{\rho_{sep}}{52}$ and an age shock with probability $\frac{\rho_{age}}{52}$. If they hit an exit shock, then a new young unemployed worker is created (the ID is increased by 1, and the age reset to 25). Similarly, if the worker passes the age of 75 this is done automatically to save time. This panel is constructed to have $5000 \times 52 \times 40$ entries to ensure the simulated sample has at least 5000 workers. This is then converted into annual data by taking annual snapshots of the different variables (except training - which is aggregated over the previous year to be consistent with the interpretation in the data, and the job start variable which takes on the value 1 if the worker switched at any point in the previous year, again to be consistent with the data).

By this point, there is a panel that roughly resembles the structure of the data used from Understanding Society. This can then be used to analyse moments used for the calibration, such as the average log-wage growth rate, the ratio of vacancy cost to average wage, the between firm-type variation in wages, the unemployment rate and the average log-wages by education type from ages 25-30. This data is also used to produce the graphs related to the model output.

When the model is run using the tax rates, some adjustments are made. For any given tax rate, there is an endogenous “Subsidy Rate” which is applied to worker training, which is guessed. While running the main loop, the optimal rate of investment is instead computed as $\frac{a}{c}(\text{Upgrade Benefit} + \text{Subsidy-Rate})$ to reflect the altered first-order condition for the investment. When computing the realised output, this is adjusted for both the tax and the subsidy - with any initial non-negative realised output being taxed, and then allocating a payment of the $\text{Subsidy-Rate} \times \rho_{invest}$ into the realised output. Matches that have negative realised output are not taxed. Within each loop, an additional step must be added in which the subsidy rate is adjusted to try and ensure that the government’s total tax receipts are equal to its expenditure on the subsidy program. This is done by computing the total tax receipts, and the total amount of training around the economy, and dividing the former by the latter to construct a “Subsidy-RateNew”. When computing the convergence measure,

$|\text{SubsidyRateNew} - \text{SubsidyRate}|$ is added. Since this new rate would be at risk of overshooting, a tuning parameter tpolicy ($=0.001$) is introduced, and the new subsidy rate for the next iteration is overwritten with:

$$\text{Subsidy-RateNew} = (1 - \text{tpolicy}) \times \text{Subsidy-Rate} + \text{tpolicy} \times \text{Subsidy-RateNew}.$$

B Mechanisms for Deciding the Level of Investment

B.1 The Baseline Model

Many abuses of notation are used in this appendix to save space. Let J^+ represent the match's joint value if human capital is upgraded (which could involve splitting up), and J its current value. The pair wish to maximise:

$$y - \frac{c}{2a} \rho_{invest}^2 + \dots + \rho_{invest}(J^+ - J).$$

Taking first order conditions, the solution to ρ_{invest} is:

$$\rho_{invest} = \frac{a}{c}(J^+ - J).$$

B.2 Simultaneous Wage and Training Bargaining

Let W^e denote the worker's value in a particular match, and V^e the value of the firms. Let W^+ and V^+ be the values of the worker and firm in the match, should the human capital be upgraded (this might mean leaving). The worker and firm "inside options" are:

$$\begin{aligned} \rho_{dis} W^e &= w + \dots + \rho_{invest}(W^+ - W^e) \\ \rho_{dis} V^e &= y - \frac{c}{2a} \rho_{invest}^2 + \dots + \rho_{invest}(V^+ - V^e), \end{aligned}$$

where w and y denote the wage and potential output respectively. The worker and firm's "outside options" are

$$\begin{aligned} \rho_{dis} W \\ \rho_{dis} V \end{aligned}$$

Nash product with bargaining parameter η :

$$\left(w + \dots + \rho_{invest}(W^+ - W^e) - \rho_{dis} W \right)^\eta \left(y - w + \frac{c}{2a} \rho_{invest}^2 + \dots + \rho_{invest}(V^+ - V^e) - \rho_{dis} V \right)^{1-\eta}$$

First-order conditions for the wage and training are:

$$\begin{aligned} \frac{W^e - W}{\eta} &= \frac{V^e - V}{1 - \eta} \\ \frac{W^e - W}{\eta(W^+ - W^e)} + \frac{V^e - V}{(1 - \eta)(-\frac{c}{a}\rho_{invest} + V^+ - V^e)} &= 0 \end{aligned}$$

Solving for ρ_{invest} delivers:

$$\rho_{invest} = \frac{a}{c} (W^{+e} + V^{+e} - W - V) \equiv \frac{a}{c} (\hat{J}^+ - J).$$

B.3 Firm Chooses Training, Then Nash Bargaining Determines the Wage

The firm chooses ρ_{invest} to maximise:

$$y - w - \frac{c}{2a} \rho_{invest}^2 + \dots + \rho_{invest} (V^{+e} - V^e)$$

but now the firm anticipates that a change in the level of training will cause a change in the wage. The FOC is therefore:

$$-\frac{\partial w}{\partial \rho_{invest}} - \frac{c}{a} \rho_{invest} + (V^{+e} - V^e) = 0$$

What is $\frac{\partial w}{\partial \rho_{invest}}$? We know that Nash bargaining over the wage ensures the condition

$$\frac{W^e - W}{\eta} = \frac{V^e - V}{1 - \eta}.$$

Deriving this expression by ρ_{invest} , this implies:

$$\frac{\frac{\partial w}{\partial \rho_{invest}} + W^{+e} - W^e}{\eta} = \frac{-\frac{\partial w}{\partial \rho_{invest}} - \frac{c}{a} \rho_{invest} + V^{+e} - V^e}{1 - \eta}$$

$$\Rightarrow \frac{\partial w}{\partial \rho_{invest}} = -(1 - \eta)(W^{+e} - W^e) + \eta(-\frac{c}{a} \rho_{invest} + V^{+e} - V^e)$$

Plugging this into the FOC for the training decision gives the optimal rate:

$$\rho_{invest} = \frac{a}{c} (W^{+e} - W^e + V^{+e} - V^e) \equiv \frac{a}{c} (\hat{J}^+ - J).$$

B.4 Worker Chooses Training, Then Nash Bargaining Determines the Wage

The worker chooses ρ_{invest} to maximise:

$$w + \dots + \rho_{invest} (W^{+e} - W^e)$$

but the worker anticipates that a change in the level of training will cause a change in the wage. The FOC is:

$$\frac{\partial w}{\partial \rho_{invest}} + (W^{+e} - W^e) = 0$$

The previous section derived this derivative; plugging this into the FOC for the training decision leads to the same solution:

$$\rho_{invest} = \frac{a}{c} (W^{+e} - W + V^{+e} - V) \equiv \frac{a}{c} (\hat{J}^+ - J).$$

B.5 State Contingent Contracts With Bargaining upon Matching and Full Commitment

If state-contingent contracts are possible, and the pair can commit (including not leaving the match even when they would like) then the solution can be found recursively using promised utility. From the firm's point of view, let's suppose they currently promise to deliver U to the worker, but if the human capital is upgraded, they may either split or offer U^+ . The firm would choose both ρ_{invest} , w , and U^+ to maximise:

$$y - w - \frac{c}{2a} \rho_{invest}^2 + \dots + \rho_{invest}((J^+ - U^+) - V^e),$$

$$\text{s.t. } w + \dots + \rho_{invest}(U^+ - U) = \rho_{dis}U.$$

Substituting in the constraint and solving for ρ_{invest} yields:

$$\rho_{invest} = \frac{a}{c} (J^+ - U - V^e) \equiv \frac{a}{c} (J^+ - J).$$

The firm could also choose to terminate the contract upon upgrading the human capital. Then the optimal investment level is given by maximising:

$$y - w - \frac{c}{2a} \rho_{invest}^2 + \dots + \rho_{invest}(V - V^e),$$

$$\text{s.t. } w + \dots + \rho_{invest}(W - U) = U,$$

in which case the optimal investment is given by:

$$\rho_{invest} = \frac{a}{c} (W + V - U - V^e).$$

In both cases, due to transferable utility and perfect commitment, the firm can arbitrarily front-load or back-load the wage, so wages are indeterminate. Since the firm can choose the better of the two schemes (terminating the contract upon upgrading or not), the solution to investment turns out to be the same:

$$\rho_{invest} = \frac{a}{c} (\hat{J}^+ - J).$$

B.6 Different Threat Points

So far it has been shown that a wide range of mechanisms result in a bilaterally-efficient investment choice. Any adjustment that is made to the level of training alters the wage that is paid, and ultimately the share of the surplus is the same. Since the investment

rate has no bearing on the outside option, there is thus always the motive to maximise the joint value of the firm and worker, irrelevant of who makes the decision. Even if the firm worries about training causing the worker to have a better outside option in the future, this can be factored into the current wage. Such may not be true if the worker is risk averse, or there is a minimum wage or some other constraint. This would undo the perfect transferability of utility. It is worth considering if there may be other inefficient arrangements that might happen even within the perfect transferability world. Suppose that the choice of one variable (either training or wage) affects the threat point that is used to bargain over the other. Under a scenario like this, there may exist hold-up issues. Consider what happens if the pair uses *no training* as the outside option for bargaining over the level of training. This style of decision-making is remarkably different; the Nash product is:

$$\left(\rho_{invest}(W^{+e} - W^e)\right)^\eta \left(-\frac{c}{2a}\rho_{invest}^2 + \rho_{invest}(V^{+e} - V^e)\right)^{1-\eta}$$

The first order condition for investment here is:

$$\rho_{invest} = \frac{1}{1 - \frac{1}{2}\eta} \frac{a}{c} (V^{+e} - V^e).$$

Notice that the worker's values do not even enter the solution if no training is used as the outside option. Let's suppose, for simplicity, that the surplus-sharing rule applies. This would arise if the pair Nash bargains over the wage using unemployment/vacancy as the outside option, but Nash bargains over training using zero training as the outside option. In that sense, the level of training would be:

$$\rho_{invest} = \frac{1-\eta}{1 - \frac{1}{2}\eta} \frac{a}{c} (\hat{J}^+ - J - (W^+ - W)),$$

where W^+ is the worker's outside option in the event of an upgrade. Since $\frac{1-\eta}{1 - \frac{1}{2}\eta} < 1$, and $(\hat{J}^+ - J - (W^+ - W)) < \hat{J}^+ - J$, this implies less investment than the bilateral efficient level. While the mechanism to get to this point is slightly crude, one may think of reasons why zero training as a bargaining point may occur in practice. Firms may for example be constrained to pay workers equally or by a minimum wage, but can negotiate training on a worker-by-worker basis for the purpose of promotion for example. While the model used in this paper sticks to joint value maximisation as the determinant of investment, this appendix is used to briefly demonstrate that the use of a different threat point to bargain over training could lead to different results.

