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A Class of Fibonacci Matrices, Graphs, and Games

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Abstract: In this paper, we define a class of Fibonacci graphs as graphs whose adjacency matrices are obtained by alternating binary Fibonacci words. We show that Fibonacci graphs are close in size to Turán graphs and that their size-stability tradeoff defined as the product of their size and stability number is very close to the maximum possible over all bipartite graphs. We also consider a combinatorial game based on sequential vertex deletions and show that the Fibonacci graphs are extremal regarding the number of rounds in which the game can terminate.

Keywords: Fibonacci array; Fibonacci matrix; Fibonacci graph; bipartite graph; Turán graph

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1. Introduction

Fibonacci sequence is among the most interesting and popular mathematical objects. The interest in this sequence is shared by experts and practitioners from different areas because of its numerous applications in different sciences and sectors of human practice, as diverse as mathematics, computer science, physics, biology, and economics, among other natural and social sciences. Various results pertinent to the Fibonacci sequence, the reporting of which goes beyond the scope of this paper, are available in the literature. Since 1963, hundreds of such works have appeared in the journal *The Fibonacci Quarterly*—the primary publication of the Fibonacci Association—as well as in many other esteemed professional journals. For an outline of the basic related findings the reader is referred to the monographs [1,2].

Apart from the results presenting properties of the Fibonacci sequence and its applications, a large number of new mathematical structures and developments have been devoted to the subject. Perhaps the most notable among these are the number sequences of Lucas [2], Pell [3], Padovan [4,5], and Perrin [6], along with the polynomials associated with them. In particular, some remarkable properties and applications are associated with the Pell-Lucas polynomials (see, e.g., [7,8]).

In addition to studying of sequences of numbers, the Fibonacci sequence influenced investigations of objects of different nature. In all likelihood, of a principal importance among these is the sequence of Fibonacci words, which are constructed using the Fibonacci recurrence, and which are closely related to the results of the present paper.

Fibonacci words belong to the class of Sturmian words and are known to have certain extremal properties that have been useful in obtaining lower bounds for the running time of various algorithms in combinatorial pattern matching (see, e.g., [9–15]).

Fibonacci words were generalized to two dimensions in [11]; 2D Fibonacci words were then used to obtain tight bounds on the number of certain 2D repetitions in two dimensional arrays, and in turn for proving tight bounds on the complexity of algorithms for their detection.

Another example of utilizing the Fibonacci sequence to a mathematical discipline different from number theory is seen in defining and studying 'Fibonacci graphs.' To our knowledge, the first study of graphs associated with Fibonacci sequence appeared in [16]. The introduction of those graphs, called 'generalized Fibonacci maximum path graphs', aimed at investigating the structure of an acyclic directed graph maximizing the number of distinct paths between two given vertices. Supposedly, the best-known class of Fibonacci graphs is the one of the Fibonacci cubes (alternatively known as Fibonacci networks) [17–20]. These are similar to the hypercube graphs, but having a Fibonacci number of vertices. Applications of these graphs have been found in parallel computing [17], routing and broadcasting in distributed computations [21] and chemical graph theory [22]. Another recent work introducing a class of Fibonacci graphs is [23]. In that paper, Fibonacci graphs are defined as ones having degree sequence consisting of n consecutive Fibonacci numbers, and the authors provide necessary and sufficient conditions for the realizability of such a sequence.

In the present paper we study a new class of graphs with Fibonacci adjacency matrices, the definition of the latter being adapted from one of the Fibonacci arrays mentioned above. We investigate various characteristics of these graphs, in particular they appear to be complete bipartite graphs possessing certain extremal or close to extremal properties. Thus, similar to Fibonacci numbers and words, these graphs are potentially applicable to analysis of the performance of algorithms for solving problems on graphs. Moreover, being bipartite, the considered Fibonacci graphs may find many of the various applications realized by bipartite graphs, such as in cryptology and secured communications [24,25], in anti-theft networks [26], in cloud computing [27], in biology and medicine [28], in data transfer [29], and in many other fields [30].

In the next section we define the classes of Fibonacci matrices and graphs which are the subject of this work and explore their properties. In particular, we show that they are complete bipartite graphs, which are close in size to the celebrated Turán graphs, and that the product of their size and stability number (called size-stability tradeoff) is very close to the maximum possible over all bipartite graphs. In Section 3, we consider a combinatorial game based on sequential vertex deletions and show that the Fibonacci graphs are extremal regarding the number of rounds in which the game can terminate. We conclude with some final remarks in Section 4.

