CHARACTERIZATIONS OF DING INJECTIVE COMPLEXES

GANG YANG[1] AND SERGIO ESTRADA[2,*]

Abstract. Let R be a ring and X a chain complex of R-modules. It is proven that if each term X_i is Ding injective in R-Mod for all $i \in \mathbb{Z}$, and there exists an integer k such that each $\mathrm{Z}_i X$ is Ding injective in R-Mod for all $i \geq k$, then X is Ding injective in $\mathrm{Ch}(R)$. If R is a left coherent ring, then a chain complex X is Ding injective if and only if each term X_i is Ding injective in R-Mod for all $i \in \mathbb{Z}$.

1. Introduction

Ding injective modules were defined in [15], and they were initially introduced as Gorenstein FP-injective modules in [9]. Several authors have studied the properties of Ding injective modules and their dual notions, see [9, 15, 23]. It is well-known that a very natural way to study homological algebra is extending the homological theory on the modules category to one on the chain complexes category. Based on this idea, Ding homological chain complexes have been already described in [16, 18, 24, 25]. Gillespie proved in [16] that if R is a Ding- Chen ring, then Ding injective complexes are precisely the complexes X for which each term X_i is a Ding injective R-module for all $i \in \mathbb{Z}$. The aim of this paper is to generalize this result.

Throughout the paper R denotes a general ring with identity. An R-module will mean a left R-module, unless stated otherwise. The category of R-modules will be denoted R-Mod. As usual, we use $\mathrm{Ch}(R)$ to denote the category of chain complexes

$$X =: \quad \cdots \longrightarrow X_{i+1} \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \longrightarrow \cdots$$

of R-modules. Let X be a chain complex of R-modules. It is proven that if each term X_i is Ding injective in R-Mod for all $i \in \mathbb{Z}$, and there exists an integer k such that each $\mathrm{Z}_i X$ is Ding injective in R-Mod for all $i \geq k$, then X is Ding injective in $\mathrm{Ch}(R)$. If R is a left coherent ring, then a chain complex X is Ding injective if and only if each term X_i is Ding injective in R-Mod for all $i \in \mathbb{Z}$.

Key words and phrases. Chain complexes, FP-injective modules, FP-injective chain complexes, Ding injective modules, Ding injective chain complexes.

2010 Mathematics Subject Classification. 18G25, 18G35, 55U15, 55U35.

The first named author is partially supported by NSF of China (Grant Nos. 11561039; 11761045; 11861055), NSF of Gansu Province (Grant Nos. 18JR3RA113; 17JR5RA091), and the Foundation of A Hundred Youth Talents Training Program of Lanzhou Jiaotong University.

The second named author is supported by the grant MTM2016-77445-P and FEDER funds and by grant 19880/GERM/15 from the Fundación Séneca-Agencia de Ciencia y Tecnología de la Región de Murcia.

* Corresponding author.

2. Preliminaries

A *chain complex* (complex for short) X of R-modules is a sequence $\cdots \to X_{i+1} \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \to \cdots$ of R-modules and R-homomorphisms such that $d_i d_{i+1} = 0$ for all $i \in \mathbb{Z}$. A chain complex X is said to be *bounded above* if $X_i = 0$ holds for $i \gg 0$, *bounded below* if $X_i = 0$ holds for $i \ll 0$, and *bounded* if it is bounded above and below, i.e. $X_i = 0$ holds for $|i| \gg 0$. Let X be a chain complex and let m be an integer. The mth *cycle module* is defined as $\mathrm{Ker}(d_m)$ and is denoted $\mathrm{Z}_m X$. The mth *boundary module* is $\mathrm{Im}(d_{m+1})$ and is denoted $\mathrm{B}_m X$. The mth *homology module* of X is the module $\mathrm{H}_m(X) = \mathrm{Z}_m X/\mathrm{B}_m X$. The chain complex X is said to be *acyclic (exact)* if its homology module $\mathrm{H}_i(X) = 0$ for all $i \in \mathbb{Z}$. The *m-fold shift* of X is the chain complex $\Sigma^m X$ given by $(\Sigma^m X)_i = X_{i-m}$ and $d_i^{\Sigma^m X} = (-1)^m d_{i-m}^X$. Usually, $\Sigma^1 X$ is denoted simply by ΣX.

Let X and Y be two chain complexes. We will let $\mathrm{Hom}_R(X, Y)$ denote the hom-complex of abelian groups with mth term $\mathrm{Hom}_R(X, Y)_m = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(X_i, Y_{i+m})$ and differential

$$(d(g))_i = d_{i+m}^Y g_i - (-1)^m g_{i-1} d_i^X \text{ for } g = (g_i)_{i\in\mathbb{Z}} \in \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(X_i, Y_{i+m}).$$

By a *morphism* (*chain map*) $f : X \to Y$ we mean a sequence $f_i : X_i \to Y_i$ such that $d_i^Y f_i = f_{i-1} d_i^X$ for all $i \in \mathbb{Z}$. A morphism $f : X \to Y$ is said to be *null-homotopic* if there exists a sequence $s_i : X_i \to Y_{i+1}$ such that $f_i = dY_{i+1} s_i + s_{i-1} d_i^X$ for all $i \in \mathbb{Z}$. The *mapping cone* $\mathrm{Cone}(f)$ of a morphism $f : X \to Y$ is defined as $\mathrm{Cone}(f)_i = Y_i \oplus X_{i-1}$ with $d_i^{\mathrm{Cone}(f)} = \begin{pmatrix} d_i^Y & f_{i-1} \\ 0 & -d_{i-1}^X \end{pmatrix}$.

