Birnbaum importance measure for reliability systems with dependent components

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***Abstract*—**Component importance measures are relevant to improve the system design and to develop optimal replacement policies. Birnbaum's importance measure is one of the most relevant measures. If the components are (stochastically) independent, this measure can be defined using several equivalent expressions. However, in many practical situations, the independence assumption is unrealistic. It also turns out that in the case of dependent components different Birnbaum's measure definitions lead to different concepts. In this paper we extend Birnbaum's importance measure to the case of dependent components in a way allowing us to obtain relevant properties including connections and comparisons with other measures proposed and studied recently. The dependence is modeled through copulas and the new measure is based on the contribution of the component to the system reliability.

***Index terms*—**Coherent system; Birnbaum importance measure; Barlow-Proschan importance measure; copula; likelihood ratio order.

Notation

n	number of components
$\boldsymbol{p} = (p_1, \dots, p_n)$	an element of $[0,1]^n$
$\mathbf{0}_n = (0, \dots, 0)$	n-dimensional vector of zeros
$\mathbf{1}_n = (1, \dots, 1)$	n-dimensional vector of ones
T	system lifetime
$T_1, \dots, T_n$	component lifetimes
$\overline{F}_T(t)$	reliability function of T
$\overline{F}_i(t)$	reliability function of T_i
C	copula of $(T_1, \dots, T_n)$
$\hat{C}$	survival copula of $(T_1, \dots, T_n)$
ID	identically distributed
$I_{BP}(i)$	Barlow-Proschan importance measure
$I_B(i; \boldsymbol{p})$	Birnbaum importance measure
t.d.l.	time-dependent lifetimes
t.i.l.	time-independent lifetimes
i.m.	importance measure

I. Introduction

A number of component importance measures have been proposed and studied in the literature in the case of independent components. Some of these have prevailed and some others have disappeared. A good survey can be seen in Kuo and Zhu (2012a) (see also Kuo and Zhu, 2012b). The Birnbaum's measure for independent components was proposed in 1969 (see Birnbaum, 1969) and it is still one of the most popular component importance measures. It is closely related with the impact of the components in the system reliability. It can be defined by using different expressions (see, (1.10a)-(1.10c) in Barlow and Proschan (1975), p. 26, and Chapter 4 in Kuo and Zhu (2012b)). If the components are independent, these expressions are equivalent (i.e., they coincide). If they are dependent, then these options may lead to different concepts. Extensions of Birnbaum's measure to non-coherent systems and risk models were studied in Aliee et al. (2017); Andrews and Beeson (2003) and Vaurio (2016).

In many practical situations, the assumption of independence is unrealistic (consider, for example, the wheels in a car or the engines in a plane). The dependence is usually a consequence of the common environment for the components. However, as far as we know, only two importance measures have been recently studied in the case of dependent components.

Iyer (1992) extended the Barlow-Proschan importance measure through

$$I_{BP}(i) = \Pr(T = T_i), \tag{1}$$

where T is the lifetime of the system and T_i is the lifetime of the ith component. This measure has a clear meaning and depends on the joint distribution of the component lifetimes. However, it is not easy to compute it in the case of dependent components (both in practice and in theoretical cases). Some properties of this measure and connections with signatures were given in Marichal and Mathonet (2013).

More lately, Zhang and Wilson (2017) studied the following alternative measure, proposed in Barlow and Proschan (1975), p. 27, for systems with dependent components

$$I_h(i) = E(\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X})),$$

where ϕ is the structure function of the system, X_j represents the state of the jth component at time t, for $j = 1, \dots, n$, $(1_i, \boldsymbol{X}) = (X_1, \dots, X_{i-1}, 1, X_{i+1}, \dots, X_n)$ and $(0_i, \boldsymbol{X}) = (X_1, \dots, X_{i-1}, 0, X_{i+1}, \dots, X_n)$. They studied this measure in the case of coherent systems with discrete marginals when the component states are dependent and positively associated. They characterize the influence of dependence structure on system reliability and component importance. They used a copula approach to represent the dependence between the components. For more details and reviews on importance measures of independent and dependent components we refer the reader to Kuo and Zhu (2012b). Recent properties can be seen in Eryilmaz (2016); Lin et al. (2016); Zhu et al. (2016).

The dependence between the component lifetimes can be represented by the copula of the random vector $(T_1, \dots, T_n)$. This is a very good way to represent the dependence relationships between the components due to the common environment. The copula representation allows us to separate the structure dependence which is usually fixed (by a given copula) or that just depends on a dependence parameter (included in the copula) from the different marginal distributions which represent the different units available for a given system. A short introduction on copula theory is given in Subsection II.A. Additional results can be seen in the books Durante and Sempi (2016) and Nelsen (2006). Then, it is well known that the system reliability at a given time t can be written as

$$\overline{F}_T(t) := \Pr(T > t) = R(\overline{F}_1(t), \dots, \overline{F}_n(t)) \qquad (2)$$

(see the details given in Subsection II.B) where $\overline{F}_i(t) = \Pr(T_i > t)$ is the reliability of the ith component and $R : [0,1]^n \to [0,1]$ is a continuous aggregation function which only depends on the structure of the system and on the copula (see, e.g., Samaniego and Navarro, 2016; Miziuła and Navarro, 2017). The function R is increasing (in each variable) and satisfies $R(0,\dots,0) = 0$ and $R(1,\dots,1) = 1$. This representation holds for any kind of joint distributions (continuous, discrete, etc.) and can be used to compute the system reliability and to compare different systems (see Navarro, 2018; Navarro et al., 2016; Navarro and del Águila, 2017; Navarro and Durante, 2017) and to obtain bounds for the system performance (see Miziuła and Navarro, 2017, 2018). If the components are independent, then the function R is a multinomial called the *reliability function of the system structure* (see Barlow and Proschan, 1975, p. 21).

In this paper, we use the copula-representation (2) to propose and study an extension of classic Birnbaum component importance measure to the case of dependent components. The considered measure is defined by

$$I_B(i; p_1, \dots, p_n) = \partial_i R(p_1, \dots, p_n), \qquad (3)$$

where $\partial_i R$ represents the partial derivative of R with respect to its ith variable (we assume that this derivative exists). This measure has also a clear meaning based on (2) since it represents how an increment in the reliability of the ith component, increments the system reliability. Note that it does not depend on the component reliability functions. If the components are independent, then this measure coincides with that studied in Zhang and Wilson (2017) but if they are dependent, then they are different (see Example 1). Moreover, we show that the importance measure given in (1) can be obtained from that in (3) as

$$I_{BP}(i) = \int_0^\infty I_B(i; \overline{F}_1(t), \dots, \overline{F}_n(t)) dF_i(t),$$

where $F_i(t) = 1 - \overline{F}_i(t)$ for $i = 1, \dots, n$, extending a similar well known property for systems with independent components.

The rest of the paper is organized as follows. In the following section we introduce the preliminary results needed in the present paper which include (Subsection 2.1) the copula representation for coherent systems with dependent components and some ordering properties and (Subsection 2.2) the importance measures used in the article with their basic properties in the case of independent components. The main results are given in Section 3 where we study the basic properties of the importance measure defined in (3) when the components are dependent. Some illustrative examples are included in Section 4 and the conclusions can be found in Section 5. The technical proofs are placed in the Appendix.

Throughout the paper we say that a function g is increasing (decreasing) if $g(x_1, \dots, x_n) \le g(y_1, \dots, y_n)$ ($\ge$) for all $x_i \le y_i$, $i = 1, \dots, n$.

II. Preliminaries

A. Introduction to Copula Theory.

The component lifetimes can be represented by a (nonnegative) random vector $(T_1, \dots, T_n)$. The different models are represented by the joint distribution function

$$\boldsymbol{F}(t_1, \dots, t_n) = \Pr(T_1 \le t_1, \dots, T_n \le t_n).$$

The function $\boldsymbol{F}$ contains both the information about the dependence structure and the marginals distributions which can be obtained as

$$F_i(t) = \Pr(T_i \le t) = \boldsymbol{F}(\infty, \dots, \infty, t, \infty, \dots, \infty)$$

where t is placed at the ith position, for $i = 1, \dots, n$.

The copula theory (Sklar's Theorem) allows us to represent the joint distribution function $\boldsymbol{F}$ as

$$\boldsymbol{F}(t_1, \dots, t_n) = C(F_1(t), \dots, F_n(t)), \qquad (4)$$

for a copula function C. A **copula** is a continuous multivariate distribution function having uniform marginals over the interval $(0,1)$. The main advantage of expression (4) is that we can separate the structure dependence (contained in C) from the marginal distributions. By changing C (or the parameter values included in C) or the marginals we obtain new models. This is especially useful when we represent the joint distribution of the component lifetimes in a system.