2. Fibonacci Matrices and Graphs

A sequence of Fibonacci words on an alphabet (a, b) is defined by $f_1 = a$, $f_2 = ab$, and $f_{k+1} = f_k f_{k-1}$ for $k \geq 2$, where $f_k f_{k-1}$ denotes the concatenation of words f_k and f_{k-1}. For example,

$$W_1 = 0, 01, 010, 01001, 01001010, 0100101001001, \ldots$$
$$W_2 = 1, 10, 101, 10110, 10110101, 1011010110110, \ldots$$

are the sequences of Fibonacci words on the alphabets $(0, 1)$ and $(1, 0)$, respectively. With this construction, the length of the kth Fibonacci word equals the $(k+1)$st Fibonacci number F_{k+1}.

A Fibonacci matrix $F^{(k)}$, $k \geq 1$, is an $F_{k+1} \times F_{k+1}$ matrix. The first row and the first column of $F^{(k)}$ is the kth Fibonacci word f_k from the sequence W_1 defined above. Then, for $2 \leq i \leq F_{k+1}$, row i (resp. column i) is the word f_k from W_1 if the ith entry of the first row/column is 0, and it is the word f_k from W_2 if the ith entry of the first row/column is 1. Thus, the first four Fibonacci matrices are:

$$F^{(1)} = \begin{bmatrix} 0 \end{bmatrix}, F^{(2)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, F^{(3)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, F^{(4)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Recall that the adjacency matrix of an undirected graph $G = (V, E)$ is a symmetric $|V| \times |V|$ matrix A whose (i, j)th entry equals 1 if there is an edge in E connecting vertices $v_i, v_j \in V$, and 0 otherwise. Note that by construction, $F^{(k)}$ is a symmetric matrix for all $k \geq 1$. We define a *Fibonacci graph* $G_{F^{(k)}}$ as a graph with adjacency matrix $F^{(k)}$ for some $k \geq 1$. In the rest of this work we study some properties of Fibonacci graphs.

2.1. Structural Properties of Fibonacci Graphs

Given a graph $G = (V, E)$, $|V|$ and $|E|$ will be referred to as the *order* and *size* of G, respectively. $G = (V, E)$ is a *simple* graph if it has no loops and no parallel edges. G is a *bipartite* graph if the set of its vertices can be partitioned into two sets V_1 and V_2, called *parts*, such that every edge in E connects a vertex from V_1 and a vertex from V_2. A bipartite graph with parts V_1 and V_2 is denoted $G = (V_1, V_2, E)$. $G = (V_1, V_2, E)$ is a *complete bipartite* graph if there is an edge joining each vertex from V_1 to every vertex of V_2. A complete bipartite graph with $|V_1| = p$ and $|V_2| = q$ is denoted $K_{p,q}$.

Proposition 1. *For any $k \geq 2$, $G_{F^{(k)}}$ is a simple complete bipartite graph with $F_{k-1}F_k$ edges.*

Proof. By construction, $F^{(k)}$ is a 0/1 matrix, so $G_{F^{(k)}}$ has no parallel edges. To see why $G_{F^{(k)}}$ has no loops, consider first an arbitrary row i of $F^{(k)}$ starting with 0 in its first column, i.e., $F^{(k)}_{i1} = 0$. By the symmetry of $F^{(k)}$, element $F^{(k)}_{ii}$ belongs to the ith column which is the same as row i, i.e., $F^{(k)}_{1i} = 0$. Since the first and the ith rows are identical, their prefixes of length i are identical as well, so $F^{(k)}_{ii} = F^{(k)}_{1i} = 0$. Now let row i start with 1, i.e., $F^{(k)}_{i1} = 1$. Similar to the previous case, by the symmetry of $F^{(k)}$ we have $F^{(k)}_{1i} = 1$. By the definition of Fibonacci words, a word of type W_1 has 0 in its i'th position if and only if a word of type W_2 has 1 in the i'th position. Since row 1 starts with 0, i.e., is of type W_1, while row i is of type W_2, it follows that $F^{(k)}_{ii} = 0$. Thus, all diagonal entries of the adjacency matrix of $G_{F^{(k)}}$ are 0's, so $G_{F^{(k)}}$ has no loops.