If M is an R-module then we denote the complex $\cdots \to 0 \to M \to 0 \to \cdots$ with M in the mth degree by $S^m(M)$, and denote the complex $\cdots \to 0 \to M \xrightarrow{Id} M \to 0 \to \cdots$ with M in the $m-1$ and mth degrees by $D^m(M)$. Usually, $S^0(M)$ is denoted simply by M. We use $\mathrm{Hom}(X, Y)$ to present the group of all morphisms from X to Y. Recall that a complex I is injective if the functor $\mathrm{Hom}(-, I)$ is exact. Equivalently, I is injective if and only if I is acyclic and $\mathrm{Z}_i I$ is an injective R-module for each $i \in \mathbb{Z}$. A projective complex is defined dually. Thus the category of chian complexes $\mathrm{Ch}(R)$ of R-modules has enough projectives and injectives, we can compute right derived functors $\mathrm{Ext}^i(X, Y)$ of $\mathrm{Hom}(-, -)$. In particular, $\mathrm{Ext}^1(X, Y)$ will denote the group of (equivalent classes) of short exact sequences $0 \to Y \to Z \to X \to 0$ under the Baer sum operation. There is a subgroup $\mathrm{Ext}^1_{dw}(X, Y) \subseteq \mathrm{Ext}^1(X, Y)$ consisting of the "degreewise split" short exact sequences. That is, those for which each $0 \to Y_m \to Z_m \to X_m \to 0$ is split exact. The following lemma gives a well-known connection between $\mathrm{Ext}^1_{dw}(-, -)$ and the above hom-complex $\mathrm{Hom}_R(-, -)$.

Lemma 2.1. *Let X and Y be two chain complexes. Then there are isomorphisms*

$$\mathrm{Ext}^1_{dw}(X, \Sigma^{-m-1} Y) \cong \mathrm{H}_m \mathrm{Hom}_R(X, Y) = \mathrm{Hom}_{K(R)}(X, \Sigma^{-m} Y),$$

where $K(R)$ is the homotopy category of chain complexes. In particular, the hom-complex $\mathrm{Hom}_R(X, Y)$ is acyclic if and only if for each $m \in \mathbb{Z}$, any chain map $f : \Sigma^m X \to Y$ (or $f : X \to \Sigma^m Y$) is null homotopic.

Recall from [19] that an R-module E is called *FP-injective* if $\mathrm{Ext}^1(A,E)=0$ for all finitely presented R-modules A. The *FP-injective dimension* of a module B, denoted by $\text{FP-id}_R(B)$, is defined to be the least integer $n\geq 0$ such that $\mathrm{Ext}^{n+1}(A,B)=0$ for all finitely presented R-modules A. If no such n exists, set $\text{FP-id}(B)=\infty$. An R-module M is called *Ding injective* (resp., *Ding projective*) [6, 9] if there exists an acyclic complex of injective (resp., projective) R-modules $\cdots\longrightarrow X_1\xrightarrow{d_1} X_0\xrightarrow{d_0} X_{-1}\longrightarrow\cdots$ with $M\cong \mathrm{Z}_0X$ and which remains exact after applying $\mathrm{Hom}_R(J,-)$ (resp., $\mathrm{Hom}_R(-,J)$) for any FP-injective (resp., flat) R-module J, see [6, 9, 15, 23] for properties of Ding injective and Ding projective R-modules. It is natural that one can define the analog for chain complexes of R-modules, see [21, 24, 26] for the notions of FP-injective chain complexes, and [16, 24] of Ding injective and Ding projective chain complexes.

Definition 2.2. A chain complex J is FP-injective if $\mathrm{Ext}^1(A,J)=0$ for any finitely presented chain complex A.

Definition 2.3. A chain complex X is called Ding injective if there exists an exact sequence of injective chain complexes $\cdots\longrightarrow I_1\longrightarrow I_0\longrightarrow I^0\longrightarrow I^1\longrightarrow\cdots$ with $X\cong \mathrm{Ker}(I^0\longrightarrow I^1)$ and which remains exact after applying $\mathrm{Hom}(J,-)$ for any FP-injective chain complex J.

Definition 2.4. A chain complex Y is called Ding projective if there exists an exact sequence of projective chain complexes $\cdots\longrightarrow P_1\longrightarrow P_0\longrightarrow P^0\longrightarrow P^1\longrightarrow\cdots$ with $Y\cong \mathrm{Ker}(P^0\longrightarrow P^1)$ and which remains exact after applying $\mathrm{Hom}(-,F)$ for any flat chain complex F.

It was shown in [21] that a chain complex J is FP-injective if and only if J is acyclic and Z_iJ is FP-injective in R-Mod for all $i\in\mathbb{Z}$. A chain complex F is flat if and only if F is acyclic and Z_iF is flat in R-Mod for all $i\in\mathbb{Z}$ ([10]).

It was shown in [24] that Ding injective (resp., Ding projective) chain complexes are precisely the chain complexes X for which each term X_i is Ding injective (resp., Ding projective) and any chain map $J\to X$ (resp., $X\to J$) is null homotopic whenever J is an FP-injective (resp., a flat) chain complex. It was shown in [16] that the null homotopic condition is automatically satisfied when the ring R is Ding-Chen (that is, R is both left and right coherent, and has both left and right self FP-injective dimensions at most n for some non-negative integer n). It was shown recently in [25] that a chain complex $X\in\mathrm{Ch}(R)$ is Ding projective if and only if each term X_i is Ding projective in R-Mod for all $i\in\mathbb{Z}$.

3. Ding injective complexes over general rings

This section is devoted to studying Ding injective chain complexes over general rings. We will first show that any bounded below chain complex of Ding injective modules can be approximated by a bounded below chain complex of injective modules. For the convenience, we use $\mathcal{I}$ (resp., $\mathrm{Ch}(\mathcal{I})$) to denote the subcategory of injective R-modules (resp., the subcategory of chain complexes of injective R-modules).

