Combinatorially Thin Trees and Spectrally Thin Trees in Structured Graphs

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Given a graph $G = (V, E)$, finding simpler estimates of G with possibly fewer edges or vertices while capturing some of its specific properties has been used in order to design efficient algorithms. The concept of estimating a graph with a simpler graph is known as *graph sparsification.* Spanning trees are an important family of graph sparsifiers that maintain connectivity of graphs, and have been utilized in many applications. However, spanning trees are a wide family, and for some applications one might need the spanning tree to have specific properties. *Combinatorially thin trees* are a type of spanning trees that show up in applications such as Asymmetric Travelling Salesman Problem (ATSP). A spanning tree T of G is *combinatorially thin* if there is no cut $U \subset V$ such that T contains all the edges in $\delta(U)$, and the *thinness parameter* $\alpha_G(T)$ measures the maximum fraction of edges in $E(T) \cap \delta(U)$ compared to $\delta(U)$ over all cuts $U \subset V$.

Intuitively, combinatorial thinness measures how much edge-connectivity we lose while removing the spanning tree T from G. It is easy to verify that if G has connectivity k, then $\frac{1}{k}$ lower bounds α_G. On the other hand, Goddyn conjectured that α_G can also be upper bounded as a function of connectivity $\alpha_G = f(\frac{1}{k})$. This conjecture which is known as *thin tree conjecture*, was proved for the special case of graphs with bounded genus by Oveis-Gharan and Saberi, in 2011. However, the general case is still open. In the first part of this thesis, we study some of the known connections between edge-connectivity and α_G and investigate the result of Oveis-Gharan and Saberi for the special case of planar graphs.

For a general graph G and spanning tree T, even verifying the combinatorial thinness $\alpha_G(T)$ of T is an NP-hard problem. A natural more efficiently computable relaxation of combinatorial thinness is the notion of *spectral thinness.* For a graph G and a spanning tree T in G the spectral thinness $\theta_G(T)$ is the smallest value of θ such that $\theta\mathcal{L}_G - \mathcal{L}_T$ is a *positive semidefinite* matrix where $\mathcal{L}_G$ and $\mathcal{L}_T$ are Laplacian matrices of G and T. Additionally, we define θ_G to be the minimum value of $\theta_G(T)$ over all spanning trees T of G. Similar to combinatorial thinness and connectivity, $\theta_G(T)$ can be lower bounded by the maximum *effective resistance* of edges in T. It was also proven by Harvey and Olver in 2014 that the maximum effective resistance of edges in G asymptotically upper bounds θ_G. However, finding a mathematical characterization of $\theta_G(T)$, even for structured graphs, is still a challenge. In the second part of this thesis, we will give general lower bound and upper bound certificates for $\theta_G(T)$ and utilize these certificates for circulant matrices to estimate spectral thinness of graphs such as complete graphs, complete bipartite graphs, and prism graphs.

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Dedication

I would like to dedicate this thesis to my parents, Firouzeh Azad and Hamid Alghasi, for their love, trust, and support throughout my life.

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Chapter 1

Introduction

Given a connected graph $G = (V, E)$, a *sparse subgraph* of G is a graph with (possibly) substantially fewer edges or vertices than G, while preserving some of the main properties of G. *Graph sparsification* is the notion of approximating a given graph G by a graph with fewer edges or vertices. The goal of graph sparsification is to find a simpler graph G_s which preserves certain properties of the original graph G such as connectivity, cut-weights, and diameter. We aim to use G_s as an input of certain algorithms without imposing considerable error compared to the solution on the original graph G, while saving computational time and storage.

An important family of sparse graphs are spanning trees, which maintain connectivity of G, while only having $|V|-1$ edges. Spanning tree sparsification has been very useful for approximating fundamental problems such as *Travelling Salesman Problem(TSP)*, *Shortest Path*, and *minimum-cost weighted perfect matching*. However, spanning trees are a wide family of graphs, and in many applications, we need to specify the spanning trees so that they have additional structures and properties. For instance, a specific family of spanning trees called *thin*-trees have been very useful in understanding important problems such as *Asymmetric Travelling Salesman Problem (ATSP)* and *No-where Zero k-flow*.

More specifically, a spanning tree T of G is *combinatorially* $\alpha-$*thin* if for every proper cut-set $S \subset E$ of G at most α fraction of the cut-edges S are in T, and our goal is to find combinatorial thin-trees with small *thinness parameter* α. One of the main challenges with this goal is that computing the thinness parameter of a given tree in a general setting is an NP-hard problem. Therefore, we may instead consider a relaxation of combinatorial thinness, called *spectral thinness*. We call a tree T a $\theta-$*spectral thin tree* of G, if for all $x \in \mathbb{R}^V$ the inequality $\theta x^\top \mathcal{L}_G x \geq x^\top \mathcal{L}_T x$ holds, where $\mathcal{L}_G$ and $\mathcal{L}_T$ are the Laplacian matrices of graphs G and T, respectively.

In the following sections, we first cover basic definitions and preliminaries for this thesis.[1] Next, we define *cambinatorial* and *spectral thin*-trees more rigorously. Finally, we show two major applications of *thin spanning trees* in the literature.

1.1 Basic Definitions and Notations

In this thesis, we use notions and notations commonly used in Graph Theory and Spectral Graph Theory literature as well as Convex Optimization and Linear programming. To have a consistent understanding of the basic notions, we will provide a concise definition of them in what follows.

We use the standard notation of denoting set of real numbers, integers, and natural numbers with $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$. Additionally, we will use $\mathbb{R}_+$ and $\mathbb{R}_{++}$ to denote the set of non-negative and positive real numbers, respectively. We may use $[n]$ to denote the set $\{1, \ldots, n\}$. For $n, m \in \mathbb{N}$, $\mathbb{R}^{m \times n}$ and $\mathbb{R}^n$ denote the set of matrices with real enteries that have size $m \times n$ and $n \times n$. Similarly, $\mathbb{S}^n$ denotes the set of symmetric matrices in $\mathbb{R}^n$. For $M \in \mathbb{R}^{m \times n}$, $\mathrm{Null}(M)$ and $\mathrm{Rank}(M)$ denote the dimension of null space and rank of M, respectively.

Graph theory

We will denote an undirected simple graph with $G = (V, E)$ where V is the set of vertices and $E \subseteq \binom{V}{2}$ is the edge-set. Therefore, each edge $e \in E$ is denoted by $e = \{u, v\}$, where $u, v \in V$. We denote the vertex-set size with $|V|$ and edge-set size with $|E|$.

For a vertex $v \in V$, we use $\deg_G(v)$ to be the number of edges with v as their endpoint, and define the *neighbour set* of v, denoted by $N_G(v) \subset V$ to the set of vertices $u \in V$ such that $\{u, v\} \in E$. Moreover, maximum degree of a graph G, denoted by $\Delta(G)$, is equal to $\max_{v \in V} \deg_G(v)$.

Note that whenever the graph is obvious from the context, we may use $\deg(v)$, Δ and $N(v)$ instead of $\deg_G(v)$, $\Delta(G)$ and $N_G(v)$.

[1]The reader may skip preliminaries section if they have basic knowledge of graph theory and spectral graph theory.

Cuts and cut-sets For a proper subset $U \subset V$, we use $\delta_G(U) \subset E$ to denote the set of edges in E with exactly one endpoint in U. Whenever the graph is obvious from the context, we may use $\delta(U)$ instead of $\delta_G(U)$.

Moreover, we use $E[U]$ to denote the set of edges in E with both endpoints in U. Given subsets $U_1, U_2 \in V$ we use $E[U_1, U_2]$ to show the subset of edges with exactly one endpoint in U_1 and one endpoint in U_2.

A *cut* $U \subset V$ in G (also denoted by $(U, V \setminus U)$) is a partition of vertices of G into two disjoint sets. We call U and $V \setminus U$ the *shores* of the cut S. Moreover, a *cut-set* of a cut $U \subset V$ refers to the subset of edges $\delta_G(U) \subseteq E$. Finally, a *bond* $S \subset E$ is a minimal cut-set.

Weighted graphs An edge weight for a graph $G = (V, E)$ is a function $w : E \to \mathbb{R}$ that assigns a weight $w(e)$ to each edge $e \in E$. We might also use vector notation $w \in \mathbb{R}^E$ to denote an edge weight function. For a subset $S \subset E$ we define the total weight of S as $w(S) := \sum_{e \in S} w(e)$. Moreover, graph G is *$k-$edge connected with respect to weight* w, if and only if for all cuts $U \subset V$ we have,

$$w(\delta_G(U)) \geq k.$$

Edge deletion and contraction For a graph $G = (V, E)$ and edge $e = \{i, j\} \in E$ we define the edge deletion, denoted by $G \setminus e$, to be the graph obtained by removing the edge e from edge-set E. Similarly, for a subset $S \subset E$ of edges we define $G \setminus S$.

We can also define the edge contraction, denoted by $G/e := (V/e, E/e)$ to be the multigraph obtained by contracting endpoints of e to a single vertex uv where

$$\begin{aligned} V/e &:= (V \setminus \{u, v\}) \cup \{uv\} \\ E/e &:= \Big(E \setminus \big(\delta(u) \cup \delta(v)\big)\Big) \cup \Big(\{\{uv, w\} : w \in N(u)\} \cup \{\{uv, w\} : w \in N(v)\}\Big). \end{aligned}$$

Multi-graphs Unlike a simple graph, in a multi-graph $G = (V, E)$ the edge-set is a multi-set. I.e., we might have multiple edges $e_1, \ldots, e_k \in E$ with end-points $u, v \in V$. For each pair of vertices $u, v \in V$ we define the *edge multiplicity set* $\mathcal{M}(u, v)$ to be the multiset of all edges $e \in E$ with u and v as their endpoint. The *edge multiplicity number* of a pair of vertices $u, v \in V$ is defined by $\mu(u, v) := |\mathcal{M}(u, v)|$.

Directed graphs We denote directed graphs with $G = (V, A)$, where $A \subset V \times V$ is the set of directed edges of G. In the directed graphs, we denote $a \in A$ by an ordered pair of vertices $(u, v) \in V \times V$. For a proper subset $U \subset V$, we will use $\delta_G^+(U) = \{(a, b) \in E : b \notin U\}$ and $\delta_G^-(U) = \{(a, b) \in E : a \notin U\}$ to denote the set of out-going and in-going edges of U.

Subgraphs Given a graph $G = (V, E)$ and a subgraph $H = (U, E(H))$, where $U \subseteq V$, if the subgraph is on a strict subset of vertices i.e., $U \neq V$, we denote the vertex set of H by $V(H)$. Moreover, we always denote the edge set of subgraph H by $E(H)$.

Remark 1.1. In the rest of this thesis, we might use the term *thin trees* in order to refer to *thin spanning trees.*

1.2 Combinatorial Thinness

Definition 1.1. Given a connected graph $G = (V, E)$, edge weight function $w : E \rightarrow \mathbb{R}_+$, and $\alpha \in (0, 1]$, we call a subgraph $H = (V, E(H))$ *combinatorially $\alpha-$thin subgraph of G with respect to w*, if for all proper subsets $S \subset V$ the following inequality holds:

$$|E(H) \cap \delta(S)| \leq \alpha |w(\delta(S))|,$$

i.e. H is a combinatorially $\alpha-$*thin* subgraph, if it contains at most $\alpha-$fraction of weighted edges of each cut-set of G.

Moreover, given a subgraph H, *combinatorial thinness of H*, denoted by $\alpha_{G,w}(H)$, is the smallest $\alpha \in [0, 1]$ such that H is combinatorially $\alpha-$thin subgraph of G with respect to w, i.e.,

$$\alpha_{G,w}(H) := \min\{\alpha : \ H \text{ is } \alpha - \text{thin subgraph of } G \text{ w.r.t weights } w\}.$$

Finally, we call a subgraph $T = (V, E(T))$ of G a *combinatorially $\alpha-$thin spanning* tree of G, if it is a spanning tree and a combinatorially $\alpha-$thin subgraph of G.

Definition 1.2. Given a connected graph $G = (V, E)$ and edge weight function $w : E \rightarrow \mathbb{R}_+$, *combinatorial thinness* of G, denoted by $\alpha_{G,w} \in (0, 1]$, corresponds to the smallest combinatorial thinness of its spanning trees, i.e.,

$$\alpha_{G,w} := \min\{\alpha_{G,w}(T) : \ T \text{ is a spanning tree of } G\}.$$

Remark 1.2. Given a connected graph $G = (V, E)$, where every edge $e \in E$ has weight equal to one ($w(e) = 1$), every spanning tree is a 1-thin tree of G, as for every cut $S \subsetneq V$ we have

$$|E(T) \cap \delta_G(S)| \leq |\delta_G(S)| = 1 \times w(\delta_G(S)).$$

.

Remark 1.3. Given a graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}_+$, we state that a spanning tree T *of* G *is combinatorially thin,* if $\alpha_{G,w}(T) < 1$. We also state that *graph* G *is combinatorially thin* if $\alpha_{G,w} < 1$.

Remark 1.4. In the rest of this thesis, whenever a weight function is not specified we assume that in the given graph all edges have weight equal to one. We may also use the notation α_G and $\alpha_G(T)$ instead of $\alpha_{G,\mathbb{1}}$ and $\alpha_{G,\mathbb{1}}(T)$, respectively.

Example 1.1. $\alpha_{K_4} = \frac{3}{4}$.

Proof. There are two non-isomorphic spanning trees for K_4 :

either (1) a star spanning tree S, or (2) a path tree P of length 3. Moreover, K_4 is a symmetric graph with its group of isomorphism equal to the group of all permutations on n elements, $\mathcal{S}_n$. Thus, any spanning tree T of K_n isomorphic to S or P has *thinness parameter* equal to $\alpha_{K_n}(S)$ or $\alpha_{K_n}(P)$, respectively. Finally, as shown in Figure (1.1), we have:

$$\alpha_{K_4} = \min\{\alpha_{K_4}(S), \alpha_{K_4}(P)\} = \min\{1, \frac{3}{4}\} = \frac{3}{4}.$$

□

Note that if a graph G has connectivity k, then there exists a cut-set that has at most weight k. Since every spanning tree has at least one edge from every cut-set, we obtain the following.

Lemma 1.1. *Suppose G is a given graph with connectivity k. Then,*

$$\alpha_{G,w} \geq \frac{1}{k}.$$

On the other hand, we intuitively expect that whenever the connectivity of graphs increases their spanning trees will have smaller thinness parameters. Goddyn in [17] conjectured the following statement, which has been named as *thin tree conjecture.*

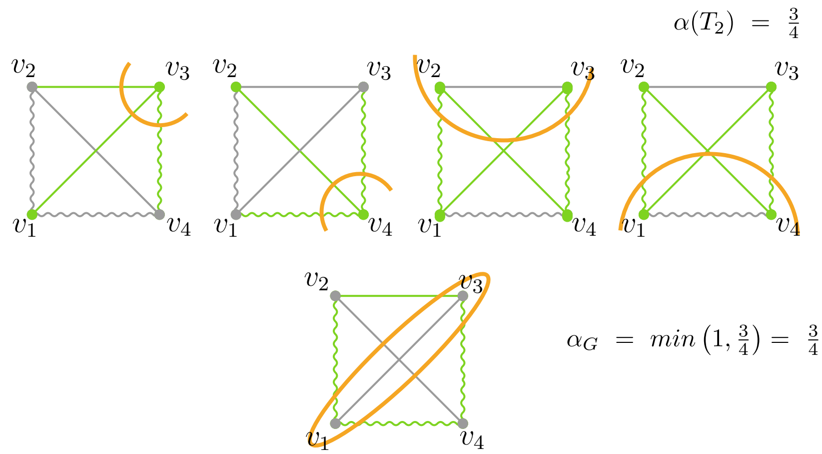


Figure 1.1: Combinatorial thinness of K_4: Each spanning tree is denoted by wavy edges. The orange curves represent the cuts and green edges are in the edges in each cuts.

Conjecture 1.1. *There exists a function $f : (0, 1] \rightarrow \mathbb{Z}_+$ such that for $\alpha \in (0, 1]$ every $f(\alpha)-$edge connected graph has a combinatorially $\alpha-$thin spanning tree.*

The thin tree conjecture has been proved for some families of structured graphs. For instance, Oveis Gharan and Saberi [16] proved that highly connected graphs with bounded genus have combinatorially thin spanning trees. However, we do not currently have any proof for the thin tree conjecture in the general setting.

In 2010, Asadpour et al. [3] suggested a stronger conjecture which is known as *strong thin tree conjecture.* The conjecture was proposed in a different setting, which can be translated into the following.

Conjecture 1.2. *There exists a constant $C \in \mathbb{R}_+$ such that every $k-$edge connected graph G has a combinatorially $\frac{C}{k}-$thin spanning tree.*

Finding efficient algorithms to obtain thin-spanning trees is also of our interest, as one of the main applications of thin trees is to utilize them in algorithms to solve other problems. Even though we can verify that a spanning tree is combinatorially thin for specific families of structured graphs, the general case is not believed to be an easy problem. In fact, verifying combinatorial $\alpha-$thinness for an arbitrary given spanning tree T is a NP-Complete problem. Therefore, introducing a relaxation of *combinatorial thinness* notion that is verifiable in polynomial time seems to be a natural next step.

1.3 Spectral Thinness

A more generalized and stronger notion of *thinness* is *spectral thinness*, which is defined by converting the information of graphs into the space of *symmetric positive semidefinite* matrices and specifically using the *Laplacian* matrices of graphs. We recall the definition of Laplacian matrices for a graph $G = (V, E)$ over edge weight functions $w : E \to \mathbb{R}_+$.

Definition 1.3. (Laplacain) Given a graph $G = (V, E)$ on n vertices, the *Laplacian* of G, denoted by $\mathcal{L}_G : \mathbb{R}^E \to \mathbb{S}^V$, takes a vector of edge weights $w \in \mathbb{R}^E$ as an input and returns the following symmetric matrix. For all $i, j \in V$ the ij-th entry of $\mathcal{L}_G(w)$ is defined as

$$(\mathcal{L}_G(w))_{ij} := \begin{cases} \sum_{\{i,\ell\}\in E} w_{i\ell} & \text{if } i = j \\ -w_{ij} & \text{if } i \neq j \text{ and } \{i,j\} \in E \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Moreover, we can also rewrite Laplacian of G as a mapping between $\mathbb{R}^E$ and space of symmetric matrices $\mathbb{S}^V$,

$$\mathcal{L}_G(w) = \sum_{\{i,j\}\in E} w_{ij}(e_i - e_j)(e_i - e_j)^\top, \tag{1.2}$$

where e_i are vectors of the standard basis for $\mathbb{R}^V$.

Definition 1.4. (Positive semidefinitness) A symmetric matrix $X \in \mathbb{S}^n$ is *positive semidefinite* if for all $v \in \mathbb{R}^n$,

$$v^\top X v \geq 0, \tag{1.3}$$

and we denote a positive semidefinite (PSD) matrix X by $X \succeq 0$.

Now, given a connected graph $G = (V, E)$, a weight function $w : E \to \mathbb{R}_+$ and a spanning tree T, the *spectral thinness* of T with respect to G and w, denoted by $\theta_{G,w}(T)$, is defined as follows.

Definition 1.5. (Spectral thinness) Let $G = (V, E)$ be a weighted graph with the weight function $w : E \to \mathbb{R}_+$, and let $\mathcal{L}_G(w)$ denote the Laplacian matrix of G. Then, a simple subgraph $H = (V, E(H))$ is $\theta-$*spectrally thin* if the following holds,

$$\mathcal{L}_H(\mathbb{1}) \preccurlyeq \theta.\mathcal{L}_G(w). \tag{1.4}$$

We define the *spectral thinness* of H with respect to G and w, denoted by $\theta_{G,w}(H)$, as the smallest value $\theta \in (0,1]$ satisfying the above inequality.

Moreover, we define *spectral thinness of* $G = (V,E)$ as the minimum spectral thinness over all spanning trees of G; i.e.,

$$\theta_{G,w} := \min\{\theta_{G,w}(T) : T \text{ is a spanning tree of } G.\}$$

Proposition 1.1. *Given a simple graph $G = (V,E)$ and $w \in \mathbb{R}_+^E$, if $T = (V, E(T))$ is a $\theta(T)-$ spectrally thin tree of G, then combinatorial thinness of T is upper bounded by θ, i.e.,*

$$\alpha_{G,w}(T) \leq \theta_{G,w}(T).$$

Proof. Note that by definition of PSD matrices for all vectors $x \in \mathbb{R}^V$ we have

$$x^\top(\theta_{G,w}(T)\mathcal{L}_G(w) - \mathcal{L}_T(\mathbb{1}))x \geq 0.$$

Therefore, the definition of spectral thinness directly implies that for any proper subsets $U \subset V$ the following holds:

$$u^\top\Big(\theta_{G,w}(T)\mathcal{L}_G(w) - \mathcal{L}_T(\mathbb{1})\Big)u \geq 0, \tag{1.5}$$

where $u \in \{0,1\}^V$ is the indicator vector of set $U \subset V$ defined as below.

$$u_i = \begin{cases} 1 & i \in U, \\ 0 & \text{otherwise.} \end{cases}$$

By rewriting the Laplacian matrix as in (1.2), we can compute the LHS of (1.5) as

$$\begin{aligned} u^\top\Big(\theta_{G,w}(T)\mathcal{L}_G(w) - \mathcal{L}_T(\mathbb{1})\Big)u &= \theta_{G,w}(T)\Big(\sum_{\{i,j\}\in E(G)} w_{ij}(u_i - u_j)^2\Big) - \sum_{\{i,j\}\in E(T)} (u_i - u_j)^2 \\ &= \theta_{G,w}(T)\Big(\sum_{\{i,j\}\in \delta_G(U)} w_{ij}(u_i - u_j)^2\Big) - \sum_{\{i,j\}\in \delta_T(U)} (u_i - u_j)^2 \\ &= \theta_{G,w}(T)w(\delta_G(U)) - |\delta_T(U)|. \end{aligned}$$

Therefore, we conclude that

$$\theta_{G,w}(T)w(\delta_G(U)) - |\delta_G(U) \cap E(T)| \geq 0.$$

Since for all $U \subset V$ the inequality holds, it implies that the graph is combinatorially $\theta_{G,w}(T)-$thin as well. Hence, we have $\alpha_{G,w}(T) \leq \theta_{G,w}(T)$. $\square$

1.4 Applications

Thin tree conjecture has some nice implications in the literature. For example, if the thin tree conjecture holds, it implies weak *Nowhere Zero 3-Flow* conjectured by Jaeger [23]. Another important implication is that if the *strong thin tree* conjecture holds, we have an $O(1)-$ approximation algorithm for the *Asymmetric Travelling Salesman Problem* (ATSP), suggested by Asadpour et al. in [3].

It is worth mentioning that the constant approximation of ATSP problem was proven by direct methods in 2017 by Svensson et al. [36]. Moreover, Lovász et al. [26] showed that Jaeger's *weak 3-Flow* conjecture holds for all $6-$edge connected graphs without using thin trees. Although confirming these implications does not give any information on whether *thin tree conjecture* holds, these confirmations may be considered a positive sign that the conjecture might hold.

In this section, we will briefly discuss applications of thin-spanning trees in solving ATSP and Nowhere Zero 3-Flow. We mainly focus on showing the deep relation between combinatorially thin trees and how utilizing thin spanning trees help solving some classical problems which had remained open for a long time.

1.4.1 ATSP and Thin Trees

This section is dedicated to the paper of Asadpour et al. [3] on $\log(n)/\log\log(n)$-approximation of the *Asymmetric Travelling Salesman* problem. For the symmetric version of *traveling salesman* problem there is the well-known Christofides algorithm, which gives a $1.5-$approximation factor for metric TSP. The general idea is to initially find a minimum cost spanning tree, augment the spanning tree into an Eulerian subgraph by finding an optimal matching of odd degree vertices of the tree, and finally, find a Hamiltonian cycle by short-cutting the Eulerian walk.

It was only recently that Karlin et al. in [24] showed that a slightly better approximation for TSP is possible. For ATSP problem, the authors in [3] presented an approximation algorithm similar to the Christofides algorithm. However, the specific choice of the underlying spanning tree, guarantees that the cost of the Eulerian augmentation of the spanning tree cannot be larger than the order of the thinness parameter.

Before we delve deeper into this discussion, we will provide the definition of *thinness* introduced in [3], and its relation to the strong thin-tree conjecture.

Definition 1.6. Let $G = (V, E)$ be an undirected connected graph. The spanning tree polytope of G, denoted by $SP(G)$ is defined as convex-hull of all spanning trees in G; i.e.,

$$SP(G) := \mathrm{Conv}\left\{ x_T \in \{0,1\}^E \ : T \text{ is a spanning tree in } G \right\}, \tag{1.6}$$

where $x_T \in \{0,1\}^E$ is the incidence vector of spanning tree T.

Proposition 1.2. *Let $G = (V, E)$ be an undirected connected graph. Then, we can rewrite the spanning tree polytope $SP(G)$ as follows*

$$SP(G) = \left\{ x \in \mathbb{R}^E \ : \ \begin{array}{c} x(E) = n-1 \\ x(E[U]) \leq |U| - 1 \ \forall\ U \subset V \\ x \geq 0 \end{array} \right\}. \tag{1.7}$$

In [3], the authors gave an algorithm to obtain an $\alpha-$thin spanning tree with α of order $\log n \ \log\log n$. Moreover, they used this algorithm to solve the Asymmetric Traveling Salesman Problem.

LP relaxation and ATSP problem

In the Asymmetric Traveling Salesman Problem, we have a complete directed graph $G = (V, A)$ on n vertices V with cost function $c \in \mathbb{R}_+^A$ which assigns a cost $c_{(u,v)}$ for every ordered pair of vertices $u, v \in V$. We also assume that the cost function satisfies the triangle inequality. Our goal is to find the minimum cost tour where each vertex is visited exactly once, or equivalently a minimum cost directed Hamiltonian cycle.

We will use $a = (u, v)$ as a directed edge $a \in A$ from $u \in V$ to $v \in V$.

In order to find an optimal solution for ATSP problem, first we consider the Held-Karp LP relaxation of ATSP (see [21]) on the directed graph G over the vertex set V:

$$
\begin{array}{lll}
\text{minimize} & \sum_{a \in A} c_a x_a & \\
\text{subject to} & x(\delta_G^+(U)) \geq 1, & \forall\, \emptyset \neq U \subsetneq V \\
& x(\delta_G^+(v)) = x(\delta_G^-(v)) = 1, & \forall v \in V \\
& x_a \geq 0, & \forall a \in A.
\end{array}
\tag{1.8}
$$

Remark 1.5. Given a feasible solution x^* for (1.8), for all subsets $U \subset V$, we have,

$$x^*(\delta_G^+(U)) = x^*(\delta_G^-(U)).$$

It is important to observe that every directed Hamiltonian cycle in G, which can also be viewed as a circulation, corresponds to an integral feasible point for (1.8). Conversely, every integral point in the feasible region of (1.8) corresponds to a Hamiltonian cycle. However, the optimal solution to Held-Karp relaxation, $x^* \in \mathbb{R}_+^A$, is not necessarily an integral solution. However, $\mathrm{OPT_{HK}} = c(x^*) = \sum_{a \in A} c_a x_a^*$ is a lower bound to the ATSP problem. Asadpour et al. in [3] gave an algorithm to approximate an integral solution for Held-Karp LP 1.8 by utilizing thin spanning trees.

To get an integral solution to ATSP, we can use a rounding method by considering the underlying subgraph driven by the support of x^*, and use it in the process of outputting a Hamiltonian cycle.

Let x^* be an optimal solution to the Held-Karp relaxation (1.8). We firstly, define the set of undirected edges in the support of $\bar{x}$ as,

$$\bar{E} := \{\{u, v\} \in E : \ x_{uv}^* > 0 \text{ or } x_{vu}^* > 0\},$$

and define graph $\bar{G} = (V, \bar{E})$ be the undirected graph with cost function $\bar{c} : \bar{E} \to \mathbb{R}_+$, such that

$$\bar{c}(\{u, v\}) := \min \Big\{ c(a) : \ a \in \mathrm{supp}(x^*) \cap \{(u, v), (v, u)\} \Big\}.$$

Since graph G has asymmetric costs, x^* is possibly asymmetric as well. To make it symmetric, we define the following vector $\bar{x} \in \mathbb{R}^{\bar{E}}$.

$$\bar{x}_{uv} := \left(\frac{n-1}{n}\right)(x_{uv}^* + x_{vu}^*) \qquad \forall\ \{u, v\} \in \bar{E}. \tag{1.9}$$

Claim 1.1. *The vector $\bar{x} \in \mathbb{R}^{\bar{E}}$ defined in (1.9) is in the relative interior of the spanning tree polytope of graph $\bar{G}$.*

Proof. See Appendix A. □

Next, Asadpour et al. in [3] introduced an algorithm that in the undirected graph $\bar{G}$ finds an $O(\frac{\log n}{\log\log n})$-combinatorially thin spanning tree with respect to edge weights $\bar{x}$, namely $\overline{T} = (V, E(\overline{T}))$, such that the cost of the spanning tree $\bar{c}(\overline{T}) := \sum_{e\in E(\overline{T})} \bar{c}_e$ satisfies

$$\bar{c}(\overline{T}) \leq 2\,\mathrm{OPT}_{\mathrm{HK}}\,. \tag{1.10}$$

By utilizing the combinatorially thin tree $\overline{T}$ in the undirected graph $\bar{G}$, they suggested a method to direct the given tree into a subgraph $T = (V, E(T))$ in original directed graph G such that

$$c(T) \leq \bar{c}(\overline{T}). \tag{1.11}$$

For each edge $\{u, v\} \in E(\overline{T})$ we put one of the arcs $(u, v) \in A$ or $(v, u) \in A$ in $E(T)$; i.e., if $c(u, v) < c(v, u)$, the edge $\{u, v\} \in E(\bar{T})$ corresponds to arc $(u, v) \in E(T)$. In the case of equality, we select one of these arcs arbitrarily.

For the rest of the discussion we aim to show that T can be extended to an Eulerian tour, and then by short-cutting we can obtain a directed Hamiltonian cycle which gives a good approximation of the minimum cost directed Hamiltonian cycle in G.

$O(\frac{log(n)}{loglog(n)})$ Approximation Algorithm for ATSP

Lemma 1.2. *Given the graph $\bar{G} = (V, \bar{E})$ and vector $\bar{x}$ as defined eirlier, let $\overline{T}$ be the $\alpha_{\bar{G},\bar{x}}(\bar{T})$-combinatorially thin tree obtained in [3], where $\alpha_{\bar{G},\bar{x}}(\bar{T}) = O(\frac{\log n}{\log\log n})$ and*

$$\bar{c}(\overline{T}) \leq 2\,\mathrm{OPT}_{\mathrm{HK}}\,.$$

Then, for the directed tree $T = (V, E(T))$ obtained from $\overline{T}$ as described earlier, there exists an integer circulation f in G such that,

i. f contains all edges of T; i.e., $f(a) \geq 1$ for $a \in E(T)$;

ii. the total circulation flow $f(A) = \sum_{a\in A} f(a)$ is upper bounded by

$$f(A) \leq c(T) + 2\alpha_{\bar{G},\bar{x}}(\bar{T})\,\mathrm{OPT}_{\mathrm{HK}} \tag{1.12}$$

$$\leq (2 + 2\alpha_{\bar{G},\bar{x}}(\bar{T}))\,\mathrm{OPT}_{\mathrm{HK}}\,. \tag{1.13}$$

In order to prove this lemma, we need to use *Hoffman Circulation Theorem*, which we state in the following. To find the proof see [11].