Let V_1 be the subset of vertices of $G_{F^{(k)}}$ corresponding to rows of $F^{(k)}$ whose first entry is 0 and let V_2 be the subset of vertices corresponding to rows of $F^{(k)}$ whose first entry is 1. It is easily seen that with this partitioning the graph $G_{F^{(k)}}$ is isomorphic to the complete bipartite graph $H^{(k)}_A$ with adjacency matrix $A^{(k)}$ whose entries are defined by

$$A^{(k)}_{ij} = \begin{cases} 0, & \text{when } i, j \leq F_k \text{ or } i, j > F_k \\ 1, & \text{otherwise} \end{cases}$$

For example, a graph with adjacency matrix $A^{(4)} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$ is isomorphic to the graph with adjacency matrix $F^{(4)}$ shown above.

Finally, by construction, the kth word in the sequence W_1 has F_k 0's and F_{k-1} 1's; thus, the matrix $F^{(k)}$ has F_k rows comprised of the kth word of W_1, which contains F_{k-1} 1's, and F_{k-1} rows comprised of the kth word of W_2, which contains F_k 1's. Thus, the total number of 1's in $F^{(k)}$ is $2F_{k-1}F_k$, and hence the number of edges of $G_{F^{(k)}}$ is $F_{k-1}F_k$. □

2.2. Fibonacci Graphs, Turán Graphs, and Size-Stability Tradeoff

A *triangle* is a simple graph on three vertices such that every two of them are adjacent. A graph is *triangle-free* if it contains no triangles. A triangle-free graph is *maximal* if adding any new edge creates a triangle subgraph. It is well-known that bipartite graphs are triangle-free, and complete bipartite graphs are maximal triangle-free. Thus, for any $k \geq 2$, $G_{F^{(k)}}$ is a maximal triangle-free graph. Turán's theorem [31] (see also [32–34]) establishes that the Turán graph $T(n, 2) = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is the unique triangle-free graph with maximum number of edges $t(n,2) = \lfloor \frac{n^2}{4} \rfloor$. The following proposition shows that asymptotically, the Fibonacci graph $G_{F^{(k)}}$ is close in size to the Turán graph $T(F_{k+1}, 2)$. We will denote the size of $G = (V, E)$ by $e(G)$.

Proposition 2.

$$\lim_{k\to\infty} \frac{t(F_{k+1},2)}{e(G_{F^{(k)}})} = \frac{\phi^3}{4} \approx 1.059,$$

where ϕ denotes the golden ratio.

Proof. By definition, the size of the Turán graph of order F_{k+1} is $\left\lfloor \frac{F_{k+1}^2}{4} \right\rfloor$, while by Proposition 1, the size of $F^{(k)}$ is $F_{k-1}F_k$. Thus, we have

$$\lim_{k\to\infty} \frac{t(F_{k+1},2)}{e(G_{F^{(k)}})} = \lim_{k\to\infty} \frac{\lfloor F_{k+1}^2/4 \rfloor}{F_{k-1}F_k}.$$

Since $x \leq \lfloor x \rfloor < x + 1$ for any $x \in \mathbb{R}$, we have

$$\begin{aligned}
\lim_{k\to\infty} \frac{t(F_{k+1},2)}{e(G_{F^{(k)}})} &= \lim_{k\to\infty} \frac{F_{k+1}^2/4}{F_{k-1}F_k} = \lim_{k\to\infty} \frac{(F_{k-1}+F_k)^2}{4F_{k-1}F_k} = \\
&= \lim_{k\to\infty} \frac{F_{k-1}^2 + 2F_{k-1}F_k + F_k^2}{4F_{k-1}F_k} = \\
&= \lim_{k\to\infty} \left(\frac{F_{k-1}}{4F_k} + \frac{1}{2} + \frac{F_k}{4F_{k-1}} \right) = \\
&= \frac{1}{4\phi} + \frac{1}{2} + \frac{\phi}{4} = \frac{(\phi+1)^2}{4\phi} = \frac{\phi^4}{4\phi} = \frac{\phi^3}{4} \approx 1.059.
\end{aligned}$$

