Balanced pairs, cotorsion triplets and quiver representations

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Abstract. Balanced pairs appear naturally in the realm of Relative Homological Algebra associated to the balance of right derived functors of the Hom functor. A natural source to get such pairs is by means of cotorsion triplets. In this paper we study the connection between balanced pairs and cotorsion triplets by using recent quiver representation techniques. In doing so, we find a new characterization of abelian categories having enough projectives and injectives in terms of the existence of complete hereditary cotorsion triplets. We also give a short proof of the lack of balance for derived functors of Hom computed by using flat resolutions which extends the one showed by Enochs in the commutative case.

1. Introduction

Let $\mathcal{C}$ be an abelian category and $\mathcal{F}$ be a precovering class. This means that for each object $M \in \mathcal{C}$ there exists a (not necessarily exact) complex

$$\cdots \to F_1 \to F_0 \to M \to 0,$$

usually called an $\mathcal{F}$-resolution of M, where $F_i \in \mathcal{F}$ for every $i \geq 0$, which is exact after applying the functor $\mathsf{Hom}_{\mathcal{C}}(F, -)$ for each $F \in \mathcal{F}$. The corresponding deleted complex is unique up to homotopy, so we can compute right derived functors of Hom, denoted by $\mathcal{F}\text{-}\mathsf{Ext}^n$, associated with such $\mathcal{F}$-resolutions [1]. In many cases there is "balance" in the computation of such functors, meaning that there exists a preenveloping class $\mathcal{L}$ such that $\mathcal{F}\text{-}\mathsf{Ext}^n(M, N)$ can be also obtained from the right derived functors $\mathcal{L}\text{-}\mathsf{Ext}^n$ computed from of a coresolution of N,

$$0 \to N \to L_0 \to L_1 \to \cdots,$$

where $L_i \in \mathcal{L}$ for every $i \geq 0$. This phenomenon can be summarized by saying that the pair $(\mathcal{F}, \mathcal{L})$ is a balanced pair (in the sense of Chen [3]) or equivalently that the functor Hom is right balanced by $\mathcal{F} \times \mathcal{L}$ (see Enochs and Jenda [8, Section 8.2]).

Thus balanced pairs have gained attention in the last years in the context of Relative Homological Algebra (see for instance [3, 5, 8, 10, 11]). Our goal in this paper is to deepen in the relation between balanced and cotorsion pairs or, to be more precise, between balanced pairs and complete and hereditary cotorsion triplets. Recall that a triplet $(\mathcal{F}, \mathcal{G}, \mathcal{L})$ is called a cotorsion triplet provided that $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{G}, \mathcal{L})$ are cotorsion pairs. The reader can have in mind the trival cotorsion triplet $(\mathsf{Proj}(R), \mathsf{Mod}(R), \mathsf{Inj}(R))$ in the category $\mathsf{Mod}(R)$ of left R-modules (where $\mathsf{Proj}(R)$ and $\mathsf{Inj}(R)$ denote the classes of projective and injective left R-modules respectively) as the canonical example of a complete and hereditary cotorsion triplet. But there are many other instances of such triplets occuring in practice (see Example 4.5).

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[1] The reader can consult, for instance, Enochs and Jenda [8, Proposition 8.1.3 and Section 8.2] as a reference for these claims.

Complete hereditary cotorsion triplets are defined in Definition 4.1. They are a natural source to providing with balanced pairs. This is so, by a result of Enochs, Jenda, Torrecillas and Xu [10, Theorem 4.1] (see also Chen [3, Proposition 2.6]).

Theorem. (Enochs, Jenda, Torrecillas and Xu'98 - Chen'10) Let $\mathcal{C}$ be an abelian category with enough injectives and projectives. If $(\mathcal{F}, \mathcal{G}, \mathcal{L})$ is a complete hereditary cotorsion triplet in $\mathcal{C}$, then $(\mathcal{F}, \mathcal{L})$ is and admissible balanced pair.

Thus, it seems natural to wonder about the converse of this result. This appears explicitly as an open problem in [10, Open Problems].

Question: Find conditions for a balanced pair $(\mathcal{F}, \mathcal{L})$ to induce a complete hereditary cotorsion triplet $(\mathcal{F}, \mathcal{G}, \mathcal{L})$.

One of our motivations in this paper is to shed any light on this question. We give in Proposition 4.6 sufficient conditions to prove the converse of the previous result.

Proposition. Let $\mathcal{C}$ be an abelian category with enough projectives and injectives. Let $\mathcal{F}$ and $\mathcal{L}$ be two classes of objects in $\mathcal{C}$ closed under direct summands such that:

(1) The class $\mathcal{F}$ is resolving and special precovering, and the class $\mathcal{L}$ is coresolving and special preenveloping.
(2) $\mathcal{F} \cap \mathcal{F}^{\perp} \subseteq {}^{\perp}\mathcal{L}$ and ${}^{\perp}\mathcal{L} \cap \mathcal{L} \subseteq \mathcal{F}^{\perp}$.
(3) The pair $(\mathcal{F}, \mathcal{L})$ is balanced.

Then, there is a complete hereditary complete cotorsion triplet $(\mathcal{F}, \mathcal{G}, \mathcal{L})$ in $\mathcal{C}$. In this case, we have $\mathcal{F} \cap \mathcal{F}^{\perp} = \mathsf{Proj}(\mathcal{C})$ and ${}^{\perp}\mathcal{L} \cap \mathcal{L} = \mathsf{Inj}(\mathcal{C})$.[2]

Let us point out that we cannot expect to get such triplet from *any* balanced pair. For instance, given any ring R with identity, the pair $(\mathsf{Mod}(R), \mathsf{Mod}(R))$ is trivially a balanced pair, but the triplet $(\mathsf{Mod}(R), \mathcal{G}, \mathsf{Mod}(R))$ is complete if and only if R is quasi-Frobenius.

However, in case $\mathcal{C} = \mathsf{Mod}(R)$, the category of left R-modules over an associative ring R with identity, we can find a 1-1 correspondence between complete cotorsion triplets in $\mathsf{Mod}(R)$ and certain balanced pairs in the abelian category $\mathrm{Rep}(Q, \mathsf{Mod}(R))$ of $\mathsf{Mod}(R)$-valued representations over a quiver Q with at least one arrow. The precise formulation of our result is the following. The proof is in Corollary 6.5.

Theorem. If $(\mathcal{F}, \mathcal{H})$ and $(\mathcal{G}, \mathcal{L})$ are complete hereditary cotorsion pairs in $\mathsf{Mod}(R)$, then the following are equivalent:

(a) $\mathcal{H} = \mathcal{G}$ (that is, $(\mathcal{F}, \mathcal{G}, \mathcal{L})$ is a complete and hereditary cotorsion triplet in $\mathsf{Mod}(R)$).
(c) $(\Phi(\mathcal{F}), \Psi(\mathcal{L}))$ is a balanced pair in $\mathrm{Rep}(Q, \mathsf{Mod}(R))$ for some left and right rooted quiver Q with at least one arrow.

The classes $\Phi(\mathcal{F})$ and $\Psi(\mathcal{L})$ are defined by Holm and Jørgensen in [21]. We recall in Section 6 their definition.

Notice that one easy example of left and right rooted quiver is the 1-arrow quiver $Q : \bullet \to \bullet$, and so in this case $\mathrm{Rep}(Q, \mathsf{Mod}(R))$ is nothing but the category $\mathsf{Mor}(R)$ of morphisms of R-modules. But there are many other (possibly infinite) quivers satisfying this condition[3]. In short,

[2]Notations $\mathcal{F}^{\perp}$ and ${}^{\perp}\mathcal{L}$ are specified in the definition of hereditary cotorsion pairs in Section 2.

[3]An example of an infinite quiver with this condition is displayed in the paragraph before Corollary 6.5.

the previous Theorem assures that in order to look for conditions to get an equivalence between balanced pairs and cotorsion triplets, we need to move to a "bigger" category. This result allows to characterize quasi-Frobeinus rings (Corollary 6.6) in terms of the so-called *monomorphism category* and *epimorphism category* as considered by Li, Luo and Zhang in [24, 25]. And also we recover and extend the recent characterization of virtually Gorenstein rings given by Zareh-Khoshchehreh, Asgharzadeh and Divaani-Aazar in [30, Theorem 3.10].

While studying cotorsion triplets, we found the following interesting result of independent interest (see Theorem 4.4).

Theorem. An abelian category $\mathcal{C}$ has enough projectives and injectives if, and only if, there exists a hereditary and complete cotorsion triplet in $\mathcal{C}$.

This theorem allows us to present a slightly stronger version of the aforementioned result by Enochs, Jenda, Torrecillas and Xu. Namely, we do not require the existence of enough projectives and injectives to prove the statement (see Proposition 4.2).

Proposition. Let $\mathcal{C}$ be an abelian category. If $(\mathcal{F}, \mathcal{G}, \mathcal{L})$ is a complete hereditary cotorsion triplet in $\mathcal{C}$, then $(\mathcal{F}, \mathcal{L})$ is an admissible balanced pair in $\mathcal{C}$.

Finally, we give in Theorem 5.2 a short and categorical proof about the lack of balance with respect to the class of flat modules over a left Noetherian non-perfect ring. Our method is different from the one used by Enochs in [5, Theorem 4.1] for the commutative case. As a consequence we give a negative answer in Corollary 5.3 to the question 6 posted in [5, Section 6]. Namely, we show in Corollary 5.3 that there is no balance for the class of flat quasi-coherent modules on a Noetherian and semi-separated scheme.

2. Preliminaries

Throughout, $\mathcal{C}$ will denote an abelian category. A class of objects in $\mathcal{C}$ will be always assumed to be closed under isomorphisms and under finite direct sums.

Cotorsion pairs in abelian categories. Two classes of objects $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{C}$ form a *cotorsion pair* $(\mathcal{Y}, \mathcal{X})$ if the following two equalities hold:

$$\mathcal{Y} = {}^{\perp_1}\mathcal{X} := \{C \in \mathcal{C} : \mathsf{Ext}^1_{\mathcal{C}}(C, X) = 0 \text{ for every } X \in \mathcal{X}\},$$
$$\mathcal{X} = \mathcal{Y}^{\perp_1} := \{D \in \mathcal{C} : \mathsf{Ext}^1_{\mathcal{C}}(Y, D) = 0 \text{ for every } Y \in \mathcal{Y}\}.$$

Since $\mathcal{C}$ does not necessarily have enough projectives and/or injectives, the extension groups $\mathsf{Ext}^i_{\mathcal{C}}(A, B)$ are defined via its Yoneda description as certain equivalent classes of i-fold extensions.

A cotorsion pair $(\mathcal{Y}, \mathcal{X})$ in $\mathcal{C}$ is called:

(1) *Complete* if for every object $C \in \mathcal{C}$ there exist short exact sequences

$$0 \to X \to Y \to C \to 0 \quad \text{and} \quad 0 \to C \to X' \to Y' \to 0$$

with $Y, Y' \in \mathcal{Y}$ and $X, X' \in \mathcal{X}$.

(2) *Hereditary* if $\mathsf{Ext}^i_{\mathcal{C}}(Y, X) = 0$ for every $Y \in \mathcal{Y}$ and $X \in \mathcal{X}$, and $i > 0$.