C The Sources of Inefficiency

The social planner's problem is rather complex, so it is easier to consider externalities associated with various decisions that the agents make. Decisions can be characterised by the match indicators as well as the investment rates. First, consider the match indicator. If a worker-firm pair decides that they should be matched rather than not, externalities are created through two channels. The first effect is that equilibrium unemployment will be lower. Since the model takes market tightness as exogenous, the

stock of vacancies is always taken to be $\frac{\hat{\rho}_{meet}}{\rho_{meet}}$, which means the total stock of firms in ratio to the workers is:

$$\text{employment rate} + \text{unemployment rate} \times \frac{\hat{\rho}_{meet}}{\rho_{meet}}.$$

In the model, $\frac{\hat{\rho}_{meet}}{\rho_{meet}}$ is calibrated to be 1.577, which being greater than one means the higher unemployment is, the more firms per worker there are in the model. Since, in the baseline calibration, the value of vacancies are positive this implies that reducing the unemployment rate will have a negative effect by reducing the amount of profits per worker generated overall in this economy. The sign of the externality is local and not global; if all workers were unemployed then the value of vacancies would be negative, in which case there would be positive externalities to reducing unemployment. The other effect the decision to match has is on the distribution of vacancies and unemployed workers. Accepting a match makes both the worker's type and firm's type less common in the pool of unmatched agents. This in turn affects the other agents' expected surpluses upon meeting a potential match. Whether worker i and firm j accepting a match will have a positive or negative externality on worker k will depend on how the surplus of a match involving k and j would compare to the expected surplus that k would gain through the random search. If j was particularly attractive to k , the match of k and j would produce a better surplus than the average for k , then k would be negatively impacted by this decision since the overall expected gain from searching is worse. It is by no means trivial whether these distributional effects are net positives or negatives.

When it comes to investment decisions, suppose we consider a match making a large investment instead of a small investment. Eventually, if this worker becomes unemployed again, the distribution of general human capital in the unemployed workers will, overall, be "higher" (in the stochastic dominance sense). This in turn impacts the expected gain that vacancies receive by meeting a worker. This gain should generally be positive. To see how this works, consider the behaviour of the surplus function in the baseline calibration.

Since increasing a worker's human capital increases both their value of unemployment and the joint value of a match, it is not immediately obvious if the surplus of any given match should rise. For lower firm types, in particular, surplus tends to decrease when the human capital gets very high. However, some of this is in the negative region, which means it is irrelevant, as these matches would never form to begin with. To show the average effect of general human capital on the expected surplus that a firm will receive, a weighted average must be taken over the firm types that is truncated at zero i.e.

$$\text{weighted \& truncated surplus} = \sum_{j=1}^{15} \frac{Vac(a_j)}{\text{sum}(Vac)} \max \left\{ 0, J(h, 1, a_j, \text{age}) - W(h, \text{age}) - V(a_j) \right\}.$$

Other than at the extremely high levels of human capital, the expected surplus from a meet is increasing in the general human capital of the worker (in the baseline model, only 0.39% of unemployed workers have human capital over the 90th rung). This means that there is generally a positive externality to investment via the fact that future employers gain from having a high-skilled pool of workers to search for by being able to extract more surplus; figures 19-22 show this.

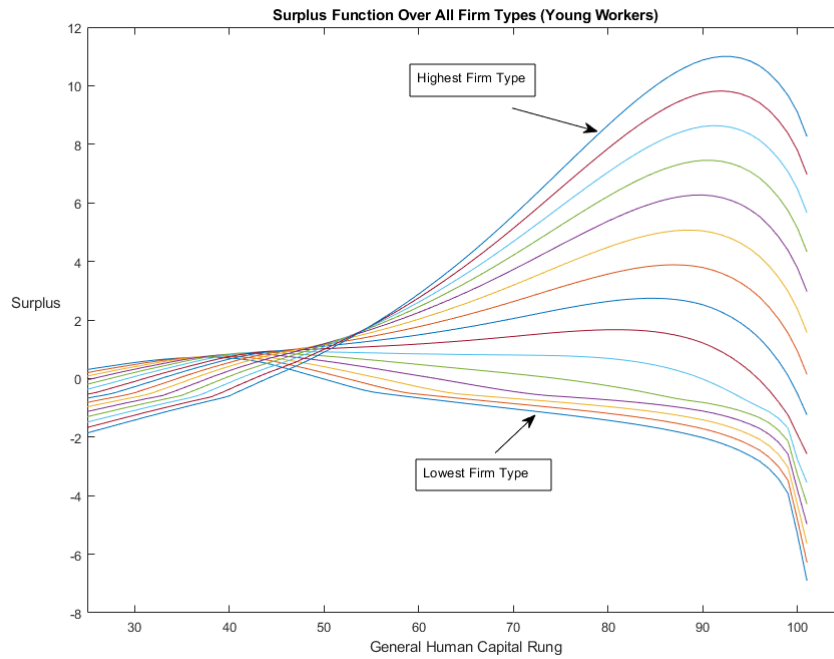


Figure 19: This shows the surplus that a new match (with no specific capital) will gain if the worker is young. Each line represents a different firm type. Some of the region is negative - these pairings would not form in the equilibrium.

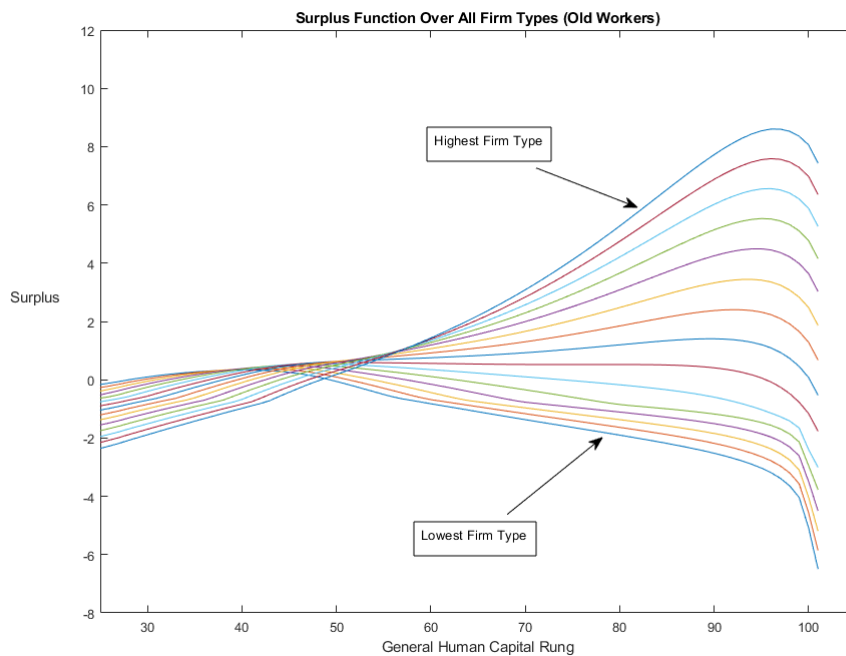


Figure 20: This shows the surplus that a new match (with no specific capital) will gain if the worker is old. Each line represents a different firm type. Some of the region is negative - these pairings would not form in the equilibrium.

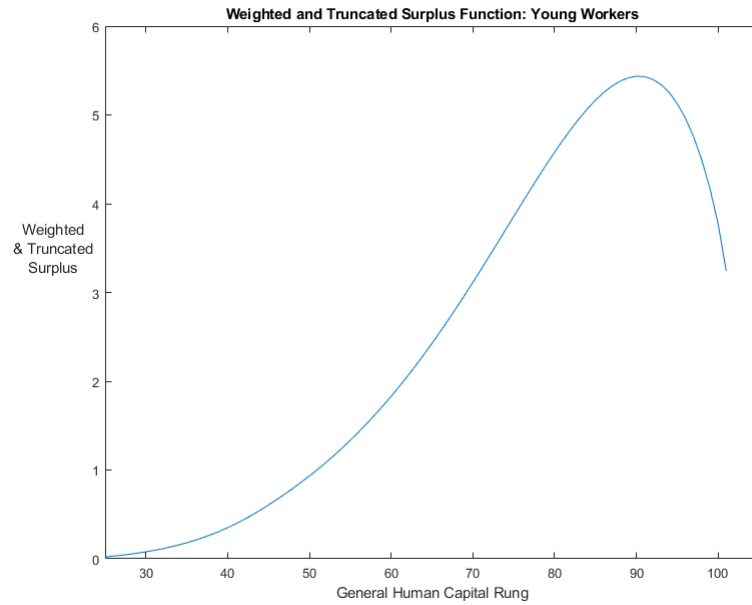


Figure 21: This shows the average surplus that a young worker is expected to generate conditional upon getting a meet.

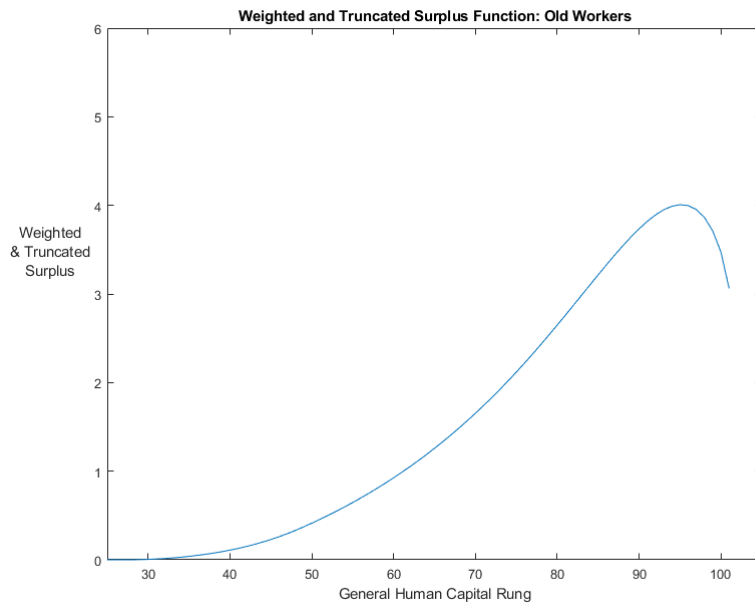


Figure 22: This shows the average surplus that an old worker is expected to generate conditional upon getting a meet.

Implicit Contracts and Asymmetric Pass-Through of Productivity Shocks

Breslin, Stuart A. Snell, Andy Stüber, Heiko

Thomas, Jonathan P.

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Abstract

We document distinctive empirical features of wage pass-through in Germany that are consistent with Thomas-Worrall wage contracting in the presence of both idiosyncratic and nonstationary aggregate productivity components. These empirical features are hard to reconcile with the predictions of search models based on period-by-period Nash bargaining over match surplus and with the predictions of financial models where risk neutral firms may costlessly shield risk averse workers from idiosyncratic shocks (Guiso, Pistaferri et al. 2005).