Lemma 3.1. *Let G be a bounded below chain complex with all terms Ding injective. Then there exists an exact sequence $0\to K\to E\to G\to 0$ of chain complexes such that the following conditions are satisfied*

(1) $E \in \mathrm{Ch}(\mathcal{I})$ *is bounded below,*
(2) K *is acyclic and bounded below with all terms Ding injective,*
(3) K *is* $\mathrm{Hom}_R(\mathcal{J},-)$*-exact, where* $\mathcal{J}$ *denotes the class of all FP-injective modules.*

Proof. Without loss of generality, we let

$$G =: \ \cdots \longrightarrow G_{n+1} \xrightarrow{d_{n+1}} G_n \longrightarrow \cdots \longrightarrow G_1 \xrightarrow{d_1} G_0 \longrightarrow 0\ .$$

Now if we denote $G(n)$ the hard truncation above of G at n, that is,

$$G(n) =: \ \cdots \longrightarrow 0 \longrightarrow G_n \xrightarrow{d_n} \cdots \longrightarrow G_1 \xrightarrow{d_1} G_0 \longrightarrow 0\ ,$$

then $\{(G(n),\alpha_{nm})|0 \le n \le m\}$ forms a direct system in $\mathrm{Ch}(R)$ and $G = \varinjlim G(n)$, where $\alpha_{nm} : G(n) \to G(m)$ is the following natural injection for any $n \le m$.

$$\begin{array}{ccccccccccccccccc}
G(n) =: & \cdots \to & 0 & \to & 0 & \to \cdots \to & G_n & \to \cdots \to & G_1 & \to & G_0 & \to & 0 \\
\downarrow{\scriptstyle \alpha_{nm}} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \\
G(m) =: & \cdots \to & 0 & \to & G_m & \to \cdots \to & G_n & \to \cdots \to & G_1 & \to & G_0 & \to & 0.
\end{array}$$

We will show the result by induction on $n \geq 0$. For the case $n = 0$, since G_0 is Ding injective, there exists an exact sequence of R-modules

$$\cdots \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0 \xrightarrow{f} G_0 \longrightarrow 0$$

with each E_i injective and which remains exact after applying the functor $\mathrm{Hom}_R(J,-)$ for any FP-injectivve R-module J. Now consider the following morphisms of complexes

$$\begin{array}{cccccccccc}
K(0) =: & \cdots \to & E_2 & \to & E_1 & \to & \overline{G}_0 & \to & 0 \\
\downarrow{\scriptstyle \rho(0)} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \rho} & & \\
E(0) =: & \cdots \to & E_2 & \to & E_1 & \to & E_0 & \to & 0 \\
\downarrow{\scriptstyle \phi(0)} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \\
G(0) =: & \cdots \to & 0 & \to & 0 & \to & G_0 & \to & 0,
\end{array}$$

where $\overline{G}_0 = \mathrm{Ker}(f)$ is Ding injective, and ρ is the natural injection. This gives an exact sequence

$$0 \longrightarrow K(0) \xrightarrow{\rho(0)} E(0) \xrightarrow{\phi(0)} G(0) \longrightarrow 0$$

of bounded below complexes. It is easily seen that $K(0)$ is acyclic with all terms Ding injective, and $\mathrm{Hom}_R(J, K(0))$ is acyclic for any FP-injective R-module J.

Now suppose that for $n \geq 0$, we have an exact sequence of bounded below complexes

$$0 \longrightarrow K(n) \xrightarrow{\rho(n)} E(n) \xrightarrow{\phi(n)} G(n) \longrightarrow 0$$

which is given by the following commutative diagram

$$\begin{array}{ccccccccccccc}
K(n) =: & \cdots \longrightarrow & E_{n+1} & \longrightarrow & K_n & \longrightarrow \cdots \longrightarrow & K_1 & \longrightarrow & K_0 & \longrightarrow & 0 \\
\downarrow{\scriptstyle \rho(n)} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \rho_n} & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_0} & & \\
E(n) =: & \cdots \longrightarrow & E_{n+1} & \longrightarrow & E_n & \longrightarrow \cdots \longrightarrow & E_1 & \longrightarrow & E_0 & \longrightarrow & 0 \\
\downarrow{\scriptstyle \phi(n)} & & \downarrow & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \\
G(n) =: & \cdots \longrightarrow & 0 & \longrightarrow & G_n & \longrightarrow \cdots \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & 0.
\end{array}$$

Where $E(n)$ is in $\mathrm{Ch}(\mathcal{I})$, $K(n)$ is acyclic with all terms Ding injective, and $\mathrm{Hom}_R(J, K(n))$ is acyclic for any FP-injective R-module J.

The next is to show that the result holds for $n+1$. Since G_{n+1} is Ding injective, there exists an exact sequence of R-modules

$$\cdots \longrightarrow I_{n+3} \longrightarrow I_{n+2} \longrightarrow I_{n+1} \xrightarrow{g} G_{n+1} \longrightarrow 0$$

with each I_i injective and which remains exact after applying the functor $\mathrm{Hom}_R(J, -)$ for any FP-injectivve R-module J. This gives an exact sequence

$$0 \longrightarrow \mathrm{Ker}(\pi) \xrightarrow{\iota} I \xrightarrow{\pi} S^{n+1}(G_{n+1}) \longrightarrow 0$$

of bounded below complexes. Where

$$I =: \cdots \longrightarrow I_{n+3} \longrightarrow I_{n+2} \longrightarrow I_{n+1} \longrightarrow 0 \longrightarrow \cdots,$$

$\pi_{n+1} = g$ and $\pi_i = 0$ for $i \neq n+1$, and ι is the natural injection. Clearly, $\mathrm{Ker}(\pi)$ is acyclic with all terms Ding injective, and $\mathrm{Hom}_R(J, \mathrm{Ker}(\pi))$ is acyclic for any FP-injective R-module J.