Theorem 1.1. *(Hoffman's circulation[11]) Given a directed graph $G = (V, A)$, $\ell \in \mathbb{R}_+^A$, $u \in (\mathbb{R}_+ \cup \{\infty\})^A$, such that $\ell \leq u$, there is a circulation f with $\ell \leq f \leq u$ if and only if every $U \subset V$ and $\bar{U} = V \setminus U$ satisfy*

$$u(\delta^+(U)) \geq \ell(\delta^+(\bar{U})) = \ell(\delta^-(U).$$

Moreover, if ℓ, u are integral, then there also exists an integral circulation for G.

Given the above theorem, we now proceed to prove Lemma 1.2:

proof of Lemma 1.2. We define capacities ℓ, u as follows,

$$\ell(i,j) := \begin{cases} 1, & \text{if } (i,j) \in E(T) \\ 0, & \text{otherwise.} \end{cases}$$

Also,

$$u(i,j) := \begin{cases} 1 + 2\alpha_{\bar{G},\bar{x}} x^*(ij), & \text{if } (i,j) \in E(T) \\ 2\alpha_{\bar{G},\bar{x}}, & \text{otherwise.} \end{cases}$$

We claim that the defined ℓ, u satisfy the Hoffman's circulation theorem, or equivalently, for every proper subset $U \subset V$ the inequality $u(\delta^+(\bar{U})) \geq \ell(\delta^+(U))$ holds:

$$\begin{aligned}
\ell(\delta^+(U)) &= \ell(\delta^+(U) \cap E(T)) + \ell(\delta^+(U) \setminus E(T)) \\
&= |\delta^+(U) \cap E(T)| && \text{(by definition of } \ell) \\
&\leq |\delta_{\bar{G}}(U) \cap E(\bar{T})| && (\bar{G}, \bar{T} \text{ are underlying graph of } G, T) \\
&\leq \alpha_{\bar{G},\bar{x}} \bar{x}(\delta_{\bar{G}}(U)) && \text{(by thinness condition)} \\
&= \alpha_{\bar{G},\bar{x}} (\frac{n-1}{n})(x^*(\delta^+(U)) + x^*(\delta^-(U))) && \text{(by definition of } \bar{x}) \\
&= 2\alpha_{\bar{G},\bar{x}} (\frac{n-1}{n}) x^*(\delta^-(U)) && \text{(by Remark 1.5)} \\
&\leq 2\alpha_{\bar{G},\bar{x}} x^*(\delta^-(U)) \\
&\leq u(\delta^-(U)) = u(\delta^+(\bar{U})) && \text{(by definition of } u).
\end{aligned}$$

Note that with the above inequality holding, by Theorem 1.1, we have a circulation f in the directed graph G where the total cost of f is at most

$$c(f) := \sum_{a \in A} c(a) f(a) \leq \sum_{a \in A} c(a) u(a).$$

However, note that this circulation is not necessarily integral as u is not an integral function. To overcome this problem, we consider the following LP.

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{a \in A} c(a) f(a) \\ \text{subject to} & A_G f = \mathbf{0}, \\ & \ell(a) \leq f(a), \quad \forall a \in A \end{array} \tag{P1}$$

where A_G is the adjacency matrix of original graph G. It is well known that if $\ell(\cdot)$ is integral then (P1) has an integral optimal solution f^* (see Corollary 12.2 in [34]). Also, note that f is a feasible solution for (P1) as well. Thus, we have

$$\begin{aligned} c(f^*) &\leq c(f) \\ &\leq \sum_{a \in A} c(a) u(a) \\ &= \sum_{a \in E(T)} c(a) + 2\alpha_{\bar{G},\bar{x}} \sum_{a \in A} c(x^*(a)) \\ &= c(T) + 2\alpha_{\bar{G},\bar{x}}(\bar{T}) \, \mathrm{OPT}_{\mathrm{HK}} \\ &\leq (2 + 2\alpha_{\bar{G},\bar{x}}(\bar{T})) \, \mathrm{OPT}_{\mathrm{HK}} . \end{aligned}$$

□

1.4.2 Nowhere Zero 3-Flow and Thin Trees

Definition 1.7. Let $G = (V, A)$ be a directed graph and X be an Abelian group. A *flow assignment* function $f : A \to X$ is a function that assigns values from group X to arcs of graph. Moreover, for a vertex $v \in V$, we define the *flow boundary of* v with respect to f to be

$$\Delta v := \sum_{e \in \delta^+(v)} f(e) - \sum_{e \in \delta^-(v)} f(e),$$

where the summation is with respect to the group X.

In our discussion, a flow assignment for a given graph $G = (V, A)$ is valid if the flow boundary of each vertex $v \in V$ is equal to zero. Let $X^* = X \setminus \{0\}$ be the set of nonzero elements of the group X. We define flow assignment $f : A \to X^*$ a *nowhere zero $X-$flow* if we have flow boundary of all vertices $v \in V$ to be equal to zero, i.e. $\Delta v = 0$. If graph $G = (V, E)$ is undirected, then G has a nowhere zero $X-$flow if the graph has a *nowhere zero $X-$flow* for some fixed orientation of edges in E.

Definition 1.8. Let $G = (V, E)$ be an undirected graph. Then, a flow assignment $f : E \to \{1, \ldots, k-1\}$ is a *nowhere zero $k-flow$* if there exists an edge orientation of G such that the flow boundary of all vertices is equal to zero, with respect to sum in $\mathbb{Z}$.

Tutte [39, 40] showed a strong connection between nowhere zero $k-$flow and $\mathbb{Z}_k$ $-$flow.

Theorem 1.2. *Let $G = (V, E)$ be an undirected graph. Then, G has a nowhere zero $\mathbb{Z}_k$ $-$flow if and only if G has nowhere zero k-flow.*

An important implication of the above theorem is that if a graph G has $k-$flow, then it has $k'-$flow for all $k' \in \mathbb{N}$ such that $k' \geq k$.
Finding the relation between edge connectivity and the existence of smallest values $k \in \mathbb{N}$ such that nowhere zero $k-$flow exists is one of the most interesting questions in the literature. One of the main open questions was specifically how much connectivity is needed for nowhere zero $3-$flows to exist. Tutte conjectured in [39, 40] the following.

Conjecture 1.3. *(Tutte's $3-$flow conjecture) Every $4-$edge connected graph has a nowhere zero $3-$flow.*

This conjecture which is also known as *Tutte's $3-$flow conjecture* remained open for decades and is open to this day. A weakened version of Tutte's conjecture was proposed by Jaeger [23], which is known as *weak $3-$flow conjecture*.

Conjecture 1.4. *(weak $3-$flow conjecture) There exists a constant k such that every $k-$edge connected graph has a nowhere zero $3-$flow.*

Thomassen [37] proved weak $3-$flow conjecture by showing every 8-edge connected graph has $3-$flow. Later Lovász et al. [26] improved the connectivity by proving every 6-edge connected graphs has a nowhere zero $3-$flow. Although weak $3-$flow conjecture has been proven affirmatively, it is important for our discussion to see its relation to thin tree conjecture. We will briefly discuss that if graph G has a thin spanning tree T with small thinness $\alpha_G(T) \leq \frac{1}{3}$, then it also has a nowhere zero $3-$flow (the idea is from Goeman's talk [19].)

Definition 1.9. (O-join) Given a graph $G = (V, E)$ and a set $O \subseteq V$, where $|O|$ is even, an *O-join* of G is a subset of edges $J \subseteq E$ such that the odd degree vertices of subgraph $H = (V, J)$ are exactly the vertices in O.

Theorem 1.3. *Let a simple graph $G = (V, E)$ be given. Suppose G is combinatorially thin with $\alpha_G \leq \frac{1}{3}$, and let thin spanning tree $T = (V, E(T))$ have $\alpha_G = \alpha_G(T)$. Then, we claim that G has a nowhere-zero 3-flow.*

Proof. Let $H = (V, E(H))$ be a subgraph of G, where $E(H) = E \setminus E(T)$. Suppose $O_H \subseteq V$ is the set of odd vertices in H.

Then, it is possible to find an O_H-join $J \subseteq E(T)$ in G. Let $G_0 = (V, E_0)$, be the graph which is defined by

$$E_0 = E(H) \cup J,$$

and let $E_1 = E(T) \setminus J$. Since all vertices of G_0 have even degrees, it has an Eulerian tour.

Now, in the original graph G, let A_0 be the directed version of edges $E_0 \subseteq E$ clockwise in the order of the given Eulerian tour. Also, we will direct edges of E_1 arbitrarily into edges graph A_1.

Let $\vec{G} = (V, A)$ be the directed version of G. We define the capacity functions $u, \ell : A \to \{1, 2\}$ as follows,

$$\ell(a) = \quad 1 \qquad \forall a \in A,$$
$$u(a) = \begin{cases} 2, & \text{if } a \in A_0 \\ 1, & \text{if } a \in A_1. \end{cases}$$

Next, we will show that by Hoffman's circulation theorem (Theorem 1.1) the suggested capacity functions ensure that $\vec{G}$ has an integral circulation, and since the circulation has only flow 1 or 2 on the arcs, it is indeed a nowhere zero 3-flow.

To show that an integral circulation exists, we need to show that for all $U \subset V$, we have

$$u(\delta^+(U)) \geq \ell(\delta^-(U)).$$

Note that set of edges in A_0 form an Eulerian walk. Fix any $U \subset V$, and let $t := |A_0 \cap \delta^+(U)|$. Then,

$$|A_0 \cap \delta^+(U)| = |A_0 \cap \delta^-(U)| = t. \tag{1.14}$$

Therefore, we have

$$|A_0 \cap (\delta^+(U) \cup \delta^-(U))| = 2t$$

Further, we know that $E_1 \subset E(T)$, and therefore,

$$\begin{aligned}\frac{|E_1 \cap \delta_G(U)|}{|\delta_G(U)|} &= \frac{|A_1 \cap (\delta^-(U) \cup \delta^+(U))|}{|A_0 \cap (\delta^-(U) \cup \delta^+(U))| + |A_1 \cap (\delta^-(U) \cup \delta^+(U))|} \\ &= \frac{|A_1 \cap (\delta^-(U) \cup \delta^+(U))|}{2t + |A_1 \cap (\delta^-(U) \cup \delta^+(U))|}.\end{aligned}$$

Since $\frac{|E_1 \cap \delta_G(U)|}{|\delta_G(U)|} \leq \alpha_G(T) \leq \frac{1}{3}$ we get,

$$|A_1 \cap (\delta^-(U) \cup \delta^+(U))| \leq t.$$

Thus, we can upper bound $\ell(\delta^-(U))$ as

$$\begin{aligned}\ell(\delta^-(U)) &= |A_0 \cap \delta^-(U)| + |A_1 \cap \delta^-(U)| \\ &= t + |A_1 \cap \delta^-(U)| \\ &\leq 2t.\end{aligned}$$

On the other hand, since $U(a) = 2$ for all $a \in A_0$, it also follows that

$$\begin{aligned}u(\delta^+(U)) &= 2|A_0 \cap \delta^+(U)| + |A_1 \cap \delta^+(U)| \\ &= 2t + |A_1 \cap \delta^+(U)| \\ &\geq \ell(\delta^-(U)).\end{aligned}$$

Thus, Hoffman's condition holds and G has a nowhere-zero 3-flow. □

1.5 Thesis Outline

In Chapter 2, we show combinatorial lower bounds and upper bounds for existence of combinatorial thin trees in specific families of structured graph. More precisely, we first show that in the graphs without two edge disjoint spanning trees, combinatorially thin trees do not exists. Moreover, by using a technique of [29], we present families of graphs that have two edge disjoint spanning trees but do not have any combinatorially thin spanning trees. Then, we will investigate a special case of result by Oveis-Gharan and Saberi [16] on finding combinatorially thin trees in highly connected graphs with bounded genus. We will slightly extend their algorithm to work on weighted planar graphs, provide a fairly different approach on showing the correctness, and getting a slight improvement in the thinness parameter. Finally, we will describe an equivalent algorithm and show its equivalence to the primal algorithm.

In Chapter 3, we will focus on spectral properties of graphs and their relation to thin spanning trees. We start by showing a slightly better upper bound on the combinatorial thinness of some families of distance regular graphs by utilising spectral and expansion properties of graphs. Next, we will begin our discussion on spectrally thin spanning trees by presenting general methods to lower bound and upper bound spectral thinness. Finally, we investigate the spectral thinness of some families of structured graphs. More specifically, we will focus on circulant graphs and exploit their symmetries to obtain bounds on the spectral thinness.

Chapter 2

Combinatorially Thin Trees in Structured Graphs

If thin tree conjecture (Conjecture 1.1) holds, it suggests that having a highly connected graph $G = (V, E)$, we can prove the existence of a combinatorially thin tree T in G, only by using information on the edge connectivity of the graph G. Although this is a strong hope which has not been proven yet, there are relevant studied questions in the literature, some of which are still open.

Some interesting questions were suggested by Goemans [19] are as follows.

1. How small the connectivity of the graph can be while having a combinatorially thin spanning tree?

2. Can we find counterexamples to show specific families of graphs that do not have combinatorial thin trees?

3. Is there any family of graphs that we can prove the thin tree conjecture for?

To address these questions, we will provide some interesting existing results in this chapter. Firstly, by elementary reasoning, we will show that families of graphs without edge-disjoint spanning trees do not have combinatorially thin trees. Next, we will show results in [29] that suggest a family of planar graphs $\mathcal{G}$ in which all graphs $G_n \in \mathcal{G}$ have two edge-disjoint spanning trees, but they do not have a combinatorially $\alpha-$thin tree for $\alpha \leq 1 - \epsilon$, where ϵ is a small constant. In fact, Merker and Postle in [29] showed that there exists a family of $4-$regular and $4-$edge connected planar graphs, denoted by $\mathcal{MP}$ such that for any $\alpha < 1$ there exists a graph $G \in \mathcal{MP}$ with $\alpha_G > \alpha$. Using Merker and Postle's ideas, we will introduce some other families of $4-$regular graphs with same property.

Next, we will present a result by Oveis Gharan and Saberi [16] which gives an algorithm to find a combinatorially thin tree in highly connected graphs with bounded genus. We provide a slightly different and more careful analysis for the case of planar graphs. In particular, we only provide their result for planar graphs, but for these graphs we will work with a more general setting, which allows arbitrary edge weights.

2.1 Graphs Without Edge-Disjoint Spanning Trees

Firstly, we show that the family of graphs without two edge-disjoint spanning trees are not combinatorially thin.

Theorem 2.1. *Let $G = (V, E)$ be a connected graph without two edge-disjoint spanning trees. Then, G is not combinatorially-thin.*

Proof. For the sake of contradiction, suppose $G = (V, E)$ has a combinatorially thin tree T with thinness parameter $\alpha_G(T) < 1$. Therefore, for each cut $\delta_G(U) \subset E$ we have at least one edge $e \in \delta_G(U)$ is not an edge of tree T. As a result, the graph $G \setminus T$ is a connected graph, and contains a spanning tree T' none of its edges are in $E(T)$. Since T' is a spanning tree in G as well, T and T' are two edge-disjoint spanning trees of G, which is a contradiction. $\square$

2.2 Families of $4-$Edge Connected Graphs Without Combinatorially Thin Trees

We proceed to the next step, and we present existing results on the combinatorial thinness of some graphs that have two edge-disjoint spanning trees. Merker and Postle [29] introduced a family of $4-$edge connected planar graphs, $\mathcal{MP}$, where for each $\alpha < 1$ there exists a graph G in the family with no combinatorially $\alpha-$thin spanning tree.f It is worth noting that as a corollary of the Nash-Williams theorem, every $4-$edge-connected graph has two edge-disjoint spanning trees as well.

Theorem 2.2. *(Nash-Williams [31]) A given graph $G = (V, E)$ has k edge-disjoint spanning trees if and only if for every partitioning $P = \{V_1, \ldots, V_p\}$ of G, partitioning graph $G/P = (P, E_P)$ obtained from P has the following property:*

$$|E_P| \geq k(p-1). \tag{2.1}$$

Proof. See Theorem 1 in [31]. $\square$

Theorem 2.3. *For every $\alpha < 1$, there exists a graph $G = (V, E)$ with two edge-disjoint spanning trees such that G does not have any $\alpha-$thin tree.*

The goal in this section is to prove the given theorem. In order to present the proof of Theorem 2.3, as described in [29] in detail, we first present some tools and facts which were used in the proof explicitly.

2.2.1 Preliminaries

Tools and Facts on $4-$Regular $4-$edge Connected Graphs

Lemma 2.1. *Let $G = (V, E)$ be a 4-regular graph on n vertices, and let $T_0 = (V, E(T_0))$ and $T_1 = (V, E(T_1))$ be two edge-disjoint spanning trees of G. Then,*

$$|E \setminus (E(T_0) \cup E(T_1))| = 2.$$

Proof. Since G is a 4-regular graph, we have $d_G(v) = 4$, for all $v \in V$. Thus, the number of edges in G is equal to

$$|E| = \frac{1}{2} \sum_{v \in V} d_G(v) = 2|V|.$$

Moreover, since T_0 and T_1 are edge-disjoint spanning trees, the total number of edges in the union of T_0 and T_1 is equal to $|E(T_0) \cup E(T_1)| = 2(|V| - 1)$. Thus, the number of remaining edges in the graph, if we remove edges of T_0 and T_1, is equal to

$$|E| - |E(T_0) \cup E(T_1)| = 2|V| - 2(|V| - 1) = 2.$$

$\square$

Lemma 2.2. *Let $G = (V, E)$ be a connected graph with maximum degree $D \in \mathbb{N}$, and let $T = (V, E(T))$ be an arbitrary spanning tree of G.Then, there exists an edge $f \in E(T)$ and a partitioning $[V_1, V_2]$ of V such that $\{f\} = \delta_T(V_1)$ and*

$$\min\{|V_1|, |V_2|\} \geq \frac{|V| - 1}{D}. \tag{2.2}$$

Proof. See Appendix A.

□

Corollary 2.1. *Given a spanning tree $T = (V, E(T))$ of a 4-regular graph $G = (V, E)$, there exists an edge $e \in E(T)$ such that both connected components of $T \setminus e$ have size at least $\frac{|V|-1}{4}$.*

Intersection of cuts and paths

Lemma 2.3. *Let $G = (V, E)$ be a simple graph and let $A \subset V$ be an $s-t$ cut for vertices $s, t \in V$ with $s \in A$. Then,*

i. *$\delta_G(A)$ contains at least one edge of any $s-t$ path P.*

$$|\delta(A) \cap E(P)| \geq 1.$$

ii. *$\delta_G(A)$ contains at least two edge of any cycle C containing s, t.*

$$|\delta(A) \cap E(C)| \geq 2.$$

Proof. (i.) Let P be a path with the sequence of vertices $s = v_1, \ldots, v_\ell = t$. We have $v_1 \in A$ and $v_\ell \notin A$; therefore, there exists an index $j \in \{1, \ldots, \ell - 1\}$ such that $v_j \in A$ and $v_{j+1} \notin A$. Then,

$$\{v_j, v_{j+1}\} \in \delta(A) \cap E(P).$$

(ii.) This part is a direct result of (i.) as every cycle containing s and t has two edge-disjoint $s-t$ paths in G. □

2.2.2 $4-$Edge-Connected Planar Family of Graphs

Definition 2.1. (Cartesian product) Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *Cartesian Product of G_1 and G_2,* denoted by $G_1 \,\square\, G_2$, is the graph with vertex set $V(G_1 \square G_2) := V_1 \times V_2$ and edge set $E(G_1 \square G_2)$. The edges of Cartesian product are of form $\{\{x_1, y_1\}, \{x_2, y_2\}\} \in E(G_1 \square G_2)$, where either $x_1 = x_2$ and $\{y_1, y_2\} \in E_2$ or $y_1 = y_2$ and $\{x_1, x_2\} \in E_1$.

For a given $k \in \mathbb{N}$, the Cartesian product of cycle C_{4k} and path P_{4k} is a planar graph, but it is neither 4-regular nor 4−edge connected. In order to make it 4−edge connected and 4-regular, the authors in [29] suggested adding k *side* vertices to each side of the grid with degree 3 vertices and connecting each side vertex to 4 distinct vertices with degree three. We define $MP_{4k} := (V, E)$ to be the Merker-Postle suggested graph when the length and the height of its grid are equal to $4k$. Finally, lets denote the family of MP_{4k} by

$$\mathcal{MP} := \{MP_{4k} : k \in \mathbb{N}\}.$$

Figure 2.1 shows the MP_8 graph.

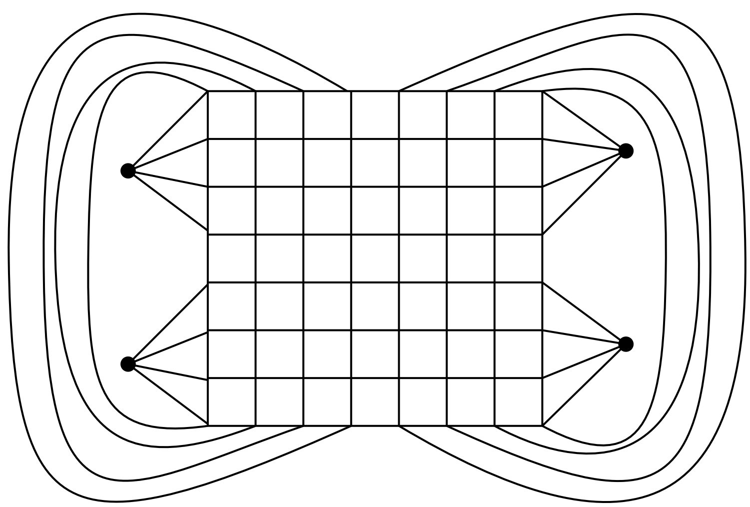


Figure 2.1: Graph MP_8 with 68 vertices

Lemma 2.4. *Let $MP_{4k} := (V, E)$, be a graph in $\mathcal{MP}$. Then, any subset $A \subset V$ with $k^2 \leq |A| \leq |V| - k^2$ has at least k edges in the boundary $\delta_{MP_{4k}}(A)$.*

We will postpone the proof of Lemma 2.4 to the end of this section. Instead, we first show that the family of $\mathcal{MP}$ graphs' combinatorial thinness cannot be bounded from above by any $\alpha \in (0, 1)$.

Theorem 2.4. *For any given value $\alpha \in (0, 1)$, there exists a graph $MP_{4k} \in \mathcal{MP}$ for some k depending on α with two edge-disjoint spanning trees such that MP_{4k} does not have any combinatorially $\alpha-$thin tree.*

Proof. Let $k > \frac{3}{1-\alpha}$ and let $MP_{4k} = (V, E) \in \mathcal{MP}$ be as described, and suppose MP_{4k} has a combinatorially $\alpha-$thin tree $T_0 = (V, E(T_0))$. Then, $MP_{4k} \setminus E(T_0)$ is still connected and has another spanning tree T_1. By Lemma 2.2, there exists an edge $f \in E(T_1)$ such that $\{f\} = \delta_{T_1}(S)$ for a cut $S \subset V$ with $\min\{|S|, |V \setminus S|\} \geq \frac{|V|-1}{4}$. Note that

$$\frac{|V|-1}{4} = \frac{16k^2 + 2k - 1}{4} \geq k^2.$$

Therefore, by Lemma 2.4, we have that the number of edges in $\delta_{MP_{4k}}(S)$ is at least k. Moreover, we have

$$|\delta_{MP_{4k}}(S) \cap E(T_1)| = |\{f\}| = 1.$$

Finally, by Lemma 2.1, we know that $|E \setminus (E(T_0) \cup E(T_1))| = 2$. Thus,

$$|E(T_0) \cap E(S, \bar{S})| \geq |E(S, \bar{S})| - 3,$$

and $\alpha_{MP_{4k}}(T_0)$ is lower bounded by,

$$\alpha_{MP_{4k}}(T_0) \geq \frac{|E(T_0) \cap E(S, \bar{S})|}{|E(S, \bar{S})|} \tag{2.3}$$

$$\geq 1 - \frac{3}{|E(S, \bar{S})|} \tag{2.4}$$

$$\geq 1 - \frac{3}{k}. \tag{2.5}$$

Therefore, $\alpha_{MP_{4k}}(T_0) > \alpha$ for $k > \frac{3}{1-\alpha}$. $\square$

Now, we will present the proof of Lemma 2.4.

Proof. (Lemma 2.4) Let $A \subset V$ satisfy the given condition, and suppose $\delta(A) < k$.

We define the set of *right vertices* $R \subset V$ to be the set of side vertices that are in right side of MP_k,

$$R = \{r_i : \ i \in [k]\},$$

the *left vertices* $L \subset V$ as

$$L = \{\ell_i : \ i \in [k]\},$$

and *Grid vertices* $G \subset V$ to be the set of vertices on the $4k \times 4k$ grid, where v_{ij} is the vertex in the row i and column j.

$$\mathcal{G} = \{v_{ij} : i, j \in [4k]\}.$$

Note that $V = R \cup L \cup \mathcal{G}$. We call a column $j \in [4k]$ a *full column* if for all $i \in [4k]$ the vertices v_{ij} are in A, an *empty column* if no vertex $v_{ij} \in A$, and an *alternating column* if there exists i and i' such that $v_{ij} \in A$ and $v_{i'j} \notin A$.

Moreover, we define a row $i \in [4k]$ to be a *full row* if the vertices

$$\{v_{ij} : j \in [4k]\} \cup \{r_{\lceil \frac{i}{4} \rceil}, \ell_{\lceil \frac{i}{4} \rceil}\} \subset A,$$

an *empty row* if

$$\{v_{ij} : j \in [4k]\} \cup \{r_{\lceil \frac{i}{4} \rceil}, \ell_{\lceil \frac{i}{4} \rceil}\} \subset V \setminus A,$$

and otherwise, an *alternating row.*

Note that if the column $j \in [4k]$ is alternating then there are vertices $v_{ij} \in A$, $v_{i'j} \notin A$ for $i, i' \in [4k]$. Since every column is a cycle, by Lemma 2.3 (ii.) there are least two edges of column j that are in $\delta(A)$. Similarly, by Lemma 2.3 (i.) if the row $i \in [4k]$ is alternating then at least on edge of row i is in $\delta(A)$. As a result, we immediately get that we have at most $\lceil \frac{k}{2} \rceil - 1$ many alternating columns, and at most $k - 1$ alternating rows.

Therefore, at least $3k + 1$ rows and $\lfloor \frac{7k}{2} \rfloor + 1$ columns must be either empty or full. However, the following conditions must holds.

1. The rows (columns) which are not alternating rows (columns) must be either all empty or all full, since otherwise, all columns(rows) will be *alternating.* In other words, there are no indices $i, i' \in [4k]$, such that row (column) i is *full* and row (column) i' is *empty.*

2. There cannot be an *empty (full)* row i and a *full (empty)* column j in MP_{4k} at the same time, since vertex v_{ij} is either in A or not.

Therefore, the columns and rows that are not *alternating* are either all *full* or all *empty.* Suppose all the remaining rows and columns are *full.* Define

$$\begin{aligned} F_r := \{v_{ij} :\ i, j \in [4k] \ \&\ i \text{ is a full row}\} \cup \{r_{\lceil \frac{i}{4} \rceil} : i \in [4k] \ \&\ i \text{ is a full row}\} \\ \cup \{\ell_{\lceil \frac{i}{4} \rceil} : i \in [4k] \ \&\ i \text{ is a full row}\}, \end{aligned}$$

and

$$F_c := \{v_{ij} : \ i, j \in [4k] \ \& \ j \text{ is a full column}\}.$$

Also, let

$$T := F_r \cap F_c.$$

By our definition of full rows and columns, we have that $F_r \cup F_c \subset A$. As mentioned earlier the number *full* rows and columns is at least $3k + 1$ and $\lfloor \frac{7k}{2} \rfloor + 1$, respectively. Moreover, at least $2\left\lceil \frac{3k+1}{4} \right\rceil$ of the side vertices also in A. Thus, the total number of vertices in A is at least,

$$\begin{aligned}
|A| &\geq (|F_r| + |F_c| - |T|) \\
&= \left(4k(3k+1) + 2\left\lceil \frac{3k+1}{4} \right\rceil\right) + \left(4k(\left\lfloor \frac{7k}{2} \right\rfloor + 1)\right) - \left((\left\lfloor \frac{7k}{2} \right\rfloor + 1)(3k+1)\right) \\
&\geq \left(4k(3k+1) + 2\left\lceil \frac{3k+1}{4} \right\rceil\right) + 4k(\frac{7k}{2} + \frac{1}{2}) - \left((\frac{7k}{2} + 1) \times (3k+1)\right) \\
&= 15.5k^2 + 2\left\lceil \frac{3k+1}{4} \right\rceil - \frac{k}{2} - 1 \\
&> 15k^2 + 2k \\
&= |V| - k^2,
\end{aligned}$$

which contradicts our assumption that $|A| \leq |V| - k^2$.
The case that all non-alternating rows and columns are empty gives a contradiction with the same method as well. □

2.2.3 $C_{4k} \,\square\, C_{4k}$

Another family of 4-regular graphs with similar properties is the family of graphs obtained from the Cartesian Product of cycles of size $4k$, denoted by $G_{4k} = C_{4k} \square C_{4k}$, for $k \in \mathbb{N}$. Figure 2.2 shows an instance of $C_8 \square C_8$ graph.

We denote this family of graphs by

$$\mathcal{C} := \{G_{4k} \ : \ k \in \mathbb{N}\}.$$

These graphs share all the properties of $\mathcal{MP}$, except they are not planar graphs. However, this is not much of a barrier as we do not use planar properties in the analysis.

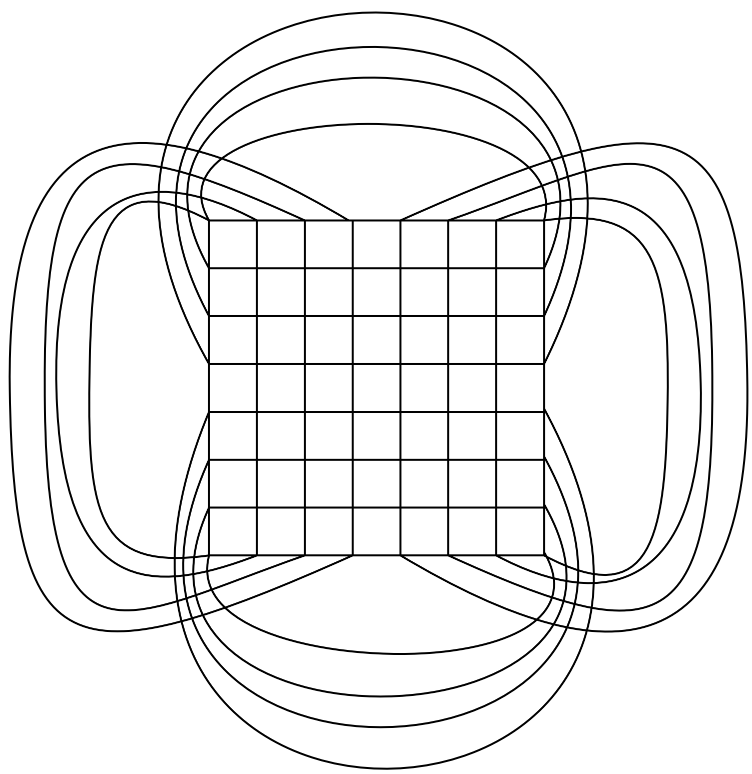


Figure 2.2: Graph $C_8 \,\square\, C_8$

Lemma 2.5. *Suppose $G_{4k} = (V, E)$ is given for $k \in \mathbb{N}$, and let $A \subset V$ such that $k^2 \leq |A| \leq |V| - k^2$, then we have $\delta_G(A) \geq 2k$.*

Proof. For the sake of contradiction, suppose $A \subset V$ and $k^2 \leq |A| \leq |V| - k^2$, but we have $\delta_G(A) < 2k$. Let the set v_{ij} for $i, j \in [4k]$ be the set of vertices of G_{4k} where v_{ij} is the vertex in the row i and column j. Similar to the MP_{4k} graphs, we call the row i a *full row* if for all $j \in [4k]$ the vertices v_{ij} are in A, an *empty row* if no vertex $v_{ij} \in A$, and an *alternating row* if there exists j and j' such that $v_{ij} \in A$ and $v_{ij'} \notin A$. Full, empty and alternating columns are defined similarly.