Summary

We construct a simple model in which risk-neutral firms can hire risk-averse workers on a competitive market by offering long-term contingent contracts. Each firm can hire at most one worker, and is subject to an exogenous level of productivity which consists of an idiosyncratic and a common (aggregate) component. There is complete information. Either party can, after observing the current productivity level, quit the relationship. For a worker this involves incurring a mobility cost and joining a new firm, and receiving the current market (lifetime-) utility. The firm's outside option is zero profits. We study theoretically and in simulations the pass-through from firm productivity to wages. Wages increase only when the worker's participation constraint binds, in which case her outside option determines future lifetime utility within the firm. Symmetrically the wage falls only when the firm's participation constraint binds, so that future expected profits are zero. Idiosyncratic shocks to a firm's productivity only affect wages when they are negative (i.e., smaller than in the previous period holding the aggregate level constant) because an idiosyncratic positive movement in productivity does not tighten the worker's participation constraint. On the other hand a positive movement in aggregate productivity may increase wages by improving the worker's outside option, whilst a negative one may reduce wages if the firm's participation constraint binds. Overall the effect of the former is expected to outweigh the latter on average. Finally the overall effect of aggregate productivity movements on wages within firms is expected to be muted rather than one for one. Using matched employer-employee data from Germany and proxying for productivity with estimated firm value-added data, we estimate pass-through of asymmetric idiosyncratic and aggregate changes in these proxies to wages. We get results broadly in line with the theoretical and simulation predictions. The empirical features we find are at odds with the predictions of a model based on period-by-period Nash bargaining over match surplus suggesting that long-term implicit contracts may play an important role, although it should be emphasised that they do not directly speak to the issue of whether the labour market is competitive or whether contracts are the outcome of ex-ante bargaining. They are also inconsistent with the predictions of financial models where risk-neutral firms may costlessly shield risk averse workers from idiosyncratic shocks (see Guiso, Pistaferri and Schivardi, 2005, for an examination of such a model).

1 Introduction and Overview

There is now a burgeoning theoretical and empirical literature on the extent to which a firm's performance impacts the wages of its workers. When workers are not committed to contracts firms must trade-off the desire to insure risk-averse workers with the desire to preserve the match. This trade off leads to interesting features of wage setting which have been taken to the data most notably by Lagakos & Ordonez (2011). Other work has looked at the pass-through from the point of view of rent sharing, most notably Lemieux et al. (2009) and Card et al. (2018). More recent and related work has expanded the ambit of the theory to include search frictions with on the job search, worker effort and stochastic heterogenous match and worker productivity (see for example Balke & Lamadon (2020)).

In this paper we extend the literature to consider separately the impact of changes in aggregate and idiosyncratic productivity on wages. Explicitly, we document important and distinctive empirical features of wage pass-through from productivity using German data. We find that these features are intrinsic to a simple parsimonious model of wage contracting which is subject to these two types of productivity movements. As with much of the recent literature our results are at odds with the predictions of macro search models with period-by-period Nash bargaining and with the predictions of financial models where risk neutral firms may costlessly shield risk-averse workers from idiosyncratic shocks (Guiso et al. (2005)).

The theoretical model in this paper is an extension of Thomas & Worrall (1988). It predicts that wages will only be changed when it is necessary to do so either to retain the worker (i.e. raise the wage to match her outside option) or to save the match's viability from the firm's perspective (i.e., lower the wage to prevent the match being destroyed). A direct implication is that in states when the firm is near to its participation constraint, adverse idiosyncratic productivity shocks may well have to be passed through into wages in order to allow the match to continue. By contrast the firm has no problem insulating wages from positive idiosyncratic shocks because these do not impact the worker's outside options. We find these asymmetric features to be strongly present in German data; workers are significantly exposed to adverse idiosyncratic shocks but do not seem to significantly benefit from positive ones. The model also predicts muted responses to aggregate shocks of the wages of stayers; within jobs wages are stabilised and only respond to large changes in the outside option.

The outline of the paper is as follows. The next section describes our data, shows how we obtain measures of aggregate and idiosyncratic firm productivity and then presents regression results detailing pass-through of these components to wages. Section 3 establishes that our empirical findings are qualitatively intrinsic to a fully specified general equilibrium model of wage contracting a la Thomas and Worrall. The section starts with the model set-up and its main results. We then offer a partial characterisation of equilibrium wage dynamics with further properties established via numerical simulations of the model. Under realistic parameter scenarios the model readily mirrors the stylised facts of pass-through we find in the German data. Section 4 concludes.

2 Estimating Productivity Pass-through to Wages

2.1 Overview

In our empirics we expose distinctive features (“stylised facts”) of productivity pass through to wages. We show that the pass through of idiosyncratic firm productivity is asymmetric; positive changes have a small and insignificant effect whilst negative changes have a relatively large and significant effect. We also find that aggregate pass through is substantially below unity for our sample of stayers. In a world of constant (or at least very slow moving) labour share we would expect the wages of all workers to keep pace with aggregate productivity. This last result therefore is indicative of within job wage stabilisation relative to the aggregate; a corollary is that wages must rise faster than the aggregate for workers switching jobs¹. In an additional exercise we show that persistent components of productivity impact wages more than transient ones. Finally we relate these findings to the existing empirical literature.

Before proceeding we must motivate and justify our focus on the data moments we referred to above. To this end we outline the expected predictions of the theory. The theory in this paper extends Thomas-Worrall contracting to allow for both idiosyncratic and aggregate job productivity. Firms (in the theory “firms” are separate CRS jobs) set wages which in equilibrium are characterised by upper and lower bounds. These bounds are determined by the current productivity state within the job (which impact the firm’s participation constraint) and within the economy (which impinge on the worker’s participation constraint). If the pre-existing wage lies within the current bounds then wages will not move. Each job has its own job history so at any point in time the bounds are heterogeneous across firms. However we may still anticipate regularities in wage pass-through to emerge and we would expect these regularities to be sign-asymmetric. The intuition for this expectation – which at this point is purely heuristic – is as follows. If the firm suffers a negative idiosyncratic shock the upper bound will fall as the job has become less viable. If the shock is severe enough the bound will fall sufficiently far to exclude the pre-existing wage and the job wage will have to fall for the relationship to survive. By contrast positive idiosyncratic shocks – which will not affect a worker’s outside option – will be absorbed by the firm and not passed through to wages. Hence asymmetric pass-through may be expected here.

The previous discussion motivates the estimation of an empirical model with asymmetric pass through something to which we now turn.

2.2 Empirical Model

In what follows we drop the reference to jobs and discuss only “firms” (which, as noted above, are actually establishments in the data). In reality of course firms play host to a number of jobs but as already noted extending the theory to allow groups of workers with identical within group productivity is easily achieved and does not change the characterisation of equilibrium wages.

¹We do not estimate pass through for job switchers or those transitioning from unemployment to employment or non employment to employment; the sample selection issues in relation to worker quality are quite severe in this context and the solution of these would take us beyond the scope of this paper.

Following the discussion above we wish to examine possible asymmetric pass-through effects to wages of idiosyncratic productivity and of aggregate productivity movements impacting on each firm.² To begin with we need a model of firm productivity which here under CRS we take to be output per worker. We adopt the following structure

$$\Delta \mathbf{a}_{jt} = \Delta \mathbf{y}_t + \Delta \mathbf{r}_{jt}$$

where \mathbf{a}_{jt} , \mathbf{y}_t and \mathbf{r}_{jt} are the log of firm j 's total, aggregate and idiosyncratic productivity in year t respectively. This is a simple short-run decomposition that takes capital as fixed. It is motivated in part by the desire for tractability when we take it to the theory and in part by data limitations. Additionally it is easy to show that under CRS and fully adjustable capital then output per worker measures labour augmenting productivity. One natural estimate of $\Delta \mathbf{y}_t$ might be the within year cross firm unweighted average of $\Delta \mathbf{a}_{jt}$. However the firms in the German data account for a tiny proportion of those in the economy as a whole. Furthermore the uncensored data is unrepresentative and heavily skewed towards manufacturing. We therefore use GDP per worker employed as an estimate of the aggregate ($\Delta \mathbf{y}_t$) for Germany.

The previous discussions suggest we should investigate separately the effects of rises and falls in aggregate and idiosyncratic productivity respectively. To this end we estimate the following regression equation

$$\Delta \mathbf{w}_{ijt} = \alpha + \beta \Delta \mathbf{y} + \gamma^+ \Delta \mathbf{r}_{jt}^+ + \gamma^- \Delta \mathbf{r}_{jt}^- + \text{controls} + \text{error} \quad (1)$$

where $\Delta \mathbf{y}$ is equal to the first differenced aggregate and $\Delta \mathbf{r}_{jt}^+$ ($\Delta \mathbf{r}_{jt}^-$) is equal to the first differenced idiosyncratic productivity when it is positive (negative) and is equal to zero otherwise. The controls are quartics in worker-firm tenure and age (proxies for firm specific and non firm specific human capital respectively). Although under the assumption of CRS our productivity measure is exogenous we do not treat (1) as a causal relationship and nor do we try and explicitly map its parameters into those that underpin the primitives of the theory (or vice versa). Instead we treat the estimates as data moments that will form stylised facts about the pass-through of idiosyncratic and aggregate productivity components to wages. We then examine the extent to which a Thomas-Worrall model of wage contracting may account for these "stylised facts" by comparing these data moments with their counterparts obtained via model simulation.

Finally, in order to examine the potentially differential impact of persistent productivity changes on wages (see for example the seminal paper of Guiso et al. (2005)) we re-estimate (1) using changes taken over two and then three years respectively rather than one; taking longer differences averages out temporary components and increases the contribution of persistent components to the variance of the RHS variables in (1). Henceforth and purely for reasons of brevity we refer to our productivity proxies merely as "productivity".

²The scarcity of annual data points precludes the exploration of potential asymmetries in aggregate pass through; we have only 15 aggregate observations and very few downturn years.

2.3 The Data

We form a matched panel dataset of workers and German establishments by merging information from the LIAB firm survey and BeH. The former is a survey of establishments for the years 1993 to 2018 that contains our productivity measure for Germany: value added (output) per employed worker. The data contains information on establishments' workers and in particular their wages, tenure, gender, experience, age and occupation. The BeH is a well used administrative worker-establishment dataset so we offer only outline detail here (for more information on matters such as top coding etc see for example Snell et al. (2018)).