Let $\mu : S^n(G_{n+1}) \to G(n)$ be the following morphism

$$\begin{array}{cccccccccccc}
S^n(G_{n+1}) =: & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & G_{n+1} & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mu} & & \downarrow & & \downarrow{\scriptstyle d_{n+1}} & & \downarrow & & \downarrow & & \\
G(n) =: & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & G_n & \xrightarrow{d_n} & G_{n-1} & \longrightarrow \cdots \longrightarrow & G_0 & \longrightarrow & 0.
\end{array}$$

It follows from [7, Lemma 2.3] that any chain map $\Sigma^{-1}I \longrightarrow K(n)$ is null-homotopic since the complex of abelian groups $\mathrm{Hom}_R(I_i, K(n))$ is acyclic for all i and the abelian group $\mathrm{Hom}_R(I_m, K(n)_m)$ is trivial for all $m \ll 0$. It gets by Lemma 2.1 ([14, Lemma 2.1]) that the subgroup of all degreewise split extensions $\mathrm{Ext}^1_{dw}(\Sigma^{-1}I, K(n)) \cong \mathrm{H}_{-2}[\mathrm{Hom}(I, K(n))] = 0$. This implies that the composition chain map

$$\Sigma^{-1}I \xrightarrow{\Sigma^{-1}\pi} S^n(G_{n+1}) \xrightarrow{\mu} G(n)$$

can factor through $\phi(n) : E(n) \longrightarrow G(n)$ as $\mu\Sigma^{-1}\pi = \phi(n)\nu$ because

$$\mathrm{Ext}^1_{\mathrm{Ch}(R)}(\Sigma^{-1}I, K(n)) = \mathrm{Ext}^1_{dw}(\Sigma^{-1}I, K(n)) = 0.$$

So one gets by the factor lemma a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma^{-1}\mathrm{Ker}(\pi) & \xrightarrow{\Sigma^{-1}\iota} & \Sigma^{-1}I & \xrightarrow{\Sigma^{-1}\pi} & S^n(G_{n+1}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\nu} & & \downarrow{\scriptstyle\mu} & & \\ 0 & \longrightarrow & K(n) & \xrightarrow{\rho(n)} & E(n) & \xrightarrow{\phi(n)} & G(n) & \longrightarrow & 0 \end{array}$$

with exact rows. Note that $G(n+1) = Cone(\mu)$. If we put $E(n+1) = Cone(\nu)$ and $K(n+1) = Cone(\omega)$ then it is easily seen that there is an induced exact sequence of complexes

$$0 \longrightarrow K(n+1) \xrightarrow{\rho(n+1)} E(n+1) \xrightarrow{\phi(n+1)} G(n+1) \longrightarrow 0$$

with $\rho(n+1) = \begin{pmatrix} \rho(n) & 0 \\ 0 & \iota \end{pmatrix}$ and $\phi(n+1) = \begin{pmatrix} \phi(n) & 0 \\ 0 & \pi \end{pmatrix}$.

Therefore, by the construction above, we get a commutative diagram of bounded below complexes

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K(n) & \xrightarrow{\rho(n)} & E(n) & \xrightarrow{\phi(n)} & G(n) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\gamma_{n(n+1)}} & & \downarrow{\scriptstyle\beta_{n(n+1)}} & & \downarrow{\scriptstyle\alpha_{n(n+1)}} & & \\ 0 & \longrightarrow & K(n+1) & \xrightarrow{\rho(n+1)} & E(n+1) & \xrightarrow{\phi(n+1)} & G(n+1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ker}(\pi) & \xrightarrow{\iota} & I & \xrightarrow{\pi} & S^{n+1}(G_{n+1}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

with exact rows and columns. One gets easily from the middle column of the above diagram that $E(n+1) \in \mathrm{Ch}(\mathcal{I})$ is bounded below. In fact,

$$E(n+1)_k = \begin{cases} E(n)_k \oplus I_k, & k \geq n+1; \\ E(n)_k, & 0 \leq k \leq n; \\ 0, & k < 0. \end{cases}$$

The left column implies that the complex $K(n+1)$ has all its terms Ding injective because the subcategory of Ding injective modules is injectively resolving by [23, Theorem 2.8], and $K(n+1)$ is acyclic and $\mathrm{Hom}_R(J, K(n+1))$ is acyclic for any FP-injective R-module J since $K(n)$ and $\mathrm{Ker}(\pi)$ are so. In particular,

$$K(n+1)_k = \begin{cases} K(n)_k \oplus I_k, & k > n+1; \\ K(n)_{n+1} \oplus \mathrm{Ker}(g), & k = n+1; \\ E(n)_k, & 0 \leq k \leq n; \\ 0, & k < 0. \end{cases}$$

Since the functor $\varinjlim$ is exact, one has an exact sequence of complexes

$$0 \longrightarrow \varinjlim K(n) \xrightarrow{\varinjlim \rho(n)} \varinjlim E(n) \xrightarrow{\varinjlim \phi(n)} G = \varinjlim G(n) \longrightarrow 0 .$$

Let $E = \varinjlim E(n)$, and $K = \varinjlim K(n)$. Then one gets clearly that

$$E_i = \begin{cases} E(i)_i, & i \geq 0; \\ 0, & i < 0. \end{cases}$$

and

$$K_i = \begin{cases} K(i)_i, & i \geq 0; \\ 0, & i < 0. \end{cases}$$

Thus it is easily seen that the complex K is acyclic with all terms Ding injective, and $\mathrm{Hom}_R(J, K)$ is acyclic for any FP-injective R-module J. This proves that the exact sequence $0 \longrightarrow K \longrightarrow E \longrightarrow G \longrightarrow 0$ is what we need. □