Recall that a class $\mathcal{Y}$ of objects in $\mathcal{C}$ is *resolving* if $\mathcal{Y}$ is closed under extensions and under kernels of epimorphisms with domain and codomain in $\mathcal{Y}$, and if $\mathcal{Y}$ contains the class of projective objects in $\mathcal{C}$. Dually, one has the notion of *coresolving class*. We say that a cotorsion pair $(\mathcal{Y}, \mathcal{X})$ in $\mathcal{C}$ is *quasi-hereditary* if $\mathcal{Y}$ is resolving and $\mathcal{X}$ is coresolving. In some references, quasi-hereditary cotorsion pairs are called hereditary, but the two notions are not the same in general. Indeed, the condition defining hereditary cotorsion pairs in (2) above is stronger than asking $\mathcal{Y}$ and $\mathcal{X}$ to be resolving and coresolving, respectively. This can be appreciated in the following result, whose proof is well-known.

Proposition 2.1. *Every hereditary cotorsion pair in $\mathcal{C}$ is quasi-hereditary. If in addition, $\mathcal{C}$ has enough projectives and injectives, then every quasi-hereditary cotorsion pair in $\mathcal{C}$ is hereditary.*

If $(\mathcal{Y}, \mathcal{X})$ is a hereditary cotorsion pair in $\mathcal{C}$, we actually have that:

$$\mathcal{Y} = {}^{\perp}\mathcal{X} := \{C \in \mathcal{C} : \mathsf{Ext}^i_{\mathcal{C}}(C, X) = 0 \text{ for every } X \in \mathcal{X} \text{ and } i > 0\},$$
$$\mathcal{X} = \mathcal{Y}^{\perp} := \{D \in \mathcal{C} : \mathsf{Ext}^j_{\mathcal{C}}(Y, D) = 0 \text{ for every } Y \in \mathcal{Y} \text{ and } j > 0\}.$$

Precovering and preenveloping classess. Let $\mathcal{F}$ be a class of objects in $\mathcal{C}$. A morphism $\phi \colon F \to M$ in $\mathcal{C}$ is called an *$\mathcal{F}$-precover* of M if $F \in \mathcal{F}$ and

$$\mathsf{Hom}_{\mathcal{C}}(F', F) \to \mathsf{Hom}_{\mathcal{C}}(F', M) \to 0$$

is a right exact sequence of abelian groups for every object $F' \in \mathcal{F}$. Further, if $\phi \colon F \to M$ is an $\mathcal{F}$-precover and $\ker(\phi) \in \mathcal{F}^{\perp_1}$ then ϕ is called a *special $\mathcal{F}$-precover*. If every object in $\mathcal{C}$ has a (special) $\mathcal{F}$-precover, then the class $\mathcal{F}$ is called *(special) precovering*.

The dual notions are *(special) preenvelope* and *(special) preenveloping classes*. It is easy to observe that, if $(\mathcal{Y}, \mathcal{X})$ is a complete cotorsion pair in $\mathcal{C}$, then $\mathcal{Y}$ is special precovering and $\mathcal{X}$ is special preenveloping.

By using a standard argument (known as Salce's trick) we get the following lemma.

Lemma 2.2. *Suppose that $\mathcal{C}$ has enough projectives and injectives. Then, the following hold:*

(1) Let $\mathcal{F}$ be a special precovering class in $\mathcal{C}$ which is also resolving and closed under direct summands. Then, $(\mathcal{F}, \mathcal{F}^{\perp})$ is a complete hereditary cotorsion pair in $\mathcal{C}$.

(2) Let $\mathcal{L}$ be a special preenveloping class in $\mathcal{C}$ which is also coresolving and closed under direct summands. Then, $({}^{\perp}\mathcal{L}, \mathcal{L})$ is a complete hereditary cotorsion pair in $\mathcal{C}$.

Resolutions and coresolutions. Let $\mathcal{X}$ be a class of objects in $\mathcal{C}$ and M an object in $\mathcal{C}$. An *$\mathcal{X}$-resolution* $X_\bullet \to M$ of M is a (not necessarily exact) complex

$$\cdots \to X_1 \to X_0 \to M \to 0,$$

with each $X_i \in \mathcal{X}$, which is exact when applying the functor $\mathsf{Hom}_{\mathcal{C}}(X, -)$, for every $X \in \mathcal{X}$. In this case, we will say that the complex $X_\bullet \to M$ is *$\mathsf{Hom}_{\mathcal{C}}(\mathcal{X}, -)$-acyclic*. Dually, we have the notion of *$\mathcal{X}$-coresolution* $M \to X^\bullet$ of M.

If $\mathcal{X}$ is precovering (respectively, $\mathcal{X}$ is preenveloping) it is easy to see that every M in $\mathcal{C}$ has an $\mathcal{X}$-resolution (respectively, an $\mathcal{X}$-coresolution). See, for instance, Enochs and Jenda [8, Proposition 8.1.3].

Balanced pairs. A pair $(\mathcal{F}, \mathcal{L})$ of classes in $\mathcal{C}$ is called a *balanced pair* if the following conditions are satisfied:

(BP0) $\mathcal{F}$ is precovering and $\mathcal{L}$ is preenveloping.

(BP1) For each object $M \in \mathcal{C}$, there is an $\mathcal{F}$-resolution $F_\bullet \to M$ which is $\mathsf{Hom}_{\mathcal{C}}(-, \mathcal{L})$-acyclic.

(BP2) For each object $M \in \mathcal{C}$, there is a $\mathcal{L}$-coresolution $M \to L^\bullet$ which is $\mathsf{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-acyclic.

A balanced pair $(\mathcal{F}, \mathcal{L})$ is called *admissible* provided that each $\mathcal{F}$-precover is an epimorphism and each $\mathcal{L}$-preenvelope is a monomorphism.

3. Relation between balanced pairs and cotorsion pairs

Let us begin this section with the following useful characterization of balanced pairs:

Lemma 3.1. *Let $\mathcal{F}$ and $\mathcal{L}$ be a precovering and a preenveloping class in $\mathcal{C}$, respectively. Then, the following conditions are equivalent.*

(a) The pair $(\mathcal{F}, \mathcal{L})$ is balanced.

(b) Each $\mathsf{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-acyclic and left exact sequence in $\mathcal{C}$ is also $\mathsf{Hom}_{\mathcal{C}}(-, \mathcal{L})$-acyclic, and each $\mathsf{Hom}_{\mathcal{C}}(-, \mathcal{L})$-acyclic and right exact sequence in $\mathcal{C}$ is also $\mathsf{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-acyclic.

(c) For each object $M \in \mathcal{C}$, there is a left exact sequence

$$0 \to K \to F \to M \to 0$$

and a right exact sequence

$$0 \to M \to L \to C \to 0,$$

which are both $\mathsf{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-acyclic and $\mathsf{Hom}_{\mathcal{C}}(-, \mathcal{L})$-acyclic, where $F \in \mathcal{F}$ and $L \in \mathcal{L}$.

Proof. The implication (a) $\Rightarrow$ (b) follows from Chen's [3, Proposition 2.2], while (b) $\Rightarrow$ (c) is clear. Let us finish the proof showing (c) $\Rightarrow$ (a). By the assumption (c), for each object $M \in \mathcal{C}$ there is a left exact sequence

$$0 \to K_0 \to F_0 \to M \to 0$$

in $\mathcal{C}$ with $F_0 \in \mathcal{F}$ which is $\mathsf{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-acyclic and $\mathsf{Hom}_{\mathcal{C}}(-, \mathcal{L})$-acyclic. Now, by applying (c) again to the object K_0 we get a left exact sequence

$$0 \to K_1 \to F_1 \to K_0 \to 0$$

with $F_1 \in \mathcal{F}$ which is $\mathsf{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-acyclic and $\mathsf{Hom}_{\mathcal{C}}(-, \mathcal{L})$-acyclic. Continuing this process, we obtain an $\mathcal{F}$-resolution $F_\bullet \to M$ which is $\mathsf{Hom}_{\mathcal{C}}(-, \mathcal{L})$-acyclic. The construction of a $\mathsf{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-acyclic $\mathcal{L}$-coresolution of M is dual. Hence, (a) follows. $\square$

Balanced pairs vs. cotorsion pairs. As a first consequence of the previous result, we can infer the following relation between cotorsion pairs and balanced pairs. From now on, we will denote by $\mathsf{Proj}(\mathcal{C})$ and $\mathsf{Inj}(\mathcal{C})$ the classes of projective and injective objects of $\mathcal{C}$, respectively.

Proposition 3.2. *Let $(\mathcal{F}, \mathcal{H})$ and $(\mathcal{G}, \mathcal{L})$ be cotorsion pairs in $\mathcal{C}$ such that the pair $(\mathcal{F}, \mathcal{L})$ is balanced. Then, $\mathcal{F} \cap \mathcal{G} = \mathsf{Proj}(\mathcal{C})$ and $\mathcal{H} \cap \mathcal{L} = \mathsf{Inj}(\mathcal{C})$.*

Proof. Let us only prove the equality $\mathcal{F} \cap \mathcal{G} = \mathsf{Proj}(\mathcal{C})$. The corresponding statement with injectives follows in a dual manner. Since $(\mathcal{F}, \mathcal{H})$ and $(\mathcal{G}, \mathcal{L})$ are cotorsion pairs, the containment $\mathsf{Proj}(\mathcal{C}) \subseteq \mathcal{F} \cap \mathcal{G}$ always holds. Conversely, let $P \in \mathcal{F} \cap \mathcal{G}$ and $C \in \mathcal{C}$ be an arbitrary object. Let us consider an element in $\mathsf{Ext}^1_{\mathcal{C}}(P, C)$ represented by an exact sequence

$$0 \to C \to D \to P \to 0. \tag{i}$$

Since $P \in \mathcal{G}$, the sequence (i) is $\mathsf{Hom}_{\mathcal{C}}(-, \mathcal{L})$-acyclic. But then by Lemma 3.1, we have that this sequence is also $\mathsf{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-acyclic. This in turn implies that (i) splits, since $P \in \mathcal{F}$. Finally, being C arbitrary, we conclude that P is projective. $\square$

Uniqueness of balanced pairs. Given a preenveloping class $\mathcal{L}$ in $\mathcal{C}$, there might be two different classes $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $(\mathcal{F}_1, \mathcal{L})$ and $(\mathcal{F}_2, \mathcal{L})$ are balanced pairs. For instance, take the category $\mathcal{C} = \mathsf{Mod}(R)$ of left R-modules and $\mathcal{L}$ the class of all injective left R-modules. Then, we have two balanced pairs $(\mathcal{F}_1, \mathcal{L})$ and $(\mathcal{F}_2, \mathcal{L})$, where $\mathcal{F}_1$ is the class of all free left R-modules and $\mathcal{F}_2$ consists of all projective left R-modules. In this example we notice that $\mathsf{Smd}(\mathcal{F}_1) = \mathsf{Smd}(\mathcal{F}_2)$ (where the notation $\mathsf{Smd}(\mathcal{F})$ stands for the class of direct summands of objects in $\mathcal{F}$). The second consequence of Lemma 3.1 shows that this sort of uniqueness property holds for any admissible balanced pair.