The BeH is organised by spells of continuous work at an establishment. We collate these spells for each year of our basic sample to obtain an estimated hourly wage of each full time worker in each of the surveyed establishments. Whilst actual hours worked are not documented, there is evidence that the variation in weekly hours of full time workers in Germany is fairly minimal (see for example Snell et al. (2018)). Worker tenure, measured in days, is obtained by adding the number of days worked ignoring periods of absence because of e.g. maternity or sickness.

As noted above the dataset is based on establishments not firms. Decisions on wages are almost certainly made at the firm level however and this raises some interesting issues. Under our maintained CRS technology we might expect a cost minimising firm to rearrange its productive resources so as to equate marginal (and hence average) costs across establishments. If so each firm would merely be a scaled up version of each of its establishments. An additional interesting issue worthy of empirical examination is the extent to which multi-establishment firms or large firms are able to diversify idiosyncratic productivity shocks across workers (or rather jobs) better than single establishment or small ones can and therefore more able to offer their workers more insurance. Interestingly we find that the standard deviation of estimated establishment value added per worker is as high in large (high employment) establishments as in small. This does seem to support the idea that productivity shocks occur at establishment rather than worker level. Whilst in our theory we assume that productivity shocks occur at the worker level, an extension that allows groups of workers (in establishments) to receive identical productivity realisations and have identical wages does not change anything (as long as the establishment is small relative to the economy as a whole). In the light of the previous discussion we refer to establishments as "firms".

Turning to the LIAB, we use its survey data to estimate each firm's value added for the year, deflate by a CPI deflator and divide by the number of full time (equivalent) workers in that year. Whilst the survey documents the amount of intermediate inputs used by the firm it does not offer data on inventory changes. There is also a concern that intermediates themselves are poorly estimated by the responder. We take up the issue of measurement error below in section 2.5. There we use an IV method which under certain assumptions would give approximately consistent estimates in some of the sectors we look at. We then use these estimates to calibrate the likely size of the measurement error variance and to quantify the bias in other sectors.

Modulo measurement error and under the constant returns (CRS) assumption adopted by our theory our data yields an estimate of productivity per worker that is arguably

exogenous.³ CRS is not an innocuous assumption of course but that there is a substantial body of previous work (e.g., Basu & Fernald (1997); Syverson (2004*a*); Syverson (2004*b*)) which shows that it offers a good medium run approximation for production conditions in many plants particularly in the manufacturing sector.

In terms of reliability of the LIAB its documentation claims that once a firm is selected for its survey that firm is rigorously pursued each year to answer each question. Despite this the data on value added is heavily censored with larger firms being more responsive than smaller ones. This raises a concern in the regression context that unobservables relevant to the determination of wages may drive the probability of censorship and cause “bias”⁴. However below we adopt a first differenced specification and regress the wage growth of stayers on productivity growth. It is somewhat comforting then that if the relevant unobservables driving censorship are time invariant (and hence vanish under differencing) then they will cause no issues. In terms of representativeness the tendency for larger firms to respond more regularly to the survey questions suggests that if there is pass through heterogeneity across firms then our estimates will be more representative of large than small firms.

Finally we identify six separate and mutually exhaustive sectors:- 1) Mining, Agriculture, etc., 2) Manufacturing, 3) Utilities, 4) Construction, 5) Retail, and 6) Non retail services. This allows us to examine parameter heterogeneity and as we explain later the extent and impact of measurement error.

2.4 Measurement Error

Our proxy for firm value added is reported sales multiplied by one minus the reported proportion of intermediates used in production. As with any survey data, measurement errors will be present. A further issue is that we do not have data on the proportion of sales met by inventory changes. We can however make some headway to assess the likely biases in OLS estimates caused by these two problems if we make further assumptions.

First of all we assume that sales are reported without error (or at least with negligible error). Sales are a definitive and well known item in a firm’s accounts and the manager/respondent is likely to both understand this quantity and know its value well. By contrast the proportion of intermediates used is a more nebulous item. It requires substantial data gathering from different sources (purchases from various suppliers). As part of our data checking process we took a random sample of firms and analysed their responses to the intermediates question. We found occasions where the reported proportion of intermediates either doubled or halved in consecutive years (e.g., from 20% in one year to 40% in the next or vice versa). If technology is fixed in the short run this may occur if there was a huge swing in input prices but that is unlikely. It is more probable that such large changes are a consequence of having different responders in two consecutive years, respondents that may have different perceptions of the

³We do not have data on the number of hours worked by part-timers. Therefore we use estimates of the number of full time worker equivalents to obtain per worker productivity.

⁴As we have already noted, we do not attempt to identify any deep parameters but merely try and estimate data moments. “Bias” here and henceforth then refers to a deviation from the moments that would obtain from data free of measurement error.

intermediate's question's meaning⁵ or different access to data to answer that question. We have no access to accounting data with which to calibrate measurement error in this item. In the face of such ignorance we do what investigators often do in the literature and assume that the errors are classical in nature – i.e. have mean zero, are uncorrelated to the true level of intermediate inputs and to the regression's error term. If these assumptions hold then we can re-estimate (1) using the change in log sales as an instrument for our value added proxy. Then, it is easy to show that in sectors/cases where inventory changes are unimportant such as utilities, construction and non retail services the IV estimates will be consistent whilst the OLS estimates will be biased towards zero. The intuition for the consistency of IV is as follows. Consider a sector where there are no (or negligible) inventories. Here using (correctly measured) sales as a proxy for value added leads to a measurement error equal to the intermediate inputs themselves. By contrast the measurement error in our original value added proxy (sales minus reported intermediates) is just the measurement error in the reported intermediates. These two proxies for value added have respective measurement errors that – under the classical assumption – are uncorrelated. In these circumstances an IV estimator instrumenting one proxy with the other will lead to consistent parameter estimates (see for example the discussion on multiple measures in Bound et al. (2001)). In addition and again where we believe inventories to be unimportant we may compare the IV estimates to their OLS counterparts and back out estimates of the variance of true value added. We can then use the average of these estimates as a benchmark for (idiosyncratic) productivity variance in our calibration exercise.

But what of cases or sectors where inventories do vary? Here it is hard to be definitive about the nature and extent of the bias. However some insights may be gained by looking at the ratio of IV to OLS estimates across sectors. If the stochastic structure of true productivity and measurement errors were the same across sectors then the differences in the ratios of IV to OLS would be an indicator of the importance of unobserved inventories. In particular if we find these ratios to be fairly constant across sectors we might take this as some evidence that unobserved inventories were not a major cause of bias.

2.5 The Estimates

The first 3 lines of Table 1 below gives summary statistics on wage growth (Δw_{ijt}), aggregate and (measured) idiosyncratic productivity growth ($\Delta \mathbf{y}_t$ and $\Delta \mathbf{r}_{jt}^m, \Delta \mathbf{r}_{jt}^{m+}, \Delta \mathbf{r}_{jt}^{m-}$ respectively)⁶. Note that given our discussions above we are careful to annotate those variates that we believe have significant measurement error with a superscript m .

Two things stand out. Firstly the variation in wage changes is small relative to the variations in productivity growth. Second most of the latter is idiosyncratic in nature - the standard deviation of our aggregate productivity growth measure is around 2%

⁵It may be for example that the distinction between inventories of partially finished goods used in production and intermediates may not be fully understood.

⁶Productivity growth standard deviations are measured within firms across years and are not weighted by workers. This reflects our view – supported by the data – that productivity is realised at the firm level not the worker level. As noted above re-casting the theory to allow groups of workers with identical productivity (“firms”) is trivial and does not change anything.

whilst its idiosyncratic counterpart has standard deviation of the order of .5 (in log changes). Of course we believe that the latter is severely inflated by measurement error the extent of which we try to calibrate below. Nonetheless it is clear that workers are shielded from most of the productivity volatility that a firm experiences. These observations already seems to militate against period by period bargaining as a maintained hypothesis of wage determination. They are also suggestive of firms offering substantial insurance and wage smoothing to their workers.

The results of estimating (1) for the complete sample by OLS are in Table 1 below.

Table 1 Standard Deviations of Productivity and Estimates of Pass-through

Standard Deviations of Wages and the Productivity Measures									
x	$\Delta \mathbf{w}_{ijt}$	$\Delta \mathbf{y}_t$	$\Delta \mathbf{r}_{jt}^{m-}$	$\Delta \mathbf{r}_{jt}^{m+}$	$\Delta \mathbf{r}_{jt}^{m-}$				
$sd(x)$.071	.021	.533	.346	.330				
OLS and IV Estimates of (1)									
	OLS(Δ^1)			IV(Δ^1)					
	β	γ^+	γ^-	β	γ^+	γ^-	$\gamma_{IV}^-/\gamma_{OLS}^-$	N_{fy}	N_w
All	.212	-.002	.008	.203	.003	.027	3.25	96,458	5,732,574
	(.041)	(.004)	(.004)	(.039)	(.006)	(.006)			
Sector 1	.035	-.001	.012	.042	.013	.032	2.75	3,801	104,417
	(.103)	(.005)	(.006)	(.100)	(.023)	(.010)			
Sector 2	.302	-.003	.007	.283	.003	.027	3.85	35,979	4,079,112
	(.065)	(.007)	(.007)	(.061)	(.010)	(.008)			
Sector 3	.022	.003	.031	-.090	.022	.052	1.68	1,291	148.055
	(.027)	(.006)	(.017)	(.070)	(.027)	(.055)			
Sector 4	-.088	.001	.009	-.080	.008	.022	2.45	11,826	253.404
	(.053)	(.003)	(.003)	(.050)	(.008)	(.007)			
Sector 5	.153	-.002	.006	.160	.003	.019	3.17	14,587	269,055
	(.071)	(.002)	(.002)	(.072)	(.003)	(.007)			
Sector 6	-.051	-.002	.005	-.05	-.005	.013	2.60	29,170	878,531
	(.045)	(.003)	(.002)	.050	(.005)	(.004)			
OLS and IV Estimates of (1) in 3rd Differences									
	OLS(Δ^3)			IV(Δ^3)					
	β	γ^+	γ^{--}	β	γ^+	γ^{--}			
All	.287	.003	.020	.300	.010	.053		42,236	
	(.042)	(.003)	(.008)	(.007)	(.012)	(.010)			

Notes: Standard errors are clustered by firm⁷. Δ^i denotes i^{th} difference specification. N_{fy} is the number of firm-years and N_w is the number of wage observations.

⁷There is little evidence of first order residual autocorrelation which suggests we should cluster standard errors by year. However doing this would probably yield poor estimates of the standard errors due to the small number of years. If the error terms were uncorrelated across firms within years as the theory suggests they should be then clustering by firms would be appropriate. If this assumption is violated – as would be the case if we have excluded relevant aggregate factors driving wages – then we would expect downward bias on the standard errors particularly those on the aggregate term. Given this hazard it is somewhat comforting that clustering by year – despite its drawbacks – makes very little difference to our standard errors and our inferences.