Notice that if the column $j \in [4k]$ is alternating then there are vertices $v_{ij} \in A$, $v_{i'j} \notin A$ for $i, i' \in [4k]$. Since every column is a cycle, by Lemma 2.3 (ii.) there are two edges of column j that are in $\delta(A)$. Similarly, if the row $i \in [4k]$ is alternating then there are at least two edges of row i are in $\delta(A)$. From this we immediately get that we have at least $3k + 1$ rows and columns that are either full or empty.

By same argument as in Lemma 2.4, all of these $3k + 1$ rows and columns are either all full or all empty. Without loss of generality, we may assume that all the remaining rows are full rows, and the other case can be argued similarly.

By the same argument, we will have $3k + 1$ columns that are all full or empty. Since we cannot have a full row and an empty column, all the remaining $3k + 1$ columns should also be full. Let

$$\begin{aligned} F_r &:= \{v_{ij} : i, j \in [4k] \quad i \text{ is full row}\} \\ F_c &:= \{v_{ij} : i, j \in [4k] \quad j \text{ is full column}\} \end{aligned}$$

and also let

$$T = F_r \cap F_c.$$

Since $F_r \cup F_c \subset A$, we have

$$\begin{aligned} |A| &\geq |F_r| + |F_c| - |T| \\ &= (3k + 1) \times 4k + 4k \times (3k + 1) - (3k + 1) \times (3k + 1) \\ &= 24k^2 + 8k - 9k^2 - 6k - 1 \\ &= 15k^2 + 2k - 1 \\ &> 15k^2, \end{aligned}$$

which is a contradiction since we assumed $|A| \leq |V| - k^2 = 15k^2$. A similar argument holds for the case with all the remaining rows and columns being empty. $\square$

Theorem 2.5. *For any given value $\alpha < 1$, there exists a graph $G_{4k} \in \mathcal{C}$ with two edge-disjoint spanning trees such that G_{4k} does not have any $\alpha-$thin tree.*

Proof. Similar to the proof of Theorem 2.4, but utilizing Lemma 2.5. □

2.2.4 Random 4−Regular Graphs

Suppose $G(n,k)$ be the set of all $k-$regular graphs on n vertices, we aim to show that for a random $4-$regular graph G on n vertices uniformly selected from $G(n,4)$ the combinatorial thinness $\alpha_G \to 1$ as $n \to \infty$, with *high probability* which means the probability tends to one as $n \leftarrow \infty$. To have a uniform distribution over all simple $4-$regular graphs, we must specify a process to generate a uniformly random graph. We cannot use the classic Erdós-Renyi family of random graphs $G_{n,p}$, where the edges are selected independently at random, as we cannot guarantee a uniform distribution over all $4-$regular graph by selecting edges independently. However, Bollobás in [7] provided a process of obtaining a uniformly random $k-$regular graph from $G(n,k)$, by another model called *configuration model.*(Also, see [14] for more detailed explanation.)

Selecting random graph G from $G(n,k)$, we will proceed to show that each cut of the graph has a relatively large number of edges. Then, we will use the previously discussed techniques to show that combinatorial thinness of a random $4-$regular graph cannot be bounded by any constant $\alpha \in (0,1)$.

To show that each cut of a given graph has large number of edges, we need to introduce a way to compare number of edges in the cut set with the size of the cut itself.

Definition 2.2. Given a graph $G=(V,E)$, we define *edge expansion* of a proper subset $U \subset V$ as

$$\Phi(U) := \frac{\delta_G(U)}{|U|},$$

and *edge expansion* of graph G to be

$$\Phi(G) := \min_{U:|U|\leq \frac{n}{2}} \Phi(U).$$

We call a $G=(V,E)$ an $\epsilon-$*expander*, if its edge expansion is at least ϵ; i.e, $\Phi(G) \geq \epsilon$.

Suppose $G(n,k)$ be the set of all $k-$regular graphs on n vertices, a random element $G \in G(n,k)$ has a property $\mathcal{A}$ *with high probability* if the probability that $\mathcal{A}(G)=1$ goes to 1 as $n \to \infty$.

Bollobás [7] proved the following theorem in 1988.

Theorem 2.6. *Let $G = (V, E)$ is a random k-regular graph on n vertices selected from $G(n, k)$. Suppose $k \geq 3$ and $0 < \mu < 1$ be such that*

$$2^{\frac{4}{k}} < (1-\mu)^{1-\mu}(1+\mu)^{1+\mu}. \tag{2.6}$$

Then, with high probability the edge expansion $\Phi(G)$ is at least $\frac{(1-\mu)k}{2}$.

Proof. See Theorem 1 in [7]. □

Corollary 2.2. *For a random $4-$regular graph $G = (V, E)$ on n vertices, to satisfy (2.6) we must have $\mu \geq 0.779$. Therefore, with high probability G has edge expansion of at least 0.44.*

Theorem 2.7. *Suppose $G = (V, E)$ is a random $4-$regular graph on n vertices chosen uniformly from $G(n, 4)$. Then, we claim that with high probability $\alpha_G \to 1$ as $n \to \infty$.*

Proof. Given a random graph $G \in G(n, 4)$, with high probability, G is an $\epsilon-$expander, where by Corollary 2.2, $\epsilon \geq 0.44$. Suppose T_1 is a combinatorially thin tree of G. Since $G \setminus T$ is still connected, it contains another spanning tree T. By Corollary 2.1, there exists an edge $e \in E(T)$ with $\{e\} = \delta_T(V_1)$ for $V_1 \subset V$ such that V_1 and $V_2 = V \setminus V_1$ have size at least $\frac{n}{5}$. Suppose $|V_1| \leq \frac{n}{2}$ and consider the corresponding cut $\delta_G(V_1) \subset E$. For this cut, we have

$$|\delta_G(V_1)| \geq \epsilon.|V_1| \geq \frac{\epsilon.n}{5}.$$

The spanning tree T has only one edge in $\delta_G(V_1)$. Also, by Lemma 2.1, there are at most 2 edges in $\delta_G(V_1)$ that are not in one of the spanning trees T_1 or T. Thus, we have

$$|\delta_G(V_1) \cap E(T_1)| \geq |\delta_G(V_1)| - 3,$$

which indicates that

$$
\begin{aligned}
\alpha_G(T_1) &\geq \frac{|\delta_G(V_1)| - 3}{|\delta_G(V_1)|} \\
&= 1 - \frac{3}{|\delta_G(V_1)|} \\
&\geq 1 - \frac{3}{\frac{\epsilon . n}{5}} \\
&= 1 - \frac{15}{\epsilon . n}.
\end{aligned}
$$

Therefore, as $n \to \infty$, *a.e.* α_G also goes to 1. □

2.3 Thin Trees and Planar Graphs

In the previous section, we presented some families of graphs with small edge connectivity that do not have combinatorially thin trees. Specifically, the result of [29] gave us an explicit construction for a family of 4-edge connected planar graphs where the combinatorial thinness parameter converges to one.

In this section, we show that combinatorially thin trees can be found in planar graphs if we allow higher edge connectivity, which is a special case of [16]. In the result of Oveis-Gharan and Saberi [16] they give an algorithm that finds combinatorially thin trees on graphs with bounded genus. We will describe the special case of their algorithm for highly connected planar graphs for planar graphs with weights. Note that a simple planar graph has connectivity at most 5. Therefore, to allow higher edge connectivity, we will work with a multi-graph $G = (V, E)$ where we may have many edges $e_1, \ldots, e_s \in E$ between two endpoints $u, v \in V$.

Formally, the special case of the algorithm in [16] for planar graphs takes a non-weighted planar multi-graph $G = (V, E)$ with edge-connectivity k as an input and returns a thin tree with thinness $\frac{10}{k}$. In this section, we will provide a more careful analysis of this algorithm to generalize the thinness parameter of $\frac{10}{k}$ for k-connected weighted planar graphs. Additionally, this analysis gives us a slightly better upper bound of $\frac{1}{\lceil \frac{k}{10} \rceil + \epsilon}$ for graphs with integral weights where $\epsilon > 0$ is a positive number depending on $|V(G)|$.

Note that the algorithm in [16] uses the dual graph of a planar graph to find a spanning set of edges that are combinatorially thin. Using the tools we develop in the following sections, we will also provide an algorithm that does not use the dual graph explicitly and prove its equivalence to the algorithm of Oveis-Gharan and Saberi [16].

To get a better understanding of the algorithm, we first present basic definitions of planar graphs and their planar dual as well as their properties and the relation between them using Chapter 10 of the textbook [8]. Next, we will try to give an interpretation of thin trees of a planar graph $G = (V, E)$ in their planar dual $G^* = (V^*, E^*)$. Having this interpretation, we will define a notion of *threads* which is used in [16] to find a subset $F^* \subset E^*$ in G^* that corresponds to a thin spanning tree $F \subset E$ in G. Additionally, we will interpret back threads in the dual graph G^* to a notion of *hanks* in the original graph G. Finally, we use these interpretations to present a pair of primal and dual algorithms in the weighted setting and prove their correctness.

2.3.1 Preliminaries

Planar graphs

Planar graphs[1] are the family of graphs that can be drawn in the plane in a way that edges intersect only at the endpoints. We call any such embedding of G into the two-dimensional plane a *planar embedding* of G. It is important to note that if a graph $G = (V, E)$ is planar, its planar embedding is not unique. However, any planar embedding $\tilde{G}$ of G provides a graph isomorphic to G. To understand the family of planar graphs, we need to understand at least some basics on the topology of the plane and planar embedding of graphs.

Note that in a planar embedding G, in order to draw any edge $e = \{u, v\}$ between its endpoints we use curves. A *curve* C is an image of a continuous function $\phi : [0, 1] \to \mathbb{R}^2$. We call a curve C *closed* if $\phi(0) = \phi(1)$ and *simple* if it does not intersect itself unless at its endpoints; i.e., $\phi(a) \neq \phi(b)$ for $0 < a < b < 1$. Finally, a subset $S \subset \mathbb{R}^2$ is *arcwise-connected* if for any two points $x, y \in S$ there exists a curve between x and y that lies entirely in S. As the planar embedding of graphs has a set of simple curves which are connected at the endpoints, planar embedding of graphs separate the plane into arcwise-connected open sets, which are referred to as *faces* of plane graph. This observation is the result of one of the most fundamental theorems in topology, which we state without proof in the following.

Theorem 2.8. *(The Jordan Curve Theorem)*
Any simple closed curve $C \in \mathbb{R}^2$ partitions $\mathbb{R}^2 \setminus C$ into two disjoint arcwise-connected open sets.

[1] For more information on planar graphs and their properties view [8] Chapter 10.

One implication of Jordan Curve Theorem is that given a simple closed curve C any curve $\tilde{C}$ connecting a point inside C to a point outside C intersects C at least once. This observation is useful in order to define the notion of *dual* of a planar graph G.

Planar dual

An embedded planar graph G separates the plane into one unbounded outer face and many bounded arcwise connected faces (see Figure 2.3). The set of faces of planar graph G is denoted by $F(G)$. A given face $f \in F(G)$ is incident to a set of vertices and edges. The set of incident edges to a face f are called *boundary* of f and is denoted by $\partial(f)$.

Definition 2.3. (*Planar dual*)
Let $G = (V, E)$ be a given embedded planar graph. We define the *planar dual* multigraph of G, denoted by $G^* = (V^*, E^*)$, as follows.

- For every face $f \in F(G)$, there is a corresponding vertex $f^* \in V^*$, which gives a bijection $V^* \simeq F(G)$.
- For every edge $e = \{u, v\} \in E$, which is incident to faces $f, g \in F(G)$, there is a unique corresponding edge $e^* = \{f^*, g^*\}$ is E^* which gives a bijection $E \simeq E^*$ (see Figure 2.3).

Definition 2.4. (*dual of weight function*) Let $G = (V, E)$ be an embedded planar graph with a weight function $w : E \to \mathbb{R}_+$ and let $G^* = (V^*, E^*)$ be its planar dual. We define the dual weight function $w^* : E^* \to \mathbb{R}_+$ such that $w^*(e^*) := w(e)$ for every $e \in E$.

Definition 2.5. Two embedded planar graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ have *same embedding* if we both the pair G_1, G_2 and G_1^*, G_2^* are isomorphic. I.e., there exists a bijection ϕ that maps $(V_1, E_1, F(G_1))$ to $(V_2, E_2, F(G_2))$ such that for every pair of vertices u, v and faces f, g the following holds.

- An edge $e \in E_1$ has endpoints $u, v \in V_1$ if and only if the edge $\phi(e)$ has endpoints $\phi(u), \phi(v) \in V_2$.
- Edge $e \in E_1$ has faces $f, g \in F(G_1)$ if and only if the edge $\phi(e)$ has faces $\phi(f), \phi(g) \in F(G_2)$.

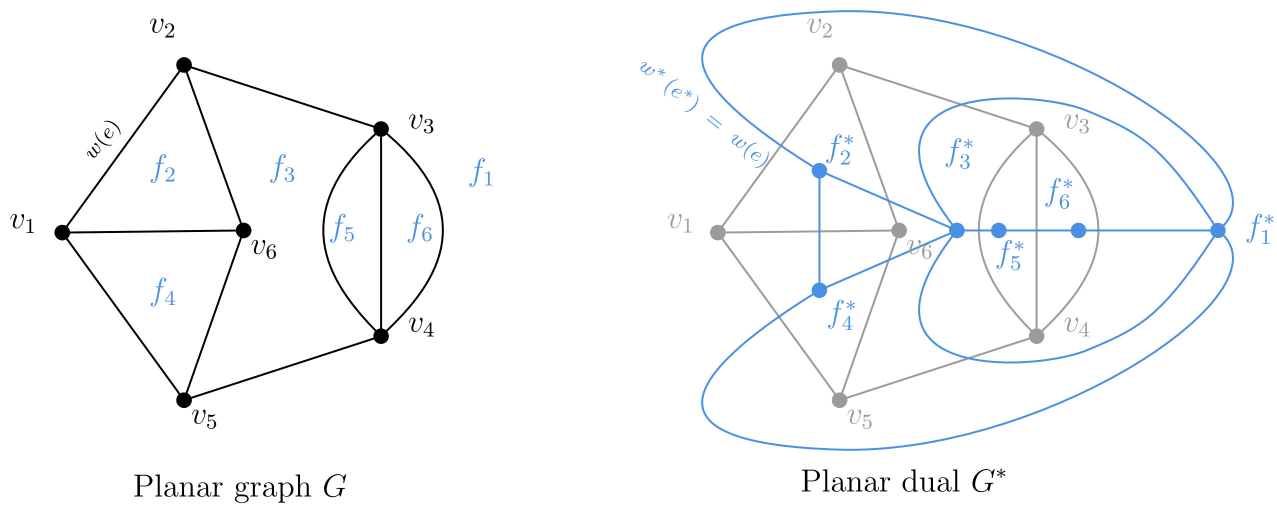


Figure 2.3: Planar Graph (G, w) and its Dual (G^*, w^*).

Notation

In the rest of this chapter, whenever we use (G, w) we are assuming that the planar embedded graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}_+$ and edge connectivity k is given. Moreover, we denote dual planar of G by $G^* = (V^*, E^*)$. Finally, we may write (G^*, w^*) to refer to the planar dual G^* with dual weight function $w^* : E^* \to \mathbb{R}_+$. For planar graphs G_1 and G_2, we denote isomorphism by $G_1 \simeq G_2$ and having a same embedding by $G_1 \equiv G_2$.

Properties of planar graphs and their dual

Next, we will state some of the useful properties of planar graphs and their duals without proof. (To read more about planar graphs see for instance [8] Chapter 10.)

Proposition 2.1. *Given a planar graph (G, w) and its dual (G^*, w^*) the following statements hold:*

i. The planar dual of G^* is equal to planar graph G; i.e., $(G^*)^* \equiv G$.

ii. The planar dual graph G^* is connected.

iii. For all vertices $f^* \in V^*$ we have $\deg_{G^*}(f^*) = |\partial(f)|$.

iv. *Let* $e \in E$ *be an arbitrary edge that is not a cut edge. Then, we have*

$$(G \setminus e)^* \equiv G^*/e^*.$$

v. *Let* $e \in E$ *be an arbitrary edge that is not a loop. Then we have*

$$(G/e)^* \equiv G^* \setminus e^*.$$

vi. *Any cut edge* $e \in E$, *corresponds to a loop* $e^* \in E^*$, *and any loop in* G *corresponds to a cut edge in* G^*.

vii. $S \subset E$ *is a bond in* G *if and only if its corresponding edges* $S^* := \{e^* : e \in S\} \subset E^*$ *is a cycle in* G^*.

viii. *Total number of incident edges of faces is equal to* $2|E|$; *i.e.*

$$\sum_{f \in F(G)} |\partial(f)| = 2|E|.$$

ix. ***(Euler's Formula)*** *The following equation holds*

$$|V(G)| + |F(G)| - |E(G)| = 2. \tag{2.7}$$

x. *For a subset of edges* $F \subset E$ *with corresponding dual edges* $F^* = \{e^* : e \in F\}$, *we have*

$$w(F) = w^*(F^*).$$

2.3.2 Combinatorially Thin Trees from the Viewpoint of the Dual Graph

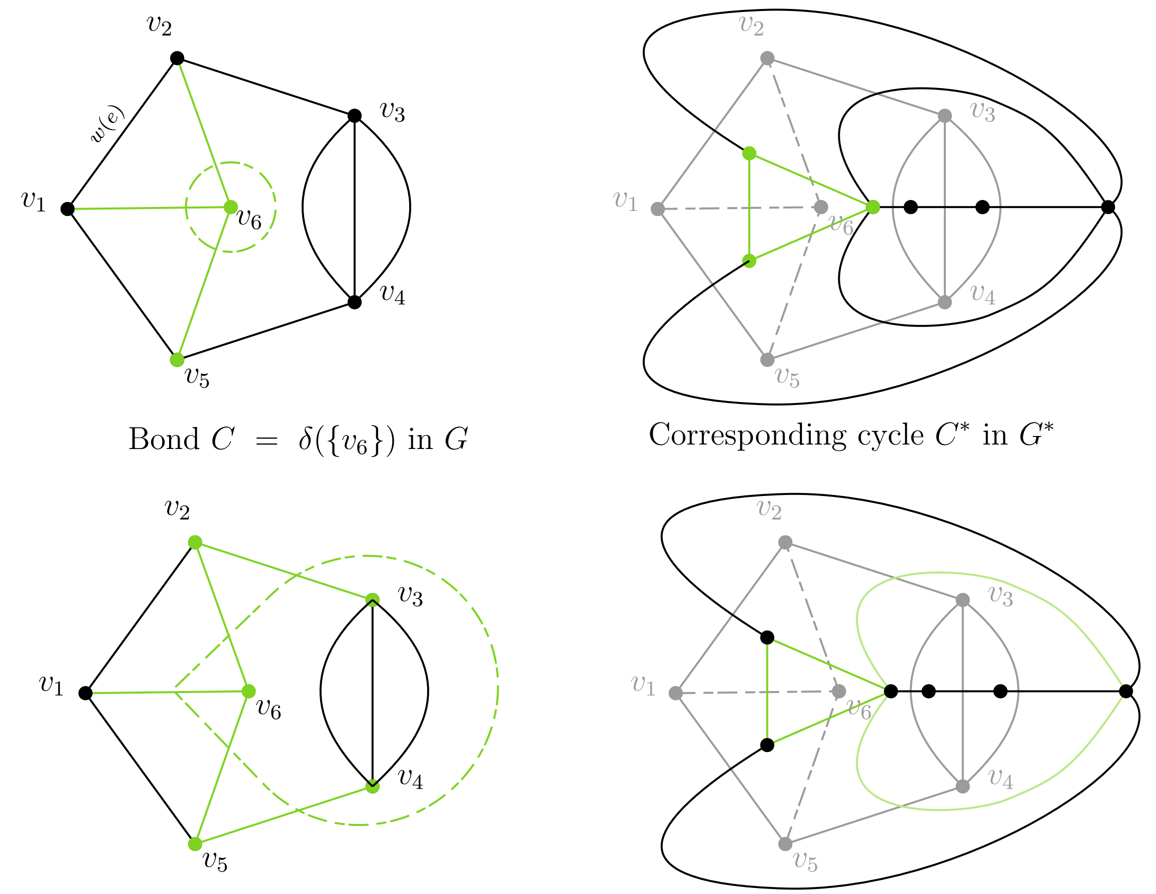
Let $G = (V, E)$ be an embedded planar graph with weight function $w : E \to \mathbb{R}_+$ and let $G^* = (V^*, E^*)$ be the planar dual of G with dual weight function $w^* : E^* \to \mathbb{R}_+$. A natural question is to ask what does a thin subgraph $F \subset E$ of G mean in the dual graph G^* and vice versa. Note that the subgraph F is α-combinatorially thin if and only if it has at most α fraction of edge weights in each cut-set $\delta_G(U) \subset E$, for all cuts $U \subset V$; i.e.,

$$\frac{|\delta_G(U) \cap F|}{w(\delta_G(U))} \leq \alpha.$$

Therefore, to understand a thin subgraph F of G from the view point of the dual graph G^*, we first need to understand the meaning of cut-sets $\delta_G(U) \subset E$ in the dual graph G^*.

The following lemma show that cut-sets in G are analogous to cycles in G^* (see Figure 2.4). This is a folklore result in planar graphs. (See Theorem 10.16 and Corollary 10.17 in Chapter 10 [8].)

Lemma 2.6. *Let a planar graph G be given. Then, $S \subseteq E$ of G is a cut-set of G if and only if the dual edges $S^* \subseteq E^*$ are union of edge-disjoint cycles in G^*.*



Bond $C = \delta(\{v_6\})$ in G

Corresponding cycle C^* in G^*

Cut-set $C = \delta(\{v_3, v_4, v_6\})$ in G

Edge disjoint cycles C_1^* and C_2^* in G^*

Figure 2.4: Bonds in G correspond to cycles in G^*. Moreover, cut-sets in G correspond to union of edge disjoint cycles in G^*.

A direct consequence of the Lemma 2.6 is the following interpretation of thin spanning subgraphs in the planar dual G^* (see Figure 2.5).

Proposition 2.2. *Let* (G, w) *and* (G^*, w^*) *be given, and let* $F \subseteq E$ *be a subset of edges of* G *and* $F^* := \{e^* : e \in F\} \subseteq E^*$ *be the corresponding edges in* G^**. Then,*

i. (V, F) *is a* α*-combinatorially thin subgraph of* G *if and only if for every cycle* $C^* \subseteq E^*$ *we have,*

$$\frac{|C^* \cap F^*|}{w^*(C^*)} \leq \alpha. \tag{2.8}$$

ii. (V, F) *is a spanning subgraph of* G *if and only if for every cycle* $C^* \subseteq E^*$*,* $C^* \cap F^* \neq \emptyset$*.*

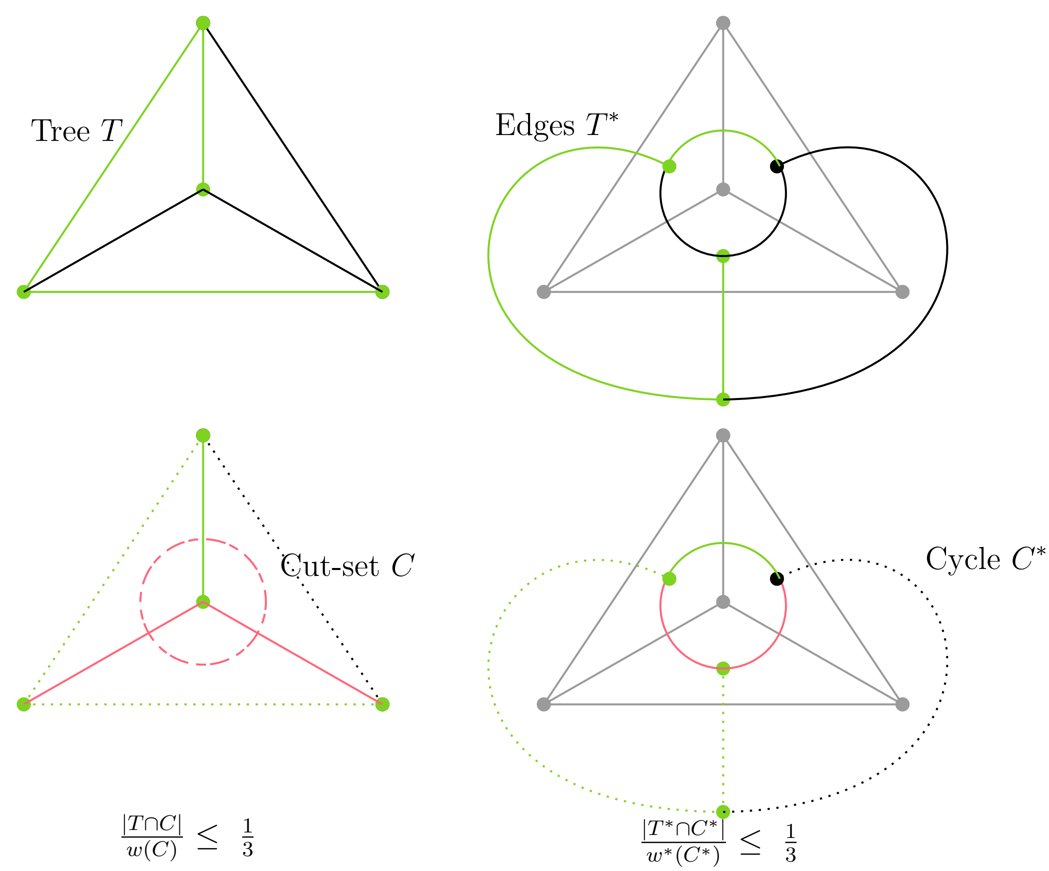


Figure 2.5: The fraction of the edges of a tree T in cut-set C^* equals the fraction of the edges of T^* in the cycle C^*.

(Proof of 2.2). By Lemma 2.6, the cut space in a planar graph G corresponds to the cycle space of its planar dual G^*. Using this fact, we prove (i.) as follows. (ii.) can be proven similarly. Suppose the subset $F \subset E$ is given. Then, (V, F) is a combinatorially thin tree of G with thinness parameter α if and only if for all cut-sets $S = \delta(U) \subset E$ we have

$$\frac{|F \cap S|}{w(S)} \leq \alpha. \tag{2.9}$$

Now, by Lemma 2.6, a subset $S \subset E$ is a cut-set in G if and only if we have $S^* = \bigcup_{i=1}^{\ell} C_i^*$ where C_i^*'s are edge-disjoint cycles in G^*. Since C_i^*'s are edge-disjoint, we can write

$$\begin{aligned}
\frac{|F \cap S|}{w(S)} &= \frac{|F^* \cap S^*|}{w(S)} \\
&= \frac{|(\bigcup_{i=1}^{\ell} C_i^*) \cap F^*|}{w(\bigcup_{i=1}^{\ell} C_i^*)} \\
&= \frac{\sum_{i=1}^{\ell} |C_i^* \cap F^*|}{\sum_{i=1}^{\ell} w(C_i^*)}.
\end{aligned}$$

Therefore, (V, F) is $\alpha-$combinatorially thin if and only if for every $\ell \in \mathbb{N}$ and set of edge disjoint cycles $C_1^*, \ldots, C_\ell^*$ we have

$$\alpha \geq \frac{\sum_{i=1}^{\ell} |C_i^* \cap F^*|}{\sum_{i=1}^{\ell} w^*(C_i^*)}. \tag{2.10}$$

Note that this is equivalent to saying that for every cycle C^* in G^* we have

$$\alpha \geq \frac{|C^* \cap F^*|}{w^*(C^*)}. \tag{2.11}$$

More specifically, if (2.10) holds then (2.11) also holds as C^* can be considered is a single cycle itself. Conversely, if (2.11) holds for every cycle C^* in G^* we have

$$\begin{aligned}
\frac{\sum_{i=1}^{\ell} |C_i^* \cap F^*|}{\sum_{i=1}^{\ell} w^*(C_i^*)} &\leq \frac{\sum_{i=1}^{\ell} (\alpha . w^*(C_i^*))}{\sum_{i=1}^{\ell} w^*(C_i^*)} \\
&= \alpha,
\end{aligned}$$

and (2.10) also holds for every set of edge-disjoint cycles $C_1^*, \dots, C_\ell^*$.

$\square$

As a result of this proposition, instead of finding a α-thin spanning tree $F \subset E$, we can find a subset $F^* \subset E^*$ where for every cycle $C^* \subset E^*$, $0 < |F^* \cap C^*| \leq \alpha w^*(C^*)$. To obtain such subset $F^* \subset E^*$ of edges, Oveis-Gharan and Saberi [16] use a notion of *threads* that allows us to upper bound the fraction of edges chosen from each cycle C^*.

2.3.3 Threads in the Dual Graph G^*

Definition 2.6. (thread) Let (G, w) be given. A *thread* T is defined as one of the following.

- *Path thread:* a vertex induced path $T = v_1 e_1 v_2 e_2 \dots e_{n-1} v_n$ of G, where the inner vertices have degree two and the endpoints v_1 and v_n have degree at least three in G.
- *Cycle thread:* a vertex induced cycle $T = v_1 e_1 v_2 e_2 \dots v_n e_n v_1$ of G, where $\deg_G(v_1) \geq 2$ and all other vertices have degree equal to two. (Figure 2.6 shows an example of threads.)

We define the weight of a thread T to be equal to its total edge weights

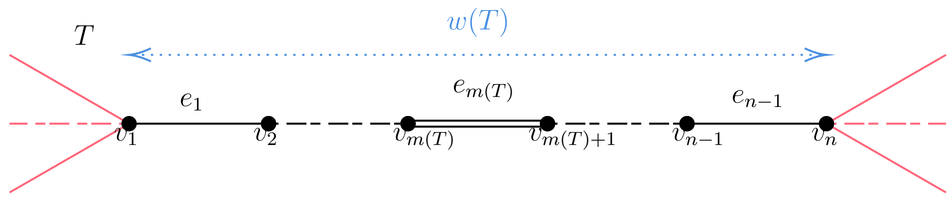
$$w(T) := \sum_{e \in E(T)} w(e).$$

Finally, we define the *median* edge $e_{m(T)} \in E(T)$ to be the first edge e_j for $j \in [n]$ such that

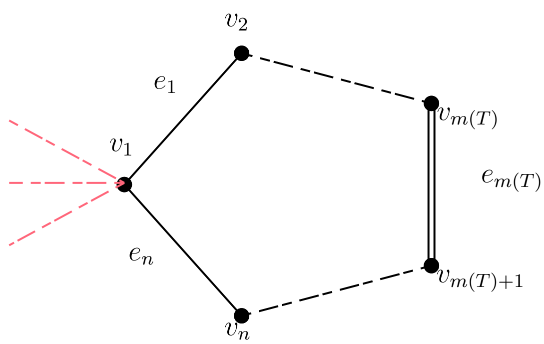
$$\sum_{i=1}^{j} w(e_i) \geq \frac{w(T)}{2}.$$

Remark 2.1. It is easy to see that we also have

$$\sum_{i=m(T)}^{|E(T)|} w(e_i) \geq \frac{w(T)}{2}.$$



a. A path thread T



a. A cycle thread T

Figure 2.6: The black edges in the figure depict edges in the thread, the red edges are in G, and the dotted red edges are possibly in G.