□

A set S of vertices of a graph G is called *stable* (or *independent*) if no two vertices from S are adjacent. The cardinality of a maximum stable set of G is the *stability number* of G, denoted $\alpha(G)$. By Proposition 1, $\alpha(G_{F^{(k)}}) = F_k$ for any $k \geq 1$. The *size-stability tradeoff* of a graph G, denoted $\psi(G)$, is the product of its size and stability number, i.e., $\psi(G) = e(G)\alpha(G)$. Increasing the size of a graph by adding new edges results in non-increasing stability number. We consider the size-stability tradeoff for triangle-free graphs, and more specifically, for bipartite graphs. Let B_n denote the class of complete bipartite graphs on n vertices and let $G^* \in B_n$ be a graph for which $\psi(G^*) = \max\{\psi(G) : G \in B_n\}$.

Lemma 1.

$$\psi(G^*) = (n - \lfloor 2n/3 \rfloor)\lfloor 2n/3 \rfloor^2 \;\text{ or }\; \psi(G^*) = (n - \lfloor 2n/3 - 1/3 \rfloor - 1)(\lfloor 2n/3 \rfloor + 1)^2. \quad (1)$$

Proof. Let $G = (V_1, V_2, E) \in B_n$ be a complete bipartite graph with $|V_1| \leq |V_2|$. We have $e(G) = (n - |V_2|)|V_2|$ and $\alpha(G) = |V_2|$. Then $\psi(G) = (n - |V_2|)|V_2|^2$. To determine the value of $|V_2|$ which maximizes $\psi(G)$, consider the function $f(x) = (n-x)x^2$, $0 \leq x \leq n$. We have $f'(x) = 2nx - 3x^2$ and $f''(x) = 2n - 6x$. Moreover, $f'(x) = 0$ for $x = 0$ and $x = 2n/3$. Since $f''(0) = 2n > 0$ and $f''(2n/3) = -2n < 0$, f has a minimum at 0 and a maximum at $2n/3$. Hence, the maximum value of f on the interval $[0, n]$ is $f(2n/3) = (n/3) \cdot (2n/3)^2 = 4n^3/27$. It follows that the maximum value of the discrete function $\psi(G)$ for $G \in B_n$ is reached for the complete bipartite graphs $G' = (V_1', V_2', E')$ with $|V_1'| = n - \lfloor 2n/3 \rfloor$ and $|V_2'| = \lfloor 2n/3 \rfloor$ or with $|V_1'| = n - \lfloor 2n/3 \rfloor - 1$ and $|V_2'| = \lfloor 2n/3 \rfloor + 1$. Then we have

$$e(G') = (n - \lfloor 2n/3 \rfloor)\lfloor 2n/3 \rfloor \quad\text{and}\quad \alpha(G') = \lfloor 2n/3 \rfloor, \text{ or}$$

$$e(G') = (n - \lfloor 2n/3 \rfloor - 1)(\lfloor 2n/3 \rfloor + 1) \quad \text{and} \quad \alpha(G') = \lfloor 2n/3 \rfloor + 1.$$

Then the possible maximum value for $\psi(G)$ on graphs from B_n is either $(n - \lfloor 2n/3 \rfloor)\lfloor 2n/3 \rfloor^2$ or $(n - \lfloor 2n/3 \rfloor - 1)(\lfloor 2n/3 \rfloor + 1)^2$. □

Theorem 1. *Let $G^* \in B_{F_k}$ be a graph for which $\psi(G^*) = \max\{\psi(G) : G \in B_{F_k}\}$. Then,*

$$\lim_{k\to\infty} \frac{\psi(G^*)}{\psi(G_{F_k})} = \frac{4}{27}\phi^4 \approx 1.015.$$