Analogously, the joint reliability (or survival) function $\overline{\boldsymbol{F}}$ of $(T_1, \dots, T_n)$ can be written as

$$\begin{aligned}\overline{\boldsymbol{F}}(t_1, \dots, t_n) &= \Pr(T_1 > t_1, \dots, T_n > t_n) \\ &= \hat{C}(\overline{F}_1(t), \dots, \overline{F}_n(t)), \qquad (5)\end{aligned}$$

where $\overline{F}_i(t) = \Pr(T_i > t) = 1 - f_i(t)$ are the marginal reliability functions and $\hat{C}$ is another copula called **survival copula**. Notice that $\hat{C}$ is not the reliability function $\overline{C}$ associated to C. Actually, $\hat{C}$ is a distribution function (it is increasing) while $\overline{C}$ is a reliability function (it is decreasing). Note that $\hat{C}$ can be obtained from C (and vice versa).

The case of independent components can also be modelled by copulas. This case is obtained when

$$C(p_1, \dots, p_n) = \hat{C}(p_1, \dots, p_n) = p_1 \dots p_n$$

for $p_1, \dots, p_n \in [0,1]$. This copula is called the **product copula**.

For more properties of copulas and applications we refer the readers to Nelsen (2006) and Durante and Sempi (2016).

B. Copula representation of system reliability

An n-component system is a Boolean function

$$\phi : \{0,1\}^n \to \{0,1\}$$

where $x_i \in \{0,1\}$ represents the state of the ith component ($x_i = 1$ means that it works) and $\phi(x_1,\dots,x_n)$ the state of the system which is completely determined by the states of the components. The system is **coherent** if its structure function ϕ is increasing and each component is relevant for the system. The ith component is **relevant** for the system if ϕ is not constant in x_i. The system is **semicoherent** if ϕ is increasing, $\phi(\mathbf{0}_n) = 0$ and $\phi(\mathbf{1}_n) = 1$, where $\mathbf{0}_n = (0,\dots,0) \in \{0,1\}^n$ and $\mathbf{1}_n = (1,\dots,1) \in \{0,1\}^n$. The basic properties of systems can be seen in, for example, the classic book Barlow and Proschan (1975).

If the component states (at a given time) are random, then they are represented by the random vector $\boldsymbol{X} = (X_1,\dots,X_n)$, $\phi(\boldsymbol{X})$ represents the state of the system and $p_i = \Pr(X_i = 1)$ is the survival probability of the ith component.

If the time is known, than the random vector $\boldsymbol{X(t)} = (X_1(t),\dots,X_n(t))$ represents the state of the components at time t. Then $\phi(\boldsymbol{X}(t))$ represents the state of the system. If, as in the preceding subsection, the random vector $\boldsymbol{T} = (T_1,\dots,T_n)$ represents the component lifetimes, then $X_i = 1$ if and only if $T_i > t$ and then $\Pr(X_i(t) = 1) = \Pr(T_i > t) = \overline{F}_i(t)$ is the ith component reliability at time t. If T represents the lifetime of the system, $\phi(\boldsymbol{X}(t)) = 1$ if and only if $T > t$ and then $\Pr(\phi(\boldsymbol{X}) = 1) = \Pr(T > t) = \overline{F}_T(t)$ is the system reliability at time t. From now on we omit t in $\boldsymbol{X}(t)$ and $X_i(t)$ to simplify the notation.

It is well known (see, e.g., Barlow and Proschan, 1975, p. 12) that $\phi(\boldsymbol{X})$ can be written as $\phi(\boldsymbol{X}) = \max_{i=1,\dots,r} \min_{j\in P_i} X_j$, where $P_1,\dots,P_r$ stand for all the minimal path sets of the system. A set $P \subseteq \{1,\dots,n\}$ is called a **path set** of a coherent system if the system operates when all the components in P do so. A **minimal path set** is a path set which does not contain other path sets. Throughout the paper we use the notation $X_P = \min_{j\in P} X_j$, i.e., X_P is the state of the series system with components in P. Hence

$$\{\phi(\boldsymbol{X}) = 1\} = \bigcup_{i=1}^{r} \{X_{P_i} = 1\}, \tag{6}$$

that is, the system works if and only if any of its minimal path sets does so. Note that, if $\hat{C}$ is the survival copula of $\boldsymbol{T}$, then

$$\Pr(X_P = 1) = \hat{C}_P(p_1,\dots,p_n), \tag{7}$$

where $\hat{C}_P(p_1,\dots,p_n) = \hat{C}(x_1^P,\dots,x_n^P)$, $x_i^P = p_i$ if $i \in P$ and $x_i^P = 1$ if $i \notin P$ (that is, $\hat{C}_P$ is the marginal survival copula of the random vector with the component lifetimes included in P). Then, by applying the inclusion-exclusion formula to the union in (6), we get

$$\begin{aligned}\Pr\big(\phi(\boldsymbol{X}) = 1\big) &= \sum_{i=1}^{r} \hat{C}_{P_i}(\boldsymbol{p}) - \sum_{1\le i<j\le r} \hat{C}_{P_i\cup P_j}(\boldsymbol{p}) + \dots \\ &\quad + (-1)^{r+1}\hat{C}_{P_1\cup\ldots\cup P_n}(\boldsymbol{p}) \\ &=: R(\boldsymbol{p}) \end{aligned}\tag{8}$$

for $\boldsymbol{p} = (p_1,\dots,p_n) \in [0,1]^n$, where R is an increasing continuous function such that $R(\mathbf{0}_n) = 0$ and $R(\mathbf{1}_n) = 1$. Function R depends on the structure of the system (the minimal path sets) and on the dependence structure between the components (the survival copula). When the components are independent this function is called the *reliability function of the structure* ϕ in Barlow and Proschan (1975), p. 21. The function R in expression (8) can be seen as an extension of this function to the case of dependent components.

Analogously, for the respective lifetimes we have

$$\Pr\big(T > t\big) = \sum_{i=1}^{r} \Pr(T_{P_i} > t) - \sum_{1\le i<j\le r} \Pr(T_{P_i\cup P_j} > t) \tag{9}$$

$$\begin{aligned} &+ \ldots + (-1)^{r+1}\Pr(T_{P_1\cup\cdots\cup P_j} > t) \\ &= R(\overline{F}_1(t),\dots,\overline{F}_n(t)),\end{aligned} \tag{10}$$

where $T_P = \min_{j\in P} T_j$. Note that $\bar{F}_T(t) := \Pr\big(T > t\big)$ is the reliability function associated to the system lifetime T. It is also called the *availability at time* t of the system (see, e.g., Kuo and Zhu, 2012b, p. 31). The respective distribution functions satisfy

$$\Pr\big(T \le t\big) = Q(F_1(t),\dots,F_n(t)), \tag{11}$$

where $Q(q_1,\dots,q_n) = 1 - R(1-q_1,\dots,1-q_n)$. Representations (9) and (11) are equivalent. Function Q can also be obtained from the minimal cut sets of the system (see, e.g., Lemma 1 in Navarro and del Águila, 2017). Also note that these representations hold for any kind of coherent systems (we do not need additional assumptions). Similar representations hold for semicoherent systems and mixed systems.

The distributions that can be written in this way are called **generalized distorted distributions** (see Navarro et al., 2016). The functions R and Q are called **dual distortion** and **distortion** functions, respectively. Ordering properties and bounds for this kind of distributions can be seen in Navarro et al. (2016); Miziuła and Navarro (2017, 2018); Navarro and del Águila (2017); Navarro and Durante (2017). Theorem 1 in Zhang and Wilson (2017) is extended in the following theorem. We use there the stochastic (st) order defined as follows: if X and Y are two random variables, then $X \le_{st} Y$ if and only if $\Pr(X > t) \le \Pr(Y > t)$ for all t. Its proof is immediate by (9) and (11). More ordering properties can be found in Navarro et al. (2016).

Theorem 1. *Let T_1 and T_2 be the lifetimes of two coherent systems with n components having distribution functions $F_1,\dots,F_n$. Let R_1, R_2 and Q_1, Q_2 be the respective distortion functions in representations* (9) *and* (11). *Then:*

1. *$T_1 \ge_{st} T_2$ for all $F_1,\dots,F_n$ if and only if $R_1 \ge R_2$.*
2. *$T_1 \ge_{st} T_2$ for all $F_1,\dots,F_n$ if and only if $Q_1 \le Q_2$.*

For a parallel system, we have $T = \max(T_1,\dots,T_n)$ and

$$\Pr(T \le t) = \Pr(T_1 \le t,\dots,T_n \le t) = C(F_1(t),\dots,F_n(t)),$$

that is, $Q = C$. A copula (distribution function) C_1 is said to be more positive lower orthant dependent (PLOD) than another C_2 if $C_1 \ge C_2$ (see, e.g. Zhang and Wilson, 2017). So, for parallel systems, the condition in item 2 of the preceding theorem can also be written as C_2 is more PLOD than C_1, obtaining item 2 of Theorem 1 in Zhang and Wilson (2017).