Proposition 3.3. *If $(\mathcal{F}_1, \mathcal{L})$ and $(\mathcal{F}_2, \mathcal{L})$ are two admissible balanced pairs in $\mathcal{C}$, then the equality $\mathsf{Smd}(\mathcal{F}_1) = \mathsf{Smd}(\mathcal{F}_2)$ holds. Dually, if $(\mathcal{F}, \mathcal{L}_1)$ and $(\mathcal{F}, \mathcal{L}_2)$ are two admissible balanced pairs in $\mathcal{C}$, then $\mathsf{Smd}(\mathcal{L}_1) = \mathsf{Smd}(\mathcal{L}_2)$.*

Proof. Let us see that $\mathsf{Smd}(\mathcal{F}_1) \subseteq \mathsf{Smd}(\mathcal{F}_2)$. The other inclusion follows by the same argument. It is easy to observe that it suffices to show $\mathcal{F}_1 \subseteq \mathsf{Smd}(\mathcal{F}_2)$. First, note that since $\mathcal{F}_2$ is a precovering class in $\mathcal{C}$, for any $F_1 \in \mathcal{F}_1$ we have a $\mathsf{Hom}_{\mathcal{C}}(\mathcal{F}_2, -)$-acyclic left exact sequence

$$0 \to K \to F_2 \to F_1 \to 0. \tag{ii}$$

in $\mathcal{C}$ with $F_2 \in \mathcal{F}_2$. In fact, since $(\mathcal{F}_2, \mathcal{L})$ is admissible, the sequence (ii) is exact. By Lemma 3.1 along with the fact that $(\mathcal{F}_2, \mathcal{L})$ is balanced, the sequence (ii) is also $\mathsf{Hom}_{\mathcal{C}}(-, \mathcal{L})$-acyclic. But then, using now that $(\mathcal{F}_1, \mathcal{L})$ is balanced, (ii) is also $\mathsf{Hom}_{\mathcal{C}}(\mathcal{F}_1, -)$-acyclic. This implies that (ii) splits, since $F_1 \in \mathcal{F}_1$. Hence, $F_1 \in \mathsf{Smd}(\mathcal{F}_2)$, which completes the proof. $\square$

4. Relation between balanced pairs and cotorsion triplets

It is not in general an easy task to check whether or not a pair of classes $(\mathcal{F}, \mathcal{L})$ form a balanced pair in an abelian category. A common source to provide with such pairs is by means of *cotorsion triplets*. This section is thus devoted to define such triplets and to explore their relation with balanced pairs. In summary, every complete and hereditary cotorsion triplet gives rise to a balanced pair. Cotorsion triplets were introduced by A. Beligiannis and I. Reiten in [2, Section 3 of Chapter VI.], where they study necessary and sufficient conditions for the existence of such triplets. The concept is also studied by Enochs and Jenda in [9, Section 4.2] in the context of chain complexes of modules over an associative ring with identity.

Definition 4.1. Three classes $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{L}$ of objects in $\mathcal{C}$ form a **cotorsion triplet** $(\mathcal{F}, \mathcal{G}, \mathcal{L})$ if $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{G}, \mathcal{L})$ are cotorsion pairs in $\mathcal{C}$. Moreover, a cotorsion triplet $(\mathcal{F}, \mathcal{G}, \mathcal{L})$ in $\mathcal{C}$ is:

(1) **Complete** if $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{G}, \mathcal{L})$ are complete cotorsion pairs.
(2) **Hereditary** if $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{G}, \mathcal{L})$ are hereditary cotorsion pairs.

From cotorsion triplets to balanced pairs. The relation between cotorsion triplets and balanced pairs is summarized in the next proposition. It was originally outlined by Enochs, Jenda and Torrecillas in [10, Theorem 4.1], but the precise formulation we state below is due to Chen [3, Proposition 2.6].

Proposition 4.2. *If $(\mathcal{F}, \mathcal{G}, \mathcal{L})$ is a complete hereditary cotorsion triplet in $\mathcal{C}$, then $(\mathcal{F}, \mathcal{L})$ is an admissible balanced pair in $\mathcal{C}$.*

Remark 4.3. Chen's original statement and proof in [3, Proposition 2.6] requires that $\mathcal{C}$ has enough projectives and injectives. However, these hypotheses are actually not necessary. This fact has to do with an interesting characterization of abelian categories with enough projectives and injectives in terms of complete hereditary cotorsion triplets, presented in Theorem 4.4 below.

In particular, this result shows that it is hopeless to look for complete hereditary cotorsion triplets in Grothendieck categories without enough projectives, such as some interesting categories studied in Algebraic Geometry. For example, if T is a non-trivial topological space and $\mathcal{O}$ is a sheaf of commutative rings with 1 on T, then $\mathbf{Sh}(\mathcal{O})$, the category of sheaves of $\mathcal{O}$-modules, does not have enough projective $\mathcal{O}$-modules. This is also the case of the category $\mathfrak{Qcoh}(X)$ of quasi-coherent sheaves on a non-affine scheme X, considered in Section 5. Thus, it will follow that neither $\mathbf{Sh}(\mathcal{O})$ nor $\mathfrak{Qcoh}(X)$ have complete and hereditary cotorsion triplets.

Theorem 4.4. *The following conditions are equivalent.*

(a) $\mathcal{C}$ has enough projectives and injectives.

(b) There exists a complete hereditary cotorsion triplet $(\mathcal{F}, \mathcal{G}, \mathcal{L})$ in $\mathcal{C}$.

Proof. For the implication (a) $\Rightarrow$ (b) it suffices to consider the complete hereditary cotorsion triplet $(\mathsf{Proj}(\mathcal{C}), \mathcal{C}, \mathsf{Inj}(\mathcal{C}))$.

Let us now prove (b) $\Rightarrow$ (a). So suppose we are given a complete hereditary cotorsion triplet $(\mathcal{F}, \mathcal{G}, \mathcal{L})$ in $\mathcal{C}$. We show that $\mathcal{C}$ has enough projectives. For any object $C \in \mathcal{C}$, we have a short exact sequence

$$0 \to L \to G \to C \to 0$$

in $\mathcal{C}$ with $G \in \mathcal{G}$ and $L \in \mathcal{L}$, since $(\mathcal{G}, \mathcal{L})$ is a complete cotorsion pair. Now using the completeness of $(\mathcal{F}, \mathcal{G})$, we have a short exact sequence

$$0 \to G' \to F \to G \to 0$$

with $F \in \mathcal{F}$ and $G' \in \mathcal{G}$. Note that F actually belongs to $\mathcal{F} \cap \mathcal{G}$ since $\mathcal{G}$ is closed under extensions. Now taking the pullback of $L \to G \leftarrow F$, we obtain two short exact sequences of the form:

$$0 \to G' \to K \to L \to 0 \tag{iii}$$

$$0 \to K \to F \to C \to 0 \tag{iv}$$

Note that $G', L \in (\mathcal{F} \cap \mathcal{G})^{\perp}$ in (iii), and so $K \in (\mathcal{F} \cap \mathcal{G})^{\perp}$. The proof will conclude after we show that $\mathcal{F} \cap \mathcal{G} = \mathsf{Proj}(\mathcal{C})$. The containment $(\supseteq)$ is clear. Now let $W \in \mathcal{F} \cap \mathcal{G}$. From (iv) we have the long homology exact sequence

$$\cdots \to \mathsf{Ext}^i_{\mathcal{C}}(W, F) \to \mathsf{Ext}^i_{\mathcal{C}}(W, C) \to \mathsf{Ext}^{i+1}_{\mathcal{C}}(W, K) \to \cdots.$$

On the one hand, $\mathsf{Ext}^i_{\mathcal{C}}(W, F) = 0$ for every $i > 0$ since $W \in \mathcal{F}$ and $F \in \mathcal{G}$, and $(\mathcal{F}, \mathcal{G})$ is a hereditary cotorsion pair. On the other hand, $\mathsf{Ext}^{i+1}_{\mathcal{C}}(W, K) = 0$ for every $i > 0$ since $W \in \mathcal{F} \cap \mathcal{G}$ and $K \in (\mathcal{F} \cap \mathcal{G})^{\perp}$. It follows that $\mathsf{Ext}^i_{\mathcal{C}}(W, C) = 0$ for every positive integer $i > 0$. Since the object $C \in \mathcal{C}$ is arbitrary, we have that $W \in \mathsf{Proj}(\mathcal{C})$.

A dual argument shows that $\mathcal{C}$ has also enough injectives. □

From now on, unless otherwise specified, R will be an associative ring with identity, and all modules are left R-modules.

Example 4.5. We collect from the literature the following examples of complete hereditary cotorsion triplets (and hence of admissible balanced pairs):

(1) Let $\mathcal{C}$ be an abelian category. We already know from the proof of Theorem 4.4 that $(\mathsf{Proj}(\mathcal{C}), \mathcal{C}, \mathsf{Inj}(\mathcal{C}))$ is a complete cotorsion triplet if, and only if, $\mathcal{C}$ has enough projectives and injectives. If any of these two conditions holds, we have the well-known balanced pair $(\mathsf{Proj}(\mathcal{C}), \mathsf{Inj}(\mathcal{C}))$. Not all of the complete hereditary cotorsion triplets in $\mathcal{C}$ have to be of the form $(\mathsf{Proj}(\mathcal{C}), \mathcal{C}, \mathsf{Inj}(\mathcal{C}))$, as shown in the rest of the examples.

(2) Consider the category $\mathsf{Mod}(R)$ of modules. In this case, let us set $\mathsf{Proj}(\mathsf{Mod}(R)) = \mathsf{Proj}(R)$ and $\mathsf{Inj}(\mathsf{Mod}(R)) = \mathsf{Inj}(R)$, for simplicity. Recall that a ring R is quasi-Frobenius if $\mathsf{Proj}(R) = \mathsf{Inj}(R)$. We can note that R is quasi-Frobenius if, and only if, the triplet $(\mathsf{Mod}(R), \mathsf{Proj}(R), \mathsf{Mod}(R))$ is a complete cotorsion triplet.

(3) Beligiannis and Reiten [2, Section 3 of Chapter VI.]: Let Λ be an Artin algebra and $\mathsf{mod}(\Lambda)$ denote the abelian category of finitely generated left Λ-modules. Let $\mathsf{add}(\Lambda)$ denote the class of objects in $\mathsf{mod}(\Lambda)$ that are direct summands of finite direct sums of copies of Λ. The class $\mathsf{CM}(\Lambda)$ of *maximal Cohen-Macaulay modules over* Λ is defined as those $M \in \mathsf{mod}(\Lambda)$ such that there exists an exact sequence

$$0 \to M \to W^0 \xrightarrow{f^0} W^1 \xrightarrow{f^1} W^2 \to \cdots$$

with $W^k \in \mathsf{add}(\Lambda)$ and $\mathrm{Ker}(f^k) \in {}^{\perp}(\mathsf{add}(\Lambda))$ for every $k \geq 0$. The class $\mathsf{CoCM}(\Lambda)$ is defined dually. On the other hand, let $\mathsf{proj}_{\infty}(\Lambda)$ (respectively $\mathsf{inj}_{\infty}(\Lambda)$) denote the class of finitely generated Λ-modules with finite projective (respectively injective) dimension. If Λ is Gorenstein, then $(\mathrm{CM}(\Lambda), \mathsf{proj}_{\infty}(\Lambda), \mathrm{CoCM}(D(\Lambda)))$ is a complete hereditary cotorsion triplet in $\mathsf{mod}(\Lambda)$, where $D(\Lambda)$ is the minimal injective cogenerator of $\mathsf{mod}(\Lambda)$. In this case, one has $\mathsf{proj}_{\infty}(\Lambda) = \mathsf{inj}_{\infty}(\Lambda)$.