Definition 2.7. (thread removal) Let (G, w) be given, and let T be a thread in G with median edge $e_{m(T)} \in E(T)$. We define *thread removal* operation which removes thread T from G as follows.

$$G - T := \big(G \setminus e_{m(T)}\big)/\big(E(T) \setminus \{e_{m(T)}\}\big).$$

In other words, we first remove the median edge $e_{m(T)}$ of T from G to obtain $G \setminus e_{m(T)}$. Then, we contract all of the remaining edges in $E(T) \setminus \{e_{m(T)}\}$ that are now cut edges in $G \setminus e_{m(T)}$. Equivalently, if T is a path thread, we remove all vertices of degree two, and if T is a cycle thread we remove all vertices except the first vertex of T.

Moreover, we define the corresponding weight function in a natural way as

$$w'(e) = w(e) \qquad \forall e \in E(G - T).$$

Remark 2.2. With abuse of notation, we may use the same notation w to denote the new

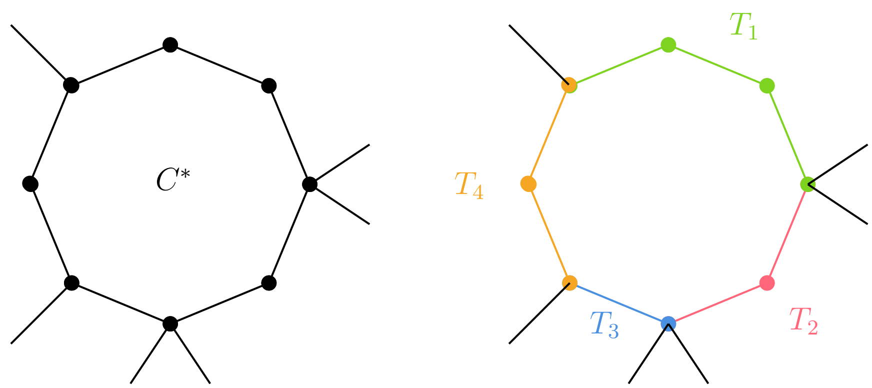


Figure 2.7: The thread $T_1, \ldots, T_4$ completely lie in cycle C^*.

obtained weight function w'.

Intuitively, finding large threads in each cycle C^* might be a good way of choosing edges from each cycle. Specifically, the following proposition shows that by adding a middle edge of a thread T to F^*, we can discard all of the remaining edges of T as every cycle intersecting T will have an edge in F^*.

Proposition 2.3. *Let (G, w) be given, and let T be a thread in G. Suppose C is a cycle in G. If C and T have at least one edge in common, the thread T lies entirely in C; i.e., $E(T) \subseteq E(C)$.*

Proof. Suppose T and C have at least one common edge. Let cycle $C = v_1e_1v_2e_2 \ldots v_ne_nv_1$.

If T is a cycle thread, we have $T = C$. Otherwise let $p = v_ie_i \ldots v_j$ be the mutual path between C and T. Then, T will have two vertices v_i, v_j of degree at least three, which cannot happen in the internal vertices of a thread. Therefore, T must entirely be in cycle C. □

The above proposition ensures that if our algorithm chooses $e_{m(T)}$, at least $w(T)$ fraction of edges from cycle C^* can be discarded for the future steps. To make the simple idea of choosing large threads work, we also need to understand the objects that remain in G^* after removing a thread T from a cycle C^* in G^*. This is vital, as the other threads of $G^* - T$, might have intersection with the remaining edges of cycle C^* in $G^* - T$. Therefore, we have to also bound the fraction of edges that we may choose from such objects which we formalize as *knotted threads* in the following (see Figure 2.10).

Definition 2.8. (knotted thread) Let (G, w) be given. A *knotted thread* in G is a path $K = v_1 e_1 v_2 \dots e_{n-1} v_n$, where $v_i \in V$ has degree $\deg_G(v_i) \geq 2$ and $v_1, \dots, v_n$ are distinct. We denote the edges and vertices of a knotted thread as $E(K) \subset E$ and $V(K) \subset V$. A vertex $v_i \in V(K)$ is a *knot* in K if $\deg_G(v_i) \geq 3$ (see Figure 2.8). Finally, the weight of a knotted thread K is denoted by

$$w(K) := \sum_{e \in E(K)} w(e).$$

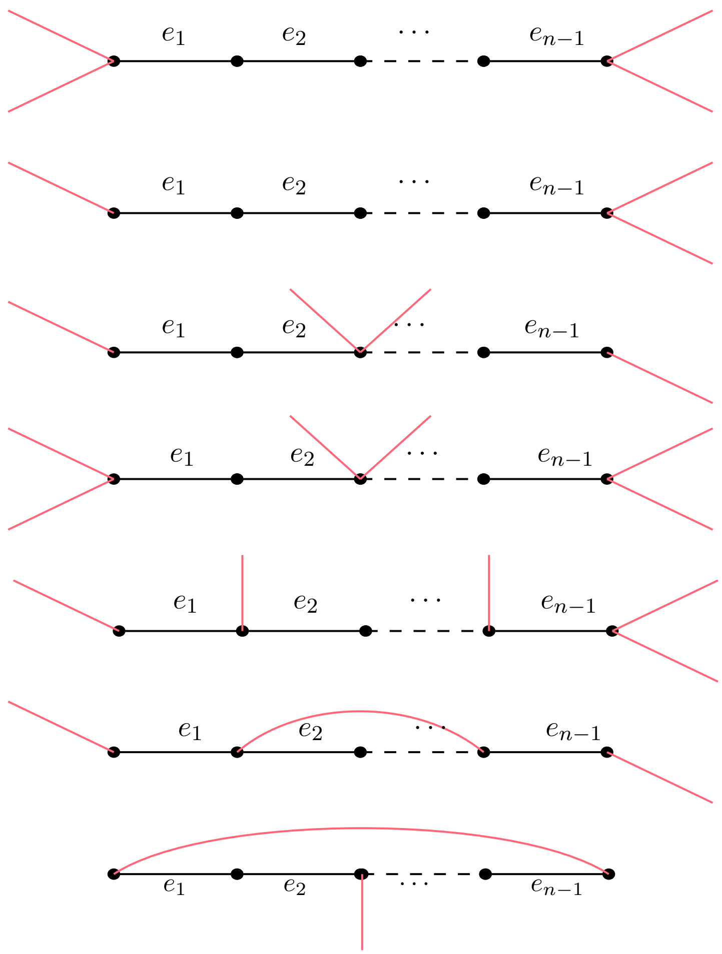


Figure 2.8: Different variants of knotted threads. Black edges are in the knotted thread and red edges are other possible edges in the graph.

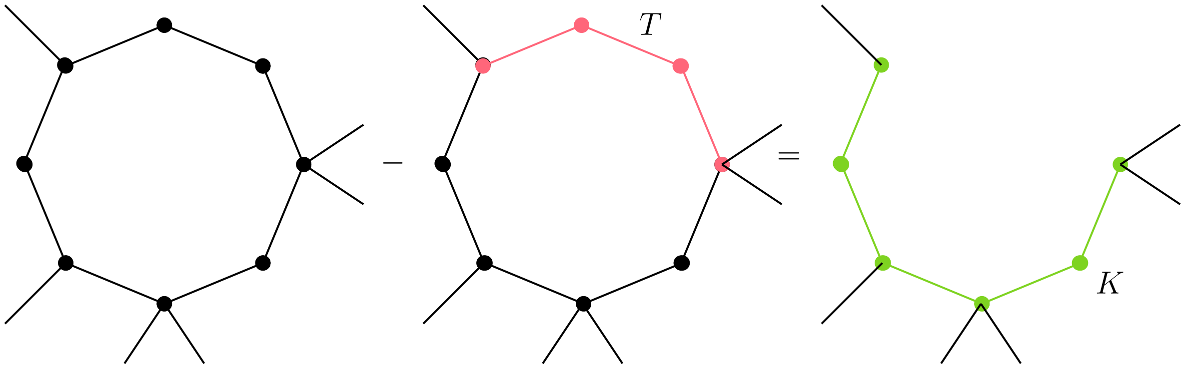


Figure 2.9: Removing a thread T from a cycle C^* gives a knotted thread K.

Remark 2.3. A path thread $T \subset E$ with corresponding vertices $V(T) = \{v_1, \cdots, v_n\}$ is a knotted thread with knot vertices only at its endpoints v_1, v_n.

We can know formalize our observation that whatever remains from C^* in $G^* - T$ is a knotted thread (see Figure 2.9).

Proposition 2.4. *Let (G, w) be given. Suppose C is a cycle in G and thread T is a path thread that lies entirely in C. Then, there exists a knotted thread K of $G - T$ such that $E(K) = C \setminus E(T)$.*

Proof. Let $C = v_1e_1v_2e_2 \dots v_ne_nv_1$ and $T = v_ie_i \dots e_{j-1}v_j$. Then, to show

$$K = v_je_j \dots v_ne_nv_1e_1 \dots e_{i-1}v_i$$

is a knotted thread in $G - T$, we only need to show

$$\deg_{G-T}(v_i) \geq 2 \qquad \text{and} \qquad \deg_{G-T}(v_j) \geq 2.$$

Note that both inequalities holds as if $\deg_{G-T}(v_i) = 1$, we have $\deg_G(v_i) = 2$. This means v_i cannot be endpoint of a thread T as both endpoints of a path thread have degree at least three. $\square$

In the following proposition, we will show that removing T from G^* reduces knotted threads to smaller knotted threads. We also show that if the median edge of a thread T is inside a knotted thread K, then many edges of K are discarded in $G^* - T$. As a result, by removing the median edge of a thread T and removing the remaining edges we ensure that not many remaining edges of each cycle (which become knotted threads and even smaller knotted threads) are chosen in each step (see Figure 2.10).

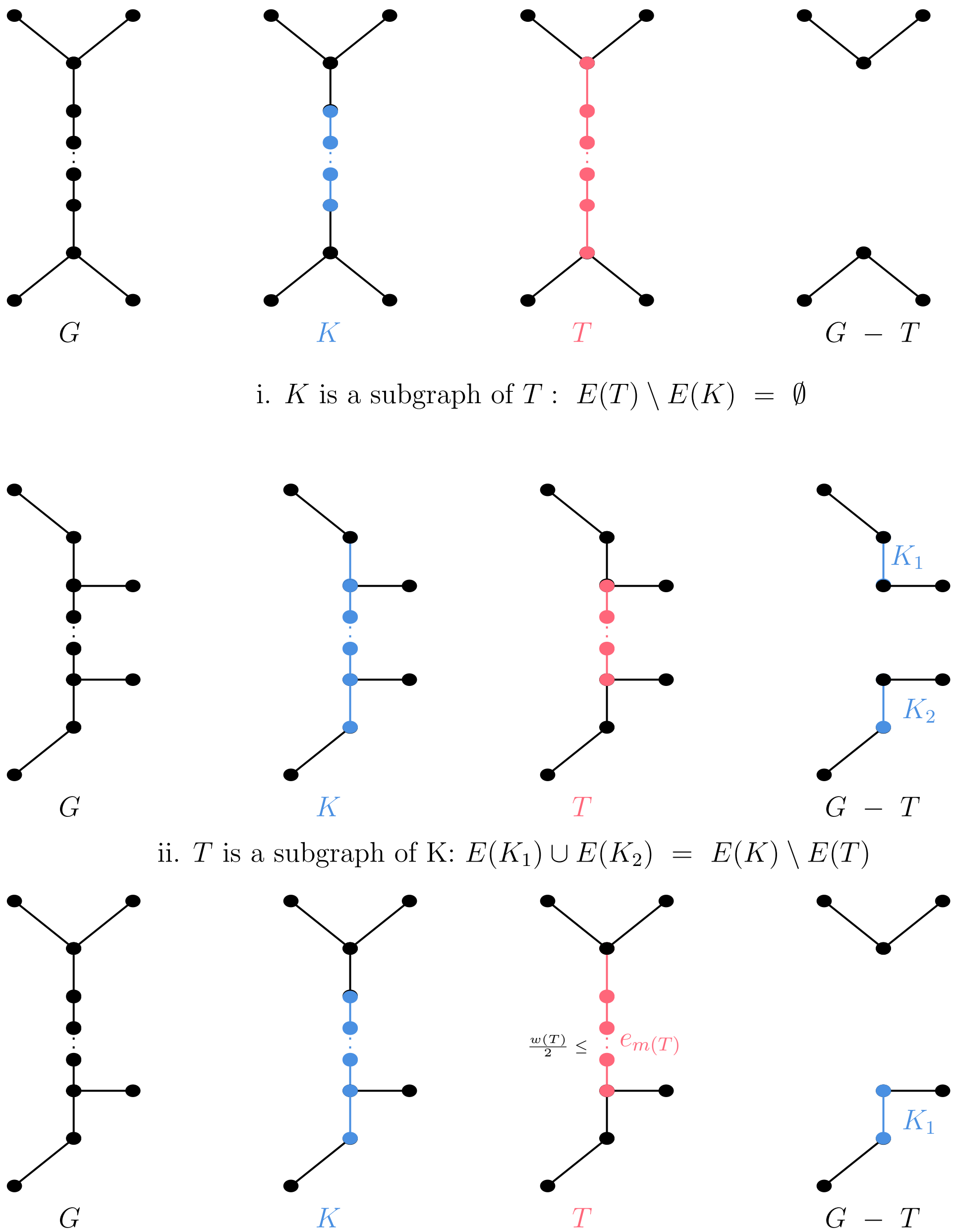
Proposition 2.5. *Let (G, w) be given, and let T be a thread in G with median edge $e_{m(T)} \in E(T)$. Also, let $K = v_1 e_1 \dots e_{n-1} v_n$ be a knotted thread in G. Then, the following statements hold.*

i. *If K is a subgraph of T, then $E(K) \setminus E(T)$ is an empty set.*

ii. *If T is a subgraph of K, there exists subgraphs K_1, K_2 of $G - T$ such that $E(K_1) \cup E(K_2) = E(K) \setminus E(T)$ and K_1, K_2 are either knotted threads or single vertices in $G - T$.*

iii. *If neither K is a subgraph of T nor T is a subgraph of K, there exists a subgraph K_1 of $G - T$ such that $E(K_1) = E(K) \setminus E(T)$ and K_1 is a knotted thread in $G - T$. Moreover, if $e_{m(T)} \in E(K)$, then*

$$w(K \cap T) \geq \frac{w(T)}{2}.$$



i. K is a subgraph of T : $E(T) \setminus E(K) = \emptyset$

ii. T is a subgraph of K: $E(K_1) \cup E(K_2) = E(K) \setminus E(T)$

iii. T and K are not subgraphs of each other: $E(K_1) = E(K) \setminus E(T)$
$e_m(T) \in E(K) \Rightarrow w(E(K) \cap E(T)) \geq \frac{w(T)}{2}$

Figure 2.10: Consequences of removing a thread T from graph G on a knotted thread K.

Proof. We prove each item as follows.

i. Trivially true, since $E(K) \subset E(T)$.

ii. Suppose T is a subgraph of K, since both T and K are paths in G, there exist indices $i, j \in [n]$, where $T = v_i e_i \dots e_{j-1} v_j$. Now, we define K_1 and K_2 as follows:

$$K_1 = \begin{cases} v_1 e_1 \dots e_{i-1} v_i, & \text{if } i > 1 \\ v_1, & \text{otherwise} \end{cases}$$
$$K_2 = \begin{cases} v_j e_j \dots v_n, & \text{if } j < n \\ v_j, & \text{otherwise.} \end{cases}$$

Note that

$$\begin{aligned} E(K_1) \cup E(K_2) &= \{e_1, \dots, e_{i-1}, e_j, \dots, e_n\} \\ &= \{e_1, \dots, e_n\} \setminus \{e_i, \dots, e_{j-1}\} \\ &= E(K) \setminus E(T). \end{aligned}$$

It only remains to show that K_1 and K_2 are either knotted threads or single vertices in $G - T$. If $i = 1$ then K_1 is a single vertex. So suppose $i > 1$. Since K is a knotted thread we have $\deg_G(v_1) \geq 2$. Also, since $T = v_i \dots v_j$ is a thread we have $\deg_G(v_i) \geq 3$, where one of the edges is inside $E(T)$. As a result both v_1 and v_i have degree at least two in the graph $G - T$ as all edges $e_i, \dots, e_{j-1}$ and vertices $v_{i+1}, \dots, v_{j-1}$ are removed in $G - T$. Therefore, K_1 is a knotted thread. Similarly, K_2 is either an single vertex or a knotted thread in G_T.

iii. Suppose neither K is a subgraph of T nor T is a subgraph of K. If T and K are disjoint, then we can define $K_1 = K$ which is also a subgraph of $G - T$. So, suppose T and K have at least one common edge.

Since T is not fully in K but it has intersection with K, if an edge $e_i \in E(T) \cap E(K)$, either $P = v_1 e_1 \dots v_i e_i v_{i+1}$ or $P = v_i e_i \dots e_{n-1} v_n$ is fully part of the thread T. This implies that at least one of the vertices v_1 or v_n have degree two in G, and either $e_1 \in E(T)$ or $e_n \in E(T)$ or both.

a. If $\{e_1, e_n\} \subset E(T)$, both endpoints of thread T lie in $V(K)$. Let $v_i, v_j \in V(K)$ (where $i < j$) be the knots in K which are endpoints of thread T. Then, $E(K) \setminus E(T) = \{e_i, \dots e_{j-1}\}$, and the path
$$K_1 = v_i e_i \dots e_{j-1} v_j$$

is a knotted thread in $G - T$, as $\deg_{G-T}(v_i) \geq 2$ and $\deg_{G-T}(v_i) \geq 2$. Moreover, if $e_{m(T)} \in E(K)$, then either $e_{m(T)} \in \{e_1 \dots e_{i-1}\}$ or $e_{m(T)} \in \{e_j \dots e_n\}$. Without loss of generality, suppose the first case happens, then

$$w(T \cap K) \geq \sum_{\ell=m(T)}^{i-1} w(e_\ell) \geq \frac{w(T)}{2},$$

where the last inequality holds by our choice of median edge.

b. In the former case, without loss of generality, suppose $e_1 \in E(T)$, and let v_i be the knot vertex in K which is the endpoint of T ($i < n$ since K is not fully in T). Similarly, the path

$$K_1 = v_i e_i \dots e_{n-1} v_n,$$

is a knotted thread in $G - T$ with

$$E(K_1) = E(K) \setminus E(T).$$

Similar to the previous case, if $e_{m(T)} \in E(K)$, then

$$w(T \cap K) \geq \frac{w(T)}{2}.$$

□

As previously mentioned, to find a thin spanning tree in a planar graph G our main goal is to specify a subset of edges $F^* \subset E^*$ in the dual graph G^*, such that it only contains an $\alpha-$fraction of edges in each cycle $C^* \subset E^*$. Also, we saw that finding long threads in G^* and selecting the median edge from each thread into F^* seems to be a good idea. However, we still do not know if such long threads exist.

To understand this process in the original graph G, we define a new notion of *hanks* and their removals in G. We establish their correspondence to threads and thread removal in G^*. Then, we show that in planar graphs with high edge connectivity k, hanks of size $\frac{k}{5}$ exist. Moreover, we will see that removing a hank can only increase the edge connectivity of a graph.

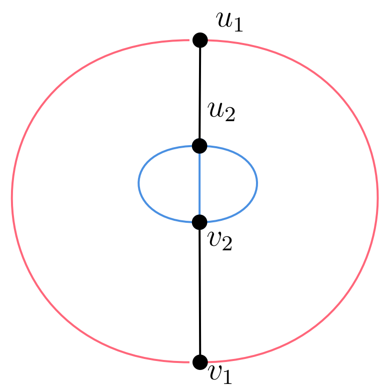


Figure 2.11: Blue edges and red edges both are parallel edges between u_1, v_1 and u_2, v_2. However, only blue edges are strictly parallel.

2.3.4 Hanks in Primal Graph G

Definition 2.9. (strictly parallel edges) Let $G = (V, E)$ be an embedded planar graph. We call two edges $e_1, e_2 \in E$ *parallel edges* if they share the same endpoints $u, v \in V$. Moreover, we say e_1, e_2 are *strictly parallel* if they are parallel and share an adjacent face $f \in F(G)$ that forms a digon; i.e., $\partial(f) = \{e_1, e_2\}$ (see Figure 2.11).

Definition 2.10. (hank) Let (G, w) be given, and let $e_1, \ldots, e_n \in E$ be a maximal set of distinct edges with distinct endpoints $\{u, v\} \subset V$, where every consequent edges e_j, e_{j+1} form a face $f_j \in F(G)$ that is a digon; i.e., e_j, e_{j+1} are strictly parallel. Suppose, $f_0, f_n \in F(G)$ be the other adjacent faces of e_1 and e_n, respectively. Then, we define $f_0 e_1 f_1 e_2 \cdots e_n f_n$ to be a *hank* in G (see Figure 2.12). We denote the endpoints of a Hank H by $V(H)$ and its edges with $E(H)$. We denote the weight of hank H with

$$w(H) = \sum_{i=1}^{n} w(e_i).$$

Moreover, let $m(H) \in [n]$ be the smallest index j such that,

$$\sum_{i=1}^{j} w(e_i) \geq \frac{w(H)}{2}.$$

Then, $e_{m(H)}$ is the *median edge* of H.

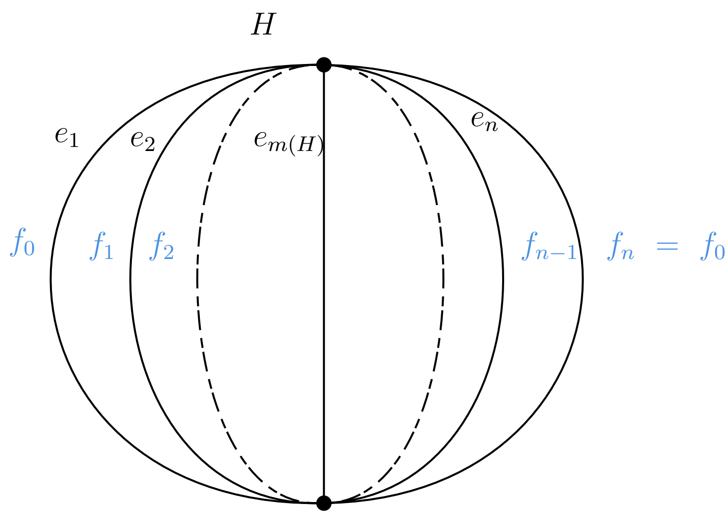


Figure 2.12: A hank H in G.

Remark 2.4. In graph G, a single edge $e \in E$ can be interpreted as a hank $H = feg$, where $f, g \in F(G)$ are the incident faces of e.

Definition 2.11. (hank removal) Let (G, w) be given, and let H be a hank of G with edge set $E(H)$ and endpoints $V(H) = \{u, v\}$. We define the operation of hank removal as follows.

$$G - H := (G/e_{m(H)}) \setminus (E(H) \setminus \{e_{m(H)}\}).$$

In other words, in the operation, we first contract the median edge $e_{m(H)} = \{u, v\}$ in G to obtain the graph $G/e_{m(H)}$. Next, we remove all of the remaining edges in $E(H)$ that are now loops in $G/e_{m(H)}$.

Moreover, we define the corresponding weight function in a natural way where

$$w'(\{x, y\}) := \begin{cases} w(\{u, y\}), & \text{if } x = uv \text{ and } \{u, y\} \in E \\ w(\{v, y\}), & \text{if } x = uv \text{ and } \{v, y\} \in E \\ w(\{x, y\}), & \text{if } x, y \neq uv \\ w(\{u, v\}), & \text{if } x = y = uv. \end{cases}$$

Remark 2.5. With abuse of notation, we may use the same notation w to denote the new obtained weight function w'.

The following lemma will be a crucial to prove existence of large hanks in graphs with high edge connectivity.

Lemma 2.7. *Let $G = (V, E)$ be an embedded planar graph. Suppose G does not have any pair of strictly parallel edges. Then, G has a vertex $v \in V$ of degree at most five.*

Proof. Suppose graph G does not have strictly parallel edges, then for each face $f \in F(G)$, we have $|\partial(f)| \geq 3$. Moreover, by Proposition 2.1 (viii.) we have

$$2|E| = \sum_{f \in F(G)} |\partial(f)|.$$

Suppose all vertices in G have degree at least six, Then, we the followings hold,

$$2|E| = \sum_{f \in F(G)} |\partial(f)| \geq 3|F(G)|$$
$$2|E| = \sum_{v \in V} \deg(v) \geq 6|V|.$$

Now, by considering Euler's formula we get

$$\begin{aligned} |E| + 2 &= |F(G)| + |V| \\ &\leq \frac{2}{3}|E| + \frac{2}{6}|E| \\ &= |E|, \end{aligned}$$

which is a contradiction. Therefore, at least one vertex $v \in V$ must have $\deg(v) \leq 5$. $\square$

Proposition 2.6. *Let (G, w) be given with edge connectivity k. Then, G always has a hank H of size at least $w(H) \geq \frac{k}{5}$. Moreover, if w is integral, then we have $w(H) \geq \lceil \frac{k}{5} \rceil$.*

Proof. Let $\tilde{G} = (V, \tilde{E})$ and $\tilde{w}$ be the planar graph which we get by replacing each set of strictly parallel edges in G by a single edge of equal weight until no pair of strictly parallel edges remain. Then, by Lemma 2.7, since $\tilde{G}$ does not contain any set of strictly parallel edges, it has a vertex $v \in V$ of degree at most 5, i.e., $\deg_{\tilde{G}}(v) \leq 5$.

Additionally, note that since G is a k edge connected graph, we have

$$w(\delta_{\tilde{G}}(v)) = w(\delta_G(v)) \geq k.$$

Since in the underlying graph $\tilde{G}$ vertex v is incident to at most 5 edges, there is an edge $e \in \delta_{\tilde{G}}(v)$ which corresponds to a hank of size at least $\frac{k}{5}$. Note that this edge e is a set of strictly parallel edges $F \subset E$ which is replaced by e or is a single edge $e \in E$ with $w(e) \geq \frac{k}{5}$. In both scenarios G has a hank of weight at least $\frac{k}{5}$. In the case that w is integral, we can find a hank of weight at least $\lceil \frac{k}{5} \rceil$. □

Lemma 2.8. *Let (G, w) be given with edge connectivity k, and let H be a hank in G. Then, $G - H$ has edge connectivity at least k.*

Proof. Let $U \subset V$ be an arbitrary cut in $G - H$ with cut set $S = \delta(U)$. Since, S is also a cut set of G and G has connectivity at least k, we have $w(S) \geq k$. This proves that connectivity of $G - H$ is at least k as well. □

2.3.5 Duality between Hanks of G and Threads of G^*

Lemma 2.9. *Let (G, w) be given. Let $f_0, \ldots, f_n \in F(G)$ and $e_1, \ldots, e_n \in E$ be faces and edges in G. Then,*

$$H = f_0 e_1 \ldots e_n f_n$$

is a hank in G if and only if

$$H^* = f_0^* e_1^* \ldots e_n^* f_n^*$$

is a thread in G^* where $e_{m(H^*)}$ is not a cut-edge of G^*. Moreover, we have $(e_{m(H)})^* = e_m(H^*)$ and $w(H) = w^*(H^*)$ (see Figure 2.13).

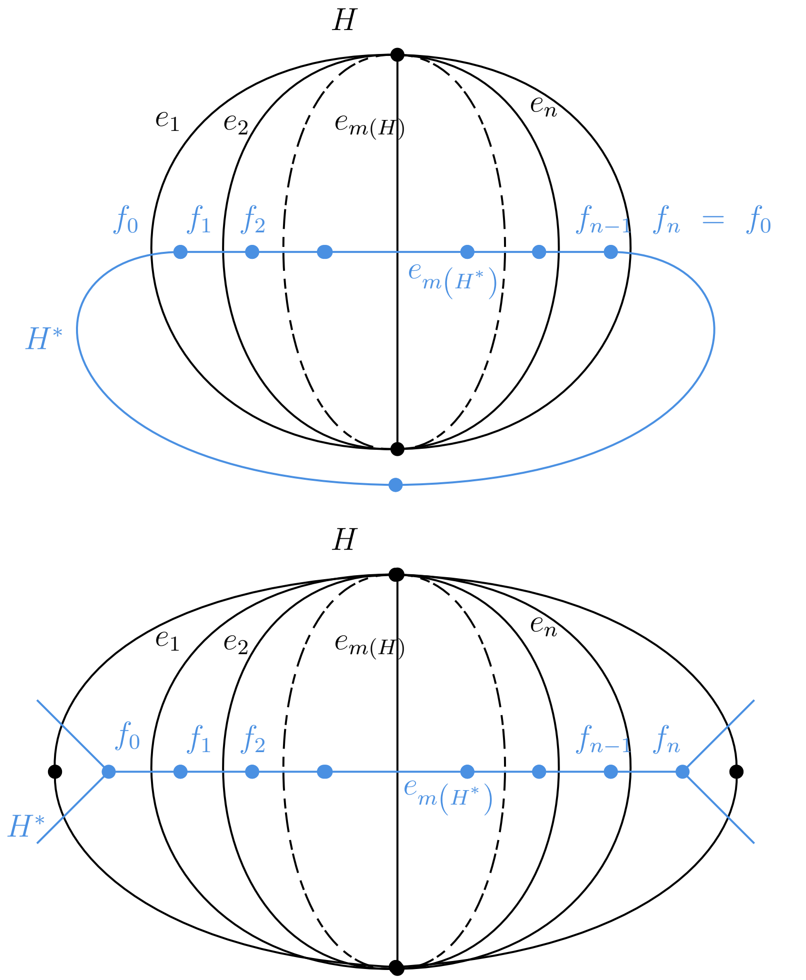


Figure 2.13: A hank H in G corresponds to a thread $T = H^*$ in G^*.

Proof. We prove each direction seperately as follows.

(left to right) Suppose $H = f_0 e_1 f_1 e_2 \cdots e_n f_n$ is a hank in G. Note that by the properties of planar dual graph G^*, the vertices $f_i^*, f_{i+1}^* \in V^*$ are connected by the edge $e_i^* = \{f_i^*, f_{i+1}^*\}$ in G^*. Additionally, since $f_1, \ldots, f_{n-1}$ are digons in G with $\partial(f_i) = \{e_i, e_{i+1}\}$, the corresponding vertices $f_1^*, \ldots, f_{n-1}^*$ in G^* have degree two and $\delta_{G^*}(f_i^*) = \{e_i^*, e_{i+1}^*\}$. We consider the following two cases:

a. If $f_0 = f_n$, then $H^* = f_0^* e_1^* f_1^* e_2^* \cdots e_n^* f_n^*$ will be a cycle in G^* where $\deg_{G^*}(v_i) = 2$ for $1 < i < n$. As a result, H^* is a cycle thread in G^*.

b. If $f_0 \neq f_n$, then $H^* = f_0^* e_1^* f_1^* e_2^* \cdots e_n^* f_n^*$ will be a path in G^* where $\deg_{G^*}(v_i) = 2$ for $1 < i < n$. Additionally, the end vertices f_0^*, f_n^* have degree at least three, as otherwise f_0, f_n also would form a digon face by parallel edges, and therefore, the hank H would not be the maximal set of parallel edges.