Looking at the OLS estimates for the entire sample ("All") we see that aggregate pass through is significant but the effect is small and in particular well below unity. On the idiosyncratic side there is a significant effect of negative productivity movements but wholly insignificant effects of positive ones. Qualitatively the OLS results accord with the heuristic intuition above that anticipated the predictions of the theory. When we estimate separately for 6 sectors we see that the effects of idiosyncratic productivity are consistent across the economy; downward movements are significant whilst upward ones are quantitatively far smaller (often negative) and generally wholly insignificant. The effect of aggregate productivity is not consistent across the sectors however and is only significantly positive in two sectors. This may be a result of poor precision⁸; the only variation in the regressand is year to year so precision requires a large number of within year data points to average out non macro effects in wages such as idiosyncratic movements in human capital. Interestingly the only two sectors displaying positively significant β' s also have the first and third largest number of data points (both in terms of number of wages and number of firm-years), giving some support to this argument. An alternative explanation is that some sectors such as services have a separate (segmented) labour market and for these sectors the outside option is mismeasured. Exploring these different hypotheses is beyond the scope of the paper however it is comforting to note that using SUR we cannot reject the hypothesis that the pass through parameters are constant across sectors. One other thing that stands out from these estimates is their size; the coefficients on idiosyncratic productivity are quantitatively very small compared with those found in other studies (see for example the survey in Card et al. (2018)). This may well be a result of measurement error in our value added proxies discussed above and we now turn to examine this issue.

Following the arguments of section 2.4 we re-estimated (1) using the change in the log of sales⁹ as an instrument for the change in our value added proxy. The results are in columns 5 to 7 of Table 1. If the assumptions about measurement error made earlier are correct then the IV estimates of the γ' s in sectors where inventories play no (or little) role- in particular, utilities, construction, retail and services (sectors 3,4 and 6) - will be consistent (or approximately consistent). We would also expect estimates of the γ' s to be larger in magnitude than their OLS counterparts. We see that the latter is borne out in a striking way; IV estimates of the key γ^- parameters are two to three times their OLS counterparts. The γ^+ 's are also somewhat larger but remain wholly insignificant. By contrast the parameters on aggregate productivity are virtually unchanged and display the same characteristics as their OLS counterparts.

Turning to the results on three year first differences here we see much larger pass through estimates of all coefficients. This is another anticipated result of our theory. In a sense persistent shocks are "larger" than transient ones because they reoccur. It is not surprising therefore that a three year difference specification which averages out transient components of productivity displays larger pass through estimates.

Finally we revisit the task of estimating the variance of true idiosyncratic produc-

⁸Although it appears to be significantly perverse in two other sectors this significance disappears when we switch to clustering by year - something we discussed above. These are the only two cases where switching the clustering to years changes the result of a coefficient's significance.

⁹More specifically we use Δs^+ and Δs^- as instruments for Δr^+ Δr^- where $\Delta s^+ = I^+(\Delta s)$ and $\Delta s^- = I^-(\Delta s)$ and I^+ (I^-) are sign dummies and where Δs is the change in the log of sales minus Δy .

tivity using a simplified version of (1) and adopting the classical measurement error structure alluded to above. Given these simplifications and assumptions we should treat this as an indicative rather than definitive exercise.

We simulate synthetic "sector" wage data from the following simplified wage model

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$$\begin{aligned}\Delta \mathbf{w}_{ijt} &= \gamma_i^+ \Delta \mathbf{r}_{ijt}^+ + \gamma_i^- \Delta \mathbf{r}_{ijt}^- \quad i = 1, 2, 3 \quad \text{and} \quad j = 1, 2, N_i \\ \text{where } \Delta \mathbf{r}_{jt}^+ &= \Delta \mathbf{r}_{jt} \cdot I^+ \quad \text{and} \quad \Delta \mathbf{r}_{jt}^- = \Delta \mathbf{r}_{jt} \cdot I^- \\ \text{where } I^+ \quad \text{and} \quad I^- &\text{ are sign dummies} \\ \text{and where } \Delta \mathbf{r}_{jt} &\sim N(0, s_i)\end{aligned}$$

The three sectors (indexed by i) are meant to represent utilities, construction and non retail services - the sectors where we believe IV delivers consistent estimates. The number of data points in each sector N_i is taken to be large. The object of the exercise is to estimate s_i and we proceed as follows. For some initial value of s_i we generate wage and productivity data and then add (mean zero normal) measurement error to $\Delta \mathbf{r}_{jt}$. This will create a measured counterpart of the synthetic productivity variable. The standard deviation of the error is set so that the standard deviation of $\Delta \mathbf{r}_{jt}^m$ equals that found in the data for the relevant sector. We then regress our generated $\Delta \mathbf{w}_{ijt}$ on an intercept and $\Delta \mathbf{r}_{ijt}^{m+}$ and $\Delta \mathbf{r}_{ijt}^{m-}$ to get OLS estimates of γ^+ and γ^- . The idea is to iterate over different values of s_i until we find the value that delivers the OLS estimate of γ^- found in the actual data¹¹. Doing this exercise for utilities, construction and non retail services yielded values for s of .39, .36 and .35 respectively - about two-thirds of the standard deviation of their respective measured counterparts. Even if we ignore the result for utilities due to its poorly determined γ^- , these results still indicate that there is substantial measurement error in our productivity proxies. The exercise also suggests that when calibrating the theoretical model a standard deviation of the order of .35 for $\Delta \mathbf{r}_{jt}$ would be appropriate.

Finally we argued above that if the ratios of the IV to OLS estimates of γ^- were fairly uniform across sectors we might tentatively conclude that unobserved inventories were not substantially impacting our estimates. Column 8 of Table 1 shows that - ignoring sector 3 (where γ^- is poorly determined) the ratios lie within the range 2.45 to 3.85. This is not a tight span but nor is it wide. We do not formally test for equality of these ratios but given the standard errors on the estimates it is not obvious that such a test would reject equality. We may tentatively conclude that our failure to observe inventories does not substantially affect our inferences.

We summarise by saying that there is strong evidence in German data for significant pass-through from negative idiosyncratic productivity shocks to wages but little to

¹⁰An orthogonal mean zero measurement error does not affect the simulation results so we omit it. Note also that we ignore other regressors in the exercise, implicitly assuming that the estimates of aggregate pass through and of the controls are not affected by the measurement error. There is some support for this; the IV and OLS estimates of these effects appear to be very close.

¹¹The justification for ignoring the match with the γ^+ s is that they are quantitatively and statistically insignificant in both OLS and IV regressions. By contrast the γ^- s are - with the exception of the utilities sector - very well determined.

no evidence of pass-through to wages of positive idiosyncratic shocks. Aggregate pass-through to wages (for stayers) is substantially and significantly below unity. Persistent components are passed through with larger coefficients in both countries. Finally there is evidence to suggest that unobserved inventories do not impact our estimates very much. In section 3 below we assess the ability of Thomas-Worrall contracting to reproduce the empirical features we have identified here.

2.6 Relationship to the Empirical Literature

As noted already there is now a huge body of empirical work examining the extent and nature of pass through from a firm's performance to its wages. Space constraints prohibit a review of all of this literature here (but see Card et al. (2018) for an excellent summary and overview). Instead we focus only on those papers that examine asymmetric pass through. There are two recent (and concurrently written) papers in this vein. Using Danish data Chan et al. (2020) extract measures of idiosyncratic firm productivity from a dynamic production function using a nonparametric approach. They find the pass through to wages of negative idiosyncratic TFP shocks is larger than for positive ones but only if a Heckman correction for selection is employed.¹² Rather than offering a simple macroeconomic explanation of the canonical features of the data as we do their focus is on explaining the heterogeneity in pass through across firm types. In particular they model firms that are heterogeneous with respect to market power, size, pecuniary benefits to workers and productivity levels. On the worker side there are worker specific shocks to the value of non employment. A paper whose theoretical approach is more related to ours - a theory based on recursive labour contracts - is that of Azzalini (2021). Using Swedish data from 2004-18 he finds asymmetric pass through from idiosyncratic value added per worker to wages of the kind we document here. However a key difference is that he finds this asymmetry only exists in the years of the Great Recession. He develops a model of directed search with recursive contracts to explain this phenomena. Whilst this is an interesting finding, it is unclear how much of it is down to the focus on one, possibly very special, economic episode - the Great Recession. Whilst our data also includes the Great Recession it spans many more years and our analysis does not revolve around the Great Recession.

Other papers have examined the sign of asymmetric pass-through. For example, Juhn et al. (2018) find some degree of asymmetry. However their results are hard to compare with ours because they use firm revenue rather than value added (output) and do not distinguish between aggregate and firm specific shocks.

3 A Non-Stationary Model of Wage Contracting

In this section we outline a version of Thomas-Worrall firm-employee wage contracting without commitment. We then show that the empirical stylised facts presented above are intrinsic features of such contracting.

¹²We do not interpret our estimates as causal parameters as these authors do. Their need to control for selection effects to deliver the asymmetry may be down to the high labour mobility that exists in Denmark; average firm tenure is just over one half of what it is in Germany.

Time is discrete, $t = 0, 1, 2, \dots$. There is a fixed large number of infinitely-lived identical workers. There is free entry of firms at each date, each of which can employ at most one worker. Productivity at a firm j at time t is

$$a_{jt} = \hat{y}_t y_t r_{jt}$$

where \hat{y}_t and y_t are aggregate shocks and r_{jt} is an idiosyncratic shock to firm j . $\hat{y}_t \in \widehat{\mathcal{Y}}$ is assumed to follow a geometric random walk, $\hat{y}_t = \xi_t \hat{y}_{t-1}$ where $\xi_t \in \{\xi_1, \dots, \xi_{\hat{Y}}\} =: \Xi$ with distribution $\Phi_{\hat{Y}}$; both $y_t \in \mathcal{Y}$ and each $r_{jt} \in \mathcal{R}$ follow independent Markov chains where $\mathcal{Y}, \mathcal{R} \subset \mathbb{R}_{++}$ are finite sets, with $\mathcal{S} = \widehat{\mathcal{Y}} \times \mathcal{Y} \times \mathcal{R}$ the state space.

Firms are assumed to be risk-neutral and workers risk-averse with per-period utility given by $u(w)$, where w is the wage received (they can neither borrow nor save); $u(\cdot)$ is assumed to be differentiable and strictly concave. Both workers and firms discount future payoffs with discount factor β , $0 < \beta < 1$.

Assume an exogenous separation rate of $1 - \sigma$, whereupon the firm exits, and the worker starts a new match.