Proof. Let G_{F_k} be a Fibonacci graph. We have $e(G_{F_k}) = F_k F_{k-1}$ and $\alpha(G_{F_k}) = F_k$, so $\psi(G_{F_k}) = F_k^2 F_{k-1}$. From relation (1) of Lemma 1 and its proof it follows that for $n = F_{k+1}$

$$\psi(G^*) = \left(F_{k+1} - \left\lfloor \frac{2F_{k+1}}{3} \right\rfloor\right) \left\lfloor \frac{2F_{k+1}}{3} \right\rfloor^2 \text{ or } \psi(G^*) = \left(F_{k+1} - \left\lfloor \frac{2F_{k+1}}{3} \right\rfloor - 1\right) \left(\left\lfloor \frac{2F_{k+1}}{3} \right\rfloor + 1\right)^2,$$

as in either case

$$\psi(G^*) \le 4F_{k+1}^3/27$$

and the value of $\psi(G^*)$ converges to $4F_{k+1}^3/27$. Consequently, we obtain

$$\begin{aligned}
\lim_{k\to\infty} \frac{\psi(G^*)}{\psi(G_{F_k})} &= \lim_{k\to\infty} \frac{4F_{k+1}^3/27}{F_k^2 F_{k-1}} = \lim_{k\to\infty} \frac{4}{27} \frac{(F_k + F_{k-1})^3}{F_k^2 F_{k-1}} = \\
&= \lim_{k\to\infty} \frac{4}{27} \frac{F_k^3 + 3F_k^2 F_{k-1} + 3F_k F_{k-1}^2 + F_{k-1}^3}{F_k^2 F_{k-1}} = \\
&= \frac{4}{27}(\phi + 3 + 3/\phi + 1/\phi^2) = \frac{4}{27} \frac{\phi^3 + 3\phi^2 + 3\phi + 1}{\phi^2} = \\
&= \frac{4}{27} \frac{(\phi + 1)^3}{\phi^2} = \frac{4}{27} \frac{\phi^6}{\phi^2} = \frac{4}{27}\phi^4 \approx 1.015.
\end{aligned}$$

□

Erdős et al. [35] proved that almost all triangle-free graphs are bipartite. Thus, Theorem 1 holds for almost all triangle-free graphs. An interesting question is whether Theorem 1 holds for all triangle-free graphs. To compare the size-stability tradeoff of Turán's and Fibonacci graphs, recall that the number of edges $t(n,2)$ and the stability number $\alpha(T(n,2))$ of a Turán's graph $T(n,2)$ approach as n increases $\frac{n^2}{4}$ and $\frac{n}{2}$, respectively, (see, e.g., [31,32]). Thus,

$$\lim_{k\to\infty} \psi(T(F_{k+1}, 2)) = \lim_{k\to\infty} (t(F_{k+1}, 2)\alpha(F_{k+1}, 2)) =$$

$$= \lim_{k\to\infty} \frac{F_{k+1}^2}{4} \cdot \frac{F_{k+1}}{2} = \lim_{k\to\infty} \frac{1}{8} {F_{k+1}}^3.$$

Then

$$\lim_{k\to\infty} \frac{\psi(G^*)}{\psi(T(F_{k+1}, 2))} = \frac{\frac{4}{27}F_{k+1}^3}{\frac{1}{8}F_{k+1}^3} \approx 1.185.$$

Thus, asymptotically the size-stability tradeoff for Fibonacci graphs is significantly closer to the maximum possible value than for $T(F_{k+1}, 2)$.

3. Fibonacci Graphs and Games

Various mathematical games involving Fibonacci numbers have been studied. In this section, we introduce a combinatorial game, called the *vertex deletion game*, whose analysis is related to Fibonacci graphs. For recent studies of other similar games, see [36–39] and the bibliography therein.

The game is defined on an arbitrary simple graph $G = (V, E)$ and is played as follows. The vertices of G are partitioned into subsets V_1 and V_2. *Cleaning* the graph means that all vertices in V_1 that do not have a neighbor in V_2 are deleted, and all vertices in V_2 that do not have a neighbor in V_1 are deleted. The following steps are repeated until no vertex can be deleted:

(1) The graph is cleaned, and for each $v \in V_1$, Player 1 deletes a neighbor of v in V_2.
(2) The graph is cleaned, and for each $v \in V_2$, Player 2 deletes a neighbor of v in V_1.

The player who deletes all the opponent's vertices first wins. We now address the following questions:

1. What is the maximum possible number of steps that one of the players must perform in order to win the game, over all graphs on n vertices and regardless of the strategy followed by the opponent?
2. For which graphs is that maximum achieved?