Additionally, all the hank edges e_i have different endpoints $\{u, v\} = V(H)$ and are not loops in G. As a result, none of the edges e_i^* in G^* are cut-edges.

(right to left) Suppose $H^* = f_0^* e_1^* f_1^* e_2^* \cdots e_n^* f_n^*$ is a thread in G^*, where $e_{m(H^*)}$ is not a cut edge in G^*. Since for every $1 < i < n$ we have $\deg_{G^*}(f_i^*) = 2$ with $\delta_{G^*}(f_i^*) = \{e_i^*, e_{i+1}^*\}$, the corresponding faces f_i form a digon with $\partial(f_i) = \{e_i, e_{i+1}\}$. As a result, e_i, e_{i+1} must be strictly parallel edges in G that have same set of end-points $\{u, v\} \subset V$. Additionally, none of e_i's can be a loop in G as $e_{m(H^*)}$ is not a cut-edge, and its two adjacent faces are distinct in G^*. Again we consider the following two cases:

a. If H^* is a cycle thread, then $f_0^* = f_n^*$ and H is a set of strictly parallel edges between $\{u, v\}$, and therefore, is a hank in G.

b. If $f_0^* \neq f_n^*$, then H^* is a path thread and we have

$$|\partial_G(f_0)| = \deg_{G^*}(f_0^*) \geq 3,$$
$$|\partial_G(f_0)| = \deg_{G^*}(f_0^*) \geq 3.$$

Therefore, f_0, f_n are not digons and the set of strictly parallel edges $e_1, \ldots, e_n$ between $\{u, v\}$ cannot be extended.

Finally, to see why $(e_{m(H)})^* = e_m(H^*)$ note that for every $j \in [n]$ we have

$$\sum_{i=1}^{j} w(e_i) = \sum_{i=1}^{j} w^*(e_i^*).$$

$\square$

From Proposition 2.6 and Lemma 2.9, we directly obtain the following corollary,

Corollary 2.3. *Let (G, w) be given. Then, G^* has a thread of size at least $\frac{k}{5}$, where k is the edge connectivity of G. Moreover, if the weight function w is integer ($w(e) \in \mathbb{N}$), then G^* has a thread of size at least $\lceil \frac{k}{5} \rceil$.*

Lemma 2.10. *Let (G, w) be given. If H is a hank of G, then the graph obtained by removing the thread H^* from the dual planar G^* has similar embedding to dual of the graph obtained by removing the corresponding hank H from G (see Figure 2.14). I.e.,*

$$(G - H)^* \equiv G^* - H^*.$$

Proof. Let H be a given hank with edge set $E(H) = \{e_1, \ldots, e_{m(H)}, \ldots, e_n\} \subset E$. The corresponding edges in thread H^* are

$$E(H^*) = E(H)^* = \{e_1^*, \ldots, e_{m(H)}^*, \ldots, e_n^*\} \subset E^*,$$

where by Lemma 2.9 $e_{m(H^*)} = e^*_{m(H)}$ is the median edges in H^*.

Note that by definition of hank removal we have

$$\begin{aligned} G - H &= (G/e_{m(H)}) \setminus (E(H) \setminus \{e_{m(H)}\}) \\ &= (G/e_{m(H)}) \setminus \{e_1, \ldots e_{m(H)-1}, e_{m(H)+1}, \ldots e_n\}. \end{aligned}$$

The edges in $E(H) \setminus \{e_{m(H)}\}$ are loops in $(G/e_{m(H)})$, by Proposition 2.1 (iv. and v.) we have

$$\begin{aligned} (G - H)^* &= (G/e_{m(H)})^* / \{e_1^*, \ldots e^*_{m(H)-1}, e^*_{m(H)+1}, \ldots e_n^*\} \\ &= (G^* \setminus e^*_{m(H)}) / \{e_1^*, \ldots e^*_{m(H)-1}, e^*_{m(H)+1}, \ldots e_n^*\} \\ &= (G^* \setminus e_{m(H^*)}) / (E(H^*) \setminus \{e_{m(H^*)}\}) \\ &= G^* - H^*, \end{aligned}$$

where the second equality holds since $e_{m(H)}$ is not a loop in G. □

2.3.6 Finding Thin Spanning Trees in a Planar Multi-graph with High Edge Connectivity

Previously, we showed an alternative way of finding a α-thin subset of edges $F \subset E$ of a planar graph $G = (V, E)$. Specifically, we can find subset of edges $F^* \subset E^*$ in the planar dual graph $G^* = (V^*, E^*)$ such that for every cycle C^*, F^* intersects at most $\alpha w^*(C^*)$ many edges of C^*. Later, we defined threads and motivated why it would be a good idea to choose median edges of threads for this task. Finally, we proved existence of large threads using their corresponding dual objects, hanks. Now, we will adjust the algorithm in [16]

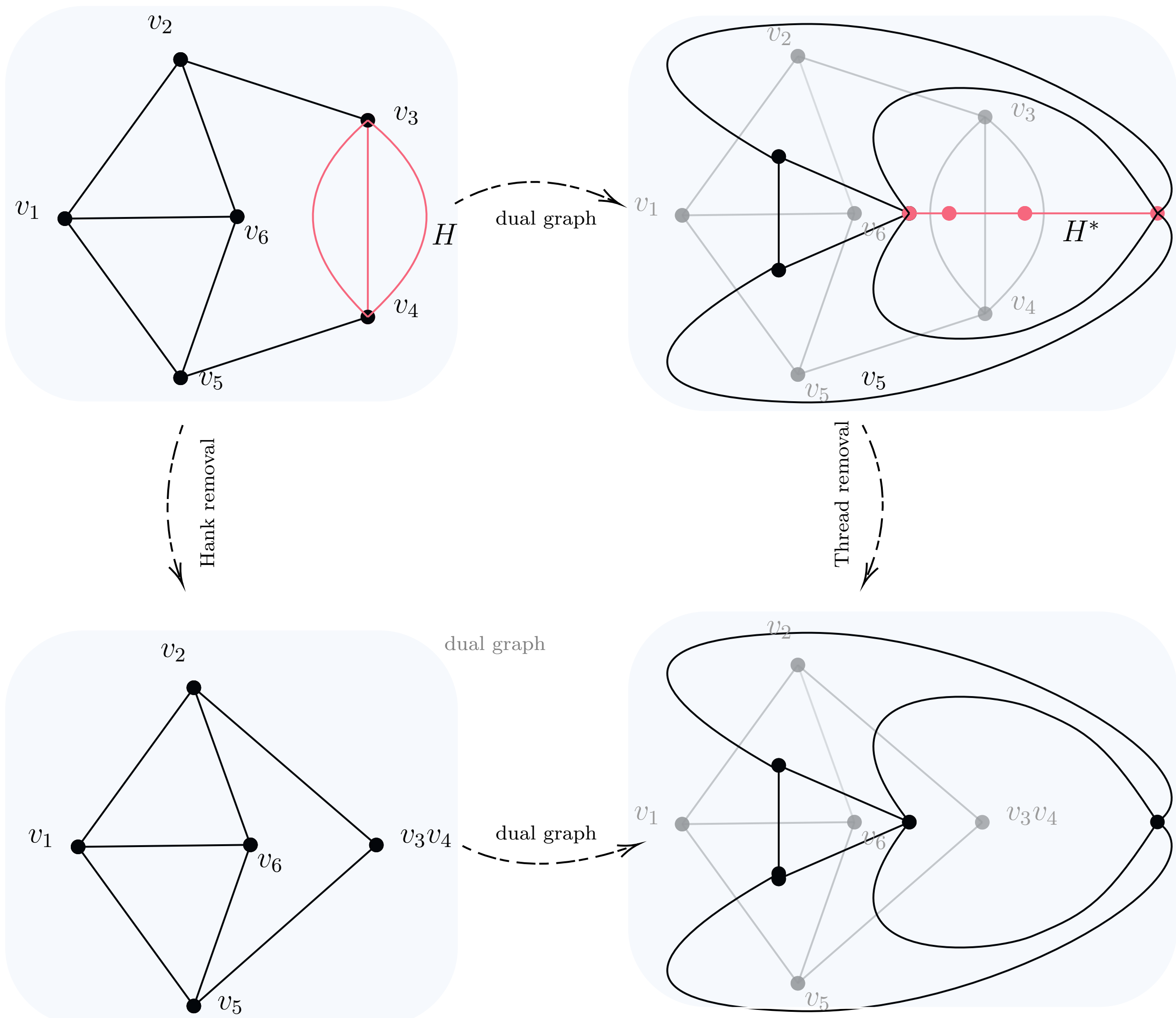


Figure 2.14: The dual of $G - H$ corresponds to the graph $G^* - H^*$.

and show two equivalent algorithms one of which runs on the original graph $G = (V, E)$ and the other on the planar dual $G^* = (V^*, E^*)$. Finally, we prove the equivalence as well as the correctness of both algorithms using the tools developed in the previous sections.

Algorithm 1: Finding thin trees in Planar Graphs with high edge connectivity

Input: planar multi-graph $G = (V, E)$
weight function $w : E \to \mathbb{R}_+$
Output: subset $F \subset E$

Function $\mathcal{A}(G)$:
while *G has a non-loop edge* **do**
  $H \leftarrow$ largest *hank* in G;
  $e_{m(H)} \leftarrow$ median edge of H;
  remove hank H to obtain,
  $G \leftarrow G - H$
  return $\mathcal{A}(G) \cup \{e_m\}$
end

Input: planar dual $G^* = (V^*, E^*)$
dual weight function $w^* : E^* \to \mathbb{R}_+$
Output: subset $F^* \subset E^*$

Function $\mathcal{A}^*(G^*)$:
while *G^* has a non-cut edge* **do**
  $H^* \leftarrow$ largest thread in G^* with no cut-edge $e \in E(H^*)$;
  $e_{m(H^*)} \leftarrow$ median edge of H^*;
  remove thread H^* to obtain,
  $G^* \leftarrow G^* - H^*$
  return $\mathcal{A}^*(G^*) \cup \{e_{m(H^*)}\}$
end

In the following we first prove that the output $\mathcal{A}(G)$ of the primal algorithm is a spanning tree. Next, we will prove that the primal and dual algorithms are equivalent and therefore, $(\mathcal{A}(G))^* \equiv \mathcal{A}(G^*)$, and the set of edges $(\mathcal{A}(G^*))^* \subset E$ is a spanning tree of G as well. We will dedicate the remaining part of this chapter to prove that the outputted spanning tree of both algorithms is $\frac{10}{k}$-combinatorially thin, where k is the edge connectivity of G.

Correctness of $\mathcal{A}$

Proposition 2.7. *Let (G, w) be given. Then, the output of algorithm $\mathcal{A}(G)$ is a spanning tree of G.*

Proof. Let $F = \mathcal{A}(G)$ be the set of edges algorithm outputs. By induction on the size of V, we will show that the algorithm returns $F \subset E$ with $|F| = |V| - 1$. For the base case, if G has only two vertices the algorithm will return an edge $F = \{e\} \in E$ which is spanning and of size $|F| = 1$.

Now, suppose we run the algorithm for G with $|V| = k$. we choose an edge $e = \{u, v\} \in E$ and contract it to obtain the graph $G' = (V', E') = G/e$ where uv is the corresponding vertex obtain by contraction of e. By I.H., the algorithm $\mathcal{A}(G')$ outputs a spanning subset $F' \subset E'$ of G', with $|F'| = k-2$. Now, by the algorithm $\mathcal{A}$ we have $F = F' \cup \{e\}$ and $|F| = k-1$.

Moreover, since F' is a spanning subset of G/e, for every pair of vertices $x, y \in V'$, there exists a path P' from x to y. Now, if P does not have vertex uv then it is contained in G as well. Otherwise, we can break P' into two parts as

$$P' = xe_1 \dots e_i uve_{i+1} \dots y$$

Now, either e_i is adjacent to u or v in G. Without loss of generality, suppose the former is true. Then, we define the path P as

$$P = xe_1 \dots e_i ueve_{i+1} \dots y$$

where all of the edges are in $F = F' \cup \{e\}$. □

Equivalence of $\mathcal{A}$ and $\mathcal{A}^*$

Proposition 2.8. *Let (G, w) be given. For every execution of $\mathcal{A}(G)$, there exists a corresponding execution of $\mathcal{A}^*(G^*)$ such that $\mathcal{A}(G) = (\mathcal{A}^*(G^*))^*$ and vice versa.*

Proof. Again, we will prove the statement by induction on $|V|$. For the base case, if $|V| = 1$, both algorithms output an empty set $\mathcal{A}(G) = \emptyset = (\mathcal{A}^*(G^*))^*$. Now, suppose we run the algorithm for graph $G = (V, E)$, where $|V| = k$.

Let H be the chosen hank of maximum size in graph G with median edge $e_{m(H)}$ when running $\mathcal{A}(G)$. By Lemma 2.9, H^* is a thread of same size in G^* with median edge

$$e_{m(H^*)} = (e_{m(H)})^*.$$

By Lemma 2.9 H^* is also a maximum size thread in G^* without a cut-edge, and the algorithm $\mathcal{A}^*(G^*)$ can choose H^*. Furthermore, by Lemma 2.10, we have

$$(G - H)^* \equiv (G^* - H^*)$$

Now, by I.H., there exists an execution of $\mathcal{A}^*(G^* - H^*)$ where

$$\mathcal{A}^*(G^* - H^*) = \mathcal{A}^*((G - H)^*) = (\mathcal{A}(G - H))^*.$$

Therefore, we have

$$\begin{aligned}(\mathcal{A}(G))^* &= (\mathcal{A}(G-H) \cup \{e_{m(H)}\})^* \\ &= (\mathcal{A}(G-H))^* \cup \{(e_{m(H)})^*\} \\ &= \mathcal{A}^*(G^* - H^*) \cup \{e_{m(H^*)}\} \\ &= \mathcal{A}^*(G^*).\end{aligned}$$

The other direction can be done similarly. □

Corollary 2.4. *Let (G, w) be given and let $F^* = \mathcal{A}^*(G^*)$. Then, the corresponding set $F := (F^*)^*$ in G is a spanning tree of G.*

Upper bounding the thinness of $\mathcal{A}(G) = (\mathcal{A}^*(G^*))^*$.

Proposition 2.9. *Let (G, w) with connectivity k be given and suppose $w(e) \geq 1$ for all $e \in E$. Then, for every knotted thread K^* in G^* we have*

$$w^*(K^*) \geq \Big(|E(K^*) \cap \mathcal{A}^*(G^*)| - 1\Big)d + 1$$

where $2d := \frac{k}{5}$ is a lower bound on the size of threads we pick in each iteration of the algorithm. Moreover, if $w : E \to \mathbb{N}$,

$$d := \lceil \frac{\lceil \frac{k}{5} \rceil}{2} \rceil.$$

Proof. Let $F^* = \mathcal{A}^*(G^*)$ be the output of the algorithm and K^* be a knotted thread in G^*. We will prove the theorem by induction on $a = |F^*|$ which is the number of threads removed from G^* in the total execution of $\mathcal{A}^*(G^*)$.

- *Base case*: suppose the number of removed threads is $a = |F^*| = 1$. Therefore,

  $$t := |F^* \cap E(K^*)| \leq |F^*| \leq 1.$$

  Therefore, either $t = 0$ or $t = 1$. Note that in both cases,

  $$w(E(K^*)) \geq 1 \geq (t-1)d + 1,$$

  since by our assumption $w(e) \geq 1$ for all $e \in E^*$.

- *Induction step:* suppose $a > 1$ threads are removed during the execution of $\mathcal{A}^*(G^*)$. Let T be the first thread removed in $\mathcal{A}^*(G^*)$ and $G_1^* = G^* - T$ be the graph obtained by removing T from G^*. Also, let $t = |F^* \cap E(K^*)|$ be given. One of the following three possibilities can happen:

  1. K^* is a subgraph of T
  2. T is a subgraph of K^*
  3. Neither K^* is a subgraph of T and nor T is a subgraph of K^*.

  Using Proposition 2.5, we prove that the inequality holds in each case as follows.

  1. K^* is a subgraph of T: Therefore, $E(K^*) \setminus E(T) = \emptyset$, and we must have $t \leq 1$ as after removing T no edges from K^* remain to be chosen in $\mathcal{A}^*(G^* - T)$. Similar to above, we have

  $$w(E(K^*)) \geq 1 \geq (t-1)d + 1.$$

  2. T is a subgraph of K^*. Then, by Proposition 2.5, there exist subgraphs K_1, K_2 in $G^* - T$ such that

  $$E(K_1) \cup E(K_2) = E(K^*) \setminus E(T),$$

  where K_1 and K_2 are either knotted threads in $G^* - T$ or are single vertices. Suppose in running of $\mathcal{A}^*(G^* - T)$ we have,

  $$r = |E(K_1) \cap \mathcal{A}^*(G^* - T)| \qquad s = |E(K_2) \cap \mathcal{A}^*(G^* - T)|.$$

  Since T is a subgraph of K^*, we have $e_{m(T)} \in F^*$. Therefore, we must have $r + s = t - 1$. If K_1 and K_2 are both knotted threads in $G^* - T$, then by I.H., we have

  $$\begin{aligned} w(E(K^*)) &= w(E(K^*) \cap E(T)) + w((E(K^*) \setminus E(T)) \\ &\geq w(T) + w(E(K_1)) + w(E(K_2)) \\ &\geq w(T) + (r-1)d + 1 + (s-1)d + 1 \\ &\geq (t-1)d + 1, \end{aligned}$$

  where the last inequality holds since at each step of the algorithm, size of chosen threads are lower bounded by $w(T) \geq 2d = \frac{k}{5}$. Cases where either K_1 or K_2 is a single vertex can be proven similarly.

3. Neither K^* is a subgraph of T and nor T is a subgraph of K^*. Then, by Proposition 2.5, there exists a subgraph K_1 in $G^* - T$ such that

$$E(K_1) = E(K^*) \setminus E(T),$$

where K_1 is either knotted threads in $G^* - T$. If $e_{m(T)} \notin E(K^*)$, then all the edges choosen from K^* are in K_1. Therefore,

$$t = |E(K^*) \cap F^*| = |E(K_1) \cap \mathcal{A}^*(G^* - T)|$$

and by induction step we have

$$\begin{aligned} w(E(K^*)) &= w(E(K^*) \setminus E(T)) + w(E(K^*) \cap E(T)) \\ &\geq w(E(K_1)) \\ &\geq (|E(K^*) \cap F^*| - 1) + 1 \\ &= (t-1)d + 1. \end{aligned}$$

If $e_{m(T)} \in E(K^*)$, then $|E(K_1) \cap F^*| = t - 1$ and by Proposition 2.5 we have

$$w(E(K^*) \cap E(T)) \geq \frac{w(T)}{2}.$$

Therefore,

$$\begin{aligned} w(E(K^*)) &= w(E(K^*) \setminus E(T)) + w(E(K^*) \cap E(T)) \\ &\geq w(E(K_1)) + \frac{w(T)}{2} \\ &\geq (t-2)d + 1 + \frac{w(T)}{2} \\ &\geq (t-1)d + 1, \end{aligned}$$

where the last inequality holds since at each step of the algorithm size of chosen threads are lower bounded by $w(T) \geq 2d = \frac{k}{5}$.

□

Theorem 2.9. *Let (G, w) with connectivity k be given. Then, for every cycle C^* in G^* we have*

$$w(C^*) \geq (|C^* \cap \mathcal{A}^*(G^*)|)d + 1$$

where $2d := \frac{k}{5}$ is a lower bound on size of threads we pick in each iteration of the algorithm. Moreover, if $w : E \to \mathbb{N}$, then

$$d := \lceil \frac{\lceil \frac{k}{5} \rceil}{2} \rceil.$$

Proof. Let $F = \mathcal{A}^*(G^*)$ be the output of the algorithm and C^* be a cycle in G^*. Again, we prove the statement by induction on $a = |F|$ the number of threads removed during the execution of $\mathcal{A}^*(G^*)$.

- *Base case:* if $a = 1$, then

$$t := |C^* \cap F^*| \leq |F^*| \leq 1.$$

But we know $w(C^*) \geq d + 1$ as the girth of G^* is equal to k, the connectivity of G. This is because as by Proposition 2.6 the bonds in G correspond to cycles in G^* and vice versa. Therefore,

$$w(C^*) \geq k \geq \frac{k}{10} + 1.$$

Now, suppose $a \geq 2$ threads are removed during execution of $\mathcal{A}^*(G^*)$. Let T be the first thread removed in $\mathcal{A}^*(G^*)$, and let $t = |C^* \cap F^*|$ be given. We consider the following two cases:

1. T is disjoint from C^*: in this case, C^* is also a subgraph of $G^* - T$. As a result, no edges of C^* are chosen in the first step and

$$t = |C^* \cap F^*| = |C^* \cap \mathcal{A}^*(G^* - T)|.$$

Now, by induction hypothesis on $G^* - T$ we have,

$$w(C^*) \geq td + 1.$$

2. Otherwise, T and C^* have at least one common edge. By Proposition 2.3, T lies entirely in C^*. If T is a cycle thread, $C^* \setminus E(T)$ is an empty set and no other edges will be chosen from C^* in the future steps. As a result, $t \leq 1$ and we obtain the inequality as above.

If T is a path thread, we have

$$w(C^* \cap E(T)) = w(T) \geq 2d.$$

suppose $T = v_i e_i \dots e_{j-1} v_j$, where $1 \le i < j \le n$. Then, by Proposition 2.4

$$K^* = v_j e_j v_{j+1} e_{j+1} \dots v_n e_n v_1 \dots e_{i-1} v_i$$

is a knotted thread in $G - T$, and

$$E(K^*) = C^* \setminus E(T).$$

Moreover, all other edges chosen from C^* must be in K^* and

$$|E(K^*) \cap \mathcal{A}^*(G^* - T)| = t - 1.$$

Thus, by Proposition 2.9, we have

$$w(E(K^*)) \ge (t-2)d + 1.$$

Finally, we have

$$\begin{aligned} w(C^*) &= w(C^* \cap E(T)) + w(C^* \setminus E(T)) \\ &\ge w(T) + w(K^*) \\ &\ge 2d + (t-2)d + 1 \\ &\ge td + 1. \end{aligned}$$

$\square$

Corollary 2.5. *The Algorithm 1 outputs an $\frac{1}{d+\epsilon}$-combinatorially thin spanning tree T of G, where $\epsilon \ge \frac{1}{w(E)}$, and $2d := \frac{k}{5}$ is a lower bound on the size of threads we pick in each iteration of the algorithm. Moreover, if $w : E \to \mathbb{N}$, then*

$$d := \lceil \frac{\lceil \frac{k}{5} \rceil}{2} \rceil.$$

Chapter 3

Spectrally Thin Trees in Structured Graphs

Even though there exist an algorithm to find combinatorially thin trees in graphs with bounded genus, designing efficient algorithms to find combinatorially thin trees in general still seems to be a hard problem. Nevertheless, there are families of graphs which we can prove the existence of combinatorially thin spanning trees in them by using spectral properties of graphs.

Further, as we discussed in Chapter 1, spectral thinness is a more general notion than combinatorial thinness. In fact, spectral thinness upper bounds combinatorial thinness of a given graph, and if a subgraph is spectrally thin, then it is combinatorially thin as well. Moreover, for a given graph $G = (V, E)$ answering whether a spanning tree T is $\theta-$spectrally thin is computationally checkable in polynomial time by confirming whether $\theta\mathcal{L}_G - \mathcal{L}_T \succeq 0$ or not. Although deciding if a specific spanning tree of a given graph is spectrally thin is computationally easy, characterizing spectrally thin graphs in general is not fully understood to this day.

As we mainly focus on the spectral properties of graphs, in this chapter we first state some necessary background information on positive semidefinite matrices as well as properties of Laplacian matrices[1].

Next, we will provide a general upper bound certificate which we can potentially use to prove existence of combinatorially thin trees in certain families of regular graphs. Although the given upper bound is not applicable to all families of graphs, similar to Mousavi in

[1]The reader may skip preliminaries section if they have basic knowledge of spectral graph theory.

[30], we will use it to slightly improve the combinatorial thinness upper bounds for Johnson Graphs, Crown Graphs, and Hamming graphs.

Then, we will switch our discussion into spectrally thin spanning trees. Given a graph $G = (V, E)$ and spanning tree T in G, we first discuss some certificates that prove upper and lower bounds on spectral thinness $\theta_G(T)$. Finally, we utilize these certificates in order to find spectral thinness of some simple families of graphs, such as complete graphs K_n, complete bipartite graphs $K_{n,n}$, and Prism graphs.

3.1 Preliminaries

3.1.1 Linear Algebra and Matrix Theory

For a pair of natural numbers $n, m \in \mathbb{N}$, we use $\mathbb{R}^{n\times m}$ and $\mathbb{S}^n$ to denote vector spaces of $n \times m$ matrices and $n \times n$ symmetric matrices with real entries.

Definition 3.1. (Adjoint) Let V and W be two vector spaces equipped with inner products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$. We define the *adjoint* of a linear map $T : V \to W$ as the unique function $T^* : W \to V$ satisfying,

$$\langle T(v), w\rangle_W = \langle v, T^*(w)\rangle_V \tag{3.1}$$

for all $v \in V$ and $w \in W$.

For symmetric matrices $X, Y \in \mathbb{S}^n$ with X, we define the *Frobenius inner product* $\langle X, Y\rangle = \sum_{i,j\in[n]} X_{ij}Y_{ij}$. We define linear operations $\mathrm{Diag} : \mathbb{R}^n \to \mathbb{S}^n$ and $\mathrm{diag} : \mathbb{S}^n \to \mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$ and $X \in \mathbb{S}^n$ we have,

$$(\mathrm{Diag}(x))_{ij} = \begin{cases} x_i & i = j \\ 0 & \text{otherwise;} \end{cases} \qquad\qquad (\mathrm{diag}(X))_i = X_{ii}.$$

Remark 3.1. Note that linear transformations Diag(.) and diag(.) are adjoint of each other under the Frobenius inner product in $\mathbb{S}^n$ and the standard inner product in $\mathbb{R}^n$.

Proof. To show $\mathrm{Diag}^* = \mathrm{diag}$ we need to prove that for all $Y \in \mathbb{S}^n$ and $x \in \mathbb{R}^n$ we have

$$\langle \mathrm{Diag}(x), Y\rangle = \langle x, \mathrm{diag}(Y)\rangle. \tag{3.2}$$

Note that

$$\begin{aligned}
\langle \mathrm{Diag}(x), Y\rangle &= \sum_{i,j\in[n]} \mathrm{Diag}(x)_{ij} Y_{ij} \\
&= \sum_{i\in[n]} \mathrm{Diag}(x)_{ii} Y_{ii} && (\text{ since } \mathrm{Diag}(x)_{ij} = 0 \text{ for } i \neq j) \\
&= \sum_{i\in[n]} x_i \times \mathrm{diag}(Y)_i \\
&= \langle x, \mathrm{diag}(Y)\rangle.
\end{aligned}$$

Hence, we have $\mathrm{Diag}^* = \mathrm{diag}$. □

Definition 3.2. (Pseudo-Inverse) Let $X \in \mathbb{R}^{n\times m}$ be a matrix. A matrix $X^\dagger \in \mathbb{R}^{m\times n}$ is called the *pseudo-inverse* of X if it satisfies all of the following properties:

i. $XX^\dagger X = X$,

ii. $X^\dagger XX^\dagger = X^\dagger$,

iii. $(XX^\dagger)^\top = XX^\dagger$,

iv. $(X^\dagger X)^\top = X^\dagger X$.

Remark 3.2. Let $X \in \mathbb{R}^{n\times m}$ be an arbitrary given matrix. Pseudo-inverse of X always exists and is unique. (See Theorem 1 in [32].)

Spectral decomposition and symmetric matrices

Proposition 3.1. *(Spectral theorem) Let $X \in \mathbb{S}^n$ be a symmetric matrix. Then, X has real eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Moreover, X has a set of orthonormal eigenvectors $v_i \in \mathbb{R}^n$ such that $Xv_i = \lambda_i v_i$. Additionally, we can write the* spectral decomposition *of X as,*

$$X = \sum_{i=1}^{n} \lambda_i v_i v_i^\top.$$

Equivalently, we have

$$X = VDV^\top,$$

where D is a diagonal matrix with diagonal entries equal to $\lambda_1, \ldots, \lambda_n$ and V is a $n \times n$ matrix where the i-th column of V equals v_i.

Proof. (For proof see for instance Chapter 7 and specifically Theorem 7.29 in [5].) □

Theorem 3.1. *(Reyleigh Quotient) Let $X \in \mathbb{S}^n$ be a symmetric matrix, with real eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and corresponding eigenvectors $v_1, \ldots, v_n$. For $i \in [n]$ we can charecterize the i-th largest eigenvalue λ_i as*

$$\begin{aligned}\lambda_i &= \max_{v \perp v_1, \ldots, v_{i-1}, v \neq 0} \frac{v^\top X v}{v^\top v} \\ &= \min_{v \perp v_{i+1}, \ldots, v_n, v \neq 0} \frac{v^\top X v}{v^\top v}.\end{aligned}$$

Additionally, the corresponding eigenvector v_i is a maximizer of the Rayleigh quotient $\frac{v^\top X v}{v^\top v}$ in the above.

Proof. By spectral theorem we can write,

$$X = \sum_{j=1}^{n} \lambda_j v_j v_j^\top.$$

Therefore, for a vector v orthogonal to the set of vectors $\{v_1, \ldots, v_{i-1}\}$, where $v \neq \mathbf{0}$, the Rayleigh quotient $\frac{v^\top X v}{v^\top v}$ can be computed as

$$\begin{aligned}\frac{v^\top X v}{v^\top v} &= \frac{\sum_{j=1}^{n} \lambda_j v^\top v_j v_j^\top v}{v^\top v} \\ &= \frac{\sum_{j=i}^{n} \lambda_j v^\top v_i v_i^\top v}{v^\top v} \\ &= \sum_{j=i}^{n} \lambda_j \frac{\langle v, v_i \rangle^2}{||v||^2} \\ &\leq \lambda_i \sum_{j=i}^{n} \frac{\langle v, v_i \rangle^2}{||v||^2} \\ &\leq \lambda_i.\end{aligned}$$

On the other hand, if we choose $v = v_i$, we have

$$\frac{v^\top X v}{v^\top v} = \lambda_i.$$

The other characterization can be proved in a similar fashion. $\square$

Trace and determinant For a square matrix $X \in \mathbb{R}^{n\times n}$, we denote its trace and determinant as $\mathrm{Tr}(X)$ and $\det(X)$, respectively. In the following, we state some of the important properties of trace and determinant without proof. (To read the proofs and related materials, see Chapter 8 section D and Chapter 9 section C in [5] for instance.)

Proposition 3.2. *Let $X, Y \in \mathbb{S}^n$ be symmetric matrices with real eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_n$, respectively, and let $c \in \mathbb{R}$ be a constant. Then,*

- $\mathrm{Tr}(X) = \sum_{i=1}^n \lambda_i$,
- $\det(X) = \prod_{i=1}^n \lambda_i$,
- $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$,
- $\det(XY) = \det(X)\det(Y)$,
- $\det(cX) = c^n \det(X)$,
- $\mathrm{Tr}(cX) = c\,\mathrm{Tr}(X)$.