For a series system, we have $T = \min(T_1, \ldots, T_n)$ and

$$\Pr(T > t) = \Pr(T_1 > t, \ldots, T_n > t) = \hat{C}(\overline{F}_1(t), \ldots, \overline{F}_n(t)),$$

that is, $R = \hat{C}$. Hence the condition in item 1 of the preceding theorem can also be written as: $\hat{C}_1 \geq \hat{C}_2$ or as $\hat{C}_1$ is more PLOD than $\hat{C}_2$. A copula (distribution function) C_1 is said to be more positive upper orthant dependent (PUOD) than another C_2 if the respective reliability functions satisfy $\overline{C}_1 \geq \overline{C}_2$ (see, e.g., Zhang and Wilson, 2017). Hence, the condition $\hat{C}_1 \geq \hat{C}_2$ is equivalent to C_1 is more PUOD than C_2, obtaining item 1 of Theorem 1 in Zhang and Wilson (2017).

C. *Component importance measures*

In this paper we adapt the classification of component importance measures given in Kuo and Zhu (2012a,b). Measures based on only system structure ϕ and survival copula $\hat{C}$ are called **structure importance measures** and denoted by $I(i; \phi, \hat{C})$. If we additionally consider components' reliabilities $p_j = \Pr(X_j = 1), j = 1, \ldots, n$, at a fixed unspecified time point, we say that we work with **reliability importance measures**, denoted by $I(i; \boldsymbol{p})$, where $\boldsymbol{p} = (p_1, \ldots, p_n)$. If the actual value of t matters, we put $\overline{F}_j(t)$ instead of p_j for $j = 1, \ldots, n$ and talk about **time-dependent lifetime (t.d.l.) importance measures**, denoted by $I(i; \overline{F}_1(t), \ldots, \overline{F}_n(t))$ or $I(i; t)$ for short. Finally, measures based on components' reliability functions $\overline{F}_j, j = 1, \ldots, n$ for all the positive t are called **time-independent lifetime (t.i.l.) importance measures** and denoted by $I(i)$.

Birnbaum (1969) originally defined the reliability importance measure of the ith component for systems with independent component as follows

$$I_B(i; \boldsymbol{p}) = \Pr\big(\phi(\boldsymbol{X}) = 1 | X_i = 1\big) - \Pr\big(\phi(\boldsymbol{X}) = 1 | X_i = 0\big). \tag{12}$$

By using the notation introduced in the preceding section, it can also be written as:

$$I_B(i; \boldsymbol{p}) = R(1_i, \boldsymbol{p}) - R(0_i, \boldsymbol{p}),$$

where $(1_i, \boldsymbol{p}) = (p_1, \ldots, p_{i-1}, 1, p_{i+1}, \ldots, p_n)$ and $(0_i, \boldsymbol{p}) = (p_1, \ldots, p_{i-1}, 0, p_{i+1}, \ldots, p_n)$, or as

$$I_B(i; \boldsymbol{p}) = \mathrm{E}\big(\phi(1_i, \phi(\boldsymbol{X})) - \phi(0_i, \phi(\boldsymbol{X}))\big).$$

This last expression was used in Barlow and Proschan (1975), p. 27, to extend this measure to the case of dependent components (the three preceding expressions lead to the same measure). Zhang and Wilson (2017) studied properties of this extension in the case of discrete marginals.

Another equivalent expression for independent components is

$$I_B(i; \boldsymbol{p}) = \partial_i R(\boldsymbol{p}). \tag{13}$$

However, (12) and (13) are not longer equivalent when the components are dependent (see Example 1). In all these cases, we can assume that the component lifetimes are identically distributed (ID), that is, $p_1 = \cdots = p_n = p$. Then we just write $I_B(i; p) := I_B(i; p, \ldots, p)$. This allow us to compare the importance of the components without including the component reliability functions. Birnbaum constructed a structural measure by putting $p = 1/2$, that is, $I_B(i) := I_B(i; 1/2)$. The Birnbaum time-dependent importance measure is obtained by replacing $\boldsymbol{p}$ with $(\overline{F}_1(t), \ldots, \overline{F}_n(t))$ in (12), cf. Lambert (1975) and Natvig (1979).

The Barlow-Proschan t.i.l. importance measure for systems with independent components can be defined (cf. Boland and El-Neweihi, 1995) as

$$I_{BP}(i) = \Pr(T = T_i). \tag{14}$$

It is the probability that the system lifetime coincides with the lifetime of component i. If the components are independent with absolutely continuous distributions, then

$$I_{BP}(i) = \int_0^\infty I_B\big(i; \overline{F}_1(t), \ldots, \overline{F}_n(t)\big) dF_i(t). \tag{15}$$

Again, to compare the components we can assume that they are ID, that is, $\overline{F}_1 = \ldots \overline{F}_n = \overline{F}$. In this case, I_{BP} does not depend on $\overline{F}$ and we get (see Marichal and Mathonet, 2013)

$$I_{BP}(i) = \int_0^1 I_B(i; p) dp. \tag{16}$$

Iyer (1992) used expression (14) to extend Barlow-Proschan measure to the case of dependent components. In this case $I_{BP}(i)$ depends on the structure of the system, the dependence between the components (copula) and on the component reliability functions.

III. Main results

We propose the following extension of Birnbaum importance measure for systems with dependent components based on formula (13). In this section and in the following ones we use the notation and representations introduced in the preceding section.

Definition 1. *The Birnbaum reliability importance measure of the ith component in a coherent system with dual distortion function R is defined by*

$$I_B(i; \boldsymbol{p}) = \partial_i R(\boldsymbol{p}) \tag{17}$$

(where $\partial_i R$ represents the partial derivative of R with respect to its ith variable) whenever this partial derivative exists for $\boldsymbol{p} = (p_1, \ldots, p_n) \in [0,1]^n$.

For $\boldsymbol{p} = (p, \ldots, p)$ we just write $I_B(i; p)$. Analogously the time-dependent lifetime importance measure is defined by $I_B(i; t) := I_B(i; \overline{F}_1(t), \ldots, \overline{F}_n(t))$ for $t \geq 0$. The preceding definitions can also be applied to semicoherent or mixed systems (i.e., mixtures of coherent systems). If the ith component is irrelevant, then $I_B(i; \boldsymbol{p}) = 0$ (since R is constant in p_i). By (9), $I_B(i; \boldsymbol{p})$ measures how the system reliability increases when p_i increases and the other component reliabilities are fixed. Since $R(\boldsymbol{p})$ is increasing in each variable, measure $I_B(i; \boldsymbol{p})$ is nonnegative. Note that the importance of a component may depend on the dependence structure (copula), which is an expectable property. So it could also be written as $I_B(i;\boldsymbol{p},\hat{C})$. Moreover, Example 1 proves that this measure is different to that studied in Zhang and Wilson (2017) denoted here (for dependent components) as

$$I_B^*(i;\boldsymbol{p}) = E\big(\phi(1_i,\phi(\boldsymbol{X}))\big) - E\big(\phi(0_i,\phi(\boldsymbol{X}))\big).$$

It can also be written as $I_B^*(i;\boldsymbol{p}) = R(1_i,\boldsymbol{p}) - R(0_i,\boldsymbol{p})$.

From now on we assume that the random vector with the component lifetimes $(T_1,\dots,T_n)$ has an absolutely continuous joint distribution. In the first theorem we show that measure $I_B\big(i;\boldsymbol{p}\big)$ defined in (17) is naturally related to the measure $I_{BP}(i)$ proposed by Iyer.

Theorem 2. *For any $i \in \{1,\dots,n\}$ we have*

$$I_B\big(i;\overline{F}_1(t),\dots,\overline{F}_n(t)\big) = \Pr(T = T_i | T_i = t).$$

The technical proof of Theorem 2 is placed in the Appendix. As we see, the time-dependent importance measure that we propose has a natural interpretation: it is the probability that the failure of the ith component at moment t causes the failure of the entire system. As a consequence, we also infer that $I_B\big(i;\boldsymbol{p}\big) \in [0,1]$. Also, the following result is immediate from Theorem 2 and equality

$$\Pr(T = T_i) = \int_0^\infty \Pr(T = T_i | T_i = t) dF_i(t).$$

Theorem 3. *For any $i \in \{1,\dots,n\}$ we have*

$$I_{BP}(i) = \int_0^\infty I_B\big(i;\overline{F}_1(t),\dots,\overline{F}_n(t)\big) dF_i(t). \tag{18}$$

Theorem 3 is one of the main results of the paper since it proves that the measure $I_{BP}(i)$ proposed by Iyer can be obtained from measure $I_B\big(i;\boldsymbol{p}\big)$ defined in (17) (as in (15) for the case of independent components). Also, expression (18) could be used to compute the importance measure $I_{BP}(i)$. The measure $I_B^*(i;\boldsymbol{p})$ does not satisfy this property.