Corollary 3.2. *Let G be a bounded below chain complex with all terms Ding injective. Then there exists an exact sequence of bounded below chain complexes $0 \to K \to E \to G \to 0$ such that K is Ding injective in $\mathrm{Ch}(R)$, where E is in $\mathrm{Ch}(\mathcal{I})$.*

Proof. Let $0 \to K \to E \to G \to 0$ be the sequence obtained in 3.1. Then it is easily seen that any chain map from a complex J with all terms FP-injective to such K is null-homotopic by [7, Lemma 2.3]. In particular, any chain map $f : F \longrightarrow K$ is of course null-homotopic, where F is any FP-injective chain complex. So K is Ding injective in $\mathrm{Ch}(R)$ by [24, Theorem 3.20]. □

Lemma 3.3. *Let G be a bounded above chain complex. If each term G_i is Ding injective in R-Mod for all $i \in \mathbb{Z}$, then G is Ding injective in $\mathrm{Ch}(R)$.*

Proof. It is dual to [24, Proposition 3.14]. □

Recall that a chain complex X is called *DG-injective* (resp., *DG-projective*), if each module X_i is injective (resp., projective) in R-Mod for all $i \in \mathbb{Z}$ and $\mathrm{Hom}_R(E, X)$ (resp., $\mathrm{Hom}_R(X, E)$) is exact for all acyclic complexes E (see [1, 13]). In the next we will give our main results in this section.

Theorem 3.4. *The following conditions are equivalent for any ring R.*

1. *(1) Every bounded below chain complex of Ding injective R-modules is Ding injective in $\mathrm{Ch}(R)$.*
2. *(2) Every bounded below chain complex of injective R-modules is Ding injective in $\mathrm{Ch}(R)$.*
3. *(3) Every chain complex of Ding injective R-modules is Ding injective in $\mathrm{Ch}(R)$.*
4. *(4) Every chain complex of injective R-modules is Ding injective in $\mathrm{Ch}(R)$.*
5. *(5) Every acyclic chain complex of Ding injective R-modules is Ding injective in $\mathrm{Ch}(R)$.*
6. *(6) Every acyclic chain complex of injective R-modules is Ding injective in $\mathrm{Ch}(R)$.*

Proof. (1)⇒(2), (3)⇒(1), (3)⇒(5), and (4)⇒(6) are trivial.

(2)⇒(1). Let G be a bounded below complex of Ding injective modules, and J an FP-injective complex. Then it follows from Lemma 3.1 and Corollary 3.2 that there exists a short exact sequence $0 \to K \to E \to G \to 0$ such that E is a bounded below chain complex of injective R-modules, and K is a Ding injective chain complex. Now applying the functor $\mathrm{Hom}_R(J, -)$ to the short exact sequence will yield the short exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(J, K) \longrightarrow \mathrm{Hom}_R(J, E) \longrightarrow \mathrm{Hom}_R(J, G) \longrightarrow 0$$

of chain complexes. Also the chain complex of abelian groups $\mathrm{Hom}_R(J,G)$ must be acyclic since the chain complexes of abelian groups $\mathrm{Hom}_R(J,K)$ and $\mathrm{Hom}_R(J,E)$ are acyclic. This implies that any chain map $J \to G$ is null-homotopic by Lemma 2.1. Now we get by [24, Theorem 3.20] that G is Ding injective in $\mathrm{Ch}(R)$.

(1)⇒(3). Let G be a complex of Ding injective modules, and J an FP-injective complex. Then there is an induced short exact sequence

$$0 \longrightarrow G_{\leq n} \longrightarrow G \longrightarrow G_{\geq n+1} \longrightarrow 0$$

of chain complexes, where $G_{\geq n+1}$ is a bounded below complex of Ding injective R-modules, $G_{\leq n}$ is a bounded above complex of Ding injective R-modules which is Ding injective in $\mathrm{Ch}(R)$ by Lemma 3.3. Applying the functor $\mathrm{Hom}_R(J,-)$ to the short exact sequence will yield the short exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(J,G_{\leq n}) \longrightarrow \mathrm{Hom}_R(J,G) \longrightarrow \mathrm{Hom}_R(J,G_{\geq n+1}) \longrightarrow 0$$

of chain complexes. It is easy to see that the chain complex of abelian groups $\mathrm{Hom}_R(J,G)$ must be acyclic since $\mathrm{Hom}_R(J,G_{\leq n})$ and $\mathrm{Hom}_R(J,G_{\geq n+1})$ are so. This implies that any chain map $J \to G$ is null-homotopic by Lemma 2.1. Now we get again by [24, Theorem 3.20] that G is Ding injective in $\mathrm{Ch}(R)$.

(2)⇔(4) is similar to the proof of (1)⇔(3).

(5)⇒(3). Let G be a complex of Ding injective modules, and J an FP-injective complex. Then there is a short exact sequence $0 \longrightarrow I \longrightarrow E \longrightarrow G \longrightarrow 0$ with E acyclic and I DG-injective ([13, Theorem 2.2.4]). It is clear that E is a complex of Ding injective R-modules, and so E is Ding injective in $\mathrm{Ch}(R)$ by the assumption. Also, any DG-injective complex is Ding injective [24, Corollary 3.21]. Applying the functor $\mathrm{Hom}_R(J,-)$ to the short exact sequence will yield the short exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(J,I) \longrightarrow \mathrm{Hom}_R(J,E) \longrightarrow \mathrm{Hom}_R(J,G) \longrightarrow 0$$

of chain complexes. Similarly, one can see easily that the chain complex of abelian groups $\mathrm{Hom}_R(J,G)$ is acyclic and then G is Ding injective in $\mathrm{Ch}(R)$.

(6)⇒(4) is similar to the proof of (5)⇒(3). □

Proposition 3.5. *Let G be a chain complex of Ding injective R-modules. If there exists an integer k such that Z_iG is Ding injective in R-Mod for all $i \geq k$, then G is Ding injective in* $\mathrm{Ch}(R)$.