The time t shocks are observable at the beginning of the period. We assume that there is an outside option, available to any worker at t whose value $\chi(\hat{y}, y)$ depends only on the aggregate state at t , and after observing the current state a worker can leave and take the outside option and the firm can costlessly exit.

Consider a bilateral match formed at time t , with aggregate shock (\hat{y}_t, y_t) and initial idiosyncratic shock r_{jt} known, so that the current state relevant to the match is $s_t \equiv (\hat{y}_t, y_t, r_{jt})$. Firm j and the worker agree on a wage contract $(w_\tau(h_\tau))_{\tau=t}^\infty$, $w_\tau(h_\tau) \geq 0$, where $h_\tau \equiv (s_t, s_{t+1}, \dots, s_\tau)$.¹³ The value of the contract to the worker at each date $\tau \geq t$, after observing the current state, is

$$U_\tau(h_\tau) = E \left[\sum_{t'=\tau}^{\infty} \beta^{t'-\tau} \sigma^{t'-\tau} u(w_{t'}(h_{t'})) + \sum_{t'=\tau+1}^{\infty} \beta^{t'-\tau} \sigma^{t'-\tau-1} (1 - \sigma) \chi(\hat{y}_{t'}, y_{t'}) | h_\tau \right],$$

where the second summation captures the assumption that after a separation the worker gets the outside option value. The corresponding firm value is

$$V_\tau(h_\tau) = E \left[\sum_{t'=\tau}^{\infty} \beta^{t'-\tau} \sigma^{t'-\tau} (a_{t'}(h_{t'}) - w_{t'}(h_{t'})) | h_\tau \right],$$

given that after separation the firm ceases to exist. We assume that a constrained efficient contract is negotiated at t to solve:

$$f_{s_t}(U) := \max_{(w_\tau(h_\tau) \geq 0)_{\tau=t}^\infty} \{V_t(h_t)\} \text{ s.t. } U_t(h_t) \geq U \quad (\text{Problem A})$$

(where the determination of U is discussed below), and for all h_τ , $\tau > t$,

$$U_\tau(h_\tau) \geq \chi(y_\tau), \quad (2)$$

and

$$V_\tau(h_\tau) \geq 0. \quad (3)$$

¹³This is w.l.o.g. Conditioning on the entire history from $t = 1$ would lead to the same contract provided the equilibrium is Markovian as defined below.

The constraints (2) and (3) are the limited commitment constraints reflecting the assumption that either party can quit the relationship at any time. Note that the constraint applies *ex post* so the worker can quit after the current (date- τ) state is realized and get her outside option $\chi(\hat{y}_\tau, y_\tau)$, and likewise the firm can shut down immediately.¹⁴

A contract is *feasible* if it satisfies (2), (3) for all histories h_τ , $\tau \geq t$.

The following characterises the evolution of wages in response to the shocks a firm faces. It states that for each state s there is an interval $[\underline{w}_s, \bar{w}_s]$ of potential wages, and the updating rule is that wages change by the minimum amount to belong to the interval corresponding to the current state. If wages rise between $\tau - 1$ and τ , w_τ will be at the bottom of the interval $[\underline{w}_{s_\tau}, \bar{w}_{s_\tau}]$ and this corresponds to the worker's participation constraint binding, and at the top if wages fall, with the firm's participation constraint binding.¹⁵

Proposition 1 (Thomas & Worrall (1988)) *For any history h_τ , the wage of an efficient contract starting at date t , $w_\tau \equiv w(h_\tau)$ is contained in a closed non-empty interval $[\underline{w}_{s_\tau}, \bar{w}_{s_\tau}]$. Moreover, $w(h_\tau)$, $\tau > t$, satisfies*

$$w_\tau = \begin{cases} \underline{w}_{s_\tau} \text{ and } U_\tau(h_\tau) = \chi(\hat{y}_\tau, y_\tau) & \text{if } w_{\tau-1} < \underline{w}_{s_\tau} \\ w_{\tau-1} & \text{if } w_{\tau-1} \in [\underline{w}_{s_\tau}, \bar{w}_{s_\tau}] \\ \bar{w}_{s_\tau} \text{ and } V_\tau(h_\tau) = 0 & \text{if } w_{\tau-1} > \bar{w}_{s_\tau} \end{cases}.$$

Using standard arguments, we can express $f_s(U)$, firm profit, as a function of life-time utility U promised to the worker in state $s \equiv (\hat{y}, y, r_j)$, as the solution to the following recursive problem. We write χ_s for the outside option in state s , i.e., $\chi(\hat{y}, y)$. f_s is strictly decreasing, strictly concave and differentiable by standard arguments. Let E_s denote expectation conditional on the current state being s .

$$f_s(U) := \max_{w \geq 0, (U_q)_{q \in S}} (a_s - w + \beta \sigma E_s[f_q(U_q)]) \quad (4)$$

subject to

$$u(w) + \beta \{E_s[\sigma U_q + (1 - \sigma) \chi_q]\} \geq U \quad : \lambda \quad (5)$$

$$U_q \geq \chi_q \quad : \beta \pi_{sq} \mu_q \quad (6)$$

$$f_q(U_q) \geq 0 \quad : \beta \pi_{sq} \phi_q. \quad (7)$$

Here, U_q is the promised utility in state q next period. Note: $f_s(U)$ is potentially defined outside of the interval where $U \geq \chi_s$ and $f_s(U) \geq 0$ as the above participation constraints only apply in the future, but we define \bar{U}_s by $f_s(\bar{U}_s) = 0$ to be the highest U the firm can offer in s .

First-order conditions are

$$-1 + \lambda u'(w) = 0, \quad (8)$$

¹⁴As usual in such models, it is assumed that if $w(h_\tau)$ is not at the level agreed to in the contract, the “wronged” party is assumed to take their outside option.

¹⁵The fact that this depends only on the current state follows directly from the assumption that the outside option itself only depends on the current state. The latter will be established when the outside option is explicitly modelled below.

and

$$\beta\sigma\pi_{sq}f'_q(U_q) + \lambda\beta\sigma\pi_{sq} + \beta\pi_{sq}\mu_q + \beta\pi_{sq}\phi_q f'_q(U_q) = 0$$

or rearranging

$$f'_q(U_q)(1 + \phi_q) + \lambda + \mu_q = 0 \quad (9)$$

together with the envelope condition

$$f'_s(U) = -\lambda. \quad (10)$$

From (9), (8) and (10) (in states s and q),

$$1/u'(w_q) = (1/u'(w) + \mu_q) / (1 + \phi_q),$$

where w_q is the wage next period in state q . It follows that if $w_q > w$ then $\mu_q > 0$, so that $U_q = \chi_q$, and w_q is at the solution to (4) for $s = q$ and $U = \chi_q$, which we denote by \underline{w}_q , and from (8) and (10)

$$f'_q(\chi_q) = -1/u'(\underline{w}_q). \quad (11)$$

Also, $w_q < \underline{w}_q$ implies $f'_q(U_q) < f'_q(\chi_q)$ from (8), (10) and (11), and so $U_q < \chi_q$, violating (6). So $w_q \geq \underline{w}_q$. Hence if $w < \underline{w}_q$, then $w_q > w$ and we showed this implies $w_q = \underline{w}_q$.

Likewise if $w_q < w$, $U_q = \bar{U}_q$ and w_q is at the corresponding solution to (4), denoted by \bar{w}_q where

$$f'_q(\bar{U}_q) = -1/u'(\bar{w}_q). \quad (12)$$

By a symmetric argument with the previous case, $w > \bar{w}_q$ implies $w_q = \bar{w}_q$.

If $w \in [\underline{w}_q, \bar{w}_q]$ then $w_q = w$ as otherwise, if $w_q > w$ then from above $\mu_q > 0$ so $w_q = \underline{w}_q$, a contradiction, and symmetrically if $w_q < w$.

The following is our main theoretical characterisation. It states that higher y (hence higher revenue *and* higher outside options) is associated with increases in *both* wage-interval end-points. This implies that positive aggregate shocks may *ceteris paribus* lead to rising wages as workers' outside options are better. The effect is symmetric: negative shocks lead to wages being cut if a firm is against or close to its profit constraint. The effect of higher r_j *ceteris paribus* is that the only the upper end-point is affected: the top of the interval expands upwards as the profit constraint is relaxed, so negative idiosyncratic shocks will lead to wage falls for firms close to their profit constraints. But the opposite effect does not exist: higher r_j does not lead to wage increases as the outside option is unaffected, so it just translates into increased profits.

Proposition 2 Assume that $\chi(\hat{y}, y)$ is increasing in \hat{y} and y . Consider two states s, s' , with $s = (\hat{y}, y, r_j)$ and $s' = (\hat{y}, y', r'_j)$. (i) First assume that they differ only in y , with $y' > y$, and that $\{y_t\}$ is i.i.d. Then $\underline{w}_s < \underline{w}_{s'}$, and $\bar{w}_s < \bar{w}_{s'}$. (ii) Likewise, if they differ only in r_j , with $r'_j > r_j$, and $\{r_{jt}\}$ is i.i.d., then $\underline{w}_s = \underline{w}_{s'}$, and $\bar{w}_s < \bar{w}_{s'}$. (iii) If both $\{y_t\}$ and $\{r_{jt}\}$ are i.i.d., then, holding \hat{y} fixed, \bar{w}_s is increasing in $y r_j$, and \underline{w}_s is increasing in y .

(i) We have

$$f_{s'}(U) = f_s(U) + r_j \hat{y} (y' - y), \quad (13)$$

since if $w, (U_q)_{q \in S}$ attains the maximum in (4), it also attains the maximum in state s' as the constraint set in (4) is the same (the distribution of q conditional on s is unchanged given y_t is i.i.d. and $r_j = r'_j$). This holds for all U such that the constraint set is non-empty, so we can differentiate (13) w.r.t. U to get

$$f'_{s'}(U) = f'_s(U). \quad (14)$$

From (13):

$$f_{s'}(\bar{U}_s) = f_s(\bar{U}_s) + r_j \hat{y}(y' - y) > 0,$$

using $f_s(\bar{U}_s) = 0$, so that $f_{s'}(\bar{U}_s) > 0$ and thus $\bar{U}_{s'} > \bar{U}_s$ (by $f' < 0$ and $f_{s'}(\bar{U}_{s'}) = 0$). Consequently we have

$$f'_{s'}(\bar{U}_{s'}) < f'_{s'}(\bar{U}_s) = f'_s(\bar{U}_s),$$

by the strict concavity of f and by (14). Hence $\bar{w}_s < \bar{w}_{s'}$ from (12). Similarly, by $\chi(\hat{y}, y) < \chi(\hat{y}, y')$, so $\chi_{s'} > \chi_s$,

$$f'_{s'}(\chi_{s'}) < f'_{s'}(\chi_s) = f'_s(\chi_s),$$

so that $\underline{w}_s < \underline{w}_{s'}$ from (11). (ii) Following similar reasoning (13) holds in this case, given that again the distribution of q is unchanged, so

$$f_{s'}(U) = f_s(U) + \hat{y}y(r'_j - r_j), \quad (15)$$

and we get $f'_{s'}(\bar{U}_{s'}) < f'_{s'}(\bar{U}_s)$, so $\bar{w}_s < \bar{w}_{s'}$, but now $\chi_{s'} = \chi_s$, so $f'_{s'}(\chi_{s'}) = f'_s(\chi_s)$ and thus $\underline{w}_s = \underline{w}_{s'}$. (iii) (13) holds again, and following the reasoning above, $\bar{w}_s < \bar{w}_{s'}$ if $y'r'_j - yr_j > 0$ and $\underline{w}_s < \underline{w}_{s'}$ if (and only if) $\chi(y) < \chi(y')$.