Expressions (16) and (18) can be used to define the Barlow-Proschan structure importance measure as

$$I_{BP}(i;\phi,\hat{C}) = \int_0^1 I_B(i;p) dp \tag{19}$$

which depends on the system structure ϕ and on the survival copula $\hat{C}$, but not on the marginal distributions. The following corollary proves that, for identical marginals, measures $I_{BP}(i)$ and $I_{BP}(i;\phi,\hat{C})$ coincide (as in the case of independent components).

Corollary 1. *If $\overline{F}_1 = \dots = \overline{F}_n = \overline{F}$, then $I_{BP}(i)$ does not depend on $\overline{F}$ and $I_{BP}(i) = I_{BP}(i;\phi,\hat{C})$.*

Proof. Substitution $p = \overline{F}(t)$ in (18) gives us

$$\begin{aligned} I_{BP}(i) &= \int_0^\infty I_B\big(i;\overline{F}(t)\big) dF(t) \\ &= \int_0^1 I_B(i;p) dp = I_{BP}(i;\phi,\hat{C}). \end{aligned}$$

As an immediate consequence we obtain the following property.

Corollary 2. *For any copula $\hat{C}$, $\sum_{i=1}^n I_{BP}(i;\phi,\hat{C}) = 1$ holds.*

Remark 1. Theorem 3 can be also stated in the following form:

$$\Pr(T = T_i) = \int_0^\infty \partial_i R\big(\overline{F}_1(t),\dots,\overline{F}_n(t)\big) dF_i(t).$$

For given ϕ, $\hat{C}$ and $\overline{F}_1(t),\dots,\overline{F}_n(t)$, the function $R\big(\overline{F}_1(t),\dots,\overline{F}_n(t)\big)$ is explicit. Hence the above integral can be calculated. Thus, we receive a formula to compute $\Pr(T = T_i)$. By Theorem 1, this formula can be simplified when $\overline{F}_1 = \dots = \overline{F}_n = \overline{F}$, obtaining

$$\Pr(T = T_i) = \int_0^1 \partial_i R\big(p,\dots,p\big) dp.$$

Then, in this case, $\Pr(T = T_i)$ does not depend on $\overline{F}$.

Definitions and relations between discussed measures are gathered in Table 1. Note that all of them are valid in both independent and dependent cases.

TABLE I
DEFINITIONS AND RELATIONS BETWEEN DISCUSSED MEASURES.

Birnbaum reliability i.m.	$I_B(i;\boldsymbol{p}) = \partial_i R(\boldsymbol{p})$
Birnbaum t.d.l. i.m.	$I_B\big(i;t\big) = I_B\big(i;\overline{F}_1(t),\dots,\overline{F}_n(t)\big)$
Barlow-Proschan t.i.l. i.m.	$I_{BP}(i) = \Pr(T = T_i)$
Barlow-Proschan structure i.m.	$I_{BP}(i;\phi,\hat{C}) = \int_0^1 I_B(i;p)dp$
Relation	$I_{BP}(i) = \int_0^\infty I_B\big(i;t\big)dF_i(t)$
Relation in the ID case	$I_{BP}(i) = I_{BP}(i;\phi,\hat{C})$

The following property shows that I_B can also be used to compute the probability density function of the system lifetime.

Proposition 1. *Let f_T and $f_1,\dots,f_n$ be probability density functions of T and $T_1,\dots,T_n$, respectively. Then for all $t \geq 0$ we have*

$$f_T(t) = \sum_{i=1}^n f_i(t) I_B\big(i;\overline{F}_1(t),\dots,\overline{F}_n(t)\big). \tag{20}$$

Proof. By (9), we have

$$\begin{aligned} f_T(t) &= -\frac{d}{dt} R\big(\overline{F}_1(t),\dots,\overline{F}_n(t)\big) \\ &= \sum_{i=1}^n f_i(t) \partial_i R\big(\overline{F}_1(t),\dots,\overline{F}_n(t)\big) \end{aligned}$$

and (20) holds since

$$I_B\big(i;\overline{F}_1(t),\dots,\overline{F}_n(t)\big) = \partial_i R\big(\overline{F}_1(t),\dots,\overline{F}_n(t)\big).$$

The above formula becomes even more interesting if the components are ID.

Corollary 3. *If $f_1 = \dots = f_n = f$, then for all $t \geq 0$ we have*

$$f_T(t) = f(t) \sum_{i=1}^n I_B\big(i;\overline{F}(t)\big).$$

This allows us to study the likelihood ratio (lr) order between the ID components and the system. This order is defined as follows: Let X and Y be two random variables with probability density functions f and g, respectively, then $X \leq_{lr} Y$ if $g(t)/f(t)$ is increasing in the union of their supports. This ordering is the strongest order among the usual reliability orders since it implies the hazard rate, mean residual life, stochastic and reversed hazard rate orders (see Shaked and Shantikumar, 2007). Now we can state the following result.

Theorem 4. *If $f_1 = \ldots = f_n = f$, then $T \leq_{lr} T_1$ ($T_1 \leq_{lr} T$) holds for any f if and only if $\sum_{i=1}^n I_B(i;p)$ is increasing (decreasing) in p in $(0,1)$.*

A similar technique can be used to compare the same system with two different copulas or two different systems. The result can be stated as follows. Note that I_B contains all the information needed to perform such comparisons.

Theorem 5. *Let T^A and T^B be lifetimes of two systems with n components having a common probability density function f. Let $I_B^A(i;p)$ and $I_B^B(i;p)$ be the respective information measures. Then $T^A \leq_{lr} T^B$ holds for any f if and only if*

$$\frac{\sum_{i=1}^n I_B^B(i;p)}{\sum_{i=1}^n I_B^A(i;p)}$$

is decreasing in p in $(0,1)$.

Finally, note that the joint reliability importance measure of components i and j introduced in Hong and Lie (1993) and Armstrong (1995) for independent components can be extended to the case of dependent components as

$$I_B(i,j;p_1,\ldots,p_n) = \partial_i\partial_j R(p_1,\ldots,p_n)$$

whenever these partial derivatives exist.

IV. Examples

The first simple example shows that importance measures I_B and I_B^* are different when the components are dependent and that I_B can be used to compute I_{BP}.

Example 1. Let us consider a series system with two dependent components having an absolutely continuous joint distribution with a survival copula $\hat{C}$. Then

$$\overline{F}_T(t) = \Pr(T > t) = \Pr(T_1 > t, T_2 > t) = \hat{C}(\overline{F}_1(t), \overline{F}_2(t)),$$

that is, $R = \hat{C}$. Hence $I_B(1;p_1,p_2) = \partial_1\hat{C}(p_1,p_2)$ and $I_B^*(1;p_1,p_2) = \hat{C}(1,p_2) - \hat{C}(0,p_2) = p_2$. If the components are independent, then $\hat{C}(p_1,p_2) = p_1p_2$ and $I_B(1;p_1,p_2) = p_2 = I_B^*(1;p_1,p_2)$. However, if they are dependent, then these measures can be different. For example, for the following Clayton-Oakes copula

$$\hat{C}(p_1,p_2) = \frac{p_1p_2}{p_1 + p_2 - p_1p_2}$$

we have

$$I_B(1;p_1,p_2) = \frac{p_2^2}{(p_1+p_2-p_1p_2)^2} \neq p_2 = I_B^*(1;p_1,p_2).$$

If the components are ID, that is, $p_1 = p_2 = p$, then

$$I_B(i;p) = \frac{1}{(2-p)^2} \neq p = I_B^*(1;p).$$

The Barlow-Prochan information measure of the first component is

$$\begin{aligned} I_{BP}(1) &= \Pr(T = T_1) = \Pr(T_1 < T_2) \\ &= \int_0^\infty \int_x^\infty \boldsymbol{f}(x,y)\,dy\,dx, \end{aligned}$$

where $\boldsymbol{f}$ is the joint probability density function of (T_1,T_2) which, by (5), can be written as

$$\boldsymbol{f}(x,y) = f_1(x)f_2(y)\partial_2\partial_1\hat{C}(\overline{F}_1(x), \overline{F}_2(y)).$$

Thus

$$\begin{aligned} I_{BP}(1) &= \int_0^\infty \int_x^\infty f_1(x)f_2(y)\partial_2\partial_1\hat{C}(\overline{F}_1(x),\overline{F}_2(y))\,dy\,dx \\ &= \int_0^\infty f_1(x)\partial_1\hat{C}(\overline{F}_1(x),\overline{F}_2(x))\,dx \\ &= \int_0^\infty I_B(1;\overline{F}_1(x),\overline{F}_2(x))\,dF_1(x). \end{aligned}$$

Note that

$$\begin{aligned} I_{BP}(1) &\neq \int_0^\infty I_B^*(1;\overline{F}_1(x),\overline{F}_2(x))\,dF_1(x) \\ &= \int_0^\infty \overline{F}_2(x)\,dF_1(x) \end{aligned}$$

since the last expression does not depend on $\hat{C}$. If the components are ID, then

$$I_{BP}(1) = \int_0^\infty I_B(1;\overline{F}_1(x),\overline{F}_1(x))\,dF_1(x) = \int_0^1 I_B(1;p)\,dp.$$

In particular, for the C-O copula used above we obtain

$$I_{BP}(1) = \int_0^1 I_B(1;p)\,dp = \int_0^1 \frac{1}{(2-p)^2}\,dp = \frac{1}{2}$$

as expected since this copula is exchangeable.