Proof. It follows from the short exact sequence

$$0 \longrightarrow \mathrm{Z}_iG \longrightarrow G_i \longrightarrow \mathrm{B}_{i-1}G \longrightarrow 0$$

that each $\mathrm{B}_{i-1}G$ is also Ding injective in R-Mod for $i \geq k$ since G_i and Z_iG are Ding injective and the subcategory of Ding injective R-modules is injectively resolving ([23, Theorem 2.8]). If we put

$$\tau_{\leq n}G =: \quad \cdots \longrightarrow 0 \longrightarrow B_nG \longrightarrow G_n \xrightarrow{d_n} G_{n-1} \xrightarrow{d_{n-1}} \cdots ,$$

then there is a projective system of complexes $\{(\tau_{\leq n}G, \phi_{mn})|k \leq n < m\}$ such that each chain map

$$\begin{array}{ccccccccccccccc} \tau_{\leq n+1}G =: & \cdots \longrightarrow & 0 & \longrightarrow & \mathrm{B}_{n+1}G & \longrightarrow & G_{n+1} & \xrightarrow{d_{n+1}} & G_n & \xrightarrow{d_n} & G_{n-1} & \xrightarrow{d_{n-1}} & \cdots \\ \downarrow{\scriptstyle \phi_{(n+1)n}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle d_{n+1}} & & \| & & \| & & \\ \tau_{\leq n}G =: & \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{B}_nG & \longrightarrow & G_n & \xrightarrow{d_n} & G_{n-1} & \xrightarrow{d_{n-1}} & \cdots \end{array}$$

is surjective and each

$$\mathrm{Ker}(\phi_{(n+1)n}) =: \ \cdots \longrightarrow 0 \longrightarrow \mathrm{B}_{n+1}G \longrightarrow \mathrm{Z}_{n+1}G \longrightarrow 0 \longrightarrow \cdots$$

is easily seen Ding injective in $\mathrm{Ch}(R)$ for $n > k$ by Lemma 3.3. It is easy to see that $G = \varprojlim \tau_{\leq n}G$. If J is an FP-injective complex, then $\mathrm{Hom}_R(J, \varprojlim \tau_{\leq n}G) \cong \varprojlim \mathrm{Hom}_R(J, \tau_{\leq n}G)$. Note that each chain complex of abelian groups $\mathrm{Hom}_R(J, \tau_{\leq n}G)$ is acyclic since $\tau_{\leq n}G$ is Ding injective in $\mathrm{Ch}(R)$. Now applying the functor $\mathrm{Hom}_R(J, -)$ to each sequence of complexes

$$0 \longrightarrow \mathrm{Ker}(\phi_{(n+1)n}) \longrightarrow \tau_{\leq n+1}G \xrightarrow{\phi_{(n+1)n}} \tau_{\leq n}G \longrightarrow 0\ ,$$

we then get the projective system

$$\cdots \longrightarrow \mathrm{Hom}_R(J, \tau_{\leq k+2}G) \longrightarrow \mathrm{Hom}_R(J, \tau_{\leq k+1}G) \longrightarrow \mathrm{Hom}_R(J, \tau_{\leq k}G)$$

with each chain map $\mathrm{Hom}_R(J, \tau_{\leq n+1}G) \to \mathrm{Hom}_R(J, \tau_{\leq n}G)$ is surjective. This implies that $\varprojlim \mathrm{Hom}_R(J, \tau_{\leq n}G)$ is acyclic. It follows from Lemma 2.1 that any chain map $J \to G$ is null-homotopic. Now we get by [24, Theorem 3.20] that G is Ding injective in $\mathrm{Ch}(R)$. □

4. Applications to special rings

It was shown recently by Gillespie in [16] that Ding injective complexes are precisely the complexes X for which all terms X_m are Ding injective R-modules whenever R is a Ding-Chen ring (that is, R is both left and right coherent, and has both left and right self FP-injective dimensions at most n for some non-negative integer n). In this section we will extend the result to left coherent rings.

We recall the purity in the sense of Cohn ([3]). The reader is also referred to the monographs by Enochs-Jenda ([11]) for an elementary description. A short exact sequence $0 \to A \to B \to C \to 0$ in R-Mod is *pure* if for each finitely presented module F, the sequence of abelian groups

$$0 \to \mathrm{Hom}_R(F, A) \to \mathrm{Hom}_R(F, B) \to \mathrm{Hom}_R(F, C) \to 0$$

is also exact. An acyclic chain complex

$$\cdots \longrightarrow X_{i+1} \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \longrightarrow \cdots$$

is pure if all the induced short exact sequences

$$0 \longrightarrow \mathrm{Z}_{i+1}X \longrightarrow X_i \longrightarrow \mathrm{Z}_iX \longrightarrow 0$$

of R-modules are pure.

Theorem 4.1. *Let R be a left coherent ring and $G \in \mathrm{Ch}(R)$ a chain complex. Then G is Ding injective in $\mathrm{Ch}(R)$ if and only if each term G_m is Ding injective in R-Mod.*

Proof. The necessity follows directly from [24, Theorem 3.20] for any ring R. For the sufficiency, by Theorem 3.4 we need only to show that any chain complex I of injective modules is Ding injective in $\mathrm{Ch}(R)$. Now let J be any FP-injective chain complex. It is easy to see that J is pure acyclic. Then it follows from [20, Proposition 6.11] that $\mathrm{Ext}^1(J, \Sigma^m I) = 0$ for all $m \in \mathbb{Z}$ since R is left coherent, and so $\mathrm{Ext}^1_{dw}(J, \Sigma^{-1} I) \cong \mathrm{H}_0\mathrm{Hom}_R(J, I) = \mathrm{Hom}_{K(R)}(J, I)$ is trivial. This implies that any chain map $f : J \to I$ is null homotopic. Thus I is Ding injective in $\mathrm{Ch}(R)$ by [24, Theorem 3.20], as desired. □