(4) Enochs and Jenda [9, Proposition 4.4.5]: Let $\mathsf{Ch}(R)$ denote the category of chain complexes of modules. Recall from [9, Definition 4.2.2] that a chain complex $P = (P_m, \partial^P_m)_{m\in\mathbb{Z}}$ is *perfect* if $P_m = 0$ except for a finite number of integers $m \in \mathbb{Z}$ and if each P_m is a finitely generated projective module. If $\mathcal{S}$ is a set of perfect complexes and $\mathcal{U}$ is the set of all complexes $\Sigma^k(P)$ where $P \in \mathcal{S}$ and $k \in \mathbb{Z}$, then there exists a unique complete hereditary cotorsion triplet $(\mathcal{Y}, \mathcal{X}, \mathcal{Z})$ in $\mathsf{Ch}(R)$ where $\mathcal{X} = \mathcal{U}^{\perp}$. Here, $\Sigma^k(P)$ denotes the *k-th suspension* of P, that is, $\Sigma^k(P)_m := P_{m-k}$ for every integer $m \in \mathbb{Z}$, with boundaries given by $(-1)^k\partial^P_{m-k}$.

(5) [9, Section 4.3 of Chapter IV.]: Let $\mathcal{E}$ denote the class of exact chain complexes in $\mathsf{Ch}(R)$. Then, $({}^{\perp_1}\mathcal{E}, \mathcal{E}, \mathcal{E}^{\perp_1})$ is a complete hereditary cotorsion triplet in $\mathsf{Ch}(R)$, known as the *Dold triplet*. Here, ${}^{\perp_1}\mathcal{E}$ coincides with the class $\mathsf{dg}(\mathsf{Proj}(R))$ of differential graded projective complexes in $\mathsf{Ch}(R)$, defined as those complexes P in $\mathsf{Ch}(R)$ such that P_m is a projective module for every integer $m \in \mathbb{Z}$, and every chain map $P \to E$ is homotopic to zero whenever $E \in \mathcal{E}$. Dually, $\mathcal{E}^{\perp_1}$ coincides with the class $\mathsf{dg}(\mathsf{Inj}(R))$ of differential graded injective complexes. Here, we have the balanced pair $(\mathsf{dg}(\mathsf{Proj}(R)), \mathsf{dg}(\mathsf{Inj}(R)))$.

(6) Hovey [22, Section 8]: Let $\mathsf{GProj}(R)$ and $\mathsf{GInj}(R)$ denote the classes of Gorenstein projective and Gorenstein injective modules. Let $\mathsf{Proj}_{\infty}(R)$ (respectively $\mathsf{Inj}_{\infty}(R)$) denote the class of modules with finite projective (respectively injective) dimension. If R is an Iwanaga-Gorenstein ring, then $(\mathsf{GProj}(R), \mathsf{Proj}_{\infty}(R), \mathsf{GInj}(R))$ is a complete hereditary cotorsion triplet in $\mathsf{Mod}(R)$, where $\mathsf{Proj}_{\infty}(R) = \mathsf{Inj}_{\infty}(R)$ by [8, Proposition 9.1.7].

Here, we have the balanced pair $(\mathsf{GProj}(R), \mathsf{GInj}(R))$ comprising several properties in Gorenstein homological algebra.

(7) Gillespie [15]: Similar to (6) above, let $\mathsf{DProj}(R)$ and $\mathsf{DInj}(R)$ denote the classes of Ding-projective and Ding-injective modules, respectively. Let $\mathsf{Flat}_\infty(R)$ (respectively FP-$\mathsf{Inj}_\infty(R)$) denote the class of modules with finite flat (respectively FP-injective) dimension. If R is a Ding-Chen ring, then $(\mathsf{DProj}(R), \mathsf{Flat}_\infty(R), \mathsf{DInj}(R))$ is a complete hereditary cotorsion triplet in $\mathsf{Mod}(R)$, where $\mathsf{Flat}_\infty(R) = \text{FP-}\mathsf{Inj}_\infty(R)$ by [4, Proposition 3.16]. In this case, we have the balanced pair $(\mathsf{DProj}(R), \mathsf{DInj}(R))$ for Ding-Chen homological algebra.

From balanced pairs to cotorsion triplets. In [10, Open Problems] is asked under what conditions a converse of Proposition 4.2 holds. Namely, giving a special precovering class $\mathcal{F}$ and a special preenveloping class $\mathcal{L}$ in $\mathcal{C}$ such that the pair $(\mathcal{F}, \mathcal{L})$ is balanced, under what conditions is it true that we have a complete cotorsion triplet $(\mathcal{F}, \mathcal{G}, \mathcal{L})$?. In the next proposition, we give sufficient conditions on such $\mathcal{F}$ and $\mathcal{L}$ to ensure that they are the extremes of a complete hereditary cotorsion triplet.

Proposition 4.6. *Let $\mathcal{C}$ be an abelian category with enough projectives and injectives. Let $\mathcal{F}$ and $\mathcal{L}$ be two classes of objects in $\mathcal{C}$ closed under direct summands such that:*

(1) The class $\mathcal{F}$ is resolving and special precovering, and the class $\mathcal{L}$ is coresolving and special preenveloping.

(2) $\mathcal{F} \cap \mathcal{F}^\perp \subseteq {}^\perp\mathcal{L}$ and ${}^\perp\mathcal{L} \cap \mathcal{L} \subseteq \mathcal{F}^\perp$.

(3) The pair $(\mathcal{F}, \mathcal{L})$ is balanced.

Then, there is a complete hereditary complete cotorsion triplet $(\mathcal{F}, \mathcal{G}, \mathcal{L})$ in $\mathcal{C}$. In this case, we have $\mathcal{F} \cap \mathcal{F}^\perp = \mathsf{Proj}(\mathcal{C})$ and ${}^\perp\mathcal{L} \cap \mathcal{L} = \mathsf{Inj}(\mathcal{C})$.

Proof. Let us call $\mathcal{H} = \mathcal{F}^\perp$ and $\mathcal{G} = {}^\perp\mathcal{L}$. With the hypothesis on $\mathcal{F}$ and $\mathcal{L}$ we get from Lemma 2.2 that $(\mathcal{F}, \mathcal{H})$ and $(\mathcal{G}, \mathcal{L})$ are complete hereditary cotorsion pairs in $\mathcal{C}$. Let us see that $\mathcal{H} = \mathcal{G}$. For any $H \in \mathcal{H}$, we have a $\mathsf{Hom}_{\mathcal{C}}(\mathcal{F}, -)$ exact sequence

$$0 \to H_0 \to F \to H \to 0,$$

with $F \in \mathcal{F}$ and $H_0 \in \mathcal{H}$. It follows that $F \in \mathcal{F} \cap \mathcal{H} \subseteq \mathcal{G}$ by hypothesis. By Lemma 3.1, the above sequence is also $\mathsf{Hom}_{\mathcal{C}}(-, \mathcal{L})$ exact, so we get $H \in \mathcal{G}$. So $\mathcal{H} \subseteq \mathcal{G}$. Dually, we also have that $\mathcal{G} \subseteq \mathcal{H}$. □

Remark 4.7. As mentioned in the introduction, one cannot expect to obtain a complete hereditary cotorsion triplet from any balanced pair. After checking the statement of Proposition 4.6, it seems difficult to obtain such triplets from a balanced pair $(\mathcal{F}, \mathcal{L})$ without assuming condition (2). For example, for any ring R we have the trivial balanced pair $(\mathsf{Mod}(R), \mathsf{Mod}(R))$ by setting $\mathcal{F} = \mathcal{L} = \mathsf{Mod}(R)$. However, we know from Example 4.5 (2) that the triplet $(\mathsf{Mod}(R), \mathcal{G}, \mathsf{Mod}(R))$ is complete if, and only if, R is quasi-Frobenius. Note that in this case, we have $\mathcal{F} \cap \mathcal{F}^\perp = \mathsf{Inj}(R)$ and ${}^\perp\mathcal{L} \cap \mathcal{L} = \mathsf{Proj}(R)$, and thus condition (2) in Proposition 4.6 holds if, and only if, R is quasi-Frobenius.

As an immediate consequence of propositions 4.2 and 4.6 we get the following.

Corollary 4.8. *Let $\mathcal{C}$ be an abelian category with enough projectives and injectives. If $(\mathcal{F}, \mathcal{H})$ and $(\mathcal{G}, \mathcal{L})$ are complete hereditary cotorsion pairs in $\mathcal{C}$ with $\mathcal{F} \cap \mathcal{H} \subseteq \mathcal{G}$ and $\mathcal{G} \cap \mathcal{L} \subseteq \mathcal{H}$, then $\mathcal{H} = \mathcal{G}$ if and only if $(\mathcal{F}, \mathcal{L})$ is an admissible balanced pair in $\mathcal{C}$.*

Virtually Gorensteins rings, balanced pairs and cotorsion triplets. We close this section presenting a first application of the relation between balanced pairs and cotorsion triplets described in Propositions 4.2 and 4.6, in the context of virtually Gorenstein rings (a notion originally due to Beligiannis and Reiten in [2] for Artin algebras). More applications will be given later on for the categories of quasi-coherent sheaves and $\mathcal{C}$-valued representations of quivers. These two settings will be studied in more detail in Sections 5 and 6, respectively.

The balanced pair $(\mathsf{GProj}(R), \mathsf{GInj}(R))$ from Example 4.5 (6) can be obtained under different assumptions on R. As a matter of fact, the existence of $(\mathsf{GProj}(R), \mathsf{GInj}(R))$ as a balanced pair in $\mathsf{Mod}(R)$ is a necessary and sufficient condition for certain rings R to be virtually Gorenstein. Recall that a (non-necessarily commutative) ring R is called *virtually Gorenstein* provided that $\mathsf{GProj}(R)^{\perp} = {}^{\perp}\mathsf{GInj}(R)$. Ding-Chen rings are examples of non-Gorenstein virtually Gorenstein rings (see Gillespie [16, Theorem 1.1] and [15, Theorem 4.7]).

In the case where R is a Noetherian ring of finite Krull dimension, it is proved by Zareh-Khoshchehreh, Asgharzadeh and Divaani-Aazar in [30, Theorem 3.10] that R is virtually Gorenstein if, and only if, $(\mathsf{GProj}(R), \mathsf{GInj}(R))$ is a balanced pair in $\mathsf{Mod}(R)$. This is an important recent result for which we will present two extensions in Corollaries 4.9 and 6.8. The former adds an extra condition in this equivalence, namely the existence of a cotorsion triplet $(\mathsf{GProj}(R), \mathcal{G}, \mathsf{GInj}(R))$ in $\mathsf{Mod}(R)$. For the latter extension, on the other hand, we will require some concepts and techniques from Representation Theory of Quivers, covered in Section 6.