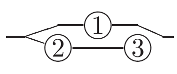
If we want to use Theorem 5 to compare the series system with the C-O copula T^B with that with the product copula (independent components) T^A, we should study the function

$$\frac{I_B^B(1;p) + I_B^B(2;p)}{I_B^A(1;p) + I_B^A(1;p)} = \frac{1}{p(2-p)^2}.$$

As it is not monotonic in $(0,1)$, T^A and T^B are not lr-ordered for all f. So, in this case, the dependence does not improve the series system (in the lr order).

In the following example we study a system with a different structure.

Example 2. Consider a 3-component coherent system with the structure function given by $\phi(x_1,x_2,x_3) = x_1 + x_2x_3 - x_1x_2x_3$. Its structure can be presented as



Its minimal path sets are $\{1\}$ and $\{2,3\}$. Hence, by (8),

$$R(p_1,p_2,p_3) = p_1 + \hat{C}(1,p_2,p_3) - \hat{C}(p_1,p_2,p_3).$$

Therefore

$$\begin{aligned}
I_B(1; p_1, p_2, p_3) &= \partial_1 R(p_1, p_2, p_3) = 1 - \partial_1 \hat{C}(p_1, p_2, p_3), \\
I_B(2; p_1, p_2, p_3) &= \partial_2 R(p_1, p_2, p_3) \\
&= \partial_2 \hat{C}(1, p_2, p_3) - \partial_2 \hat{C}(p_1, p_2, p_3), \\
I_B(3; p_1, p_2, p_3) &= \partial_3 R(p_1, p_2, p_3) \\
&= \partial_3 \hat{C}(1, p_2, p_3) - \partial_3 \hat{C}(p_1, p_2, p_3).
\end{aligned}$$

For example, consider the Farlie-Gumbel-Morgenstern copula given by

$$\hat{C}(p_1, p_2, p_3) = p_1 p_2 p_3 [1 + \alpha(1 - p_1)(1 - p_2)],$$

where $\alpha \in [-1, 1]$. For that copula, T_3 is independent from T_1, T_2 and the strength of dependence (either positive or negative) between T_1 and T_2 is controlled by the parameter α. In particular, for $\alpha = 0$ all the components are independent. For this copula we have

$$\begin{aligned}
I_B(1; p_1, p_2, p_3) &= 1 - p_2 p_3 [1 + \alpha(1 - 2p_1)(1 - p_2)], \\
I_B(2; p_1, p_2, p_3) &= p_3 - p_1 p_3 [1 + \alpha(1 - p_1)(1 - 2p_2)], \\
I_B(3; p_1, p_2, p_3) &= p_2 - p_1 p_2 [1 + \alpha(1 - p_1)(1 - p_2)].
\end{aligned}$$

In particular, if $p_1 = p_2 = p_3 = p$, then

$$\begin{aligned}
I_B(1; p) &= 1 - p^2 [1 + \alpha(1 - 2p)(1 - p)], \\
I_B(2; p) &= p - p^2 [1 + \alpha(1 - 2p)(1 - p)], \\
I_B(3; p) &= p - p^2 [1 + \alpha(1 - p)^2].
\end{aligned}$$

One can check that $I_B(1; p, \hat{C}) \geq I_B(2; p, \hat{C})$ and $I_B(1; p, \hat{C}) \geq I_B(3; p, \hat{C})$ for all $\alpha \in [-1, 1]$, $p \in [0, 1]$ and $I_B(2; p, \hat{C}) \geq I_B(3; p, \hat{C})$ for all $p \in [0, 1]$ if and only if $\alpha \geq 0$. The same order is preserved by Barlow-Proschan structure importance measure defined in (19):

$$\begin{aligned}
I_{BP}(1; \phi, \hat{C}) &= \int_0^1 I_B(1; p) dp = (2/3) + (1/60)\alpha, \\
I_{BP}(2; \phi, \hat{C}) &= \int_0^1 I_B(2; p) dp = (1/6) + (1/60)\alpha, \\
I_{BP}(3; \phi, \hat{C}) &= \int_0^1 I_B(3; p) dp = (1/6) - (1/30)\alpha.
\end{aligned}$$

We have $I_{BP}(1; \phi, \hat{C}) + I_{BP}(2; \phi, \hat{C}) + I_{BP}(3; \phi, \hat{C}) = 1$, as stated.

If the components are ID with a common probability density function f, then the function

$$\sum_{i=1}^{3} I_B(i; p) = 1 + 2p - p^2 [3 + \alpha(1 - p)(3 - 5p)]$$

is not monotonic in p for any $\alpha \in [-1, 1]$. So, by Theorem 4, we conclude that T and T_1 are not lr-ordered under this copula for all f. In a similar way we can compare the systems obtained for different values of α. Thus as the function

$$\frac{\sum_{i=1}^{n} I_B(i; p, \hat{C}_\beta)}{\sum_{i=1}^{n} I_B(i; p, \hat{C}_\alpha)} = \frac{1 + 2p - p^2 [3 + \beta(1 - p)(3 - 5p)]}{1 + 2p - p^2 [3 + \alpha(1 - p)(3 - 5p)]}$$

is not monotonic when $\alpha \neq \beta$, the system does not improve (in the lr order) when the dependence increases (decreases).

We have seen that for ID components, the first one has the greatest information measure. The situation may change when we place the 'weakest' component in the 'strongest' place. For example, if $\overline{F}_1(t) = e^{-2t}$, $\overline{F}_2(t) = \overline{F}_3(t) = e^{-t}$ and $I(i; t) := I_B\big(i; \overline{F}_1(t), \overline{F}_2(t), \overline{F}_3(t)\big)$ for $i = 1, 2, 3$, then

$$\begin{aligned}
I(1; t) &= 1 - e^{-2t} [1 + \alpha(1 - 2e^{-2t})(1 - e^{-t})], \\
I(2; t) &= e^{-t} - e^{-3t} [1 + \alpha(1 - e^{-2t})(1 - 2e^{-t})], \\
I(3; t) &= e^{-t} - e^{-3t} [1 + \alpha(1 - e^{-2t})(1 - e^{-t})].
\end{aligned}$$

In the independent case (for $\alpha = 0$), we have

$$I(1; t) = 1 - e^{-2t} > e^{-t}(1 - e^{-2t}) = I(i; t)$$

for $i = 2, 3$ and all $t > 0$. However, for $\alpha = 1$ we get $I(2; t) > I(1; t)$ for small t's and for $\alpha = -1$ we obtain $I(3; t) > I(1; t)$ for small t's. The plots can be seen in Figure 1.

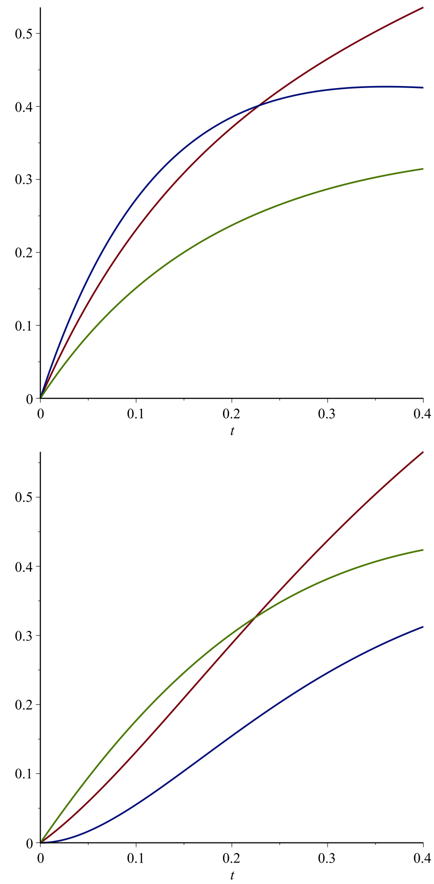
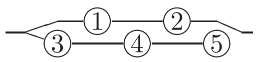


Fig. 1. Importance measures for component 1 (red), 2 (blue) and 3 (green) for the system considered in Example 2 with a FGM survival copula for $\alpha = 1$ (top) and $\alpha = -1$ (bottom).

The dependence provided by the Farlie-Gumbel-Morgenstern copula used in the preceding example is too weak to significantly change components importance measures in relation to the independent case when the components are identically distributed. In the following example we use copulas bringing much stronger dependence.