Let M be a chain complex. Recall that the *flat dimension* of M is defined as

$$\mathrm{fd}(M) = \inf\{n \mid \text{there exists an exact sequence}$$
$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0 \text{ with each } F_i \text{ flat }\}.$$

If no such n exists, set $\mathrm{fd}(M) = \infty$. In fact, $\mathrm{fd}(M) \leq n$ in $\mathrm{Ch}(R)$ if and only if M is acyclic, and $\mathrm{fd}_R(Z_i M) \leq n$ in R-Mod for each $i \in \mathbb{Z}$ ([13, Lemma 5.4.1]). The *FP-injective dimension* of M, denoted by FP-id(M), is defined dually. According to [21, Theorem 2.26], we have FP-id$(M) \leq n$ in $\mathrm{Ch}(R)$ if and only if M is acyclic, and FP-id$_R(Z_i M) \leq n$ in R-Mod for each $i \in \mathbb{Z}$ provided that R is left coherent.

The following result is of special interest when one compares with [1, Proposition 3.4], which says that if R is a ring with left global dimension gldim $R < \infty$, then any chain complex of R-modules with all terms projective (resp., injective) is DG-projective (resp., DG-injective).

Proposition 4.2. *Let R be a ring with left weak global dimension $WD(R) < \infty$.*

(1) Any chain complex of R-modules with all terms projective is DG-projective;
(2) If R is left coherent, then any chain complex of R-modules with all terms injective is DG-injective.

Proof. (1) Since projective modules are Ding projective, we would prove a more general result that the hom-complex $\mathrm{Hom}_R(G, E)$ is acyclic for any chain complex G with all terms G_m Ding projective in R-Mod and any acyclic complex E. Let G and E be such two chain complex as above. Then, by [25, Theorem 3.8], we follow that G is Ding projective in $\mathrm{Ch}(R)$, and this holds for any ring R. But E has finite flat dimension because $\mathrm{WD}(R) < \infty$, that is, there exists an exact sequence of complexes

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow E \longrightarrow 0$$

with each F_i flat in $\mathrm{Ch}(R)$. Note that the abelian group $\mathrm{Ext}^1(G_m, K) = 0$ for any module K with $\mathrm{fd}_R(K) < \infty$. Now applying the functor $\mathrm{Hom}_R(G, -)$ to the above sequence, one will get an exact sequence of chain complexes

$$0 \to \mathrm{Hom}_R(G, F_n) \to \mathrm{Hom}_R(G, F_{n-1}) \to \cdots \to \mathrm{Hom}_R(G, F_0) \to \mathrm{Hom}_R(G, E) \to 0 .$$

Since each chain complex of abelian groups $\mathrm{Hom}_R(G, F_i)$ is acyclic ([24, Theorem 3.7]), we see easily that $\mathrm{Hom}_R(G, E)$ is acyclic, as desired.

(2) Under the assumption, we follow from [5, Theorem 3.8] and [19, Proposition 3.5] that FP-id$(E) \leq \mathrm{WD}(R)$ for any acyclic complex E. Now the proof is dual to that of (1). □

We recall that an R-module C is called a *cotorsion* module ([17, 22]) provided that $\mathrm{Ext}^1_R(F, C) = 0$ for any flat R-module F. It is shown recently by Bazzoni, Izurdiaga, and Estrada in [2, Theorem 4.1] that each boundary module of an acyclic complex of cotorsion R-modules is always a cotorsion module. We get easily that every acyclic complex of injective R-modules has cotorsion cycles. This was also shown by Šťovíček [20, Corollary 5.9]. An R-module M is *pure injective* ([17, 22]) provided that any pure short exact sequence remains exact after applying the functor $\mathrm{Hom}_R(-, M)$. For the next proposition we recall that a ring R is called a *Xu* ring ([8, 12]) provided that every cotorsion R-module is pure injective.

Proposition 4.3. *Let R be a Xu ring. A chain complex G is Ding injective in* $\mathrm{Ch}(R)$ *if and only if each term G_m is Ding injective in R-Mod.*

Proof. Let G be a Ding injective complex. Again, by [24, Theorem 3.20], we get that every module G_m is Ding injective, and this holds for any ring R. Conversely, let us assume that G_m is Ding injective in R-Mod, for each $m \in \mathbb{Z}$. By Theorem 3.4 we need only to show that any acyclic complex I of injective modules is Ding injective in $\mathrm{Ch}(R)$. Now let J be any FP-injective chain complex. Then it is easy to see that J is in fact pure acyclic. If we put

$$\tau_{\leq n} I =: \quad \cdots \longrightarrow 0 \longrightarrow B_n I \longrightarrow I_n \xrightarrow{d_n} I_{n-1} \xrightarrow{d_{n-1}} \cdots,$$

then there is a projective system of complexes $\{(\tau_{\leq n} I, \phi_{mn}) | k \leq n < m\}$ with each ϕ_{mn} surjective. Note that each chain complex

$$\mathrm{Ker}(\phi_{(n+1)n}) =: \ \cdots \longrightarrow 0 \longrightarrow \mathrm{B}_{n+1} I \xrightarrow{=} \mathrm{Z}_{n+1} I \longrightarrow 0 \longrightarrow \cdots$$

has all terms pure injective, and the chain complex $\mathrm{Hom}_R(J, \mathrm{Ker}(\phi_{(n+1)n}))$ of abelian groups is clearly acyclic. Now applying the functor $\mathrm{Hom}_R(J, -)$ to each short exact sequence of chain complexes

$$0 \longrightarrow \mathrm{Ker}(\phi_{(n+1)n}) \longrightarrow \tau_{\leq n+1} I \xrightarrow{\phi_{(n+1)n}} \tau_{\leq n} I \longrightarrow 0,$$