Corollary 4.9. *Let R be a commutative Noetherian ring with finite Krull dimension. Then, the following conditions are equivalent.*

- *(a) R is a virtually Gorenstein ring.*
- *(b) $(\mathsf{GProj}(R), \mathsf{GInj}(R))$ is an admissible balanced pair in $\mathsf{Mod}(R)$.*
- *(c) There is a complete hereditary cotorsion triplet $(\mathsf{GProj}(R), \mathcal{G}, \mathsf{GInj}(R))$ in $\mathsf{Mod}(R)$.*

Proof. The equivalence (a) $\Leftrightarrow$ (b) is [30, Theorem 3.10], which also holds in the non commutative case. The implication (c) $\Rightarrow$ (b) is an immediate consequence of Proposition 4.2. So the proof will conclude after showing (b) $\Rightarrow$ (c).

Suppose that the classes $\mathsf{GProj}(R)$ and $\mathsf{GInj}(R)$ form a balanced pair $(\mathsf{GProj}(R), \mathsf{GInj}(R))$. Firstly, it is well-known for any arbitrary ring R that the classes $\mathsf{GProj}(R)$ and $\mathsf{GInj}(R)$ are resolving and coresolving, respectively, and that $\mathsf{GProj}(R) \cap \mathsf{GProj}(R)^{\perp} = \mathsf{Proj}(R)$[4] $\subseteq {}^{\perp}\mathsf{GInj}(R)$ and ${}^{\perp}\mathsf{GInj}(R) \cap \mathsf{GInj}(R) = \mathsf{Inj}(R)$[5] $\subseteq \mathsf{GProj}(R)^{\perp}$. Moreover, since R is Noetherian we have by Krause [23, Theorem 7.12] that $\mathsf{GInj}(R)$ is special preenveloping. On the other hand, since also R is commutative with finite Krull dimension, we have that $\mathsf{GProj}(R)$ is special precovering

[4]Let us prove this equality. It is clear that $\mathsf{Proj}(R) \subseteq \mathsf{GProj}(R) \cap \mathsf{GProj}(R)^{\perp}$. Conversely, let M be a module in $\mathsf{GProj}(R) \cap \mathsf{GProj}(R)^{\perp}$. Then, by the definition of Gorenstein projective module, there exists a short exact sequence

$$0 \to M \to P \to M' \to 0,$$

with P projective and M' Gorenstein projective. Since $M \in \mathsf{GProj}(R)^{\perp}$, the sequence splits, and so M is a direct summand of a projective module, hence projective.

[5]The proof is analogous to the projective case before.

(see for example [12, Proposition 6]). Thus, we are under the hypotheses of Proposition 4.6, which says that there must exist a complete hereditary cotorsion triplet $(\mathsf{GProj}(R), \mathcal{G}, \mathsf{GInj}(R))$ in $\mathsf{Mod}(R)$. □

5. Balance with flat objects

In this section, we first give a different proof to that of Enochs in [5, Theorem 4.1] about the lack of balance with respect to the class of flat modules, in case the ring R is left Noetherian and non-perfect.

Balance and closure under direct sums and products. We start with the following consequence of balance in abelian categories. We recall that an abelian category satisfies AB4 if it is cocomplete and any direct sum of monomorphisms is a monomorphism. The axiom AB4* of an abelian category is dual.

Lemma 5.1. *Let $\mathcal{F}$ and $\mathcal{L}$ be two classes of objects in $\mathcal{C}$ such that $(\mathcal{F}, \mathcal{L})$ is a balanced pair. Then, the following statements hold:*

(1) If $\mathcal{C}$ satisfies AB4, has enough injectives and any direct sum of injective objects belongs to $\mathcal{F}^{\perp_1}$, then $\mathcal{F}^{\perp_1}$ is closed under direct sums.

(2) If $\mathcal{C}$ satisfies AB4, has enough projectives and any direct product of projective objects belongs to ${}^{\perp_1}\mathcal{L}$, then ${}^{\perp_1}\mathcal{L}$ is closed under direct products.*

Proof. Let us prove (1). So let $\{C_i\}$ be a family of objects in $\mathcal{F}^{\perp_1}$ and

$$0 \to C_i \to E_i \to D_i \to 0$$

be a family of exact sequences with each E_i injective. Since each $C_i \in \mathcal{F}^{\perp_1}$, each of these sequences is $\mathsf{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact. Hence by Lemma 3.1, they will be $\mathsf{Hom}_{\mathcal{C}}(-, \mathcal{L})$-exact. So, for each i and each $L \in \mathcal{L}$, we have the exact sequence of abelian groups

$$0 \to \mathsf{Hom}_{\mathcal{C}}(D_i, L) \to \mathsf{Hom}_{\mathcal{C}}(E_i, L) \to \mathsf{Hom}_{\mathcal{C}}(C_i, L) \to 0.$$

We can take the direct product of the previous family of short exact sequences to get the exact sequence

$$0 \to \prod_i \mathsf{Hom}_{\mathcal{C}}(D_i, L) \to \prod_i \mathsf{Hom}_{\mathcal{C}}(E_i, L) \to \prod_i \mathsf{Hom}_{\mathcal{C}}(C_i, L) \to 0.$$

Now, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \prod_i \mathsf{Hom}_{\mathcal{C}}(D_i, L) \arrow[r] \arrow[d, "\simeq"] & \prod_i \mathsf{Hom}_{\mathcal{C}}(E_i, L) \arrow[r] \arrow[d, "\simeq"] & \prod_i \mathsf{Hom}_{\mathcal{C}}(C_i, L) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \mathsf{Hom}_{\mathcal{C}}\left(\bigoplus_i D_i, L\right) \arrow[r] & \mathsf{Hom}_{\mathcal{C}}\left(\bigoplus_i E_i, L\right) \arrow[r] & \mathsf{Hom}_{\mathcal{C}}\left(\bigoplus_i C_i, L\right) \arrow[r] & 0
\end{tikzcd}$$

where the columns are natural isomorphisms. The bottom row tells us that the exact sequence

$$0 \to \bigoplus_i C_i \to \bigoplus_i E_i \to \bigoplus_i D_i \to 0$$

is $\mathsf{Hom}_{\mathcal{C}}(-,\mathcal{L})$-exact. Since $(\mathcal{F},\mathcal{L})$ is balanced, by applying Lemma 3.1 again, it follows that the sequence is $\mathsf{Hom}_{\mathcal{C}}(\mathcal{F},-)$-exact. Since $\oplus_i E_i \in \mathcal{F}^{\perp_1}$ by hypothesis, it follows from the usual long exact sequence of cohomology that $\mathsf{Ext}^1_{\mathcal{C}}(F,\oplus_i C_i) = 0$ for each $F \in \mathcal{F}$, that is, $\oplus_i C_i \in \mathcal{F}^{\perp_1}$.

The proof of (2) is dual. □

Lack of balance with respect to flat modules. We are now in position to give a short proof of the aforementioned result of [5, Theorem 4.1]. In what follows, we will denote by $\mathsf{Flat}(R)$ the class of flat left R-modules.

Theorem 5.2. *Let R be a left Noetherian ring. The class of flat left R-modules is the left part of a balanced pair if, and only if, the ring R is left perfect.*

Proof. Let us first prove the implication ($\Leftarrow$). If R is left perfect, then the class of flat modules coincides with the class of projective modules (Bass' [1, Theorem P]). Hence we get the standard balanced pair $(\mathsf{Proj}(R), \mathsf{Inj}(R))$ in $\mathsf{Mod}(R)$.

In order to show the converse implication ($\Rightarrow$), suppose there is a balanced pair $(\mathsf{Flat}(R), \mathcal{L})$ for some class of modules $\mathcal{L}$. Since R is left Noetherian, any direct sum of injective modules is injective. Therefore, we are in the assumptions of part (1) of Lemma 5.1, that says that the class $(\mathsf{Flat}(R))^{\perp_1}$ of cotorsion modules is closed under direct sums. But then by Guil Asensio and Herzog's [18, Theorem 19], the ring R must be left perfect. □

Following the philosophy of [5, Section 5], we want to mention other cases for which Theorem 5.2 is also valid. First, one can state a chain complex version of Theorem 5.2 by noticing some facts. Firstly, recall that a chain complex is flat if it is exact with flat cycles. Also, projective and injective complexes have similar descriptions. So if $\mathbf{Flat}(R)$ denotes the class of flat complexes, we can note that if $(\mathbf{Flat}(R))^{\perp_1}$ is closed under direct sums, then so will be the class $(\mathsf{Flat}(R))^{\perp_1}$ of cotorsion modules. For it suffices to note that for every cotorsion module C, the complex $\underline{C} = \cdots \to 0 \to C \to 0 \to \cdots$ belongs to $(\mathbf{Flat}(R))^{\perp_1}$. This follows applying a well-known natural isomorphism appearing in [14, Lemma 4.2].

The other context we are interested in is the category of quasi-coherent sheaves on a scheme X, presented in the following section.

Lack of balance with respect to flat quasi-coherent modules on a scheme. From now until the end of this section all rings are commutative.

Let $\mathfrak{Qcoh}(X)$ denote the category of quasi-coherent sheaves on a scheme X. The corresponding version of Theorem 5.2 for $\mathfrak{Qcoh}(X)$ is formulated below in Corollary 5.3. This result answers the question (6) posted in [5, Section 6] in the negative.

For a better understanding of Corollary 5.3, we need to recall a few well-known facts about $\mathfrak{Qcoh}(X)$. First, a scheme X is called *semi-separated* if it has a *semi-separating* open affine covering $\mathfrak{U} = \{U_i : i \in I\}$, that is, for each $i,k \in I$ the intersection $U_i \cap U_k$ is also an open affine. For each $i \in I$, the canonical inclusion $\iota_i \colon U_i \to X$ gives an adjoint pair (ι_i^*, ι_*^i), where

$$\iota_i^* \colon \mathfrak{Qcoh}(X) \to \mathfrak{Qcoh}(U_i) \quad \text{and} \quad \iota_*^i \colon \mathfrak{Qcoh}(U_i) \to \mathfrak{Qcoh}(X)$$

are the inverse and direct image functors, respectively. In general, the direct image functor ι_*^i does not preserve quasi-coherence, but it does for semi-separated schemes X. So, for each U_i, we have an isomorphism

$$\mathsf{Hom}_{\mathfrak{Qcoh}(U_i)}(\iota_i^*\mathscr{H},\mathscr{T})\cong \mathsf{Hom}_{\mathfrak{Qcoh}(X)}(\mathscr{H},\iota_*^i\mathscr{T}).$$

Since, for each open affine U_i, the categories $\mathsf{Mod}(\mathcal{O}_X(U_i))$ and $\mathfrak{Qcoh}(U_i)$ are equivalent by a well-known result of Grothendieck (see for instance Hartshorne's [20, Chapter II, Corollary 5.5]), we can write the previous isomorphism as

$$\mathsf{Hom}_{\mathcal{O}_X(U_i)}(\mathscr{H}(U_i),T)\cong \mathsf{Hom}_{\mathfrak{Qcoh}(X)}(\mathscr{H},\iota_*^i(T)),$$

for any $\mathcal{O}_X(U_i)$-module T and any quasi-coherent sheaf $\mathscr{H}$. We recall that a scheme is *Noetherian* if it is quasi-compact and it possesses an open affine covering $\mathfrak{U}=\{U_1,\dots,U_n\}$ such that, for each $i=1,\dots,n$, $\mathcal{O}_X(U_i)$ is a Noetherian ring.