Example 3. Consider a 5-component coherent system with the structure function given by $\phi(x_1,x_2,x_3,x_4,x_5) = x_1x_2 + x_3x_4x_5 - x_1x_2x_3x_4x_5$. Its structure can be drawn as



Moreover, assume that T_1, T_2 are independent from T_3, T_4, T_5. In other words, we assume that the survival copula $\hat{C}$ can be expressed as

$$\hat{C}(p_1,p_2,p_3,p_4,p_5) = K_A(p_1,p_2)K_B(p_3,p_4,p_5)$$

for some copulas K_A, K_B. Since $\{1,2\}$ and $\{3,4,5\}$ are the minimal path sets, we have

$$R(\boldsymbol{p}) = K_A(p_1,p_2)+K_B(p_3,p_4,p_5)-K_A(p_1,p_2)K_B(p_3,p_4,p_5)$$

and

$$I_B(i;\boldsymbol{p}) = \begin{cases} \partial_i K_A(p_1,p_2)[1-K_B(p_3,p_4,p_5)], & i=1,2 \\ \partial_i K_B(p_3,p_4,p_5)[1-K_A(p_1,p_2)], & i=3,4,5. \end{cases} \tag{21}$$

In the independent case, i.e., for $K_A(p_1,p_2) = p_1p_2$ and $K_B(p_3,p_4,p_5) = p_3p_4p_5$, we have

$$I_B(i;p) = p - p^4 > p^2 - p^4 = I_B(j;p)$$

for $i = 1,2$ and $j = 3,4,5$ and all $p \in (0,1)$. This result is intuitive: components on the shorter path set are 'more important' than these on the longer path set. However, it is no longer true when we choose K_A and K_B in such a way that T_1 and T_2 are strongly negatively dependent and T_3, T_4, T_5 are strongly positively dependent.

The 'border' copulas (Fréchet-Hoeffding bounds), which provide the maximum possible dependence, are the counter-monotonic and comonotonic ones, given by $K_c(p_1,p_2) = 0 \vee (p_1 + p_2 - 1)$ and $K_m(p_3,p_4,p_5) = p_3 \wedge p_4 \wedge p_5$, respectively, where '$\vee$' and '$\wedge$' stand for the maximum and the minimum, respectively. Unfortunately, we cannot use them directly because $\partial_1 K_c(1/2,1/2)$ and $\partial_3 K_m(p,p,p)$ do not exist. Therefore we approximate them by the following Clayton-Oakes and Gumbel copulas

$$\begin{aligned} K_A(p_1,p_2;\varepsilon) &= [0 \vee (p_1^{1-\varepsilon} + p_2^{1-\varepsilon} - 1)]^{1/(1-\varepsilon)}, \\ K_B(p_3,p_4,p_5;\varepsilon) &= \exp\big[-\big((-\log p_3)^{1/\varepsilon} + (-\log p_4)^{1/\varepsilon} \\ &\qquad + (-\log p_5)^{1/\varepsilon}\big)^{\varepsilon}\big] \end{aligned}$$

for a small parameter $\varepsilon > 0$. Then one can see that

$$\lim_{\varepsilon\to 0} K_A(p_1,p_2;\varepsilon) = K_c(p_1,p_2)$$

and

$$\lim_{\varepsilon\to 0} K_B(p_3,p_4,p_5;\varepsilon) = K_m(p_3,p_4,p_5).$$

Furthermore, we have

$$K_A(p,p;\varepsilon) = \begin{cases} 0, & p \le (1/2)^{\frac{1}{1-\varepsilon}} \\ (2p^{1-\varepsilon}-1)^{\frac{1}{1-\varepsilon}} \simeq 2p-1, & p \ge (1/2)^{\frac{1}{1-\varepsilon}} \end{cases}$$

and

$$\partial_i K_A(p,p;\varepsilon) = \begin{cases} 0, & p < (1/2)^{\frac{1}{1-\varepsilon}} \\ (2p^{1-\varepsilon}-1)^{\frac{\varepsilon}{1-\varepsilon}} p^{-\varepsilon} \simeq 1, & p > (1/2)^{\frac{1}{1-\varepsilon}} \end{cases}$$

for $i = 1,2$ when $\varepsilon \to 0$. Also $K_B(p,p,p;\varepsilon) = p^{3^\varepsilon} \simeq p$ and $\partial_i K_B(p,p,p;\varepsilon) = 3^{\varepsilon-1}p^{3^\varepsilon-1} \simeq 1/3$ for $i = 3,4,5$ when $\varepsilon \to 0$. Hence

$$I_B(i;p) \simeq \begin{cases} 0, & p < (1/2)^{1/(1-\varepsilon)} \\ 1-p, & p > (1/2)^{1/(1-\varepsilon)} \end{cases} \tag{22}$$

for $i = 1,2$ and

$$I_B(i;p) \simeq \begin{cases} 1/3, & p \le (1/2)^{1/(1-\varepsilon)} \\ (2/3)(1-p), & p \ge (1/2)^{1/(1-\varepsilon)} \end{cases} \tag{23}$$

for $i = 3,4,5$. In particular,

$$I_B(i;p) \simeq 0 < 1/3 \simeq I_B(j;p)$$

for $i = 1,2$, $j = 3,4,5$ and $p < (1/2)^{1/(1-\varepsilon)} \simeq 1/2$. Moreover,

$$I_{BP}(i;\phi,\hat{C}) = \int_0^1 I_B(i;p)dp \simeq \begin{cases} 1/8, & i=1,2 \\ 1/4, & i=3,4,5. \end{cases}$$

Thus, if $T_1,\ldots,T_5$ are identically distributed then

$$\Pr(T = T_i) \simeq \frac{1}{8} < \frac{1}{4} \simeq \Pr(T = T_j)$$

for $i = 1,2$ and $j = 3,4,5$ and, i.e., components $1,2$ are 'less important' in the BP sense than components $3,4,5$. However,

$$I_B(i) := I_B(i;1/2) \simeq \begin{cases} 1/2, & i=1,2 \\ 1/3, & i=3,4,5. \end{cases}$$

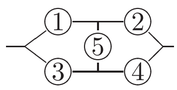
So the Birnbaum importance order between them is the same as in the independent case.

Remark 2. In order to construct an example for which the orders of the Birnbaum structural component measures in the independent and dependent case are not the same, it suffices to slightly modify Example 3. In fact, by (21), (22) and (23), we have

$$\begin{aligned} I_B(1) &= \partial_1 K_A(1/2,1/2;\varepsilon)[1-K_B(1/2,1/2,1/2;\varepsilon)] \\ &\simeq 1\cdot[1-1/2] \\ &> (1/3)\cdot[1-0] \\ &\simeq \partial_3 K_B(1/2,1/2,1/2;\varepsilon)[1-K_A(1/2,1/2;\varepsilon)] \\ &= I_B(3). \end{aligned}$$

The value $\partial_3 K_B(1/2,1/2,1/2;\varepsilon) \simeq 1/3$ is too small to reverse the inequality. But if we chose K_B^* such that $K_B^*(1/2,1/2,1/2;\varepsilon) \simeq 1/2$ and $\partial_3 K_B^*(1/2,1/2,\ 1/2;\varepsilon) > 1/2$ instead, we would obtain the desired relation $I_B(1) < I_B(3)$.

Example 4. A very relevant system structure is the bridge system given by



Its structure function is

$$\begin{aligned} \phi(\boldsymbol{x}) = \max[&\min(x_1,x_2), \min(x_3,x_4), \\ &\min(x_1,x_4,x_5), \min(x_2,x_3,x_5)] \end{aligned}$$

for $\boldsymbol{x} = (x_1,x_2,x_3,x_4,x_5) \in \{0,1\}^n$. Let us assume that only the components $\{1,4\}$ and $\{2,3\}$ are dependent with the same dependence structure, that is, the survival copula can be written as

$$\hat{C}(\boldsymbol{p}) = p_5 K(p_1, p_4) K(p_2, p_3)$$

for $\boldsymbol{p} = (p_1, p_2, p_3, p_4, p_5) \in [0,1]^n$ and a bidimensional copula K. Then by (8),

$$\begin{aligned} R(\boldsymbol{p}) &= p_1 p_2 + p_3 p_4 + p_5 K(p_1, p_4) + p_5 K(p_2, p_3) \\ &- K(p_1, p_4) K(p_2, p_3) - p_2 p_5 K(p_1, p_4) - p_1 p_5 K(p_2, p_3) \\ &- p_3 p_5 K(p_1, p_4) - p_4 p_5 K(p_2, p_3) + 2 p_5 K(p_1, p_4) K(p_2, p_3). \end{aligned}$$

Therefore, the importance index of the first component is

$$\begin{aligned} I_B(1; \boldsymbol{p}) &= p_2 - p_5 K(p_2, p_3) \\ &+ [p_5(1 - p_2 - p_3) + (2p_5 - 1) K(p_2, p_3)] \partial_1 K(p_1, p_4). \end{aligned}$$