we then get the projective system

$$\cdots \longrightarrow \mathrm{Hom}_R(J, \tau_{\leq 2} I) \longrightarrow \mathrm{Hom}_R(J, \tau_{\leq 1} I) \longrightarrow \mathrm{Hom}_R(J, \tau_{\leq 0} I)$$

with each chain map $\mathrm{Hom}_R(J, \tau_{\leq n+1} I) \to \mathrm{Hom}_R(J, \tau_{\leq n} I)$ surjective and each chain complex of abelian groups $\mathrm{Hom}_R(J, \tau_{\leq n} I)$ acyclic by the dual result of [7, Lemma 2.3]. This implies clearly that

$$\varprojlim \mathrm{Hom}_R(J, \tau_{\leq n} I) \cong \mathrm{Hom}_R(J, \varprojlim \tau_{\leq n} I) = \mathrm{Hom}_R(J, I)$$

is acyclic. Thus we get easily that I is Ding injective in $\mathrm{Ch}(R)$ by [24, Theorem 3.20], as desired. □

Remark 4.4. A cotorsion R-module is not in general pure injective. For instance cotorsion R-modules are always closed under extensions. But this is not the case for pure-injective R-modules. In fact, it was proven [22, Theorem 3.5.1] that every cotorsion R-module is pure-injective (Xu rings) if and only if the class of pure injective R-modules is closed under extensions. The reader is also referred to [8] for characterizations of Xu rings and to [4] for examples of commutative Xu rings.

References

[1] L.L. Avramov and H.-B. Foxby, Homological dimensions of unbounded complexes, *J. Pure Appl. Algebra* **71** (1991), 129-155.
[2] S. Bazzoni, M.C. Izurdiaga, and S. Estrada, Periodic modules and acyclic complexes, arXiv: **1704.06672**.
[3] P.M. Cohn, On the free product of associative rings, *Math. Z.* **71** (1959), 380-398.
[4] F. Couchot, Commutative rings whose cotorsion modules are pure-injective, arXiv: **1506.02924v5**.
[5] N.Q. Ding and J.L. Chen, The flat dimensions of injective modules, *Manuscripta Math.* **78** (2) (1993), 165-177.
[6] N.Q. Ding, Y.L. Li, L.X. Mao, Strongly Gorenstein flat modules, *J. Aust. Math. Soc.* **86** (3) (2009), 323-338.
[7] I. Emmanouil, On pure acyclic complexes, *J. Algebra* **465** (2016), 190-213.
[8] I. Herzog, P. Rothmaler, When cotorsion modules are pure injective, *J. Math. Logic* **9** (1) (2009), 63-102.
[9] L.X. Mao, N.Q. Ding, Gorenstein FP-injective and Gorenstein flat modules, *J. Algebra Appl.* **7** (4) (2008), 491-506.
[10] E.E. Enochs and J.R. García Rozas, Flat covers of complexes, J. Algebra **210** (1998), 86-102.
[11] E.E. Enochs, O.M.G. Jenda, *Relative homological algebra*, in: de Gruyter Expositions in Mathematics, Vol 30, Walter de Gruyter and Co. Berlin 2000.
[12] X.H. Fu, I. Herzog, Powers of the phantom ideal, *Proc. London Math. Soc.* **112** (3) (2016), 714-752.
[13] J.R. García Rozas, *Covers and envelope in the category of complexes of modules*, Boca Raton London New York Washington, D.C. 1999.
[14] J. Gillespie, The flat model structure on Ch(R), *Trans. Amer. Math. Soc.* **356** (8) (2004), 3369-3390.
[15] J. Gillespie, Model structures on modules over Ding-Chen rings, *Homology, Homotopy Appl.* **12** (1) (2010) 61-73.
[16] J. Gillespie, On Ding injective, Ding projective and Ding flat modules and complexes, *Rocky Mountain J. Math.* **47** (8) (2017), 2641-2673.
[17] R. Göbel, and J. Trlifaj, *Approximations and endmorphism algebras of modules*, Walter De Gruyter, Berlin, 2006.
[18] J.S. Hu, Y.X. Geng, Z.W. Xie, and D.D. Zhang, Gorenstein FP-injective dimension for complexes, *Comm. Algebra* **43** (8) (2015), 3515-3533.
[19] B. Stenström, Coherent rings and FP-injective modules, *J. Lond. Math. Soc.* **2** (2) (1970) 323-329.
[20] J. Šťovíček, On purity and applications to coderived and singularity categories, arXiv: **1412.1615**.
[21] Z.P. Wang, Z.K. Liu, FP-injective complexes and FP-injective dimension of complexes, *J. Aust. Math. Soc.* **91** (2) (2011) 163-187.
[22] J.Z. Xu, *Flat covers of modules*, Lecture Notes in Math., 1634, Springer-Verlag, Berlin, 1996.
[23] G. Yang, Z.K. Liu, L. Liang, Ding projective and Ding injective modules, *Colloq. Algebra* **20** (4) (2013), 601-612.
[24] G. Yang, Z.K. Liu, and L. Liang, Model structures on categories of complexes over Ding-Chen rings, *Comm. Algebra* **41** (1) (2013), 50-69.
[25] G. Yang, X.S. Da, On Ding projective complexes, *Acta Math. Sin. Engl. Ser.* **34** (11) (2018), 1718-1730.
[26] X.Y. Yang, Z.K. Liu, FP-injective complexes, *Comm. Algebra* **38** (1) (2010), 131-142.

1. Department of Mathematics, Lanzhou Jiaotong University, Lanzhou 730070, China
Email address: `yanggang@mail.lzjtu.cn (G Yang)`

2. Universidad de Murcia, Murcia 30100, Spain
Email address: `sestrada@um.es (S Estrada)`