Positive semidefinite and definite matrices

Definition 3.3. (Positive semidefiniteness) A symmetric matrix $X \in \mathbb{S}^n$ is called a *positive semidefinite* matrix if for all $v \in \mathbb{R}^n$,

$$v^\top X v \geq 0.$$

We denote a symmetric positive semidefinite (PSD) matrix X by $X \succeq 0$. Moreover, X is *positive definite* if for all $v \in \mathbb{R}^n \setminus \{0\}$, we have $v^\top X v > 0$. Similarly, we denote a symmetric positive definite (PD) X by $X \succ 0$. Additionally, We denote the set of all symmetric positive semidefinite and symmetric positive definite matrices by $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$, respectively.

Remark 3.3. For symmetric matrices $X, Y \in \mathbb{S}^n$, we use $X \succeq Y$ to denote $X - Y \succeq 0$.

In the following, we will state a set of equivalent conditions for a symmetric matrix $X \in \mathbb{S}^n$ to be PSD without proof. (To see the proof you may also see Chapter 7 section (C) and Theorem 7.38 in [5] or Chapter 1 in [38].)

Proposition 3.3. *For a symmetric matrix $X \in \mathbb{S}^n$, the following statements are equivalent.*

i. X is positive semidefinite.

ii. There exists a matrix $B \in \mathbb{R}^{n\times n}$ such that $X = BB^\top$.

iii. All eigenvalues λ_i of X are non-negative; i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$.

iv. For all nonempty subsets $J \subseteq [n]$, the determinant of the principal submatrix X_J is non-negative; i.e. $\det(X_J) \geq 0$.

v. X can be decomposed as the following summation

$$X = \sum_{i=1}^{n} \mu_i v_i v_i^\top,$$

where $v_i \in \mathbb{R}^n$ and $\mu_i \geq 0$ for all $i \in [n]$.

Proposition 3.4. *Let $X \in \mathbb{S}^n_+$ be a positive semidefinite matrix with spectral decomposition,*

$$X = \sum_{j=1}^{i} \lambda_i v_i v_i^\top,$$

where

$$\lambda_1 \geq \lambda_i > \lambda_{i+1} = \cdots = \lambda_n = 0$$

are its eigenvalues with corresponding orthonormal eigenvectors,

$$v_1, \ldots, v_n.$$

Then,

i. X has a symmetric positive semidefinite square root $X^{\frac{1}{2}}$, where $X^{\frac{1}{2}}X^{\frac{1}{2}} = X$, and we have

$$X^{\frac{1}{2}} = \sum_{j=1}^{i} \sqrt{\lambda_i} v_i v_i^\top.$$

ii. the pseudo-inverse $X^\dagger$ is equal to,

$$X^\dagger = \sum_{j=1}^{i} \frac{1}{\lambda_j} v_j v_j^\top,$$

and therefore, $X^\dagger$ is also positive semidefinite.

Lemma 3.1. *(Schur Complement) Let $X \in \mathbb{S}^n$ be a symmetric matrix and let $A \in \mathbb{S}^n_{++}, B \in \mathbb{S}^n$. Then, we have*

$$X := \begin{bmatrix} A & U^\top \\ U & B \end{bmatrix} \succeq 0 \iff B - UA^{-1}U^\top \succeq 0. \tag{3.3}$$

Proof. For proof, see Lemma 1.22 in [38]. □

3.1.2 Spectral Graph Theory Tools and Facts

Recall the definition of Laplacian matrices (Definition 1.3). Given a graph $G = (V, E)$ and weight $w \in \mathbb{R}^E_+$, we may also consider Laplacian of G as a mapping from $\mathbb{R}^E$ to the space of symmetric matrices $\mathbb{S}^V$ defined as

$$\mathcal{L}_G(w) = \sum_{\{i,j\} \in E} w_{ij}(e_i - e_j)(e_i - e_j)^\top, \tag{3.4}$$

where e_i are vectors of the standard basis for $\mathbb{R}^V$. We state some of the key properties of Laplacian matrices in the following without proof.

Proposition 3.5. *Given a graph $G = (V, E)$, the following properties hold for the Laplacian $\mathcal{L}_G : \mathbb{R}^E \to \mathbb{S}^V$.*

i. For every edge-weights $w \in \mathbb{R}^E$ and vector $v \in \mathbb{R}^V$,

$$v^\top \mathcal{L}_G(w) v = \sum_{\{i,j\} \in E} w_{\{i,j\}}(v_i - v_j)^2.$$

Therefore, if $w \in \mathbb{R}^E_+$, then $\mathcal{L}_G(w)$ is a positive semidefinite matrix, and for every pair $w, u \in \mathbb{R}^E$ such that $w \geq u$, we have $\mathcal{L}_G(w) \succeq \mathcal{L}_G(u)$.

ii. The adjoint *of $\mathcal{L}_G$ denoted by $\mathcal{L}^*_G : \mathbb{S}^V \to \mathbb{R}^E$ is given by*

$$[\mathcal{L}^*_G(X)]_{\{i,j\}} = X_{ii} + X_{jj} - 2X_{ij} = \mathrm{Tr}\big[(e_i - e_j)(e_i - e_j)^\top X\big],$$

for every edge $\{i, j\} \in E$.

iii. For every symmetric positive semidefinite matrix $X \in \mathbb{S}_+^V$, the vector $\mathcal{L}_G^*(X) \in \mathbb{R}^E$ is a non-negative vector; therefore, for every pair $X, Y \in \mathbb{S}^V$ such that $X \preceq Y$, we have $\mathcal{L}_G^*(X) \leq \mathcal{L}_G^*(Y)$.

Laplacian matrix $\mathcal{L}_G(w)$ has some information on the combinatorial properties of graph G, where we state some of these properties in the following.

Proposition 3.6. *Let $G = (V, E)$ be a graph with weight function $w \in \mathbb{R}^E$. Then,*

i. The vector of all ones $\mathbb{1} \in \mathbb{R}^E$ is in the null-space of $\mathcal{L}_G(w)$. As a result,

$$\operatorname{Rank}(\mathcal{L}_G(w)) \leq |V| - 1.$$

ii. The graph G on the support of the weight function w is connected if and only if

$$\operatorname{Rank}(\mathcal{L}_G(w)) = |V| - 1.$$

I.e., $\mathcal{L}_G(w)$ has eigenvalues $\lambda_1 \geq \cdots \geq \lambda_{n-1} > \lambda_n = 0$.

iii. Suppose $w \in \mathbb{R}_+^E$, and let $H = (V, E(H))$ be a subgraph of G. Then for the corresponding weight function

$$\hat{w}(e) := w(e) \qquad \forall e \in E(H)$$

we have

$$\mathcal{L}_G(w) \succeq \mathcal{L}_H(\hat{w}).$$

Specifically,

$$\mathcal{L}_G(\mathbb{1}) \succeq \mathcal{L}_H(\mathbb{1}).$$

We can also define the adjacency matrix of a graph $G = (V, E)$ as a linear map $\mathcal{A}_G$ that takes an edge weight $w \in \mathbb{R}^E$ to a symmetric matrix in $\mathbb{S}^V$; i.e.,

$$\mathcal{A}_G : \mathbb{R}^E \to \mathbb{S}^V$$
$$(\mathcal{A}_G(w))_{i,j} = \begin{cases} w(\{i,j\}), & \{i,j\} \in E \\ 0, & \text{otherwise.} \end{cases}$$

Another way of viewing the Laplacian matrix $\mathcal{L}_G(w)$ is through the adjacency matrix $\mathcal{A}_G(w)$. More specifically, let $\mathcal{D}_G : \mathbb{R}^E \to \mathbb{S}^V$ map edge weights $w \in \mathbb{R}^E$ to a diagonal matrix of degrees, where

$$(\mathcal{D}_G(w))_{ii} = \deg_{G,w}(i) = \sum_{j:\{i,j\}\in E} w(\{i,j\}).$$

Then, we have the following.

$$\mathcal{L}_G(w) = \mathcal{D}_G(w) - \mathcal{A}_G(w). \tag{3.5}$$

From this observation, we can immediately obtain the following result.

Proposition 3.7. *Let $G = (V, E)$ be a k-regular graph where $\lambda_1 \geq \cdots \geq \lambda_n$ are eigenvalues of $\mathcal{L}_G(\mathbb{1})$ and $\alpha_1 \geq \cdots \geq \alpha_n$ are eigenvalues of $\mathcal{A}_G(\mathbb{1})$. Then,*

$$\lambda_i = k - \alpha_{n-i+1}.$$

Similar to the Laplacian matrix, the adjacency matrix $\mathcal{A}_G$ has some information on the combinatorial properties of graph G.

Proposition 3.8. *Let G be a given graph where $\alpha_1 \geq \cdots \geq \alpha_n$ are the eigenvalues of $\mathcal{A}_G(\mathbb{1})$. Then, G is bipartite if and only if for all $i \in [n]$ we have,*

$$\alpha_i = -\alpha_{n-i+1}, \qquad \forall\ i \in [n].$$

3.1.3 Notation

Whenever we intend to refer to an eigenvalue of a matrix X (Laplacian of a graph G) corresponding to an eigenvector v, we use $\lambda_X(v)$ ($\lambda_G(v)$). We also denote the i-th largest eigenvalue of a matrix X (Laplacian of a graph G) as $\lambda_i(X)$ ($\lambda_i(G)$).

Finally, in the following sections, we denote the Laplacian (adjacency matrix) of a graph $G = (V, E)$ with edge weights $\mathbb{1} \in \mathbb{R}^E$ by $\mathcal{L}_G := \mathcal{L}_G(\mathbb{1})$ ($\mathcal{A}_G := \mathcal{A}_G(\mathbb{1})$). Throughout the chapter, we refer the vertices of graph G with indices such as i and j.

3.2 Combinatorial Thinness Upper Bound via Spectral Properties

One approach to prove that certain families of graphs are combinatorially thin is to show an upper bound on their thinness parameters. In this section, given a simple $k-$regular graph

$G = (V, E)$ and a spanning tree $T = (V, E(T))$, we will state a general certificate of upper bound that uses the spectral properties of the underlying graph G. More specifically, we will use the second smallest eigenvalue of $\mathcal{L}_G$ as well as the maximum degree of spanning tree T.

Having this result, we obtain a slightly improved version of combinatorial lower bounds in [30] by using the fact that Johnson, Hamming, and Crown graphs are Hamiltonian.

General upper bound for α_G We will present a result which is also used by Mousavi [30] as a certificate of upper bound on the combinatorial thinness of regular graphs.

Graph G	Degree k	$\lambda_{\lvert V\rvert-1}(G)$	Upper Bounds in [30]	Improved Upper Bounds
$J(n,k)$	$k(n-k)$	n	$\alpha_G \leq \frac{6}{n-6}$	$\alpha_G \leq \frac{4}{n}$
$Crown(n)$	$n-1$	$n-2$	$\alpha_G \leq \frac{6}{n-2}$	$\alpha_G \leq \frac{4}{n-2}$
$H(q,n)$	$q(n-1)$	n	$\alpha_G \leq \frac{6}{n}$	$\alpha_G \leq \frac{4}{n}$

Table 3.1: Upper bounds obtained for families of regular graph using Proposition 3.9 and corresponding upper bounds in [30].

Proposition 3.9. *Let $G = (V, E)$ be a $k-$regular connected graph and $T = (V, E(T))$ be a spanning tree in G with maximum degree equal to d. Let $\lambda_1 \geq \cdots \geq \lambda_{|V|-1} > \lambda_{|V|} = 0$ be the eigenvalues of $\mathcal{L}_G$. Then, we have $\alpha_G(T) \leq \frac{2d}{\lambda_{|V|-1}}$.*

To prove this proposition we will use a special case of *Cheeger's Inequality* that holds for $k-$regular graphs. Recall Definition 2.2 where we introduced notion of *edge expansion* of G, $\Phi(G)$, which is the smallest ratio between any cut set $\delta_G(S)$ and its corresponding cut shore $S \subset V$ with $|S| \leq \frac{|V|}{2}$.

Theorem 3.2. *Suppose $G = (V, E)$ is a $k-$regular graph, and let λ_{n-1} be the second smallest eigenvalue of $\mathcal{L}_G(\mathbb{1})$. Then, we have*

$$\frac{\lambda_{n-1}}{2} \leq \Phi(G). \tag{3.6}$$

Proof. See Appendix A $\square$

Proof of Proposition 3.9. To show that $\alpha_G(T) \leq \frac{2d}{\lambda_{n-1}}$, we must show that for each cut set $\delta_G(U)$ where $U \subset V$ and $|U| \leq \frac{|V|}{2}$ we have

$$|\delta_G(U) \cap T| \leq (\frac{2d}{\lambda_{n-1}})|\delta_G(U)|.$$

Since $\Delta(T) = d$, we have

$$|\delta(U) \cap T| \leq d|U|.$$

Thus,

$$\begin{aligned}
\frac{|\delta_G(U) \cap T|}{|\delta_G(U)|} &\leq \frac{d|U|}{|\delta_G(U)|} \\
&\leq \frac{d}{\Phi(G)} && \text{(Since } \Phi(G) \leq \frac{|\delta_G(U)|}{|U|} \text{)} \\
&\leq \frac{2d}{\lambda_{n-1}} && \text{(By Cheeger's inequality).}
\end{aligned}$$

Therefore, $\alpha_G(T) \leq \frac{2d}{\lambda_{n-1}}$. □

Next, we will utilize Proposition 3.9 to upper bound the combinatorial thinness of some families of regular graphs.

3.2.1 Johnson Graphs

Definition 3.4. (Johnson Graphs) *Johnson Graph* $J(n, k) = (V, E)$ is defined as the graph where its vertices are $k-$subsets $S \subset [n]$, i.e.,

$$V := \Big\{ S \subset [n] : \ |S| = k \Big\}.$$

There is an edge between two vertices $S, S' \subset [n]$ of $J(n, k)$ if their intersection has size $k - 1$ (see Figure 3.1).

$$E := \Big\{ \{S, S'\} : \ S, S' \in V \text{ and } |S \cap S'| = k - 1 \Big\}.$$

It is easy to see that *Johnson graph* $J(n, k)$ is a $k(n - k)-$regular graph, and $J(n, k)$ is isomorphic to $J(n, n - k)$, as subsets of size k and $n - k$ are complements of each other.

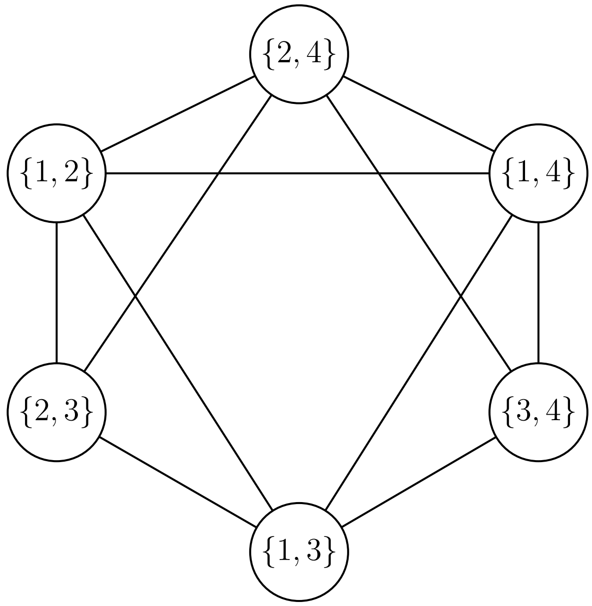


Figure 3.1: Johnson graph $J(4,2)$

In order to use Proposition 3.9 and improve the bounds given in [30], we intend to find a spanning tree T with smaller maximum degree. Note that as any spanning tree has a vertex of degree at least two, having $\Delta_T = 2$ is the best choice for a spanning tree, which means T is a Hamiltonian path. Therefore, we need to investigate whether Johnson graphs have Hamiltonian paths.

In 2012, Alspach [1] proved the following result on Johnson graphs.

Theorem 3.3. *For all $n \in \mathbb{N}$ and $k \leq n$, Johnson Graph $J(n,k) = (V,E)$ is* Hamiltonian-connected*; i.e., there exists a Hamiltonian path between each pair of distinct vertices $S_i, S_j \in V$.*

Next, to further improve results in [30], we obtain the explicit value of λ_{n-1} in terms of k and n instead of estimating a lower bound on it.

Brouwer et.al. in [10] gave an explicit formula for eigenvalues of adjacency matrix of Johnson graphs (see Theorem 9.2). Note that since Johnson graphs are regular, we can easily find the eigenvalues of their Laplacian matrix as well.

Theorem 3.4. *Let $J(n,k)$ be a Johnson graph with $2 \leq k \leq n-2$. The graph $J(n,k)$ has diameter equal to $d = \min(n-k,k)$, and its adjacency matrix $\mathcal{A}_{J(n,k)}$ has at most $d+1$*

distinct eigenvalues. The set of the distinct eigenvalues τ_j *and their multiplicity* m_j*, for* $0 \leq j \leq d$*, is given by*

$$\tau_{j+1} = (k-j)(n-k-j) - j = j^2 - (n+1)j + nk - k^2$$

and

$$m_{j+1} = \binom{n}{j} - \binom{n}{j-1} \qquad j \in \{1, \ldots, d\},$$

and $m_1 = 1$.

Corollary 3.1. *Since* $d \leq \frac{n}{2}$*, the function* τ_{j+1} *is a decreasing function in terms of* j*, and multiplicity of* τ_1 *is equal to 1. Hence, the second largest eigenvalue of* $\mathcal{A}_{J(n,k)}$ *is equal to* $\tau_2 = nk - k^2 - n$*, and the second smallest eigenvalue of* $\mathcal{L}_{J(n,k)}$ *is equal to*

$$\lambda_{|V|-1} = k(n-k) - \tau_2 = n.$$

By the given conditions in Proposition 3.9, since Johnson graph has a Hamiltonian path tree P, where $\Delta(P) = 2$ we get:

$$\alpha_{J(n,k)} \leq \alpha_{J(n,k)}(P) \leq \frac{4}{n}. \tag{3.7}$$

Therefore, Johnson graphs $J(n,k)$ are combinatorially thin, for all $n \geq 5$.

3.2.2 Crown Graphs

Definition 3.5. (*Crown graph*) $Crown(n) = (V, E)$ is a graph on a vertex set of size $2n$,

$$V = \{1, \ldots, n, n+1, \ldots, 2n\}$$

and edge set

$$E = \{\{i, n+j\} : \ i, j \in [n] \text{ and } i \neq j\}.$$

In other words, $Crown(n)$ is isomorphic to a complete bipartite graph with one set of perfect matching edges deleted (see Figure 3.2).

Lemma 3.1. *For all* $n \in \mathbb{N}$*,* $Crown(n)$ *has at least one Hamiltonian path* P*.*

Proof. We will explicitly show a Hamiltonian path of $Crown(n)$ that passes through all vertices.

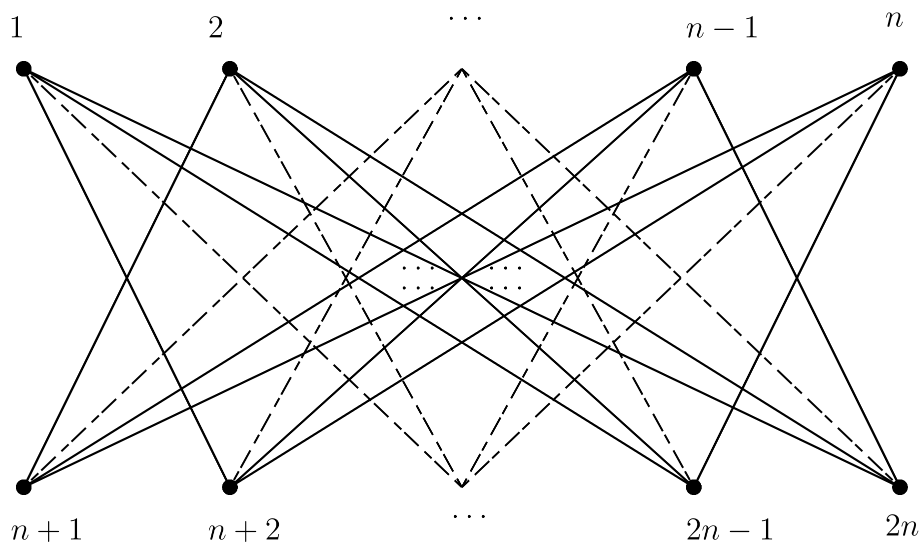


Figure 3.2: Crown graph $Crown(n)$

First, suppose $n = 2k + 1$, consider the following sequence of vertices which each two consequetive vertices are connected in $Crown(n)$:

$$P = 1, n + 2, 3, n + 4, 5, \ldots, 2n - 1, n, n + 1, 2, n + 3, \ldots, n - 1, 2n.$$

Note that P is indeed a path as vertices $\{i, n + i + 1\}$, $\{i, n + i - 1\}$, and $\{n, n + 1\}$ are edges in $Crown(n)$. Additionally, all vertices of $Crown(2k + 1)$ are in the path and therefore the path is spanning as well. Thus, P is a Hamiltonian path.
Similarly, for the case that $n = 2k$ we have the following Hamiltonian path.

$$P = 1, n + 2, 3, n + 4, 5, \ldots, n - 1, 2n, 2, n + 3, \ldots, 2n - 1, n, n + 1.$$

$\square$

Lemma 3.2. *The second smallest eigenvalue of $\mathcal{L}_{Crown(n)}$ is equal to $\lambda_{2n-1}(Crown(n)) = n - 2$.*

To prove this lemma, we will utilize definition of Cartesian product of two graphs as in Definition 2.1. Note that, $Crown(n)$ is isomorphic to the graph $\overline{K_n \Box K_2}$, which is the complement graph of Cartesian product of K_n and K_2 (see Figure 3.2). The following shows how to obtain eigenvalues of a Cartesian product of two graphs.

Proposition 3.10. *Suppose $G_1 \square G_2$ is Cartesian product of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. If $v_1 \in \mathbb{R}^{V_1}$ and $v_2 \in \mathbb{R}^{V_2}$ are eigenvectors of $\mathcal{A}_{G_1}$ and $\mathcal{A}_{G_2}$ with corresponding eigenvalues a_1 and a_2, then $v_1 \otimes v_2 \in \mathbb{R}^{V_1 \otimes V_2}$ is an eigenvector of $\mathcal{A}_{G_1 \square G_2}$ with eigenvalue $a_1 + a_2$.*

Proof. See [18], Section 9.7. □

Proof of Lemma 3.2. The adjacency matrix of the complement graph can be computed as

$$\mathcal{A}_{Crown(n)} = \mathcal{A}_{\overline{K_n \square K_2}} = (J_{2n} - I_{2n}) - \mathcal{A}_{K_n \square K_2},$$

where $J_{2n} = \mathbb{1}_{2n} \mathbb{1}_{2n}^\top$ is the all ones matrix in $\mathbb{S}^{2n}$. Moreover, as $Crown(n)$ is a $(n-1)-$regular graph, we have

$$\begin{aligned} \mathcal{L}_{crown(n)} &= (n-1)I_{2n} - \mathcal{A}_{Crown(n)} \\ &= nI_{2n} - J_{2n} + \mathcal{A}_{K_n \square K_2}. \end{aligned}$$

It is easy to calculate the set of eigenvalues of $\mathcal{A}(K_n)$ as

$$\begin{aligned} \lambda_n(\mathcal{A}(K_n)) = \cdots = \lambda_2(\mathcal{A}(K_n)) &= -1 \\ \lambda_1(\mathcal{A}(K_n)) &= n-1. \end{aligned}$$

Let

$$n = \lambda_1(\mathcal{A}_{K_n \square K_2}) \geq n-2 = \lambda_2(\mathcal{A}_{K_n \square K_2}) \geq -2 = \lambda_3(\mathcal{A}_{K_n \square K_2}) = \cdots = \lambda_{2n}(\mathcal{A}_{K_n \square K_2}).$$

be the eigenvalues of $\mathcal{A}_{K_n \square K_2}$ with corresponding eigenvectors $v_1 = \mathbb{1}, \ldots, v_{2n-1}, v_{2n}$. Then, the same set of eigenvectors are also eigenvectors of I_{2n} and J_{2n} with eigenvalues,

$$\lambda_1(I_{2n}) = \cdots = \lambda_{2n}(I_{2n}) = 1$$

and

$$\lambda_1(J_{2n}) = 2n, \lambda_2(J_{2n}) = \cdots = \lambda_{2n}(J_{2n}) = 0.$$

Therefore, we can decompose the Laplacian matrix of $Crown(n)$ as

$$\begin{aligned}
\mathcal{L}_{crown(n)} &= nI_{2n} - J_{2n} + \mathcal{A}_{K_n \square K_2} \\
&= n\sum_{i=1}^{2n} v_i v_i^\top - (2n)\,\mathbb{1}\,\mathbb{1}^\top + \sum_{i=1}^{2n} \lambda_i(\mathcal{A}_{K_n\square K_2}) v_i v_i^\top \\
&= (-n + \lambda_1(\mathcal{A}_{K_n\square K_2}))\,\mathbb{1}\,\mathbb{1}^\top + \Big(n + \lambda_2(\mathcal{A}_{K_n\square K_2})\Big) v_2 v_2^\top + \sum_{i=3}^{2n}\Big(n + \lambda_i(\mathcal{A}_{K_n\square K_2})\Big) v_i v_i^\top \\
&= 0\,\mathbb{1}\,\mathbb{1}^\top + (2n-2) v_2 v_2^\top + \sum_{i=3}^{2n} (n-2) v_i v_i^\top.
\end{aligned}$$

Thus, we can calculate the second smallest eigenvalue of $\mathcal{L}_{crown(n)}$ as

$$\lambda_{2n-1}(Crown(n)) = n - 2.$$

□

Therefore, by utilizing Lemma 3.1 and Lemma 3.2 in Proposition 3.9, we obtain an upper bound for combinatorial thinness of crown graph as follows.

$$\alpha_{crown(n)} \leq \frac{4}{n-2}. \tag{3.8}$$

Thus, all Crown graphs $Crown(n)$ with $n \geq 7$ have a combinatorially thin spanning tree.

3.2.3 Hamming Graphs

Another family of regular graphs are *Hamming graphs,* denoted by $H(q, n)$.

Definition 3.6. Hamming graph $H(q, n) = (V, E)$ is the graph where each vertex $v \in V$ is a $q-$tuple $v = (a_1, \ldots, a_n) \in \{1, \ldots, n\}^q$. Two vertices $v_i = (a_1, \ldots, a_q)$ and $v_j = (b_1, \ldots, b_q)$ are connected by an edge $e = \{v_i, v_j\} \in E$ if there exists exactly one index ℓ where $a_\ell \neq b_\ell$.
In other words, $H(q, n)$ is isomorphic to Cartesian product of q complete graphs K_n (see Figure 3.3).

In Hamming graph $H(q, n)$, each vertex $v_i = (a_1, \ldots, a_q)$ is neighbour to exactly $q(n-1)$ other vertices; thus, Hamming graphs are $q(n-1)-$regular graphs. Similar to Johnson

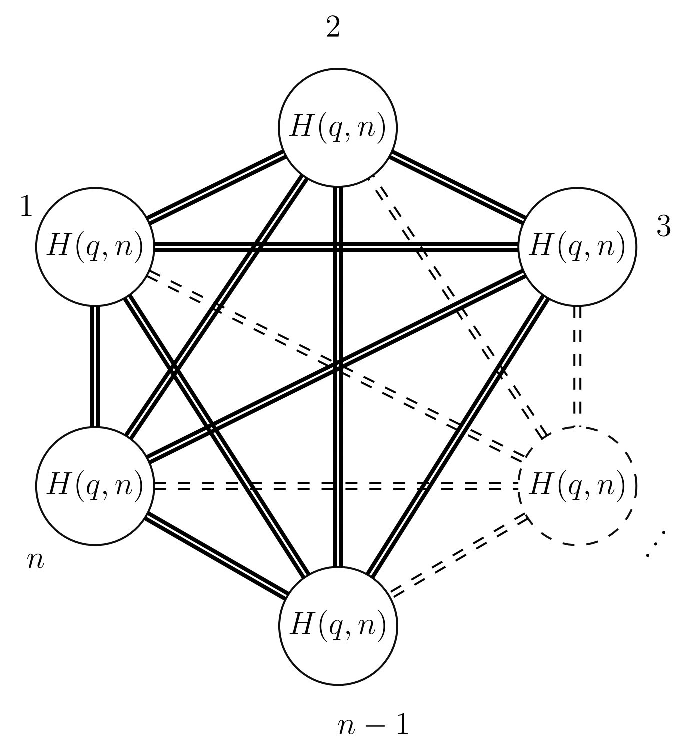


Figure 3.3: Hamming graph $H(q+1,n)$ from $H(q,n)$.

graphs, we can use Theorem 9.2.1 of [10] to convert eigenvalues of the adjacency matrix $\mathcal{A}_{H(q,n)}$ to eigenvalues of Laplacian $\mathcal{L}_{H(q,n)}$ and find the second smallest eigenvalue of $\mathcal{L}_{H(q,n)}$ as

$$\lambda_{|V|-1}(H(q,n)) = n.$$

Therefore, if we show that $H(q,n)$ has a Hamiltonian path P for all $q, n \in \mathbb{N}$, we will obtain an upper bound

$$\alpha_{H(q,n)} \leq \frac{2.d}{\lambda_{|V|-1}} \leq \frac{4}{n} \tag{3.9}$$

on the combinatorial thinness of $H(q,n)$.

Lemma 3.3. *Given $n, q \in \mathbb{N}$ with $n \geq 2$, Hamming graph $H(q,n)$ has a Hamiltonian path P.*

Proof. We will show that Hamming graphs are Hamiltonian, by induction on $q \in \mathbb{N}$.

- *Base:* For $q = 1$, clearly $H(1,n) \simeq K_n$ is Hamiltonian.
- *Induction hypothesis:* Suppose for $H(q-1,n)$ we have a Hamiltonian path $P_{q-1} = v_1, \ldots v_{n^{q-1}}$.
- *Induction step:* We will show $H(q,n)$ has also a Hamiltonian path. To do so, we will define a new notation where for a vertex $v_i = (a_1, \ldots, a_{q-1}) \in \{1, \ldots, n\}^{q-1}$ we use $jv_i := (j, a_1, \ldots, a_{q-1})$ for $1 \leq j \leq n$. Now, for odd $n = 2k+1$ we define

  $$P_q := 1v_1, \ldots, 1v_{n^{q-1}}, 2v_{n^{q-1}}, \ldots, 2v_1, 3v_1, \ldots, 3v_{n^{q-1}}, \ldots\ldots, nv_1, \ldots, nv_{n^{q-1}} \tag{3.10}$$

  and for even $n = 2k$ we define

  $$P_q := 1v_1, \ldots, 1v_{n^{q-1}}, 2v_{n^{q-1}}, \ldots, 2v_1, 3v_1, \ldots, 3v_{n^{q-1}}, \ldots\ldots, nv_{n^{q-1}}, \ldots, nv_1. \tag{3.11}$$

  Note that by definition of Hamming graphs and (I.H), all vertices in the path are connected in $H(q,n)$. Therefore. P_q is indeed a path. Moreover, path P_q is spanning and all vertices of $H(q,n)$ appeared in P_q exactly once. Hence, P_q is a Hamiltonian path in $H(q,n)$.

□

3.3 Spectral Thinness and Certificates of Bounds

Recall the notion of spectral thinness (Definition 1.5), where for a given graph $G = (V, E)$ with weights $w \in \mathbb{R}_+^E$, a subgraph $H = (V, F)$ is $\theta-$spectrally thin subgraph, if

$$\theta \mathcal{L}_G(w) - \mathcal{L}_H \succeq 0.$$

We also defined spectral thinness of a subgraph H, denoted by $\theta_{G,w}(H)$, to be smallest $\theta \in [0, 1]$ that satisfies the above inequality.