The indices of components 2, 3 and 4 are similar by the symmetry. The importance index for component 5 is

$$\begin{aligned} I_B(5; \boldsymbol{p}) &= (1 - p_2 - p_3) K(p_1, p_4) + (1 - p_1 - p_4) K(p_2, p_3) \\ &+ 2 K(p_1, p_4) K(p_2, p_3). \end{aligned}$$

Indices 1 and 5 are not ordered in general. If we assume that the components are identically distributed and $p = p_i$ for $i = 1, 2, 3, 4, 5$, then

$$I_B(1; p) = p - pK(p, p) + (1 - 2p)[p - K(p, p)] \partial_1 K(p, p)$$

and

$$I_B(5; p) = 2(1 - 2p) K(p, p) + 2\big(K(p, p)\big)^2.$$

In the independent case, we have $K(p, p) = p^2$, $\partial_1 K(p, p) = p$ and

$$I_B(1; p) = p + p^2 - 4p^3 + 2p^4 \geq 2p^2 - 4p^3 + 2p^4 = I_B(5; p).$$

So components 1, 2, 3, 4 are more important than component 5. However, when components 1, 4 are strongly positively correlated and so they are components 2, 3, the order tends to flip. For example, take the Gumbel copula $K_B(p, p; \varepsilon)$ defined in Example 3. Then, for $\varepsilon \to 0$, we have $K_B(p, p; \varepsilon) = p^{2^\varepsilon} \simeq p$, $\partial_1 K_B(p, p; \varepsilon) = 2^{\varepsilon - 1} p^{2^\varepsilon - 1} \simeq 1/2$ and

$$I_B(1; p) \simeq p(1 - p) \leq 2p(1 - p) \simeq I_B(5; p).$$

Besides, one may check that

$$I_B(1) = I_B(1; 1/2) < I_B(5; 1/2) = I_B(5)$$

for $\varepsilon < 0.44$.

V. Conclusions

The copula approach is a convenient way to express the system reliability as a function of component (marginal) reliability functions and the dependence structure between them. This representation allows us to generalize the Birnbaum component importance measure to the case of dependent components. Our generalization is consistent with the results already existing in the literature. It shares properties with its original version defined for independent components. Furthermore, the extended Birnbaum component measure is related to the version of the Barlow-Proschan measure for dependent components in the same way as their respective counterparts are in the independent components case.

Ordering the system components with respect to their importance measures is a natural way to seek the 'fragile' places of the system structure. It turns out that the order of components significantly depends on the dependence between components. Hence, one can boost the system performance by a reasonable location of dependent components in a given system structure when the dependence (copula) is known (or can be estimated).

Also, using the Birnbaum component importance measure one can easily examine the likelihood ratio order between the entire system and single component lifetimes or lifetimes of two systems with the same components, but different structures or dependences. Since the likelihood ratio order implies many other well known stochastic orders, in this way, we are able to effortlessly check if all the popular orderings occur before the further analysis.

This paper is just a first step in this direction showing which importance measures should be used in the case of dependent components. There are several open problems. Thus some specific (relevant) dependence models in reliability should be examined in detail following the key ideas given here. Also, particular (more realistic) system structures should be studied including modular systems. Furthermore, the properties of the joint reliability importance measure proposed here should be studied as well as the connections of component information measures with replacement policies.

APPENDIX: Proof of Theorem 2.

To prove Theorem 2 we need the following technical lemmas and corollaries.

Lemma 1. *For any system structure ϕ and any $i \in \{1, \dots, n\}$ we have*

$$\begin{aligned} \Pr\big(\phi(\boldsymbol{X}) = 1\big) &= \sum_{\boldsymbol{y}\colon \phi(0_i, \boldsymbol{y}) = 1} \Pr(X_j = y_j \text{ for all } j \neq i) \\ &+ \sum_{\boldsymbol{y}\colon \phi(1_i, \boldsymbol{y}) - \phi(0_i, \boldsymbol{y}) = 1} \Pr(X_i = 1, X_j = y_j \text{ for all } j \neq i). \end{aligned}$$

Proof. Since

$$\begin{aligned} A &= \{\boldsymbol{x}\colon \phi(\boldsymbol{x}) = 1\} \\ &= \{(1_i, \boldsymbol{y})\colon \phi(1_i, \boldsymbol{y}) = 1\} \cup \{(0_i, \boldsymbol{y})\colon \phi(0_i, \boldsymbol{y}) = 1\} \\ &= \{(1_i, \boldsymbol{y})\colon \phi(1_i, \boldsymbol{y}) = 1, \phi(0_i, \boldsymbol{y}) = 0\} \\ &\quad \cup \{(1_i, \boldsymbol{y})\colon \phi(1_i, \boldsymbol{y}) = 1, \phi(0_i, \boldsymbol{y}) = 1\} \\ &\quad \cup \{(0_i, \boldsymbol{y})\colon \phi(0_i, \boldsymbol{y}) = 1\} \\ &= \{(1_i, \boldsymbol{y})\colon \phi(1_i, \boldsymbol{y}) - \phi(0_i, \boldsymbol{y}) = 1\} \\ &\quad \cup \{(1_i, \boldsymbol{y})\colon \phi(0_i, \boldsymbol{y}) = 1\} \\ &\quad \cup \{(0_i, \boldsymbol{y})\colon \phi(0_i, \boldsymbol{y}) = 1\}, \end{aligned}$$

we have

$$\begin{aligned}\Pr\big(\phi(\boldsymbol{X})=1\big) &= \sum_{\boldsymbol{x}:\ \phi(\boldsymbol{x})=1} \Pr(X_i=x_i, i=1,\dots,n)\\
&= \sum_{\boldsymbol{y}:\ \phi(1_i,\boldsymbol{y})-\phi(0_i,\boldsymbol{y})=1} \Pr(X_i=1, X_j=y_j \text{ for all } j\neq i)\\
&\quad + \sum_{\boldsymbol{y}:\ \phi(0_i,\boldsymbol{y})=1} \Pr(X_i=1, X_j=y_j \text{ for all } j\neq i)\\
&\quad + \sum_{\boldsymbol{y}:\ \phi(0_i,\boldsymbol{y})=1} \Pr(X_i=0, X_j=y_j \text{ for all } j\neq i)\\
&= \sum_{\boldsymbol{y}:\ \phi(1_i,\boldsymbol{y})-\phi(0_i,\boldsymbol{y})=1} \Pr(X_i=1, X_j=y_j \text{ for all } j\neq i)\\
&\quad + \sum_{\boldsymbol{y}:\ \phi(0_i,\boldsymbol{y})=1} \Pr(X_j=y_j \text{ for all } j\neq i).\end{aligned}$$

This concludes the proof.

If $\Pr(X_i=1)=p_i$ for $i=1,\dots,n$ then, for any $\boldsymbol{y}$, the probability $\Pr(X_i=1, X_j=y_j \text{ for all } j\neq i)$ is a function of $\hat{C}$. In particular, for $\boldsymbol{y}=(1,\dots,1)$, by (5), we have

$$\begin{aligned}\Pr(X_i=1, X_j=y_j \text{ for all } j\neq i) &= \Pr(X_1=1,\dots,X_n=1)\\ &= \hat{C}(\boldsymbol{p}).\end{aligned}$$

For $i<n$ and $\boldsymbol{y}=(1,\dots,1,0)$, by (5) and (7) we get

$$\begin{aligned}\Pr(X_i=1,&X_j=y_j \text{ for all } j\neq i)\\
&= \Pr(X_1=1,\dots,X_{n-1}=1, X_n=0)\\
&= \Pr(X_1=1,\dots,X_{n-1}=1)\\
&\quad - \Pr(X_1=1,\dots,X_n=1)\\
&= \hat{C}\big((1_n,\boldsymbol{p})\big) - \hat{C}(\boldsymbol{p}).\end{aligned}$$

The analogous formulas for any i and $\boldsymbol{y}$ can be obtained explicitly in a similar way. They are sums of $\hat{C}$ at points with the coordinates p_i and 1 in some configurations. More precisely, they are the $\hat{C}$-volume of a hyperrectangle defined by i and $\boldsymbol{y}$ in the following way: the ith interval is $[0,p_i]$ and for $j\neq i$, if $y_j=1$, then choose interval $[0,p_j]$, else choose interval $[p_j,1]$. However, we do not need the exact formulas. We only note that there exists a function $\hat{K}_i$ such that

$$\Pr(X_i=1, X_j=y_j \text{ for all } j\neq i) = \hat{K}_i(\boldsymbol{p},\boldsymbol{y})$$

for all $\boldsymbol{p}$ and $\boldsymbol{y}$. If $\partial_i\hat{C}(\boldsymbol{p})$ exists, then the partial derivative of $\hat{K}_i(\boldsymbol{p},\boldsymbol{y})$ with respect to p_i, denoted by $\partial_i\hat{K}_i(\boldsymbol{p},\boldsymbol{y})$, exists as well. We can consider $\Pr(X_j=y_j \text{ for all } j\neq i)$ in the same way and conclude that

$$\Pr(X_j=y_j \text{ for all } j\neq i) = \hat{K}_i\big((1_i,\boldsymbol{p}),\boldsymbol{y}\big), \tag{24}$$

i.e, it does not depend on p_i. Hence, Lemma 1 and (8) imply the following corollary.