With CRRA preferences we have:

Proposition 3 Write $f_s(U) = f(U; \hat{y}, y, r)$. Suppose that $u(w) = w^{1-\alpha}/(1-\alpha)$, $\alpha \neq 1$, and $\chi(\hat{y}, y) = \tilde{\chi}(y) \hat{y}^{(1-\alpha)}$ for some increasing function $\tilde{\chi}(y)$. Then $f(U; \hat{y}, y, r) = \hat{y}f(\hat{y}^{-(1-\alpha)}U; 1, y, r)$, and $\bar{w}_{(\hat{y}, y, r)} = \hat{y}\bar{w}_{(1, y, r)}$, $\underline{w}_{(\hat{y}, y, r)} = \hat{y}\underline{w}_{(1, y, r)}$.

Consider first a solution $(\hat{w}_\tau(h_\tau))_{\tau=t}^\infty$ to Problem A with $s_t = (\hat{y} = 1, y, r)$. Now consider Problem A with $s_t = (\hat{y} \neq 1, y, r)$, and the contract $\tilde{w}_\tau(h_\tau) = \hat{y}\hat{w}_\tau(h'_\tau)$ where h'_τ is h_τ with each $\hat{y}_{t'}$, $t' \geq t$, replaced by $\hat{y}\hat{y}_{t'}$. It follows from the definition of a geometric random walk and the assumption $\chi(\hat{y}, y) = \tilde{\chi}(y) \hat{y}^{(1-\alpha)}$ that $\tilde{U}_\tau(h_\tau) = \hat{y}^{(1-\alpha)}\hat{U}_\tau(h_\tau)$ and $\tilde{V}_\tau(h_\tau) = y\hat{V}_\tau(h_\tau)$ (using obvious notation). Thus $(\tilde{w}_\tau(h_\tau))_{\tau \geq t}$ satisfies (6) and (7) and delivers values $\hat{y}^{(1-\alpha)}\hat{U}_\tau(h_\tau)$ and $y\hat{V}_\tau(h_\tau)$. No other feasible contract, say $(w'_\tau(h_\tau))_{\tau=t}^\infty$, Pareto-dominates this with profits strictly higher; otherwise using the same logic there would be a contract $(\hat{y}^{-1}w'_\tau(h_\tau))_{\tau=t}^\infty$ in the original problem that dominated $(\hat{w}_\tau(h_\tau))_{\tau=t}^\infty$, a contradiction. Thus $f(\hat{y}^{(1-\alpha)}U; y, r) = \hat{y}f(U; 1, y, r)$. Next, differentiating this at $\bar{U}_{(\hat{y}, y, r)} = \hat{y}^{(1-\alpha)}\bar{U}_{(1, y, r)}$ we get $f'(\bar{U}_{(\hat{y}, y, r)}; \hat{y}, y, r) = \hat{y}^\alpha f'(\hat{y}^{-(1-\alpha)}\bar{U}_{(1, y, r)}; 1, y, r) = -1/u'(\bar{w}_{(\hat{y}, y, r)})$ (using (12)) $= -\bar{w}_{(\hat{y}, y, r)}^\alpha$, so $\bar{w}_{(\hat{y}, y, r)} = \hat{y}\bar{w}_{(1, y, r)}$. Likewise $\underline{w}_{(\hat{y}, y, r)} = \hat{y}\underline{w}_{(1, y, r)}$.

We can summarise: under the IID and CRRA assumptions mentioned in the propositions, aggregate shocks, either to y or \hat{y} , can push wages both up and down. A positive shock implies both \underline{w}_s and \bar{w}_s will be higher. In view of Proposition 1, and holding

other components of s fixed, this implies that the wage may be pushed up if the outside option binds for the worker. Likewise a negative shock can push wages down if the firm's viability is at stake. By contrast, a shock to r_j can only push wages down – if it is negative and provided the the firm's viability is at stake. A positive idiosyncratic shock cannot push wages up.

3.1 Endogenizing outside options

In our simulations we will analyse the CRRA case and look for an equilibrium that conforms with Proposition 3. Thus far we haven't specified how $\tilde{\chi}(y)$ is determined, nor how the surplus is split at the start of a match between firm and employee. We assume henceforth that there is free entry of firms, with initial idiosyncratic productivity fixed at $r^* \in \mathcal{R}$ for all entrants, so the initial state for a firm entering at time t is $s_t = (\hat{y}_t, y_t, r^*)$. Because of competition between new entrants, we assume that the worker who leaves their current firm at t , either voluntarily or because of exogenous separation, is able not only to match with a new entrant immediately¹⁶ but also extract the full surplus so will receive a utility of $\bar{U}_{s_t} - c$, where c is a state-independent utility cost of mobility.¹⁷

A Markov equilibrium is a function $\tilde{\chi}(y)$ and for each y a contract $(w_\tau(h_\tau) \geq 0)_{\tau=1}^\infty$ where $h_1 = (1, y, r^*)$, such that this contract solves Problem A above where $s_t = h_1$, and where $\chi(\hat{y}, y) = \tilde{\chi}(y) \hat{y}^{(1-\alpha)}$. While wages within a match will in general be history dependent, in a Markov equilibrium in which χ depends only on (\hat{y}_t, y_t) new entrant firms face the same future for any given (\hat{y}_t, y_t) and so will agree a contract depending only on (\hat{y}_t, y_t) . This is true even at date 0 when all workers need to find employment.

3.2 Simulations

In the simulation, time is treated as quarterly. This allows for the inclusion of much less persistent shocks in the idiosyncratic process that only last for a quarter, in addition to longer shocks which may last many years. We consider an eight-state model where y takes on two possible values; $y \in \{y^l, y^h\}$ and is persistent with $\Pr[y_t = y_{t-1}] = q^{agg}$, and r consists of two independent two-point shocks, one is iid $r^i \in \{r^l, r^h\}$ and the other is persistent $r^p \in \{r^l, r^h\}$ with $\Pr[r_t^p = r_{t-1}^p] = q^{id}$, and $r = r^i \times r^p$. Assuming that the temporary and persistent processes are equal in size allows a degree of freedom to be shut down. Similarly, q^{agg} is fixed to 0.25. This makes the boom/bust process rather short at one year, but the inclusion of the random walk process allows for the series to exhibit a lot of persistence. Furthermore, a geometric random walk process that with equal probability each quarter, the log aggregate productivity grows or shrinks by $\pm \frac{1}{2}\xi$. The inclusion of a drift does not add to the state space but does complicate the solution method; this is discussed in the appendix. We assume utility is constant relative

¹⁶Our interest is with wages in ongoing matches so for simplicity we abstract from unemployment. The main impact of modelling labor search in this framework would be to modify the impact of aggregate shocks on outside options, but it would not change the qualitative predictions of our model. Rudanko (2009) shows that limited commitment contracts combined with directed search and aggregate shocks do little to amplify unemployment volatility, and we anticipate similar results here.

¹⁷If r^* is the worst idiosyncratic state then this guarantees that all states are viable in the sense that there is positive match surplus.

risk aversion with risk aversion coefficient $\alpha = 0.5$ (robustness checks try different values of α). To add some realism, jobs can only be lost when firms are in the worst state i.e. $r = \underline{r}^i \times \underline{r}^p$, which means σ is internally calibrated to ensure that average tenure matches between the data and the model. β is fixed at $0.97^{\frac{1}{4}}$. For the outside option $\chi(y)$ we assume that the worker instantly finds a new job but suffers a mobility cost c (this ensures that it is always jointly efficient to continue a match so there is no endogenous termination.)

The stochastic components of the model are set to target standard deviations and autocorrelations both the log-growth processes in GDP and idiosyncratic productivity implied by the data. This means that the model's annual log-growth in both idiosyncratic and aggregate productivity match those implied by the LIAB/BeH data and German GDP. These stochastic moments primarily inform the gap between the good and bad states ($\log(y^h) - \log(y^l)$ and $\log(r^h) - \log(r^l)$), the size of the movements in the trend process ξ , and the switch rate of the idiosyncratic process q^{id} . The job staying rate σ is primarily informed by tenure in the data; since jobs are only lost in the worst state, this is found by simulation. The mobility cost of job loss c is set to 0.002, which is able to roughly match the size of the aggregate and downwards idiosyncratic coefficients in the main regressions.

Table 2: Calibrated parameters, and fit of targeted moments

Parameter	Value	Informative Moment	Model	Data
$\log(y^h) - \log(y^l)$	0.0320	Standard Deviation of Δy	0.021	0.021
$\log(r^h) - \log(r^l)$	0.8323	Standard Deviation of Δr	0.35	0.35
ξ	0.0189	Autocorrelation of Δy	-0.05	-0.0495
q^{id}	0.02	Autocorrelation of Δr	-0.31	-0.3009
σ	0.82	Average Tenure of Worker (Years)	10.5	10.55

After simulating the model, the same regression as equation (1) is analysed (without controls) on the simulated data, using different intervals for differencing the data. This gives the coefficients described in table 4.

In general, the model does fairly well with matching the aggregate and idiosyncratic downward coefficients, with the idiosyncratic downwards coefficient being between the OLS and IV estimates, and the aggregate coefficient overshooting, but not by a large order of magnitude. The upwards idiosyncratic coefficient in general is extremely small relative to the downward coefficients, and its sign can be either positive or negative. α and c have small effects on the coefficients outside extreme changes, as can be seen by 3.2

To understand the intuition here, note first that the coefficient on negative idiosyncratic shocks r is non-negligible, in the range of the data, and positive. Lemma 1 states

Table 3: Regression Results from Model Simulation

Interval Use For Difference	β	γ^+	γ^-
1 Year	0.279	3.53×10^{-4}	0.0126
2 Years	0.332	-5.52×10^{-4}	0.0164
3 Years	0.345	-2.23×10^{-3}	0.0213

Table 4: Robustness of regression coefficients to risk aversion and mobility cost.