In this section, we aim to present some mathematical certificates of bounds for spectral thinness of subgraphs in a given graph, and we will discuss these certificates and techniques for proving lower and upper bounds.

3.3.1 Lower Bounds

One way of finding such values of θ_l is by considering the notion of extended eigenvalues and eigenvectors defined as follows.

Definition 3.7. Let $A, B \in \mathbb{S}^n$ be two real valued matrices. A real number θ is an *extended eigenvalue* of (A, B) if there exists a vector $v \in \mathbb{R}^n$ such that

$$\begin{aligned} Av &= \theta B v \\ Bv &\neq \mathbf{0}\,. \end{aligned}$$

Proposition 3.11. *Let graph $G = (V, E)$ be a connected graph and let $w \in \mathbb{R}_+^E$ be its weight function. Suppose T is a spanning tree of G, and let θ be an extended eigenvalue of $(\mathcal{L}_T, \mathcal{L}_G(w))$. Then,*

$$\theta \leq \theta_{G,w}(T).$$

Proof. Let $v \in \mathbb{R}^n$ be the corresponding vector for θ as in the above definition, and let $\varepsilon > 0$ be an arbitrary small number. Then,

$$\begin{aligned} v^\top \Big((\theta - \varepsilon)\mathcal{L}_G(w) - \mathcal{L}_T \Big) v &= v^\top \Big(\theta \mathcal{L}_G(w) - \mathcal{L}_T \Big) v + -\varepsilon v^\top \mathcal{L}_G v \\ &= -\varepsilon v^\top \mathcal{L}_G v < 0, \end{aligned}$$

where the last step is because $\mathcal{L}_G v \neq \mathbf{0}$ and $\mathcal{L}_G = UU^\top$ is a PSD matrix. $\square$

To relax the above proposition, we can consider an arbitrary value θ_l instead of an extended eigenvalue. To do so, we need to show that the difference matrix $\theta_l \mathcal{L}_G(w) - \mathcal{L}_T$ has a negative eigenvalue. Equivalently $\theta_l < \theta_{G,w}$ if and only if there exists a vector $v \in \mathbb{R}^V$ such that

$$v^\top(\theta_l \mathcal{L}_G(w) - \mathcal{L}_T)v < 0.$$

Note that in the above, we first choose a lower bound θ_l and then find a certificate v for it. The issue with this method is that it does not obtain the best lower bound possible for the certificate v. In fact, one needs to first guess a lower bound and then check whether $\theta_l \mathcal{L}_G(w) - \mathcal{L}_T \succeq 0$ or not, and then, increase the value of θ_l. However, an equivalent approach toward finding a lower bound is to fix a vector $v \in (\mathbb{R}^V \setminus \{\mathbf{0}\}) \cap \text{ span}(\mathbb{1})^\perp$, and then rule out all values $\theta_l \in (0, 1]$ such that

$$v^\top(\theta_l \mathcal{L}_G(w) - \mathcal{L}_T)v < 0.$$

Remark 3.4. Note that this condition guarantees that

$$\begin{aligned} v^\top \mathcal{L}_T v &> v^\top \theta_l \mathcal{L}_G v \\ \mathcal{L}_G v &\neq 0. \end{aligned}$$

In the following, we will discuss how to obtain the best lower bound corresponding to a certificate vector $v \in (\mathbb{R}^V \setminus \{\mathbf{0}\}) \cap \text{ span}(\mathbb{1})^\perp$.

Lower Bound via a certificate vector $v \in (\mathbb{R}^V \setminus \{\mathbf{0}\}) \cap$ span$(\mathbb{1})^\perp$

Proposition 3.12. *Let graph $G = (V, E)$ be a connected graph and let $w \in \mathbb{R}^E_+$ be its weight function such that $w(e) \geq 1$ for all edges $e \in E$. Suppose T is a spanning tree of G, then, for every $v \in (\mathbb{R}^V \setminus \{\mathbf{0}\}) \cap$ span$(\mathbb{1})^\perp$ we have,*

$$\frac{v^\top \mathcal{L}_T v}{v^\top \mathcal{L}_G(w) v} \leq \theta_{G,w}(T).$$

Additionally, there exists an optimal vector $v^* \in (\mathbb{R}^V \setminus \{\mathbf{0}\}) \cap$ span$(\mathbb{1})^\perp$ achieving equality. I.e.,

$$\theta_{G,w}(T) = \frac{{v^*}^\top \mathcal{L}_T v^*}{{v^*}^\top \mathcal{L}_G(w) v^*}.$$

Proof. Consider any fixed vector $v \in \mathbb{R}^V$, such that $v \in (\mathbb{R}^V \setminus \{\mathbf{0}\}) \cap \text{ span}(\mathbb{1})^\perp$. Now, by Proposition 3.6 (ii) we have $v^\top \mathcal{L}_G(w) v > 0$. Thus,

$$\begin{aligned} v^\top(\theta_{G,w}(T)\mathcal{L}_G(w) - \mathcal{L}_T)v \geq 0 \qquad &\Leftrightarrow \qquad \theta_{G,w}(T) v^\top \mathcal{L}_G(w) v \geq v^\top \mathcal{L}_T v \\ &\Leftrightarrow \qquad \theta_{G,w}(T) \geq \frac{v^\top \mathcal{L}_T v}{v^\top \mathcal{L}_G(w) v}. \end{aligned}$$

Since the above inequality holds for every $v \in (\mathbb{R}^V \setminus \{\mathbf{0}\}) \cap \text{ span}(\mathbb{1})^\perp$ we have

$$\sup_{v \in (\mathbb{R}^V \setminus \{\mathbf{0}\}) \cap \text{ span}(\mathbb{1})^\perp \ s.t. \ \|v\|=1} \frac{v^\top \mathcal{L}_T v}{v^\top \mathcal{L}_G(w) v} \leq \theta_{G,w}(T). \tag{3.12}$$

Additionally, since

$$f(v) := \frac{v^\top \mathcal{L}_T v}{v^\top \mathcal{L}_G(w) v} \leq \theta_{G,w}(T) \leq 1$$

is bounded from above and is continuous on the nonempty and compact set $\{v : (\mathbb{R}^V \setminus \{\mathbf{0}\}) \cap \text{ span}(\mathbb{1})^\perp \text{ and } \|v\| = 1\}$, the supremum is attained by a vector v^*.

To show that the lower bound (3.12) is tight observe that for all $v \in (\mathbb{R}^V \setminus \{\mathbf{0}\}) \cap \text{ span}(\mathbb{1})^\perp$ we have

$$\begin{aligned} v^\top(f(v^*)\mathcal{L}_G(w) - \mathcal{L}_T)v \geq 0 &\Leftrightarrow \\ f(v^*) v^\top \mathcal{L}_G v - v^\top \mathcal{L}_T v \geq 0 &\Leftrightarrow \\ f(v^*) \geq \frac{v^\top \mathcal{L}_T v}{v^\top \mathcal{L}_G v}&, \end{aligned}$$

where the last inequality holds by the choice of v^*. Therefore, we also have $f(v^*)\mathcal{L}_G(w) - \mathcal{L}_T \succeq 0$ and $f(v^*) \geq \theta_{G,w}(T)$. Hence, the equality $f(v^*) = \theta_{G,w}(T)$ holds as well. □

Lower bound via Effective Resistance

Another approach to finding a lower bound for $\theta_{G,w}$ is through a notion of *effective resistance*. We can view the graph G as an electrical circuit where each edge $e = \{i, j\} \in E$ has conductance $w(e)$, or equivalently, resistance $R(e) = \frac{1}{w(e)}$. Informally, the resistance of an edge denotes how well a flow (for instance electrical current) can pass through that edge:

$$I((i,j)) = \frac{V_i - V_j}{R(e)} = (\Delta V_e)w(e), \qquad \text{(Ohm's law)}$$

where $I((i,j))$ is the current passing from vertex i to vertex j and ΔV_e is the potential difference between vertices i and j.

Note that if we put a potential difference between two endpoints i and j a current flows from i to j through different paths in G. Then, Effective conductance $W_{G,w}(e)$ of an edge $e = \{i,j\} \in E$ is the maximum current that the circuit can flow when putting a potential difference of one between two endpoints i and j, and the effective resistance $\mathrm{Reff}_{G,w}(e) = \frac{1}{W_{G,w}(e)}$ measures the total resistance of the circuit when having endpoints i and j.

One can formalize the effective resistance with Kirchhoff's circuit laws to obtain an equivalent definition of the below (for instance, see [13]).

Definition 3.8. Let $G = (V,E)$ be a graph with the weight function $w : E \to \mathbb{R}_+^E$, and let $\{i,j\} \in E$. The *Effective Resistance* of edge e in graph G is defined as follows:

$$\mathrm{Reff}_{G,w}(\{i,j\}) := (e_i - e_j)^\top \mathcal{L}_G^\dagger(w)(e_i - e_j),$$

where $\mathcal{L}_G^\dagger$ is the pseudo-inverse of $\mathcal{L}_G$. Moreover, we define the *Effective Resistance of* G w.r.t. w to be

$$\mathrm{Reff}_{G,w} := \max\{\mathrm{Reff}_{G,w}(\{i,j\}); \;\; \text{for all } \{i,j\} \in E\}.$$

We can formalize effective resistance in an equivalent form as follows.

Proposition 3.13. *Let G be a connected graph with weight function $w : E \to \mathbb{R}_+^E$. Then,*

$$\begin{aligned}\mathrm{Reff}_{G,w}(\{i,j\}) &= \max_{x \perp \mathbb{1}} \frac{(x_i - x_j)^2}{\sum_{\{k,l\}\in E}(x_k - x_l)^2} \\ &= \max_{x \perp \mathbb{1}} \frac{(x_i - x_j)^2}{x^\top \mathcal{L}_G(w) x}.\end{aligned}$$

Proof. By definition of effective resistance we have

$$\mathrm{Reff}_{G,w}(\{i,j\}) = (e_i - e_j)^\top \mathcal{L}_G^\dagger(w)(e_i - e_j).$$

By Proposition 3.2 recall that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Additionally, $\mathrm{Reff}_{G,w}(\{i,j\}) \in \mathbb{R}$. Therefore, we have

$$\begin{aligned}\mathrm{Reff}_{G,w}(\{i,j\}) &= \mathrm{Tr}\left((e_i - e_j)^\top \mathcal{L}_G^\dagger(w)(e_i - e_j)\right)\\ &= \mathrm{Tr}\left(\mathcal{L}_G^{\frac{\dagger}{2}}(w)(e_i - e_j)(e_i - e_j)^\top \mathcal{L}_G^{\frac{\dagger}{2}}(w)\right).\end{aligned}$$

Note that $\mathcal{L}_{ij} := (e_i - e_j)(e_i - e_j)^\top$ is a rank one matrix. Therefore, $\mathcal{L}_G^{\frac{\dagger}{2}}(w)\mathcal{L}_{ij}\mathcal{L}_G^{\frac{\dagger}{2}}(w)$ is a rank one matrix as well, and it has exactly one nonzero eigenvalue λ_1.

Additionally, recall that given a matrix $X \in \mathbb{S}^n$ we have $\mathrm{Tr}(X) = \sum_{i=1}^n \lambda_i(X)$.Therefore, for a rank one PSD matrix X the trace is equal to its maximum eigenvalue λ_1. Now, by Theorem 3.1 we have

$$\begin{aligned}\mathrm{Reff}_{G,w}(\{i,j\}) &= \lambda_1(\mathcal{L}_G^{\frac{\dagger}{2}}(w)\mathcal{L}_{ij}\mathcal{L}_G^{\frac{\dagger}{2}}(w))\\ &= \max_{y \perp \mathbb{1}} \frac{y^\top(\mathcal{L}_G^{\frac{\dagger}{2}}(w)\mathcal{L}_{ij}\mathcal{L}_G^{\frac{\dagger}{2}}(w))y}{y^\top y}\\ &= \max_{x \perp \mathbb{1}} \frac{x^\top \mathcal{L}_{ij} x}{x^\top \mathcal{L}_G x},\end{aligned}$$

where the last equality holds since $\mathrm{span}(\mathbb{1}) = \mathrm{Null}(\mathcal{L}_G^{\frac{\dagger}{2}}(w))$ and $\mathcal{L}_G^{\frac{\dagger}{2}}(w)$ acts bijective on the vector space $\mathrm{Span}(\mathbb{1})^\perp$. Finally, we can rewrite $\mathrm{Reff}_{G,w}(\{i,j\})$ as

$$\begin{aligned}\mathrm{Reff}_{G,w}(\{i,j\}) &= \max_{x \perp \mathbb{1}} \frac{x^\top \mathcal{L}_{ij} x}{x^\top \mathcal{L}_G x}\\ &= \max_{x \perp \mathbb{1}} \frac{(x_i - x_j)^2}{x^\top \mathcal{L}_G x}.\end{aligned}$$

$\square$

From this equivalent definition, we obtain that the effective conductance $W_{G,w}(e)$ between two vertices i and j lower bounds the connectivity between vertices i and j. Moreover, we have the following.

Lemma 3.4. *Suppose graph $G = (V, E)$ with edge weight $w \in \mathbb{R}_+^E$ has connectivity k. Let*

$H = (V, E(H))$ *be a spanning subgraph of* G. *Then,*

$$\max_{e \in E(H)} \mathrm{Reff}_{G,w}(e) \geq \frac{1}{k}.$$

Proof. Let $U \subset V$ be the cut that specifies the connectivity of G; i.e., $w(\delta_G(U)) = k$. Let $u \in \mathbb{R}^V$ be the characteristic vector for U defined as

$$u_i = \begin{cases} 1 & i \in U \\ 0 & \text{otherwise.} \end{cases}$$

Since H is spanning, there exists at least one edge $e = \{i, j\}$ such that $e \in E(H) \cap \delta_G(U) \neq \emptyset$. Now, we have

$$\begin{aligned} \max_{e \in E(H)} \mathrm{Reff}_{G,w}(e) &\geq \mathrm{Reff}_{G,w}(\{i,j\}) \\ &= \max_x \frac{(x_i - x_j)^2}{x^\top \mathcal{L}_G x} \\ &\geq \frac{(u_i - u_j)^2}{u^\top \mathcal{L}_G u} \\ &\geq \frac{1}{w(\delta_G(U))} \geq \frac{1}{k}. \end{aligned}$$

□

Note that for a tree T combinatorial thinness is lower bounded by connectivity.

$$\theta_{G,w}(T) \geq \alpha_{G,w}(T) \geq \frac{1}{k}.$$

We will show that effective resistance obtains a better lower bound on $\theta_{G,w}(T)$, as suggested in [2] Lemma 1.4.

Proposition 3.14. *Let* $G = (V, E)$ *be a graph with the weight function* $w : E \to \mathbb{R}_+^E$ *and let* T *be a spanning tree of* G. *Then,*

$$\max_{e \in E(T)} \mathrm{Reff}_{G,w} \leq \theta_{G,w}(T). \tag{3.13}$$

Proof of Proposition 3.14. By Proposition 3.12, we can characterize $\theta_{G,w}(T)$ as,

$$\begin{aligned}\theta_{G,w}(T) &= \max_{x\in(\,\mathrm{span}(\mathbb{1})^{\perp}\setminus\{0\})} \frac{x^\top \mathcal{L}_T x}{x^\top \mathcal{L}_G x}\\ &\geq \max_{x\in(\,\mathrm{span}(\mathbb{1})^{\perp}\setminus\{0\})} \frac{(x_i-x_j)^2}{x^\top \mathcal{L}_G(w) x}.\end{aligned}$$

Further, by Proposition 3.13 for every edge $e=\{i,j\}\in E(T)$ we can write

$$\mathrm{Reff}_{G,w}(\{i,j\}) = \max_{x\in(\,\mathrm{span}(\mathbb{1})^{\perp}\setminus\{0\})} \frac{(x_i-x_j)^2}{x^\top \mathcal{L}_G(w) x}.$$

Combining these inequalities, we get

$$\begin{aligned}\mathrm{Reff}_{G,w}(\{i,j\}) &= \max_{x\in(\,\mathrm{span}(\mathbb{1})^{\perp}\setminus\{0\})} \frac{(x_i-x_j)^2}{x^\top \mathcal{L}_G(w) x}\\ &\leq \max_{x\in(\,\mathrm{span}(\mathbb{1})^{\perp}\setminus\{0\})} \frac{x^\top \mathcal{L}_T x}{x^\top \mathcal{L}_G x}\\ &= \theta_{G,w}(T).\end{aligned}$$

Since the inequality holds for every edge $\{i,j\}\in E(T)$ we have

$$\max_{e\in E(T)} \mathrm{Reff}_{G,w}(\{i,j\}) \leq \theta_{G,w}(T).$$

□

The following proposition shows that spectral thinness of simple complete graph K_n is lower bounded by

$$\theta_{K_n} \geq \mathrm{Reff}_{K_n} = \frac{2}{n}.$$

Proposition 3.15. *Let K_n be the simple complete graph on n vertices such that $n\geq 3$. Then, for any edge $e\in E(K_n)$, we have*

$$\mathrm{Reff}_{K_n} = \mathrm{Reff}_{K_n}(e) = \frac{2}{n}.$$

Proof. For any two edges $e=\{i,j\}$ and $e'=\{i',j'\}$ in $E(K_n)$, there exists an isomorphism that maps e to e'. Therefore,

$$\max_{e' \in E(K_n)} \mathrm{Reff}_{K_n}(e') = \mathrm{Reff}_{K_n}(e).$$

Additionally, by Proposition 3.13, we have

$$\mathrm{Reff}_{K_n}(\{i,j\}) = \max_{x \in \text{ span}(\mathbb{1})^{\perp}, \|x\|=1} \frac{(x_i - x_j)^2}{\sum_{1 \le \ell < k \le n} (x_\ell - x_k)^2}.$$

Now, note that for $x \in$ span$(\mathbb{1})^{\perp}$ and $\|x\| = 1$ we can compute the denominator as

$$\begin{aligned}
\sum_{1 \le \ell < k \le n} (x_\ell - x_k)^2 &= (n-1) \sum_{\ell=1}^{n} x_i^2 - 2 \sum_{1 \le \ell < k \le n} x_\ell x_k \\
&= n \sum_{\ell=1}^{n} x_i^2 - \sum_{\ell,k \in [n]} x_\ell x_k. \\
&= n \sum_{\ell=1}^{n} x_i^2 - \sum_{\ell \in [n]} x_\ell \Big(\sum_{k \in [n]} x_k \Big) \\
&= n,
\end{aligned}$$

where the last equality holds since $x \perp 1$ and $\|x\| = 1$. Thus,

$$\mathrm{Reff}_{K_n}(\{i,j\}) = \max_{x \perp \mathbb{1}, \|x\|=1} \frac{(x_i - x_j)^2}{n} = \frac{2}{n},$$

where the last equality holds since $(x_i - x_j)^2$ is maximized if all other weights are zero, $x_i^2 + x_j^2 = 1$, and $x_i = -x_j = \frac{\sqrt{2}}{2}$. □

3.3.2 Upper Bounds

Let $G = (V, E)$ be a weighted simple graph with weight function $w : E \to \mathbb{R}_+^E$ and let T be a spanning tree for G. Then, to show that a value $\theta_u \in (0, 1]$ is an upper bound for $\theta_{G,w}(T)$, it suffices to show

$$\theta_u \mathcal{L}_G(w) - \mathcal{L}_T \succcurlyeq 0.$$

Equivalently, θ_u is an upper bound for $\theta_{G,w}(T)$ if for all $x \in \mathbb{R}^V$

$$x^\top(\theta_u \mathcal{L}_G(w) - \mathcal{L}_T)x \geq 0.$$

Note that even though the above inequalities hold for the upper bound θ_u, we do not obtain a generic certificate to verify these inequalities.

General certificates of positive semidefiniteness

By Proposition 3.3, another approach for obtaining positive semidefiniteness is to show that there exists a matrix $B \in \mathbb{R}^{n\times n}$ such that

$$\theta_u \mathcal{L}_G(w) - \mathcal{L}_T = BB^\top.$$

Thus, to prove that θ_u is an upper bound, we can search for a certificate matrix $B \in \mathbb{R}^{n\times n}$ such that $BB^\top = \theta_u \mathcal{L}_G - \mathcal{L}_T$. One issue with such a general approach is that the certificate matrix $B \in \mathbb{R}^{n\times n}$ might have irrational entries. Therefore, finding such a certificate could be challenging with or without software. Additionally, due to computational errors, guessing, computing, and verifying the exact values of B may not be easy.

On the other hand, we might be able to find simpler certificates if the original matrix $\theta_u \mathcal{L}_G(w) - \mathcal{L}_T$ can be written as $\frac{1}{k}X$, where $k \in \mathbb{N}$ and $X \in \mathbb{S}^n_+$ has rational entries. This approach was utilized in [4].

UV-certificate for positive semidefiniteness

Definition 3.9. Let $X \in \mathbb{Q}^{n\times n}$ be a symmetric matrix. A pair of matrices $U, V \in \mathbb{Z}^{n\times n}$ together with a positive integer k make a $UV-$*certificate* of X if the following holds.

i. There exist an integer $k \in \mathbb{N}$ such that $k.X = U^\top U + V$.

ii. The matrix V is *diagonally dominant,* i.e., for all $i \in [n]$ we have

$$V_{i,i} \geq \sum_{j\in[n]\setminus\{i\}} |V_{i,j}|.$$

Proposition 3.16. *Let $X \in \mathbb{Q}^{n\times n}$ be a symmetric matrix with rational entries. Then, if X has a $UV-$certificate, then X is a positive semidefinite matrix.*

Proof. Let U, V be matrices in $\mathbb{Z}^{n\times n}$ that form a $UV-$ certificate for X along with $k \in \mathbb{N}$. Since V is a diagonally dominant matrix, V is positive semidefinite. Therefore, $U^\top U + V$ is a positive semidefinite matrix as well. This proves that $X = \frac{1}{k}(U^\top U + V)$ is a positive semidefinite matrix, as it is summation of two positive semidefinite matrices. □

A $UV-$certificate allows us to obtain an PSD-certificate by searching in the space of integer symmetric matrices.

Upper bound via Effective Resistance

Previously, we showed how to use effective resistance to lower bound the thinness of a spanning tree T. It is worth mentioning that the effective resistance is also a measure that can upper bound the spectral thinness parameter of graphs. In fact, Marcus, Spielman, and Srivastava [27] proved Kadison-Singer Conjecture by proving Weaver's equivalent conjecture [41]. As an application of this result Harvey and Olver in [20] (see Appendix C) showed the following theorem.

Theorem 3.5. *Let $G = (V, E)$ be a connected graph with edge weight $w \in \mathbb{R}_+^E$. Then, we have*

$$\theta_{G,w} \leq O(\mathrm{Reff}_{G,w}). \tag{3.14}$$

Figure 3.4 captures all the results regarding combinatorial and spectral thinness and their relation to edge connectivity and effective resistance, respectively.

$$
\begin{tikzcd}
O(\mathrm{Reff}_{G,w}) \arrow[r, "(3.4)"] \arrow[d, "(3.5)"'] & O\left(\frac{1}{k}\right) \arrow[dd, dashed, "(1.2)"] \\
\theta_{G,w} = \theta_{G,w}(T) \arrow[dr, "(1.1)"] \arrow[dd, "(3.14)"'] & \\
& \alpha_{G,w} = \alpha_{G,w}(T') \arrow[dd, "(1.1)"] \\
\max_{e \in E(T)} \mathrm{Reff}_{G,w}(e) \arrow[dr, "(3.4)"] & \\
& \frac{1}{k}
\end{tikzcd}
$$

Figure 3.4: Each arrow shows that the higher endpoint is possibly larger than the lower one. More specifically, black arrows show an overview of connections between connectivity, combinatorial and spectral thinness, and effective resistance, and the red arrow demonstrates the strong thin-tree conjecture.

3.4 Spectral Thinness and Hamiltonian Paths

As we saw before in Section 3.2, we were able to improve upper bounds given in [30] for combinatorial thinness of some families of structured graphs by considering Hamiltonian paths in them. Intuitively, this suggests that Hamiltonian paths, where they exists, might be among good candidates for spectrally thin trees. The following proposition shows a lower bound for the spectral thinness of non-Hamiltonian trees in a k-regular graph that formalizes this intuition to an extend.

Proposition 3.17. *Let $G = (V, E)$ be a k-regular graph with unit weights, and let T be a spanning tree of G that is not a Hamiltonian path. Then, spectral thinness of T in G is at least $\theta_G(T) \geq \frac{4}{k+2}$.*

Proof. Let T be a spanning tree which is not a Hamiltonian path. Then, the maximum degree of T is at least three; i.e., $\Delta(T) \geq 3$. Without loss of generality, we may assume that $|V| \geq 5$, since for a graph with less number of vertices $1 = \alpha_G(T) \leq \theta_G(T) \leq 1$. We will prove the statement in three cases in the following.

i. Suppose $\Delta(T) \geq 4$, and let vertex $u_1 \in V$ be the vertex having $\deg_T(u_1) = \Delta(T)$. Then, considering the cut $\delta_G(u_1)$, we will get a combinatorial lower bound

$$\alpha_G(T) \geq \frac{\Delta(T)}{k} \geq \frac{4}{k}.$$

Since combinatorial thinness lower bounds spectral thinness, we will also get

$$\theta_G(T) \geq \frac{4}{k}.$$

ii. Now, suppose $\Delta(T) = 3$, and the spanning tree T contains two adjacent vertices u_1, u_2 of degree three. By reordering the vertices in G, we assume that u_1, u_2 correspond to the first two vertices $\{1, 2\}$ in $\mathcal{L}_G$. Then, for $\theta \in (0, 1]$, the principal submatrix corresponding to $\{1, 2\}$ can be written as

$$B := (\theta\mathcal{L}_G - \mathcal{L}_T)_{\{1,2\}} = \begin{bmatrix} \theta k - 3 & -\theta + 1 \\ \theta + 1 & \theta k - 3 \end{bmatrix}.$$

Now, a necessary condition for $\theta\mathcal{L}_G - \mathcal{L}_T \succeq 0$ is to have $\det(B) \geq 0$. Therefore, we must have

$$\begin{aligned} \det(B) \geq 0 &\iff (\theta k - 3)^2 - (-\theta + 1)^2 \geq 0 \\ &\iff (\theta k - 3)^2 \geq (1 - \theta)^2 \\ &\iff \theta k - 3 \geq 1 - \theta \\ &\iff \theta \geq \frac{4}{k+1}. \end{aligned}$$

iii. The remaining case is when each degree three vertex has only neighbours of degree one or two. Suppose for vertex $u_1 \in V$ degree $\deg_T(u_1) = 3$. Then, all of its neighbours $N_T(u_1) = \{u_2, u_3, u_4\}$ have degree at most two in T. Moreover, since T is connected, at least one of the neighbours has degree two. Thus, we can assume

$$2 = \deg_T(u_2) \geq \deg_T(u_3) \geq \deg_T(u_4).$$

Also, we order the vertices of G such that in the ordering vertices $u_1, \ldots, u_4$ corresponds to first four rows of $\mathcal{L}_G$ and $\mathcal{L}_T$. Next, let vector $x \in \mathbb{R}^V$ be defined as

$$x_i = \begin{cases} 2\sqrt{2}, & \text{if } i = 1, \\ -1, & \text{if } 2 \le i \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

We aim to find a lower bound on the value $\frac{x^\top \mathcal{L}_T x}{x^\top \mathcal{L}_G x}$. For the numerator, note that by our assumption $\deg_T(2) = 2$; thus, there exists a vertex $\ell \in V$ such that $\{2, \ell\} \in E(T)$ and $\ell \notin \{1, \dots 4\}$. Now, we can lower bound the numerator by

$$\begin{aligned} x^\top \mathcal{L}_T x &= \sum_{\{i,j\} \in E(T)} (x_i - x_j)^2 \\ &\ge \sum_{j=2}^{4} (x_1 - x_j)^2 + (x_2 - x_\ell)^2 \\ &= 3(2\sqrt{2} + 1)^2 + 1 \\ &= 28 + 12\sqrt{2}. \end{aligned}$$

Also, for the denominator, we have the upper bound

$$\begin{aligned} x^\top \mathcal{L}_G x &= \sum_{\{i,j\} \in E} (x_i - x_j)^2 \\ &= \sum_{\{1,j\} \in E} (x_1 - x_j)^2 + \sum_{\ell=2}^{4} \Big(\sum_{\{\ell,j\} \in E} (x_\ell - x_j)^2 \Big) \\ &\le \Big(3(2\sqrt{2} + 1)^2 + (k-3)(2\sqrt{2})^2 \Big) + 3(k-1) \\ &= 11k + 12\sqrt{2}. \end{aligned}$$

Now, by Proposition 3.12 we have

$$
\begin{aligned}
\theta_G(T) &\geq \frac{x^\top \mathcal{L}_T x}{x^\top \mathcal{L}_G x} \\
&\geq \frac{28 + 12\sqrt{2}}{11k + 12\sqrt{2}} \\
&\geq \frac{44.97}{11k + 12\sqrt{2}} && (\text{ since } 12\sqrt{2} \geq 16.97) \\
&\geq \frac{44.97}{11k + 22} && (\text{ since } 12\sqrt{2} \leq 22) \\
&\geq \frac{4}{k+2}.
\end{aligned}
$$

Therefore, in general case, if spanning tree T is not a Hamiltonian path, then $\theta_G(T) \geq \frac{4}{k+2}$. □

For the rest of this section, we will focus on spectral thinness of Hamiltonian graph $G = (V, E)$ with weight function $w \in \mathbb{R}_+^E$ and Hamiltonian path P of G. Suppose we want to upper bound $\theta_{G,w}(P)$ with θ_u. We need to show that,

$$\theta_u \mathcal{L}_G(w) - \mathcal{L}_P \succeq 0.$$

Our main tool for doing so is to exploit the symmetric properties of the matrix $\theta\mathcal{L}_G(w) - \mathcal{L}_P$ to simplify the positive semidefiniteness condition.

Note that for simple graphs such as $G = K_n$ or $G = K_{n,n}$ the Laplacian $\mathcal{L}_G$ is a highly symmetric matrix. However, even though the Hamiltonian path P has very simple structure, it only has one isomorphism, and therefore, it does not have a lot of symmetry. As a result, the difference $\theta\mathcal{L}_G - \mathcal{L}_P$ does not capture all the symmetries available in G. To overcome this issue, we will relax the Laplacian of P to the corresponding Hamiltonian cycle C that contains P, if such a cycle exist. Then, we will aim to show

$$\theta_u \mathcal{L}_G(w) - \mathcal{L}_C \succeq 0,$$

which means θ_u is an upper bound for $\theta_{G,w}(C)$. Next, we argue that spectral thinness of C upper bounds $\theta_{G,w}(P)$.

Proposition 3.18. *Let $G = (V, E)$ with weight function $w \in \mathbb{R}_+^E$ be given. Suppose $P = (V, E(P))$ is a Hamiltonian path in G and $C = (V, E(C))$ is a Hamiltonian cycle of*

G, where $E(P) \subset E(C)$. Then, we have

$$\theta_{G,w}(P) \leq \theta_{G,w}(C).$$

Proof. In order to show the inequality holds, it is sufficient to show

$$\theta_{G,w}(C)\mathcal{L}_G(w) - \mathcal{L}_P \succeq 0.$$

Note that since P is a subgraph of C, by Proposition 3.6, we have $\mathcal{L}_C \succeq \mathcal{L}_P$. Thus, we get

$$\theta_{G,w}(C)\mathcal{L}_G(w) \succeq \mathcal{L}_C \succeq \mathcal{L}_P.$$

Therefore,

$$\theta_{G,w}(P) \leq \theta_{G,w}(C)$$

holds. □

Moreover, note that the Hamiltonian cycle C containing Hamiltonian path P only differ in one edge. Therefore, one might expect that the corresponding parameters $\theta_{G,w}(C)$ and $\theta_{G,w}(P)$ to not differ much. The following proposition formalizes this observation for some specific families of graphs such as $K_{n,n}$, where $n = 2k$ is even, and K_n for all $n \in \mathbb{N}$.