Corollary 4. *For any* $i\in\{1,\dots,n\}$ *we have*

$$\partial_i R(\boldsymbol{p}) = \sum_{\boldsymbol{y}:\ \phi(1_i,\boldsymbol{y})-\phi(0_i,\boldsymbol{y})=1} \partial_i\hat{K}_i(\boldsymbol{p},\boldsymbol{y}),$$

providing these partial derivatives exist.

Now assume that $V_1,\dots,V_n$ are random variables with uniform distributions in $(0,1)$ and with a copula $\hat{C}$, i.e.,

$$\Pr(V_1<p_1,\dots,V_n<p_n) = \hat{C}(\boldsymbol{p}) \tag{25}$$

for any $\boldsymbol{p}=(p_1,\dots,p_n)\in[0,1]^n$. Then one can see that for any i and $\boldsymbol{y}$ we have

$$\begin{aligned}\Pr(V_i<p_i,&V_j<p_j\ \forall j: y_j=1, V_j>p_j\ \forall j: y_j=0)\\ &= \hat{K}_i(\boldsymbol{p},\boldsymbol{y}).\end{aligned} \tag{26}$$

Also the following lemma holds.

Lemma 2. *For any* i *and* $\boldsymbol{y}$ *we have*

$$\begin{aligned}\Pr(V_j<p_j\ \forall j: y_j=1, V_j>p_j\ \forall j: y_j=0 | V_i=p_i)\\ = \partial_i\hat{K}_i(\boldsymbol{p},\boldsymbol{y}),\end{aligned}$$

providing this partial derivative exists.

Proof. To simplify the notation, we present the proof for $i=1$. It is analogous for every other i. Let $f_{2,\dots,n|1}(u_2,\dots,u_n|u_1)$, $f_{1,\dots,n}(u_1,\dots,u_n)$ and $f_1(u_1)$ stand for the conditional density of $(V_2,\dots,V_n)$ under the condition $V_1=u_1$, the joint density of $(V_1,\dots,V_n)$ and the density of V_1, respectively. By the formula of conditional density we have

$$\begin{aligned}f_{2,\dots,n|1}(u_2,\dots,u_n|u_1) &= \frac{f_{1,\dots,n}(u_1,\dots,u_n)}{f_1(u_1)}\\ &= f_{1,\dots,n}(u_1,\dots,u_n)\end{aligned}$$

since $f_1(u_1)=1$ for all $u_1\in(0,1)$. Let $A_{\boldsymbol{p},\boldsymbol{y}} = I_2\times\dots\times I_n$ where $I_j=[0,p_j)$ if $y_j=1$ and $I_j=(p_j,1]$ if $y_j=0$, $j=2,\dots,n$. Then, by (26), for $0<p_1<1$, we get

$$\begin{aligned}\Pr\big(&(V_2,\dots,V_n)\in A_{\boldsymbol{p},\boldsymbol{y}} \big| V_1=p_1\big)\\
&= \int\limits_{A_{\boldsymbol{p},\boldsymbol{y}}}\!\!\dots\int f_{1,\dots,n}(p_1,u_2,\dots,u_n)du_2\dots du_n\\
&= \partial_1 \int\limits_{A_{\boldsymbol{p},\boldsymbol{y}}}\!\!\dots\int\int_0^{p_1} f_{1,\dots,n}(u_1,u_2,\dots,u_n)du_1du_2\dots du_n\\
&= \partial_1\hat{K}_1(\boldsymbol{p},\boldsymbol{y}).\end{aligned}$$

Corollary 4 and Lemma 2 imply the following result.

Corollary 5. *For any* $i\in\{1,\dots,n\}$ *we have*

$$\begin{aligned}\partial_i R(\boldsymbol{p},K) = \sum_{\boldsymbol{y}\in S} \Pr(&V_j<p_j \textit{ for all } j: y_j=1,\\ &V_j>p_j \textit{ for all } j: y_j=0 | V_i=p_i),\end{aligned}$$

providing this partial derivative exists, where

$$S=\{\boldsymbol{y}\colon \phi(1_i,\boldsymbol{y})-\phi(0_i,\boldsymbol{y})=1\}.$$

Let us consider now the component lifetimes $(T_1,\dots,T_n)$ with reliability functions $\overline{F}_1,\dots,\overline{F}_n$. Then by putting $V_i=\overline{F}_i(T_i)$ and $p_i=\overline{F}_i(t)$ for $i=1,\dots,n$, we obtain the uniformly distributed random variables $V_1,\dots,V_n$ fulfilling (25). Moreover,

$$\begin{aligned}\Pr\Big(\max_{j:\ y_j=0} T_j < t < \min_{j:\ y_j=1} T_j \Big| T_i=t\Big)&\\
= \Pr(V_j<p_j \text{ for all } j: y_j=1,&\\
V_j>p_j \text{ for all } j: y_j=0 | V_i=p_i)&\end{aligned}$$

holds for any i and $\boldsymbol{y}$. Therefore we can rewrite Corollary 5 as follows.

Corollary 6. *For any $i \in \{1, \dots, n\}$ we have*

$$\partial_i R\big(\overline{F}_1(t), \dots, \overline{F}_n(t)\big) = \sum_{\boldsymbol{y}\in S} \Pr(A_{\boldsymbol{y}} | T_i = t)$$

where $A_{\boldsymbol{y}} = \{\max_{j\colon y_j=0} T_j < t < \min_{j\colon y_j=1} T_j\}$, providing this partial derivative exists.

We need one more lemma.

Lemma 3. *For any $i \in \{1, \dots, n\}$ we have*

$$\Pr(T = T_i | T_i = t) = \sum_{\boldsymbol{y}\in S} \Pr(A_{\boldsymbol{y}} | T_i = t).$$

Proof.

$$\begin{aligned} \Pr(T = T_i | T_i = t) &= \Pr(T = t | T_i = t) \\ &= \sum_{\boldsymbol{y}} \Pr(T = t, A_{\boldsymbol{y}} | T_i = t) \\ &= \sum_{\boldsymbol{y}\in S} \Pr(A_{\boldsymbol{y}} | T_i = t). \end{aligned}$$

This concludes the proof.

Proof of Theorem 2. It is obtained from Corollary 6 and Lemma 3.

We conclude with two remarks.

Remark 3. It is worth noting that Lemma 3, Theorem 2 and Corollary 3 imply that

$$\begin{aligned} I_{BP}(i; \overline{F}_1(t), \dots, \overline{F}_n(t)) &= \sum_{\boldsymbol{y}\in S} \int_0^\infty \Pr\left(A_{\boldsymbol{y}} \middle| T_i = t\right) dF_i(t) \\ &= \sum_{\boldsymbol{y}\in S} \int_0^\infty \Pr\left(A_{\boldsymbol{y}} \middle| T_i = t\right) dF_i(t) \\ &= \sum_{\boldsymbol{y}\in S} \Pr\left(A_{\boldsymbol{y}}\right). \end{aligned}$$

This formula was obtained in Theorem 3 of Marichal and Mathonet (2013).

Remark 4. For the alternative measure defined as $I_B^*(i; \boldsymbol{p}) = \mathrm{E}[\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X})]$, using the same method as in Lemma 1, by (24), we get

$$\begin{aligned} &\mathrm{E}[\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X})] \\ &\quad = \Pr\big(\phi(1_i, \boldsymbol{X}) = 1\big) - \Pr\big(\phi(0_i, \boldsymbol{X}) = 1\big) \\ &\quad = \sum_{\boldsymbol{y}\colon \phi(1_i, \boldsymbol{y}) - \phi(0_i, \boldsymbol{y}) = 1} \Pr(X_j = y_j \text{ for all } j \neq i) \\ &\quad = \sum_{\boldsymbol{y}\colon \phi(1_i, \boldsymbol{y}) - \phi(0_i, \boldsymbol{y}) = 1} \hat{K}_i\big((1_i, \boldsymbol{p}), \boldsymbol{y}\big). \end{aligned}$$

Meanwhile, by Corollary 4,

$$I_B(i; \boldsymbol{p}) = \sum_{\boldsymbol{y}\colon \phi(1_i, \boldsymbol{y}) - \phi(0_i, \boldsymbol{y}) = 1} \partial_i \hat{K}_i(\boldsymbol{p}, \boldsymbol{y}).$$

Hence, one can see that $I_B^*(i; \boldsymbol{p}) = I_B(i; \boldsymbol{p})$ for all $\boldsymbol{p} \in [0,1]^n$ only if

$$\hat{C}(p_1, \dots, p_n) = p_i K(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$$

for an $(n-1)$-dimensional copula K, i.e., when X_i is independent from all X_j, $j \neq i$.

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