Let $\mathfrak{Flat}(X)$ denote the class of flat quasi-coherent sheaves over X in the following result.

Corollary 5.3. *Let X be a Noetherian and semi-separated scheme, with semi-separating open affine covering $\mathfrak{U}=\{U_1,\dots,U_n\}$. Assume that $\mathcal{O}_X(U_i)$ is a Noetherian but not Artinian ring, for some $i\in\{1,\dots,n\}$. Then, $\mathfrak{Flat}(X)$ is not the left part of a balanced pair in $\mathfrak{Qcoh}(X)$.*

Proof. Suppose that there is such balanced pair $(\mathfrak{Flat}(X),\mathcal{L})$ in $\mathfrak{Qcoh}(X)$, for some class $\mathcal{L}$. The category $\mathfrak{Qcoh}(X)$ is well-known to be a Grothendieck category (see Grothendieck and Dieudonné's [17, Chapitre 1, §6, Corollarie 6.9.12] for the existence of a family of generators) and so it is cocomplete, satisfies AB4 and has enough injectives. Indeed, since X is Noetherian, the category $\mathfrak{Qcoh}(X)$ is locally Noetherian ([19, Chapter II, §7]), hence the direct sum of injective objects in $\mathfrak{Qcoh}(X)$ is again injective (Stenström's [29, Chapter V, Proposition 4.3]). Therefore, part (1) of Lemma 5.1 tells us that the class $(\mathfrak{Flat}(X))^{\perp_1}$ of cotorsion quasi-coherent sheaves, is closed under direct sums. Now let $\{C_k\}$ be a family of cotorsion $\mathcal{O}_X(U_i)$-modules. By Gillespie [13, Lemma 6.5] the functor $\iota_*^i\colon \mathsf{Mod}(\mathcal{O}_X(U_i))\to\mathfrak{Qcoh}(X)$ preserves cotorsion objects. Hence, the family $\{\iota_*^i(C_k)\}$ is a family of cotorsion quasi-coherent sheaves and thus, by the previous, $\oplus_k\iota_*^i(C_k)\in(\mathfrak{Flat}(X))^{\perp_1}$. We will finish the proof by showing that this implies that $\oplus_k C_k$ is a cotorsion $\mathcal{O}_X(U_i)$-module. So, by Guil Asensio and Herzog [18, Theorem 19], the ring $\mathcal{O}_X(U_i)$ must be Artinian. A contradiction.

To show what we claimed, let F be a flat $\mathcal{O}_X(U_i)$-module. We want to show that the equality $\mathsf{Ext}^1_{\mathcal{O}_X(U_i)}(F,\oplus_k C_k)=0$ holds. Firstly, notice that $F=\iota_i^*\iota_*^i(F)$. Then, the isomorphism shown in the proof of [13, Lemma 6.5] gives

$$\mathsf{Ext}^1_{\mathcal{O}_X(U_i)}(F,\oplus_k C_k)\cong \mathsf{Ext}^1_{\mathfrak{Qcoh}(X)}(\iota_*^i(F),\iota_*^i(\oplus_k C_k)).$$

The last Ext functor vanishes, because $\iota_*^i(F)$ is a flat quasi-cohent sheaf (so it belongs to $\mathfrak{Flat}(X)$) and $\iota_*^i(\oplus_k C_k)\simeq\oplus_k\iota_*^i(C_k)\in(\mathfrak{Flat}(X))^{\perp_1}$, because the functor ι_*^i commutes with direct sums. □

6. Balance in quiver representations and cotorsion triplets

Throughout this section $\mathcal{C}$ will be an abelian category with enough projectives and injectives that satisfies AB4 and AB4*.

In [27, Theorem 4.1.3] Odabaşı has recently proved that, under some conditions on a quiver Q, a complete cotorsion pair in $\mathcal{C}$ induces two complete cotorsion pairs in the abelian category $\mathrm{Rep}(Q,\mathcal{C})$ of $\mathcal{C}$-valued representations of Q. Taking into account the relation between balanced pairs and cotorsion triplets, it seems natural to expect that balanced pairs in $\mathcal{C}$ and $\mathrm{Rep}(Q,\mathcal{C})$

should be also related. Thus we will devote this section to study the relation between balanced pairs in $\mathcal{C}$ and balanced pairs in $\mathrm{Rep}(Q,\mathcal{C})$. One of the consequences of our results is that they will lead us to finding new conditions over two complete hereditary cotorsion pairs to form a cotorsion triplet.

Adjoint Functors between $\mathcal{C}$ and $\mathrm{Rep}(Q,\mathcal{C})$. A *quiver* $Q=(Q_0,Q_1,s,t)$ is a directed graph with vertex set Q_0, arrow set Q_1 and two maps s,t from Q_1 to Q_0 which associate to each arrow $\alpha\in Q_1$ its source $s(\alpha)\in Q_0$ and its target $t(\alpha)\in Q_0$, respectively. The quiver Q is said to be *finite* if Q_0 and Q_1 are finite.

A *representation* $\mathbb{X}=(\mathbb{X}_i,\mathbb{X}_\alpha)$ of Q over $\mathcal{C}$, or a *$\mathcal{C}$-valued representation*, is defined by the following data:

(1) To each vertex i in Q_0 is associated an object $\mathbb{X}_i\in\mathcal{C}$.
(2) To each arrow $\alpha: i\to j$ in Q_1 is associated a morphism $\mathbb{X}_\alpha:\mathbb{X}_i\to\mathbb{X}_j$ in $\mathcal{C}$.

A *morphism* f from $\mathbb{X}$ to $\mathbb{Y}$ is a family of morphisms $\{f_i\colon \mathbb{X}_i\to\mathbb{Y}_i\}_{i\in Q_0}$ such that $\mathbb{Y}_\alpha f_i=f_j\mathbb{X}_\alpha$ for any arrow $\alpha\colon i\to j\in Q_1$. We will denote by $\mathrm{Rep}(Q,\mathcal{C})$ the category of all $\mathcal{C}$-valued representations of a quiver Q.

Define the functor $e^i_\lambda\colon\mathcal{C}\to\mathrm{Rep}(Q,\mathcal{C})$ as

$$e^i_\lambda(M)_j:=\bigoplus_{Q(i,j)}M$$

for every vertex $j\in Q_0$ (see Mitchell's [26, Section 28]) with $Q(i,j)$ the set of paths p in Q such that $s(p)=i$ and $t(p)=j$. Moreover, for an arrow $\alpha\colon j\to k$, the morphism $e^i_\lambda(M)_\alpha$ is the canonical injection. Dually, the functor $e^\rho_i\colon\mathcal{C}\to\mathrm{Rep}(Q,\mathcal{C})$ is defined by Enochs and Herzog in [6, 7] as

$$e^\rho_i(M)_j:=\prod_{Q(j,i)}M$$

for every vertex $j\in Q_0$.

Lemma 6.1. [6, 21] *Let $i\in Q_0$ and $(\ )_i\colon\mathrm{Rep}(Q,\mathcal{C})\longrightarrow\mathcal{C}$ be the restriction functor given by $(\mathbb{X})_i=\mathbb{X}_i$ for any representation $\mathbb{X}$ of $\mathrm{Rep}(Q,\mathcal{C})$. Then, the following conditions hold:*

(1) $(\ )_i$ is a right adjoint of e^i_λ and a left adjoint of e^ρ_i.
(2) $\mathsf{Ext}^m_{\mathrm{Rep}(Q,\mathcal{C})}(e^i_\lambda(Y),\mathbb{X})\cong\mathsf{Ext}^m_{\mathcal{C}}(Y,(\mathbb{X})_i)$ for every $m\geq 0$.
(3) $\mathsf{Ext}^m_{\mathrm{Rep}(Q,\mathcal{C})}(\mathbb{X},e^\rho_i(Y))\cong\mathsf{Ext}^m_{\mathcal{C}}((\mathbb{X})_i,Y)$ for every $m\geq 0$.

For any representation $(\mathbb{X}_i,\mathbb{X}_\alpha)$ of $\mathrm{Rep}(Q,\mathcal{C})$, there are induced morphisms

$$\varphi_{\mathbb{X}_i}\colon\bigoplus_{t(\alpha)=i}\mathbb{X}_{s(\alpha)}\to\mathbb{X}_i\text{ and }\psi_{\mathbb{X}_i}\colon\mathbb{X}_i\to\prod_{s(\alpha)=i}\mathbb{X}_{t(\alpha)}.$$

We will denote by $\mathbb{c}_i(\mathbb{X})$ the cokernel of $\varphi_{\mathbb{X}_i}$ and by $\mathbb{k}_i(\mathbb{X})$ the kernel of $\psi_{\mathbb{X}_i}$. The assignments $\mathbb{c}_i(-)$ and $\mathbb{k}_i(-)$ from $\mathrm{Rep}(Q,\mathcal{C})$ to $\mathcal{C}$ are functorial.

Lemma 6.2. [21, Section 4 and Proposition 5.4] *Let $i\in Q_0$ and $\mathbb{s}_i:\mathcal{C}\to\mathrm{Rep}(Q,\mathcal{C})$ be the stalk functor given by $\mathbb{s}_i(Y)_j=\delta_{ij}Y$, where $\delta_{ii}Y=Y$ and $\delta_{ij}Y=0$ whenever $j\neq i$. Then, we have:*

(1) $\mathbb{s}_i$ is a right adjoint of $\mathbb{c}_i$ and a left adjoint of $\mathbb{k}_i$;
(2) $\mathsf{Ext}^1_{\mathrm{Rep}(Q,\mathcal{C})}(\mathbb{X},\mathbb{s}_i(Y))\cong\mathsf{Ext}^1_{\mathcal{C}}(\mathbb{c}_i(\mathbb{X}),Y)$, provided that $\varphi_{\mathbb{X}_i}$ is monic.
(3) $\mathsf{Ext}^1_{\mathrm{Rep}(Q,\mathcal{C})}(\mathbb{s}_i(Y),\mathbb{X})\cong\mathsf{Ext}^1_{\mathcal{C}}(Y,\mathbb{k}_i(\mathbb{X}))$, provided that $\psi_{\mathbb{X}_i}$ is epic.