Remark 1. *After the first cleaning the graph all neighbors of the remaining vertices in V_1 are in V_2 and vice versa. Thus, the remaining graph is bipartite and it clearly stands bipartite until the end of the game.*

Lemma 2. *Let $G = (V_1, V_2, E)$ be an incomplete bipartite graph and s be the maximal possible number of steps that one of the players must perform in order to win the game on G, regardless of the strategy followed by the opponent. Let there be no edge between $u \in V_1$ and $v \in V_2$, and let G' be the bipartite graph obtained from G by adding an edge uv. Then a player can win the game on G' in no less than s steps, regardless of the strategy followed by the opponent.*

Proof. Let us first remark that in the course of the game a vertex can be removed from the graph either if it has been deleted by a vertex of the opponent or after itself has deleted (at different steps of the game) all of its neighbors (and thus it would be removed after cleaning the graph from vertices with no neighbors).

In the framework of the game on graph G, we can distinguish between the following possibilities regarding vertices u and v.

(a) At some steps u and v are deleted by their neighbors from the other part;
(b) u is deleted at some step; at a later step v deletes its last neighbor and is removed after cleaning the graph, or survives up to the end of the game;
(c) v deletes its last neighbor and is removed after cleaning the graph; u is deleted at a later step, or survives up to the end of the game;
(d) u deletes its last neighbor and is removed after cleaning the graph; v deletes its last neighbor and is removed after the deletion of u upon cleaning the graph (the case where u is removed after v being symmetric).

Clearly, u and/or v can survive until the end of the game – either to be among the vertices which can delete neighbors at the last step, or to be among the vertices deleted at the last step.

Next we consider the game on G' if the players use the same strategies as in the game on G. In Cases (a) and (b) the game goes in exactly the same way as on G and ends in the same number of steps. In Case (c), the edge uv remains in the graph until the deletion of u, with u being the only neighbor of v, the other neighbors being deleted in the same steps as in the game on G. Before that point, edge uv has no effect on the course of the game; after uv's removal, the game continues in the same way as on G and ends after the same number of steps. If vertex u survives until the end of the game, so does edge uv. If vertex u was deleted at the last step of the game on G (then Player 2 wins), then edge uv would vanish with u's removal, as well, so the game ends in the same way and with the same number of steps as on G. If u was among vertices which remove the last opponents from V_2 (then

Player 1 wins), then after cleaning G' only the edge uv would remain and Player 2 would be in turn to play. It deletes u and wins the game in $s+1$ steps. The analysis of Case (d) is similar to the one of Case (c). Again uv remains after all neighbors of u (different from v) are deleted and the game continues in the same way as on G. At a later point all neighbors of v (different from u) are deleted in the same steps as on G. Edge uv remains in the graph and the game proceeds in the rest of the graph in the same way as on G, as edge uv does not affect the process. After the last deletion (which is identical to the last one on G) and the following graph cleaning, only the edge uv remains. Its deletion takes one more step, so the player whose turn is to play wins in $s+1$ steps. □

Theorem 2. *Let $G = (V, E)$ be a simple connected graph on n vertices. Then the vertex deletion game on G terminates after at most $\lceil \log_\phi n \rceil + 1$ stages, as the maximum number of stages is reached for Fibonacci graphs.*

Proof. By Remark 1, the graph which remains after the first cleaning is bipartite. By Lemma 2, adding a new edge to an incomplete bipartite graph cannot decrease the number of steps in which a player can win the game, hence winning the game on a complete bipartite graph cannot take fewer steps than on an incomplete one on the same parts of vertices. Therefore, without loss of generality we can suppose that the game is played on a complete bipartite graph $G = (V_1, V_2, E)$, where V_1 are the vertices of Player 1 and V_2 are the vertices of Player 2, $|V| = n$, $|V_1| = n_1$, $|V_2| = n_2$, $n = n_1 + n_2$. Note also that after each cleaning the remaining graph is still a complete bipartite graph on fewer vertices.