Change to Baseline	Interval Use For Difference	β	γ^+	γ^-
$c = 0.001$	1 Year	0.280	4.06×10^{-4}	0.0127
	2 Years	0.333	-5.07×10^{-4}	0.0164
	3 Years	0.346	-2.19×10^{-3}	0.0213
$c = 0.004$	1 Year	0.276	2.43×10^{-4}	0.0125
	2 Years	0.330	-6.48×10^{-4}	0.0163
	3 Years	0.344	-2.30×10^{-3}	0.0212
$\alpha = 0.8$	1 Year	0.278	3.57×10^{-4}	0.0126
	2 Years	0.332	-5.52×10^{-4}	0.0163
	3 Years	0.345	-2.24×10^{-3}	0.0212
$\alpha = 1.5$	1 Year	0.278	3.46×10^{-4}	0.0125
	2 Years	0.331	-5.68×10^{-4}	0.0162
	3 Years	0.346	-2.28×10^{-3}	0.0211

that wages only change when there is a binding participation constraint. Here, a cut in productivity will mean the firm's participation constraint may bind, pushing the wage down. (The outside option for the worker doesn't change if y is unchanged so there is no other effect that goes in the other direction.) However a positive idiosyncratic shock, although it relaxes the firm's participation constraint, does not lead to a wage increase if y is unchanged, because then the outside option hasn't changed, so the wage is held constant.¹⁸

For aggregate shocks the situation is reversed to a large extent. A positive aggregate shock will increase the worker's outside option, requiring a wage increase if currently the worker's constraint is binding or close to binding. A negative aggregate shock will also have some effect: if the firm is initially on or close to its participation constraint, it will cut wages.¹⁹

4 Concluding comments

The paper studies the pass-through of firm productivity changes to wages. A simple limited commitment model of wage contracting exhibits salient features that we find in German data. In particular the asymmetric nature of wage responses to positive and negative idiosyncratic productivity movements is captured in both analytical results and in a calibrated simulation, as well as the relative size of the aggregate and idiosyncratic effects.

¹⁸To be precise, if when we go from state s to s' , y remains constant, but r falls, then by Proposition 2 provided r is an iid shock, $\underline{w}_s = \underline{w}_{s'}$, and $\bar{w}_{s'} < \bar{w}_s$. Thus w_t will fall to $\bar{w}_{s'}$ if it previously was between $\bar{w}_{s'}$ and \bar{w}_s . If on the other hand y remains constant, but r rises, the interval expands only at the top so the wage does not increase.

¹⁹If when we go from state s to s' , r remains constant, but y falls, then by Proposition 2, provided y is an iid shock, $\underline{w}_{s'} < \underline{w}_s$, and $\bar{w}_{s'} < \bar{w}_s$.

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Appendices

While the model can be written recursively in terms of utility promised to the worker, and future (state-contingent) utilities promised, it is much simpler from a computational standpoint to treat the wage as a state variable. Since the structure of the policy function is already known i.e. the wage stays the same over time unless it collides with one of the state-contingent lower or upper bounds. For this reason, it is useful to solve the following dynamic program, which reformulates the problem:

$$F^*(w, S) = y(S) - w + \beta \sigma_S \mathbb{E}_{g, S'|S} \left(g \hat{F} \left(\frac{w}{g}, S' \right) \right), \quad (16)$$

where :

$$\hat{F}(w, S) = \begin{cases} F^*(w, S), & \text{if } F^*(w, S) \geq 0, \text{ and } U^*(w, S) \geq \chi^*(S) - c \\ 0, & \text{if } F^*(w, S) < 0, \text{ and } U^*(w, S) \geq \chi^*(S) - c \\ F^{extraction}(S), & \text{if } F^*(w, S) \geq 0, \text{ and } U^*(w, S) < \chi^*(S) - c \\ 0, & \text{if } F^*(w, S) < 0, \text{ and } U^*(w, S) < \chi^*(S) - c \end{cases}$$

Here, $F^*(w, S)$ denotes the firm's expected discounted profits conditional on paying wage w and being in state S . $y(S) - w$ is therefore flow profit. The continuation value must be discounted by β , weighed by the probability that the match survives, σ_S . g refers to the growth in the trend process between the current and next period, which can take on values $e^{\frac{1}{2}\xi}$ and $e^{-\frac{1}{2}\xi}$ with equal probability. Since growth is permanent, it effectively re-scales the firm value, hence the need to scale by g for comparability to the current value. Similarly, from the perspective of next period, the current wage will have to be normalised, hence the entry $\frac{w}{g}$. $\hat{F}()$ is convenient notation as it allows for various possibilities. If sticking with the same wage is such that neither worker nor firm wish to leave, then \hat{F} is the same as F^* . However, if one, or both agents wish to leave by trying to keep the wage the same, then this is like hitting a wage bound. Either the worker wishes to leave, in which case the worker will be given the wage that makes them indifferent, and the firm will get the "extraction" value, $F^{extraction}$, or the firm wishes to leave which ensures that \hat{F} will be set to zero. The extraction value is defined as the maximum amount of profit the firm is able to make if they had full market power over the wage i.e.

$$F^{extraction}(S) = \max_w \{ F^*(w, S) \}, \quad \text{subject to } U^*(w, S) \geq \chi^*(S) - c. \quad (17)$$

If, somehow both agents wish the match to end (which does not occur in the equilibrium but could occur during iterations of the value functions) then the firm value is allocated zero.

These make reference to the worker's utility under wage w and state S which must be defined:

$$U^*(w, S) = \frac{w^{1-\alpha}}{1-\alpha} + \beta \mathbb{E}_{g, S'|S} \left(g^{1-\alpha} (\sigma_S \hat{U} \left(\frac{w}{g}, S' \right) + (1 - \sigma_S)(\chi^*(S') - c)) \right), \quad (18)$$

where :

$$\hat{U}(w, S) = \begin{cases} U^*(w, S), & \text{if } F^*(w, S) \geq 0, \text{ and } U^*(w, S) \geq \chi(S) - c \\ U^{extraction}(S), & \text{if } F^*(w, S) < 0, \text{ and } U^*(w, S) \geq \chi(S) - c \\ \chi(S) - c, & \text{if } F^*(w, S) \geq 0, \text{ and } U^*(w, S) < \chi(S) - c \\ \chi(S) - c, & \text{if } F^*(w, S) < 0, \text{ and } U^*(w, S) < \chi(S) - c. \end{cases}$$

The intuition for the worker's value function $U^*(w, S)$ is similar to that of the firm's. The worker gains flow utility from the wage, and they have value $\chi(S) - c$ if they lose the job. Notice that when accounting for growth, the utility is scaled by $g^{1-\alpha}$, not g , to account for the worker's utility function. \hat{U} is then defined in a similar manner to \hat{F} ; if both wish to stay under keeping the wage the same then $\hat{U} = U^*$, if the firm wants to leave but the worker does not then the worker gets the extraction value, and if the worker wants to leave then irrelevant of the firm's wishes, the worker gets their outside value $\chi(S) - c$. The worker's extraction value and outside options are defined as:

$$U^{extraction}(S) = \max_w \{U^*(w, S)\}, \quad \text{subject to } F^*(w, S) \geq 0,$$

$$\chi(S) = \max_w \{U^*(w, S^{id:low})\}, \quad \text{subject to } F^*(w, L(S)) \geq 0. \quad (19)$$

The worker extraction value is symmetric to that of the firms. The definition of $\chi(S)$ uses the rule that upon leaving a job, the worker matches to a low-idiosyncratic productivity firm (hence the crude notation $S^{low:id}$). So far, this constitutes a dynamic program which is straightforward to solve numerically. This is done by defining a logarithmically spaced "wage" grid set with a minimum of -0.2 and maximum of 1, with spacing 10^{-4} apart (meaning each gap is \sim one hundredth of a percent). Over S there are 12 possible combinations. Value function iteration is repeated until mean root squared error over all the elements of U^* and F^* is less than 10^{-4} . When this solution is reached, now the "wage bounds" associated with each state can be found by solving:

$$w_{LB}(S) = \underset{w}{\operatorname{argmin}} \{w\}, \quad \text{subject to } U^*(w, S) \geq \chi^*(S) - \varepsilon,$$

$$w_{UB}(S) = \underset{w}{\operatorname{argmax}} \{w\}, \quad \text{subject to } F^*(w, S) \geq 0.$$

With these bounds, the simulation can now be run. This is done for 120,000 periods. This then generates a quarterly time series which has the aggregate and idiosyncratic productivity, as well as the wage of the worker. To compare this to annual data, the data is bunched and aggregated over groups of four quarters, which then is used to do the regressions. An example of the evolution of the wage bounds and the wage is given in figures 1 and 2.



Figure 1: 40 year example of simulation.

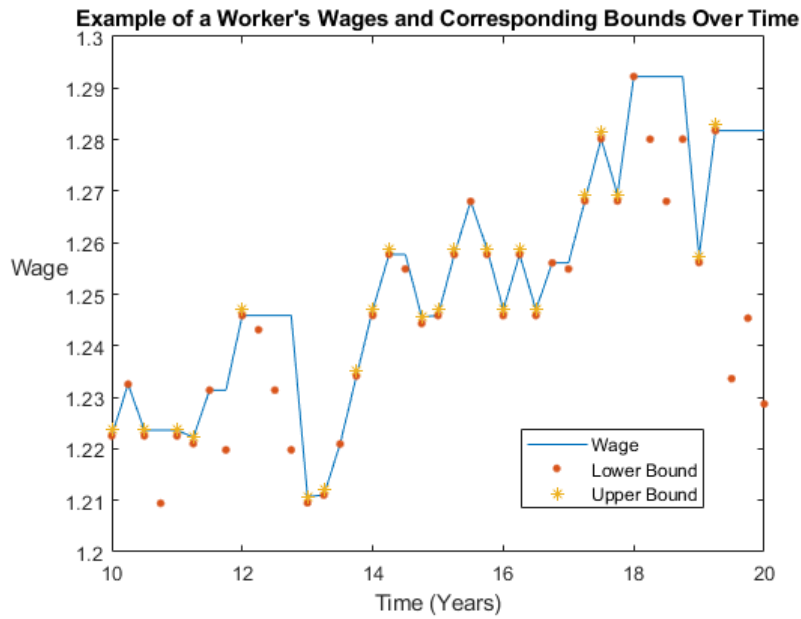


Figure 2: This shows the same series as 4 but zoomed in to the 10 - 20 year section and a rescaled y-axis. Notice sometimes the lower bound pushes the wage up and the upper bound (which often disappears off the top of the range) pushes the wage down.