Corollary 6.3. *Let Q be a quiver without oriented cycles, and let us fix a vertex $k \in Q_0$. Given a class $\mathcal{L}$ of objects of $\mathcal{C}$, for any $G \in {}^{\perp_1}\mathcal{L}$ there is an exact sequence*

$$0 \to \mathbb{K} \to e_\lambda^k(G) \xrightarrow{\widetilde{id}} \mathbb{s}_k(G) \to 0$$

in $\operatorname{Rep}(Q, \mathcal{C})$ *with* $\widetilde{id} = \delta_{ki} id_G$. *Moreover, for any* $\mathbb{X} \in \operatorname{Rep}(\mathcal{C}, Q)$, *if* $\mathbb{k}_k(\mathbb{X}) \in \mathcal{L}$ *and* $\psi_{\mathbb{X}_k}$ *is epic, then the above sequence is* $\mathsf{Hom}_{\operatorname{Rep}(Q,\mathcal{C})}(-, \mathbb{X})$ *exact.*

Proof. Clearly, $\widetilde{id}$ is surjective: For any arrow $\alpha : i \to j$, if $j = k$, then $e_\lambda^k(G)_i = 0$ since the quiver has no oriented cycles. And so we have the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ G \arrow[r, "\mathrm{id}_G"'] & G \end{tikzcd}$$

Otherwise, the diagrams

$$i = k, j \neq k \quad \begin{tikzcd} G \arrow[r, "\mathrm{id}_G"] \arrow[d, "G_\alpha"] & G \arrow[d] \\ \bigoplus_{Q(k,j)} G \arrow[r] & 0 \end{tikzcd} \qquad i \neq k, j \neq k \quad \begin{tikzcd} \bigoplus_{Q(k,i)} G \arrow[r] \arrow[d, "G_\alpha"] & 0 \arrow[d] \\ \bigoplus_{Q(k,j)} G \arrow[r] & 0 \end{tikzcd}$$

are also commutative. That is, $\widetilde{id}$ is an epimorphism in $\operatorname{Rep}(Q, \mathcal{C})$.

Moreover, $\mathsf{Ext}^1_{\operatorname{Rep}(Q,\mathcal{C})}(\mathbb{s}_k(G), \mathbb{X}) \cong \mathsf{Ext}^1_{\mathcal{C}}(G, \mathbb{k}_k(\mathbb{X})) = 0$ by Lemma 6.2 and the hypothesis on G. Therefore the sequence

$$0 \to \mathbb{K} \to e_\lambda^k(G) \xrightarrow{\widetilde{id}} \mathbb{s}_k(G) \to 0$$

is $\mathsf{Hom}_{\operatorname{Rep}(Q,\mathcal{C})}(-, \mathbb{X})$ exact. □

Induced classes in $\operatorname{Rep}(Q, \mathcal{C})$**.** Let $\mathcal{L}$ be a class of objects of $\mathcal{C}$. Following [21] we denote by

$$\begin{aligned} \operatorname{Rep}(Q, \mathcal{L}) &:= \{\mathbb{X} \in \operatorname{Rep}(Q, \mathcal{C}) \mid \mathbb{X}_i \in \mathcal{L} \text{ for all } i \in Q_0\}, \\ \Phi(\mathcal{L}) &:= \{\mathbb{X} \in \operatorname{Rep}(Q, \mathcal{L}) \mid \varphi_{\mathbb{X}_i} \text{ is monic and } \mathbb{c}_i(\mathbb{X}) \in \mathcal{L} \text{ for all } i \in Q_0\}, \\ \Psi(\mathcal{L}) &:= \{\mathbb{X} \in \operatorname{Rep}(Q, \mathcal{L}) \mid \psi_{\mathbb{X}_i} \text{ is epic and } \mathbb{k}_i(\mathbb{X}) \in \mathcal{L} \text{ for all } i \in Q_0\}. \end{aligned}$$

Proposition 6.4. *Let Q be a quiver with at least one arrow and without oriented cycles. With the notation above, assume that $(\Phi(\mathcal{F}), \Psi(\mathcal{L}))$ is a balanced pair in* $\operatorname{Rep}(Q, \mathcal{C})$ *for certain classes $\mathcal{F}$ and $\mathcal{L}$ in $\mathcal{C}$. Then, the following statements holds:*

(1) $(\mathcal{F}, \mathcal{L})$ is a balanced pair in $\mathcal{C}$.
(2) If $\mathcal{F}$ is resolving, then ${}^{\perp_1}\mathcal{L} \subseteq \mathcal{F}^{\perp_1}$.
(3) If $\mathcal{L}$ is coresolving, then $\mathcal{F}^{\perp_1} \subseteq {}^{\perp_1}\mathcal{L}$.

Proof. Let us prove (1) and (2). Part (3) is dual to (2).

(1) For any object $M \in \mathcal{C}$, there is a $\Phi(\mathcal{F})$-precover $\sigma \colon \mathbb{F} \to \mathbb{s}_i(M)$. Let $\mathbb{K} = \ker(\mathbb{s}_i)$. Thus we have an induced morphism $\widetilde{\sigma}_i \colon \mathbb{F}_i \to \mathbb{s}_i(M)_i = M$ in $\mathcal{C}$, and a left exact sequence

$$0 \to \mathbb{K}_i \to \mathbb{F}_i \xrightarrow{\widetilde{\sigma}_i} M \to 0. \tag{v}$$

We claim that $\widetilde{\sigma}_i \colon \mathbb{F}_i \to \mathbb{s}_i(M)_i = M$ is an $\mathcal{F}$-precover of M, where $\widetilde{\sigma}_i$ is induced by σ. In fact, for any $F \in \mathcal{F}$, one can note that the representation $e^i_\lambda(F)$ belongs to $\Phi(\mathcal{F})$. Then, we have an epimorphism

$$\mathsf{Hom}_{\mathrm{Rep}(Q,\mathcal{C})}(e^i_\lambda(F), \mathbb{F}) \to \mathsf{Hom}_{\mathrm{Rep}(Q,\mathcal{C})}(e^i_\lambda(F), \mathbb{s}_i(M))$$

which implies by Lemma 6.1 an epimorphism $\mathsf{Hom}_{\mathcal{C}}(F, \mathbb{F}_i) \to \mathsf{Hom}_{\mathcal{C}}(F, M)$, as desired.

Since $(\Phi(\mathcal{F}), \Psi(\mathcal{L}))$ is a balanced pair in $\mathrm{Rep}(Q, \mathcal{C})$ and $e^\rho_i(L) \in \Psi(\mathcal{L})$ for any $L \in \mathcal{L}$, we have by Lemma 3.1 an exact sequence

$$0 \to \mathsf{Hom}_{\mathrm{Rep}(Q,\mathcal{C})}(\mathbb{s}_i(M), e^\rho_i(L)) \to \mathsf{Hom}_{\mathrm{Rep}(Q,\mathcal{C})}(\mathbb{F}, e^\rho_i(L)) \to \mathsf{Hom}_{\mathrm{Rep}(Q,\mathcal{C})}(\mathbb{K}, e^\rho_i(L)) \to 0$$

with $\mathbb{K} = \ker(\sigma)$. Now by part (1) of Lemma 6.1, we have an exact sequence

$$0 \to \mathsf{Hom}_{\mathcal{C}}(M, L) \to \mathsf{Hom}_{\mathcal{C}}(\mathbb{F}_i, L) \to \mathsf{Hom}_{\mathcal{C}}(\mathbb{K}_i, L) \to 0.$$

Thus the left exact sequence (v) is $\mathsf{Hom}_{\mathcal{C}}(-, \mathcal{L})$ and $\mathsf{Hom}_{\mathcal{C}}(\mathcal{F}, -)$ exact. Similarly, we have that $\mathcal{L}$ is preenveloping and that there is a right exact sequence

$$0 \to M \to L \to C \to 0$$

in $\mathcal{C}$, which is $\mathsf{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-acyclic and $\mathsf{Hom}_{\mathcal{C}}(-, \mathcal{L})$-acyclic. Therefore, by Lemma 3.1 the pair $(\mathcal{F}, \mathcal{L})$ is balanced.

(2) Before proving the statement, we need to make some observations.

- By the hypothesis on Q, we can fix a non-sink vertex $k \in Q_0$. This means that there exists at least an arrow $k \to i$ in Q.
- Let $F \in \mathcal{F}$ and $\sigma \colon P \to F$ be an epimorphism with P projective and let us denote by $\mathbb{P}$ and $\mathbb{F}$ the induced representations $e^k_\lambda(P)$ and $\mathbb{s}_k(F)$, respectively. Then, we have an induced epimorphism $\widetilde{\sigma} \colon \mathbb{P} \to \mathbb{F}$ in $\mathrm{Rep}(Q, \mathcal{C})$ with $\widetilde{\sigma}_i = \delta_{ki}\sigma$, for any vertex $i \in Q_0$. Let $\mathbb{K} = \ker(\widetilde{\sigma})$. We will show that $\mathbb{K} \in \Phi(\mathcal{F})$.

  For each vertex $i \in Q_0$, we have the following exact commutative diagram in $\mathcal{C}$:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{t(\alpha)=i} \mathbb{K}_{s(\alpha)} \arrow[r] \arrow[d, "\varphi_{\mathbb{K}_i}"] & \bigoplus_{t(\alpha)=i} \mathbb{P}_{s(\alpha)} \arrow[r] \arrow[d, "\varphi_{\mathbb{P}_i}"] & \bigoplus_{t(\alpha)=i} \mathbb{F}_{s(\alpha)} \arrow[r] \arrow[d, "\varphi_{\mathbb{F}_i}"] & 0 \\
0 \arrow[r] & \mathbb{K}_i \arrow[r] & \mathbb{P}_i \arrow[r] & \mathbb{F}_i \arrow[r] & 0
\end{tikzcd}$$

  Since $\mathcal{F}$ is resolving it contains all the projective objects, so $P \in \mathcal{F}$. Therefore, by the definition of the functor $e^k_\lambda(-)$, we follow that $\mathbb{P} = e^k_\lambda(P)$ belongs to $\Phi(\mathcal{F})$. So, in particular, the morphism $\varphi_{\mathbb{P}_i}$ is monic for any vertex $i \in Q_0$. It follows that $\varphi_{\mathbb{K}_i}$ is monic since $\varphi_{\mathbb{P}_i}$ is monic. By the snake lemma, we get the exact sequence

$$0 \longrightarrow \ker(\varphi_{\mathbb{F}_i}) \longrightarrow \mathbb{c}_i(\mathbb{K}) \longrightarrow \mathbb{c}_i(\mathbb{P}) \longrightarrow \mathbb{c}_i(\mathbb{F}) \longrightarrow 0.$$

  Note that $\ker(\varphi_{\mathbb{F}_i}), \mathbb{c}_i(\mathbb{P}), \mathbb{c}_i(\mathbb{F}) \in \mathcal{F}$ and $\mathcal{F}$ is resolving. It follows that $\mathbb{c}_i(\mathbb{K}) \in \mathcal{F}$. Thus $\mathbb{K} \in \Phi(\mathcal{F})$.

- Moreover, for any arrow $\alpha\colon\ k \to i$ with $i \neq k$, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{K}_k = \ker(\sigma) \arrow[r, "l"] \arrow[d] & \mathbb{P}_k = P \arrow[d, "\mathbb{P}_\alpha"] \\
 & \mathbb{K}_i \arrow[r, equal] & \mathbb{P}_i = \bigoplus_{Q(k,i)} P
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{P}_k = P \arrow[r, "\sigma"] \arrow[d, "\mathbb{P}_\alpha"] & \mathbb{F}_k = F \arrow[r] \arrow[d] & 0 \\
\mathbb{P}_i = \bigoplus_{Q(k,i)} P \arrow[r] & 0 &
\end{tikzcd}$$

where l and $\mathbb{P}_\alpha$ are canonical injections.