If $n_1 \geq n_2$ and Player 1 starts first, then the game ends after the first step, as Player 1 deletes all vertices of V_2 and wins the game. Let $n_1 < n_2$ and Player 1 start first by deleting n_1 vertices from V_2. Then Player 2 remains with $n_2 - n_1$ vertices of V_2. In turn, Player 2 removes $n_2 - n_1$ vertices from V_1, so after that Player 1 remains with $n_1 - (n_2 - n_1) = 2n_1 - n_2$ vertices of V_1. Thus, we obtain the following sequence of remaining numbers of vertices of the two players after consecutive game steps:

Player 1: $n_1, 2n_1 - n_2, 5n_1 - 3n_2, 13n_1 - 8n_2, 34n_1 - 21n_2, \ldots, F_{k+1}n_1 - F_k n_2, \ldots$

Player 2: $n_2, n_2 - n_1, 2n_2 - 3n_1, 5n_2 - 8n_1, 13n_2 - 21n_1, \ldots, F_{k-1}n_2 - F_k n_1, \ldots$

Both sequences are (possibly non-strictly) decreasing until a term of one of them, say, of Player 1, becomes equal to 0 or negative. At that point Player 2 wins. If the process of sequence generation is continued after the end of the game, the following terms of the first sequence keep decreasing (possibly, non-strictly) with negative values, while those of the second sequence start increasing (possibly, non-strictly) with positive values. Then at a certain point either $F_{k-1}n_2 - F_k n_1 = 0$ or $F_{k+1}n_1 - F_k n_2 = 0$, depending on whether for a fixed integer k, $n_1 = F_{k-1}, n_2 = F_k$ or $n_1 = F_k, n_2 = F_{k+1}$. In the former case (the latter one being symmetric) the following terms of the sequence of Player 1 keep decreasing with values $-F_2, -F_4, -F_6, \ldots$, while those of the sequence of Player 2 start increasing with values $F_1, F_3, F_5, \ldots$. To see why, note that the general terms of the two sequences can, respectively, be written as $F_k F_{m+1} - F_m F_{k+1} = (-1)^m F_{k-m}$ with m even, and $-(F_k F_{m+1} - F_m F_{k+1}) = (-1)^{m+1} F_{k-m}$ with m odd. The above equalities follow from the well-known d'Ocagne's identity (see [2]).

It is easy to see that the game termination takes a maximum number $\lceil \log_\phi n \rceil + 1$ of steps when n_1 and n_2 are two consecutive Fibonacci numbers. As a matter of fact, the steps of the considered game resemble the steps of the Euclidean algorithm performed on numbers n_1 and n_2 until a quotient of division happens to be greater than or equal to 2. At the next step the game terminates, as the player in turn has at least as many vertices as the opponent, and thus can delete all opponent's vertices in one step. Still in the remote 1844 Gabriel Lamé [40] proved that the number of steps needed to compute the greatest common divisor of two integers a, b less than an integer n is less than or equal to $M = \frac{\ln n + \ln \sqrt{5}}{\ln \phi}$, as this upper bound is reached if a and b are two consecutive Fibonacci numbers. We have

$$
\begin{aligned}
M =& \frac{\log_\phi n / \log_\phi e + \log_\phi \sqrt{5}/log_\phi e}{\log_\phi \phi / \log_\phi e} = \\
&= \log_\phi n + \log_\phi \sqrt{5} = \\
&= \log_\phi n + 1.67228... < \lceil \log_\phi n \rceil + 1 + 0.673.
\end{aligned}
$$

Since for the considered vertex deletion game the maximum possible number of steps s is integer, it follows that $s \leq \lceil \log_\phi n \rceil + 1$.

Thus, vertex deletion game with a maximum number of steps is the one played on a Fibonacci graph $G_{F^{(k)}}(V, E)$ on $n = F_{k+1}$ vertices. If Player 1 starts first, then after no more than $\lceil \log_\phi n \rceil + 1$ steps one of the two players wins the game. More precisely, let $n_1 = F_{k-1}$, $n_2 = F_k$, $k = 2, 3, \ldots$. Then, if $k - 1$ is odd, Player 1 wins, otherwise Player 2 wins, as the winner has exactly one survived vertex in either case. □

4. Conclusions

In this paper, we defined Fibonacci graphs and studied their properties. In particular, we showed that Fibonacci graphs appear to be simple complete bipartite graph. We showed that the Fibonacci graphs are close in size to the notorious Turán graphs which are extremal in terms of graph size. We defined size-stability tradeoff of graphs and showed that for Fibonacci graphs this characteristic is very close to the maximum possible. We also introduced a vertex deletion game on graphs and showed that the maximum possible number of game stages is reached on Fibonacci graphs. It illustrates how Fibonacci graphs can be used to obtain bounds on the time complexity of problems defined on graphs. Studying other properties of Fibonacci graphs is seen as an engaging future task.

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