Let us prove now the claim (2). So let $G \in {}^{\perp_1}\mathcal{L}$. We want to show that $G \in \mathcal{F}^{\perp_1}$. Given $F \in \mathcal{F}$, we have the previous exact sequence

$$0 \to \ker(\sigma) \xrightarrow{l} P \xrightarrow{\sigma} F \to 0,$$

with P projective. Then to get what we claim, it suffices to show that any $f\colon\ \ker(\sigma) \to G$ can be lifted to a map $P \to G$, that is, the previous sequence is $\mathsf{Hom}_{\mathcal{C}}(-, G)$ exact. So, let $f\colon\ \ker(\sigma) \to G$ be any morphism and let $\widetilde{f}\colon\ \mathbb{K} \to \mathbb{s}_k(G)$ be the induced morphism in $\mathrm{Rep}(Q, \mathcal{C})$ with $\widetilde{f}_{ki} = \delta_{ij} f$. Note that $G \in {}^{\perp_1}\mathcal{L}$, we get that

$$\widetilde{id}\colon\ e^k_\lambda(G) \to \mathbb{s}_k(G) \to 0$$

is $\mathsf{Hom}_{\mathcal{C}}(-, \Psi(\mathcal{L}))$ exact from Corollary 6.3. It follows that $\widetilde{id}$ is $\mathsf{Hom}_{\mathcal{C}}(\Phi(\mathcal{F}), -)$ exact by the hypothesis on the balance. And we have previously proved that $\mathbb{K} \in \Phi(\mathcal{F})$. Therefore, for the map $\widetilde{f}\colon\ \mathbb{K} \to \mathbb{s}_k(G)$, there is $\widetilde{g} : \mathbb{K} \to e^k_\lambda(G)$ such that $\widetilde{f} = \widetilde{id}\widetilde{g}$. In particular, for the arrow $\alpha\colon\ k \to i$, we have the following commutative diagram

$$\begin{tikzcd}
 & & \ker(\sigma) \arrow[dl, "\widetilde{g}_k"'] \arrow[dr, "f"] \arrow[dd, "\mathbb{P}_\alpha l"'] & \\
 & G \arrow[rr, equal] \arrow[dd, "e^k_\lambda(G)_\alpha"'] & & G \arrow[dd] \\
 & & \mathbb{K}_i \arrow[dl, "\widetilde{g}_i"'] \arrow[dr] & \\
G & \bigoplus_{Q(k,i)} G \arrow[l, "\pi_\alpha"] \arrow[rr] & & 0
\end{tikzcd}$$

It follows that $\widetilde{g}_i \mathbb{P}_\alpha l = e^k_\lambda(G)_\alpha \widetilde{g}_k$. Let π_α be the canonical projection corresponding to the canonical injection $e^k_\lambda(G)_\alpha$, and so

$$f = \widetilde{g}_k = \pi_\alpha e^k_\lambda(G)_\alpha \widetilde{g}_k = (\pi_\alpha \widetilde{g}_i) \circ (\mathbb{P}_\alpha l) = (\pi_\alpha \widetilde{g}_i \mathbb{P}_\alpha) \circ l.$$

That is, the sequence

$$0 \to \ker(\sigma) \to P \to F \to 0$$

is $\mathsf{Hom}_{\mathcal{C}}(-, G)$ exact, and so $G \in \mathcal{F}^{\perp_1}$.

□

For the following results, recall (see for example [21]) that a quiver Q is said to be *left rooted* if it contains no paths of the form $\cdots \to \bullet \to \bullet \to \bullet$. Dually, Q is called *right rooted* if it contains no paths of the form $\bullet \to \bullet \to \bullet \to \cdots$.

Let us focus now in the case $\mathcal{C} = \mathsf{Mod}(R)$. If the quiver Q is *left and right rooted* (for instance the quiver $\cdots \to \bullet \leftarrow \bullet \to \bullet \leftarrow \bullet \to \bullet \leftarrow \bullet \to \cdots$) we can combine [21, Theorems A and B] and Eshraghi, Hafezi, Hosseini and Salarian [11, Theorem A] (or [27, Theorem 4.1.3]) to infer that, in case we start with two complete hereditary cotorsion pairs $(\mathcal{F}, \mathcal{H})$ and $(\mathcal{G}, \mathcal{L})$ in $\mathsf{Mod}(R)$, then we get two induced complete hereditary cotorsion pairs $(\Phi(\mathcal{F}), \mathrm{Rep}(Q, \mathcal{H}))$ and $(\mathrm{Rep}(Q, \mathcal{G}), \Psi(\mathcal{L}))$ in $\mathrm{Rep}(Q, \mathsf{Mod}(R))$. Therefore, we get the following result.

Corollary 6.5. *If $(\mathcal{F}, \mathcal{H})$ and $(\mathcal{G}, \mathcal{L})$ are complete hereditary cotorsion pairs in $\mathsf{Mod}(R)$, then the following are equivalent:*

(a) $\mathcal{H} = \mathcal{G}$

(b) $(\Phi(\mathcal{F}), \Psi(\mathcal{L}))$ *is an admissible balanced pair for any left and right rooted quiver Q with at least one arrow.*

(c) $(\Phi(\mathcal{F}), \Psi(\mathcal{L}))$ *is an admissible balanced pair for some left and right rooted quiver Q with at least one arrow.*

Proof.

- (a) $\Rightarrow$ (b). Let Q be any left and right rooted quiver with at least one arrow. By the previous comments, we have the two complete hereditary cotorsion pairs $(\Phi(\mathcal{F}), \mathrm{Rep}(Q, \mathcal{H}))$ and $(\mathrm{Rep}(Q, \mathcal{G}), \Psi(\mathcal{L}))$ in $\mathrm{Rep}(Q, \mathsf{Mod}(R))$. By hypothesis we have that $\mathcal{H} = \mathcal{G}$, and so $\mathrm{Rep}(Q, \mathcal{H})) = \mathrm{Rep}(Q, \mathcal{G})$. Hence Proposition 4.2 gives the claim (b).
- (b) $\Rightarrow$ (c). It is trivial.
- (c) $\Rightarrow$ (b). By Proposition 6.4(1), the pair $(\mathcal{F}, \mathcal{L})$ is a balanced pair. By the assumption in the corollary, the classes $\mathcal{F}$ and $\mathcal{L}$ are, in particular, resolving and coresolving, respectively (see Section 2). Hence Proposition 6.4(2) gives that

$$\mathcal{G} = {}^{\perp_1}\mathcal{L} \subseteq \mathcal{F}^{\perp_1} = \mathcal{H},$$

and Proposition 6.4 (3) gives that $\mathcal{H} = \mathcal{F}^{\perp_1} \subseteq {}^{\perp_1}\mathcal{L} = \mathcal{G}$. So (a) follows.

□

As a consequence of Corollary 6.5, we have the following characterization of quasi-Frobenius rings.

Corollary 6.6. *A ring R is quasi-Frobenius if and only if $(\Phi(\mathsf{Mod}(R)), \Psi(\mathsf{Mod}(R)))$ is an admissible balanced pair for a left and right rooted quiver Q with at least one arrow. In this case, we have the complete hereditary cotorsion triplet $(\Phi(\mathsf{Mod}(R)), \mathrm{Rep}(Q, \mathsf{Proj}(R)), \Psi(\mathsf{Mod}(R)))$ in $\mathrm{Rep}(Q, \mathsf{Mod}(R))$.*

Proof. Let us first recall that, for any ring R, we have the trivial complete hereditary cotorsion pairs $(\mathsf{Mod}(R), \mathsf{Inj}(R))$ and $(\mathsf{Proj}(R), \mathsf{Mod}(R))$. By the comments before Corollary 6.5, for a given left and right rooted quiver Q, we have the induced complete hereditary cotorsion pairs

$$(\Phi(\mathsf{Mod}(R)), \mathrm{Rep}(Q, \mathsf{Inj}(R))) \text{ and } (\mathrm{Rep}(Q, \mathsf{Proj}(R)), \Psi(\mathsf{Mod}(R))) \text{ in } \mathrm{Rep}(Q, \mathsf{Mod}(R)). \qquad \text{(vi)}$$

Now, suppose that R is quasi-Frobenius. Then $\mathsf{Inj}(R) = \mathsf{Proj}(R)$, and so Corollary 6.5 ((a) $\Rightarrow$ (c)) gives that $(\Phi(\mathsf{Mod}(R)), \Psi(\mathsf{Mod}(R)))$ is a balanced pair for some left and right rooted quiver Q with at least one arrow.

Conversely, if we assume that $(\Phi(\mathsf{Mod}(R)), \Psi(\mathsf{Mod}(R)))$ is a balanced pair for some left and right rooted quiver Q with at least one arrow, we get from Corollary 6.5 ((c) $\Rightarrow$ (a)) that $\mathsf{Inj}(R) = \mathsf{Proj}(R)$, that is the ring R is quasi-Frobenius.

Finally, if any of the equivalent conditions holds, we follow that the categories $\mathrm{Rep}(Q, \mathsf{Inj}(R)))$ and $\mathrm{Rep}(Q, \mathsf{Proj}(R))$ coincide, and so the pairs in (vi) give rise to the complete hereditary cotorsion triplet

$$(\Phi(\mathsf{Mod}(R)), \mathrm{Rep}(Q, \mathsf{Proj}(R)), \Psi(\mathsf{Mod}(R)))$$

in $\mathrm{Rep}(Q, \mathsf{Mod}(R))$. □

Remark 6.7. The category $\Phi(\mathsf{Mod}(R))$ is known in the literature as *monomorphism category*. It has been extensively studied by Li, Luo and Zhang in [24, 25]. Dually, $\Psi(\mathsf{Mod}(R))$ is called *epimorphism category*.

Our last result allows to give another extension of the characterization of virtually Gorenstein Noetherian rings of finite Krull dimension given by Zareh-Khoshchehreh, Asgharzadeh and Divaani-Aazar in [30, Theorem 3.10]. We recall that a ring R is called *left n-perfect* if every flat left R-module has finite projective dimension $\leq n$.

Corollary 6.8. *Let R be a left n-perfect and right coherent ring. Then, the following conditions are equivalent:*

- *(a) R is virtually Gorenstein.*
- *(b) $(\Phi(\mathsf{GProj}(R)), \Psi(\mathsf{GInj}(R)))$ is an admissible balanced pair in $\mathrm{Rep}(Q, \mathsf{Mod}(R))$ for some left and right rooted quiver Q with at least one arrow.*
- *(c) $(\mathsf{GProj}(R), \mathsf{GInj}(R))$ is an admissible balanced pair in $\mathsf{Mod}(R)$.*

Proof. Firstly we point out that under the assumptions on R, the pair $(\mathsf{GProj}(R), \mathsf{GProj}(R)^{\perp})$ is known to be a complete hereditary cotorsion pair (see Estrada, Iacob, Odabaşı [12, Proposition 6]). On the other hand, Šaroch and Šťovíček ([28]) have recently proved that the pair $({}^{\perp}\mathsf{GInj}(R), \mathsf{GInj}(R))$ is a perfect (so, in particular, complete) and hereditary cotorsion pair for *any* ring.

Now, (a) $\Leftrightarrow$ (c) immediately follows from Corollary 4.8 by the above and by noticing that

$$\mathsf{GProj}(R) \cap \mathsf{GProj}(R)^{\perp} = \mathsf{Proj}(R) \ \text{ and } \ {}^{\perp}\mathsf{GInj}(R) \cap \mathsf{GInj}(R) = \mathsf{Inj}(R).$$

Finally (a) $\Leftrightarrow$ (b) follows from Corollary 6.5. □

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