Proposition 3.19. *Let $G = (V, E)$ be a Hamiltonian graph with unit edge-weight where all of its Hamiltonian paths have same spectral thinness. Suppose C is a Hamiltonian cycle and P is a Hamiltonian path in G. Then, the following inequality holds;*

$$\theta_G(C) \geq \theta_G(P) \geq (\frac{n-1}{n})\theta_G(C). \tag{3.15}$$

Proof. Suppose the Hamiltonian cycle C has edge set $E(C) = \{e_1, \ldots, e_n\}$. Define the Hamiltonian path $P^i := C \setminus e_i$. By assumption the spectral thinness of all Hamiltonian paths are equal. Thus, for all $i \in [n]$ we have

$$\theta_G(P)\mathcal{L}_G - \mathcal{L}_{P^i} \succeq 0.$$

By summing all the inequalities, we should still have a PSD matrix. Thus,

$$\begin{aligned} \sum_{i=1}^{n}(\theta_G(P)\mathcal{L}_G - \mathcal{L}_{P^i}) \succeq 0 & \qquad\qquad\qquad \iff \\ (n\theta_G(P))\mathcal{L}_G - \sum_{i=1}^{n}\mathcal{L}_{P^i} \succeq 0 & \qquad\qquad\qquad \iff \\ (n\theta_G(P))\mathcal{L}_G - (n-1)\mathcal{L}_C \succeq 0, & \end{aligned}$$

where the last equivalence holds since each cycle edge appears $n-1$ times in $\sum_{i=1}^{n}\mathcal{L}_{P^i}$. Thus,

$$\Big((\frac{n}{n-1})\theta_G(P)\Big)\mathcal{L}_G - \mathcal{L}_C \succeq 0$$

holds for Hamiltonian cycle C, and by the definition of $\theta_G(C)$ we must have

$$(\frac{n}{n-1})\theta_G(P) \geq \theta_G(C).$$

By rearranging the inequality, we get

$$\theta_G(P) \geq (\frac{n-1}{n})\theta_G(C).$$

On the other hand, by Proposition 3.18 and the assumption that all Hamiltonian paths in G have the same spectral thinness,

$$\theta_G(C) \geq \theta_G(P). \qquad \square$$

As we previously mentioned, our aim is to exploit symmetries of the matrix $\theta\mathcal{L}_G - \mathcal{L}_C$ to relax the PSD condition $\theta\mathcal{L}_G(w) - \mathcal{L}_C$. Note that even if both Laplacian $\mathcal{L}_G$ and $\mathcal{L}_C$ have a lot of symmetry, it does not guarantee that their difference also inherits all of these symmetries. Therefore, we need an extra condition that assures symmetries of $\mathcal{L}_G(w)$ and $\mathcal{L}_C$ coincide with each other.

3.4.1 Circulant Matrices and Graphs

Definition 3.10. (Circulant matrix) A matrix $X \in \mathbb{R}^{n\times n}$ is a circulant matrix if there exists real numbers $c_0, \dots, c_{n-1} \in \mathbb{R}$ such that,

$$X = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & c_0 & c_1 \\ c_1 & \cdots & \cdots & c_{n-1} & c_0 \end{bmatrix}$$

For the above circulant matrix X, we call the vector $c = [c_0, \dots, c_{n-1}]$ the *essential row* of X.

Proposition 3.20. *Let $X \in \mathbb{R}^{n\times n}$ be a circulant matrix as in Definition 3.10 with essential row $x = [x_0, \dots, x_n]$. Then, the following statements hold.*

i. X has orthonormal basis of eigenvectors $\{v_j : \quad 0 \le j \le n-1\}$, where

$$v_j = \frac{1}{\sqrt{n}}[\omega_n^{0.j}, \omega_n^{1.j}, \dots, \omega_n^{(n-1).j}]^\top, \tag{3.16}$$

and $\omega_n = e^{\frac{2\pi\iota}{n}}$ is the n-th root of unity where $\iota^2 = -1$.

ii. Moreover, for each eigenvector v_j, the corresponding eigenvalue $\lambda_X(v_j)$ is computed as follows:

$$\lambda_X(v_j) := \sum_{k=0}^{n-1} x_k \omega_n^{kj} = \sqrt{n} x v_j.$$

Proof. See [12], Chapter 3, Sections 3.1 and 3.2. □

Proposition 3.21. *Let $X, Y \in \mathbb{R}^{n\times n}$ be circulant matrices as in Definition 3.10 with essential rows $x = [x_0, \dots, x_n]$ and $y = [y_0, \dots, y_n]$. Then, the following statements hold.*

i. For every constant $a \in \mathbb{R}$, the matrix aX is a Circulant matrix.

ii. The matrices $X - Y$ and $X + Y$ are circulant matrices with essential rows $x - y$ and $x + y$, respectively.

iii. The matrices X and Y commute; i.e., $XY = YX$.

Proof. Part (i.) and (ii.) are trivially true. For the proof of (iii.) see [12], page 68. □

Definition 3.11. A graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$ is a *circulant graph* if there exists a relabelling on vertices V such that Laplacian matrix $\mathcal{L}_G(w)$ with respect to the given order is a circulant matrix. We also define *essential row of* G to be the essential row of its circulant Laplacian $\mathcal{L}_G(w)$.

Proposition 3.22. *Let $C_n = (V, E)$ be a cycle graph on n vertices with unit edge weights. Then, C_n is a circulant graph with essential row $c = [2, -1, 0, \ldots, 0, -1]$. Moreover, for all $j \in \{0, \ldots, n-1\}$, the eigenvalue $\lambda_{C_n}(v_j)$ of C_n is equal to*

$$\lambda_{C_n}(v_j) = 4\sin^2(\frac{\pi j}{n}).$$

Proof. First, note that if we order the vertices as appear in the cycle, i.e., $\{i, (i \pm 1) \bmod n\} \in E(C_n)$, then the Laplacian is indeed circulant with essential vector equal to $c = [2, -1, 0, \ldots, 0, -1]$. In order to find the eigenvalue $\lambda_{C_n}(v_j)$ we have

$$\begin{aligned}
\lambda_{C_n}(v_j) &= \sum_{k=0}^{n-1} c_k \omega_n^{kj} \\
&= 2\omega_n^0 - \omega_n^j - \omega_n^{(n-1)j} \\
&= 2 - (\omega_n^j + \omega_n^{-j}), && (\text{ since } \omega_n^n = 1).
\end{aligned}$$

Now, recall the Euler's formula,

$$e^{\iota x} + e^{-\iota x} = 2\cos(x), \tag{3.17}$$

where $\iota = \sqrt{-1}$. By utilizing the above equation, we can simplify $\lambda_{C_n}(v_j)$ as

$$\begin{aligned}\lambda_{C_n}(v_j) &= 2 - (e^{(\frac{2\pi\iota}{n})j} + e^{-(\frac{2\pi\iota}{n})j}) \\ &= 2 - 2\cos\left(\frac{2\pi j}{n}\right) \\ &= 2\left(1 - \cos\left(\frac{2\pi j}{n}\right)\right) \\ &= 4\sin^2(\frac{\pi j}{n}),\end{aligned}$$

where in the last step we used $(1 - \cos(2x)) = 2\sin^2(x)$. □

3.4.2 Complete Graph K_n

Proposition 3.23. *Let $n > 3$ be a natural number, and let K_n be the simple complete graph on n vertices. Then, the Laplacian matrix $\mathcal{L}_{K_n}$ is a circulant matrix with essential row $c = [n-1, -1, \ldots, -1]$ and eigenvalues*

$$\lambda_{K_n}(v_0) = 0, \qquad \lambda_{K_n}(v_1) = \cdots = \lambda_{K_n}(v_{n-1}) = n.$$

Proof. Since K_n is a complete graph with its group of isomorphism equal to $\mathcal{S}_n$, all Laplacian matrices of K_n are circulant with same essential row $c = [n-1, -1, \ldots, -1]$.

Similar to previous lemma, we calculate all eigenvalues $\lambda_{K_n}(v_j)$ for $j \in \{0, \ldots, n-1\}$ as follows.

Firstly, suppose $j = 0$, then

$$\lambda_{K_n}(v_0) = \sum_{k=0}^{n-1} c_k \omega_n^0 = \sum_{k=0}^{n-1} c_k = (n-1) + \sum_{k=1}^{n-1} -1 = 0.$$

If $1 \leq j \leq n-1$, then we get

$$\lambda_{K_n}(v_j) = \sum_{k=0}^{n-1} c_k \omega_n^{kj} = (n-1)\omega_n^0 - \sum_{k=1}^{n-1} \omega_n^{kj} = n - \sum_{\ell=0}^{n-1} \omega_n^{j\ell} = n,$$

where the last equality holds since

$$\sum_{\ell=0}^{n-1} \omega_n^{j\ell} = \frac{\omega_n^{nj} - \omega_n^0}{\omega_n^j} = \frac{1-1}{\omega_n^j} = 0.$$

□

We can also improve the lower bound in Proposition 3.17 for the special case of $(n-1)$-regular graph K_n.

Lemma 3.5. *For $n > 3$ and any non-path tree T of K_n, the spectral thinness $\theta_{K_n}(T)$ is at least $\frac{4}{n}$.*

Proof. see Appendix A for the proof. □

Theorem 3.6. *Let $n > 3$ be a natural number, and let K_n be the simple complete graph on n vertices. For every Hamiltonian path P of K_n, we have,*

$$\frac{4(n-1)}{n^2} \leq \theta_{K_n}(P) \leq \frac{4}{n}.$$

Additionally, P achieves the spectral thinness

$$\theta_{K_n} = \theta_{K_n}(P).$$

Proof. Without loss of generality, let P be the path with ordered set of vertices $(1, 2, \ldots, n)$ where edges $\{i, i+1\} \in E(P)$ for $i \leq n-1$. Also, let C be the Hamiltonian cycle containing P. Then, with the given vertex labeling, $\mathcal{L}_{K_n}$ and $\mathcal{L}_C$ are circulant matrices. By Proposition 3.21, for $\theta \in (0, 1]$ the difference $\mathcal{D}_\theta = \theta\mathcal{L}_{K_n} - \mathcal{L}_C$ is also a circulant matrix with same set of eigenvectors $\{v_j : 0 \leq j \leq n-1\}$. Therefore, for $0 \leq j \leq n-1$ we can compute

$$\begin{aligned} \lambda_{\theta\mathcal{L}_{K_n} - \mathcal{L}_C}(v_j) &= \lambda_{\theta\mathcal{L}_{K_n}}(v_j) - \lambda_{\mathcal{L}_C}(v_j) \\ &= \theta\lambda_{\mathcal{L}_{K_n}}(v_j) - \lambda_{\mathcal{L}_C}(v_j). \end{aligned}$$

Using Proposition 3.22 and Lemma 3.23 we obtain

$$\lambda_{\theta\mathcal{L}_{K_n} - \mathcal{L}_C}(v_j) = \begin{cases} 0, & j = 0 \\ \theta n - 4\sin^2(\frac{\pi j}{n}), & \text{otherwise.} \end{cases}$$

Therefore, for $\theta = \frac{4}{n}$ the difference $\theta \mathcal{L}_{K_n} - \mathcal{L}_C$ is a PSD matrix. Since any vertex permutation is an isomorphism for K_n, the spectral thinness of all Hamiltonian paths of K_n are equal. Therefore, we can use Proposition 3.19 to get

$$\frac{4(n-1)}{n^2} \leq \theta_{K_n}(P) \leq \frac{4}{n}.$$

Now, by Lemma 3.5, for any tree T with $\Delta(T) \geq 3$, we have $\theta_{K_n}(T) \geq \frac{4}{n}$. Therefore,

$$\frac{4(n-1)}{n^2} \leq \theta_{K_n} = \theta_{K_n}(P) \leq \frac{4}{n}.$$

□

As a result of Theorem 3.6, $\theta_{K_n} \to \frac{4}{n}$ as $n \to \infty$. As seen in the Figure 3.5, computational experiment suggests that the exact thinness of a Hamiltonian P in the graph K_n is closer to the upper bound $\frac{4}{n}$.

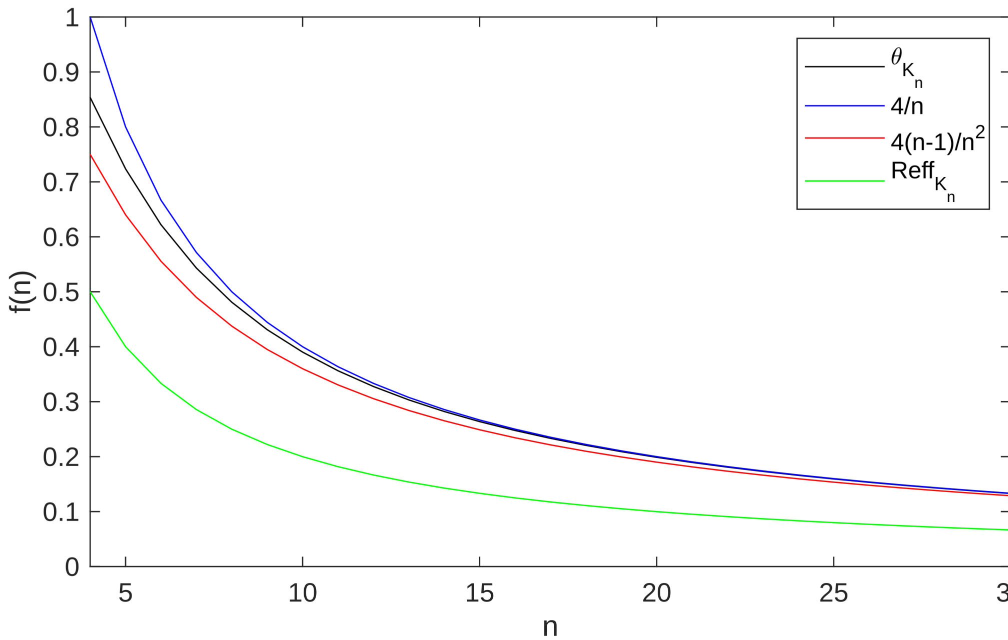


Figure 3.5: The black curve indicates the exact value of $\theta_{K_n}(P)$, and the red and blue curves are the suggested lower and upper bounds in Proposition 3.6, respectively. The green line shows that the effective resistance $\mathrm{Reff}_{K_n} = \frac{2}{n}$ lower bounds $\theta_{K_n}(P)$ (by Proposition 3.15), and also $\theta_{K_n}(P) = O(\mathrm{Reff}_{K_n})$.

3.4.3 Complete Bipartite Graph $K_{n,n}$

Proposition 3.24. *Let $K_{n,n}$ be the simple complete bipartite graph with $2n$ vertices and bipartition $A = \{2t-1 : t \in [n]\}$, $B = \{2t : t \in [n]\}$, where*

$$E(K_{n,n}) = \Big\{\{i,j\} : i \in A,\ j \in B\Big\}.$$

Then, the Laplacian $\mathcal{L}_{K_{n,n}}$ is a circulant matrix with essential row

$$c = [n, -1, 0, -1, 0, -1, \ldots, 0, -1]$$

and eigenvalues

$$\lambda_{K_{n,n}}(v_j) := \begin{cases} 0, & \text{if } j = 0 \\ 2n, & \text{if } j = n \\ n, & \text{otherwise.} \end{cases}$$

Proof. By the choice of the bipartition, every odd vertex $2t-1$ is connected to all of the even vertices and every even vertex $2t$ is connected to all of the odd vertices for all $t \in [n]$. Therefore, the rows $2t-1$ and $2t$ of $\mathcal{L}_{K_{n,n}}$ can be computed as follows.

$$\begin{aligned} (\mathcal{L}_{K_{n,n}})_{2t-1,\cdot} &= \begin{bmatrix} 0 & -1 & \dots & 0 & -1 & n & -1 & 0 & \dots & -1 \end{bmatrix}, \\ (\mathcal{L}_{K_{n,n}})_{2t,\cdot} &= \begin{bmatrix} -1 & 0 & \dots & -1 & 0 & -1 & n & -1 & \dots & 0 \end{bmatrix}. \end{aligned}$$

Therefore, $\mathcal{L}_{K_{n,n}}$ is circulant with first row

$$c = \begin{bmatrix} n & -1 & \dots & -1 & 0 & -1 & 0 & \dots & -1 \end{bmatrix}.$$

By Proposition 3.20, for $0 \leq j \leq n-1$ we can compute eigenvalues $\lambda_{\mathcal{L}_{K_{n,n}}}(v_j)$. Note that we have $\omega_{2n} = e^{\frac{2\pi\iota}{2n}} = e^{\frac{\pi\iota}{n}}$. Thus, by replacing it in the formulas for $j = 0$ we get

$$\lambda_{K_{n,n}}(v_0) = \sum_{\ell=0}^{2n-1} c_\ell e^0 = \sum_{\ell=0}^{2n-1} c_\ell = 0.$$

Also, if $j = n$ we have

$$\begin{aligned} \lambda_{K_{n,n}}(v_n) &= \sum_{\ell=0}^{2n-1} c_\ell \omega_{2n}^{n\ell} \\ &= \sum_{\ell=0}^{2n-1} c_\ell e^{\ell\pi\iota} \\ &= n - \sum_{\ell=0}^{n-1} e^{(2\ell+1)\pi\iota} \\ &= n - (e^{\pi\iota})^1 - (e^{\pi\iota})^3 - \dots - (e^{\pi\iota})^{2n-1} \\ &= 2n, \end{aligned}$$

where the last equality holds since $e^{\pi\iota} = -1$. Finally, for $j \notin \{0, n\}$

$$\begin{aligned}
\lambda_{K_{n,n}}(v_j) &= \sum_{\ell=0}^{2n-1} c_\ell \omega_{2n}^{j\ell} \\
&= \sum_{\ell=0}^{2n-1} c_\ell e^{\frac{\pi\iota}{n} j\ell} \\
&= n - \sum_{\ell=0}^{n-1} e^{\frac{\pi\iota}{n} j(2\ell+1)} \\
&= n,
\end{aligned}$$

where the last step holds since by geometric series we have

$$\begin{aligned}
\sum_{\ell=0}^{n-1} e^{\frac{\pi\iota}{n} j(2\ell+1)} &= e^{\frac{\pi\iota}{n} j} (\sum_{\ell=0}^{n-1} e^{\frac{2\pi\iota}{n} \ell j}) \\
&= e^{\frac{\pi\iota}{n} j} (\sum_{\ell=0}^{n-1} (e^{\frac{2\pi\iota j}{n}})^\ell) \\
&= e^{\frac{\pi\iota}{n} j} (\sum_{\ell=0}^{n-1} (\omega_n^j)^\ell) \\
&= 0.
\end{aligned}$$

□

Theorem 3.7. *Suppose $K_{n,n}$ is the simple complete bipartite graph on $2n$ vertices, where $n \geq 4$ is an even number. Let P be a given Hamiltonian path of $K_{n,n}$. Then, we have*

$$\frac{4}{n} \approx \frac{4\cos^2(\frac{\pi}{2n})}{n} \geq \theta_{K_{n,n}}(P) \geq \frac{(4n-2)\cos^2(\frac{\pi}{2n})}{n^2} \approx \frac{4n-2}{n^2}. \tag{3.18}$$

Additionally, for large enough n,

$$\frac{4}{n} \geq \theta_{K_{n,n}} \geq \frac{4}{n+2}.$$

Proof. Suppose $K_{n,n}$ is labelled with bipartition

$$A = \{2t-1 : t \in [n]\} \qquad \text{and} \qquad B = \{2t : t \in [n]\}.$$

Then, $\mathcal{L}_{K_{n,n}}$ is a circulant matrix as suggested in Proposition 3.24.

Note that for every two Hamiltonian paths P_1 and P_2 of $K_{n,n}$ there exists an isomorphism that maps i-th vertex of P_1 to the i-th vertex of P_2 by relabelling vertices of A and B. Therefore, $\theta_{K_{n,n}}(P_1) = \theta_{K_{n,n}}(P_2)$.

Now, let P be the path with ordered set of vertices $(1, 2, \ldots, 2n)$, where edges $\{i, i+1\} \in E(P)$ for $i \leq 2n - 1$. Also, let C be the Hamiltonian cycle containing P. Then, with the given vertex labeling, $\mathcal{L}_C$ is a circulant matrix with essential row

$$c = [n, -1, 0, -1, 0, -1, \ldots, 0, -1]$$

as defined in Proposition 3.22.

Also, by Proposition 3.21, for $\theta \in (0, 1]$, the matrix $\theta\mathcal{L}_{K_{n,n}} - \mathcal{L}_C$ is also a circulant matrix with same set of eigenvectors. Thus, for $0 \leq j \leq n - 1$, we can compute

$$\begin{aligned} \lambda_{\theta\mathcal{L}_{K_{n,n}} - \mathcal{L}_C}(v_j) &= \lambda_{\theta\mathcal{L}_{K_{n,n}}}(v_j) - \lambda_{\mathcal{L}_C}(v_j) \\ &= \theta\lambda_{\mathcal{L}_{K_{n,n}}}(v_j) - \lambda_{\mathcal{L}_C}(v_j). \end{aligned}$$

Using Proposition 3.22 and Lemma 3.23 we obtain

$$\begin{aligned} \lambda_{\theta\mathcal{L}_{K_{n,n}} - \mathcal{L}_C}(v_j) &= \begin{cases} 0, & \text{if } j = 0 \\ \theta(2n) - 4\sin^2(\frac{\pi}{2}), & \text{if } j = n \\ \theta(n) - 4\sin^2(\frac{\pi j}{2n}), & \text{otherwise} \end{cases} \\ &= \begin{cases} 0, & \text{if } j = 0 \\ \theta(2n) - 4, & \text{if } j = n \\ \theta(n) - 4\sin^2(\frac{\pi j}{2n}), & \text{otherwise.} \end{cases} \end{aligned}$$

Note that $\lambda_{\theta\mathcal{L}_{K_{n,n}} - \mathcal{L}_C}$ is PSD if and only if all of its eigenvalues are non-negative. First, we need $\theta \geq \frac{2}{n}$ as otherwise $(2n)\theta - 4 < 0$. For other values of $j \notin \{0, n\}$, we have

$$\theta n - 4\sin^2(\frac{\pi j}{2n}) \geq 0 \iff \theta \geq \frac{4\sin^2(\frac{\pi j}{2n})}{n}.$$

Therefore,

$$\theta_{K_{n,n}}(C) = \max\left\{\frac{2}{n}, \max\left\{\frac{4\sin^2(\frac{\pi j}{2n})}{n} : j \notin \{0, n\}\right\}\right\},$$

since this is the smallest value such that all eigenvalues of $\theta\mathcal{L}_{K_{n,n}} - \mathcal{L}_C$ are non-negative.

Now, $\frac{4\sin^2(\frac{\pi j}{2n})}{n}$ has its maximum value whenever $j = n+1$ or $j = n-1$, which is equal to

$$\frac{4\sin^2(\frac{\pi(n\pm 1)}{2n})}{n} = \frac{4\sin^2(\frac{\pi}{2} \pm \frac{\pi}{2n})}{n} = \frac{4\cos^2(\frac{\pi}{2n})}{n}.$$

Thus, for $n \geq 4$,

$$\theta_{K_{n,n}}(C) = \max\left\{\frac{2}{n}, \frac{4\cos^2(\frac{\pi}{2n})}{n}\right\} = \frac{4\cos^2(\frac{\pi}{2n})}{n}.$$

Since the spectral thinness of all Hamiltonian paths of $K_{n,n}$ are equal, we can use Proposition 3.19 to get

$$\frac{(4n-2)\cos^2(\frac{\pi}{2n})}{n^2} \leq \theta_{K_{n,n}}(P) \leq \frac{4\cos^2(\frac{\pi}{2n})}{n}.$$

Finally, by Proposition 3.17, any non-path tree of $K_{n,n}$ has spectral thinness at least $\frac{4}{n+2}$. Therefore, we will have

$$\theta_{K_{n,n}} \geq \min\left\{\frac{(4n-2)\cos^2(\frac{\pi}{2n})}{n^2}, \frac{4}{n+2}\right\},$$

which is equal to $\frac{4}{n+2}$ for large enough n.

Therefore, we have

$$\frac{4}{n+2} \leq \theta_{K_{n,n}} \leq \theta_{K_{n,n}}(P) \leq \frac{4\cos^2(\frac{\pi}{2n})}{n} \leq \frac{4}{n}.$$

□

Similar to K_n, by Theorem 3.7 we obtain that $\theta_{K_{n,n}} \to \frac{4}{n}$ as $n \to \infty$. As seen in the Figure 3.6, computational experiment suggests that the exact thinness of path tree P in the graph $K_{n,n}$ matches the upper bound $\frac{4\cos^2(\frac{\pi}{2n})}{n}$.

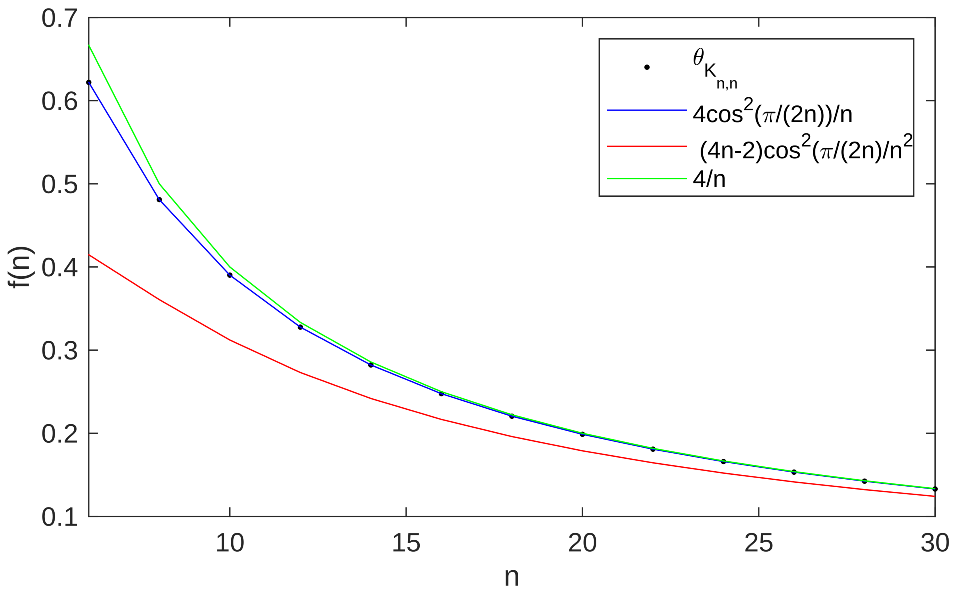


Figure 3.6: The black dots indicates the exact value of $\theta_{K_{n,n}}(P)$ and the red and blue curves are the suggested lower bound and upper bounds in Theorem 3.7. The green curve shows the function $\frac{4}{n}$.

3.4.4 Prism Graph Π_n

Prism graphs were used in the literature of Traveling Salesman Problem by Boyd and Pulleyblank [9] to show that the LP relaxation defined by the subtour elimination constraints can have fractional extreme points with a complicated strucuture. Moreover, Harvey and Olver [20] used prism graph with specific edge weights as an example of $k-$edge connected graph with spectral thinness $\Omega(\frac{\sqrt{n}}{k})$. In this part, we aim to bound spectral thinness of prism graphs Π_n defined as follows.

Definition 3.12. Let $n \in \mathbb{N}$ be a natural number. We define the prism graph as

$$\Pi_n = C_n \Box P_2,$$

where C_n is the cycle of length n, P_2 is the path of length 2, and $\Box$ denotes Cartesian product of graphs (see Definition 2.1). In other words, we can define $\Pi_n = (V_1 \cup V_2, E)$ where $V_1 = \{1, \ldots, n\}$ and $V_2 = \{n+1, \ldots, 2n\}$ are the upper and lower vertices and

$$E = C_n^1 \cup M_n \cup C_n^2$$

is the edge-set where C_n^1 and C_n^2 are the upper and lower cycle edges and M_n is the matching edges between V_1 and V_2 (see Figure 3.7). I.e.,

$$\begin{aligned}
C_n^1 &= \{\{i, i+1 \mod n\} : i \in [n]\} \\
M_n &= \{\{i, i+n\} : i \in [n]\} \\
C_n^2 &= \{\{n+i, n+(i+1 \mod n)\} : i \in [n]\}.
\end{aligned}$$

Remark 3.5. For the rest of this chapter, we will use C_n^1, C_n^2, and M_n to denote the upper cycle, lower cycle, and matching edges.

A simple analysis in the following proposition shows that the spectral thinness of unweighted prism graph is equal to one.

Proposition 3.25. *Let Π_n be the prism graph on $2n$ vertices. Then,*

$$\theta_{\Pi_n} = 1.$$

Proof. Since Π_n has $2n$ vertices and the degree of each vertex is equal to 3, we have

$$|E(\Pi_n)| = \frac{1}{2} \sum_{i \in [2n]} 3 = \frac{6n}{2} = 3n.$$

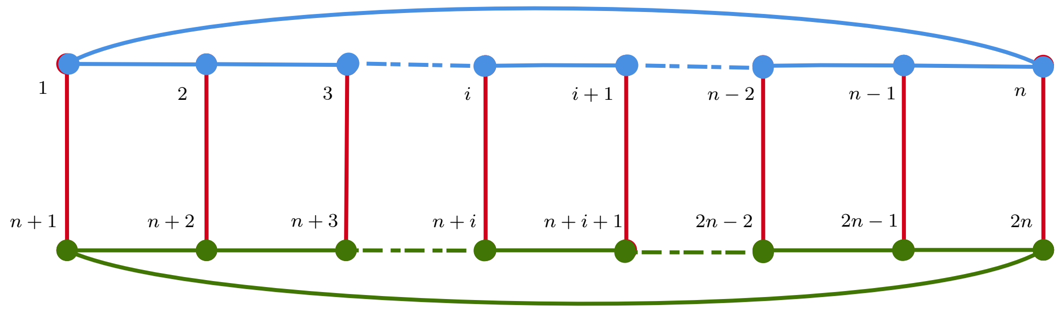


Figure 3.7: Prism graph Π_n. The blue vertices and edges denote the upper vertices $\{1, \ldots, n\}$ and upper cycle edges C_n^1. Similarly, the green vertices and edges denote the lower vertices $\{n+1, \ldots, 2n\}$ and lower cyle edges C_n^2. Red edges denote the matching edges M_n.

Next, we claim that Π_n does not have two edge-disjoint spanning trees. To see this, note that each spanning tree of Π_n has $2n-1$ edges. Therefore, for Π_n to have two edge-disjoint spanning trees we must have

$$|E(\Pi_n)| \geq 2(2n-1) = 4n-2 > |E(\Pi_n)|,$$

which is a contradiction. Thus, by Proposition 2.1, we have

$$1 = \alpha_{\Pi_n} \leq \theta_{\Pi_n} \leq 1.$$

Therefore, $\theta_{\Pi_n} = 1$. □

By the previous proposition, we need to consider edge weight function $w \in \mathbb{R}_+^{2n}$ to obtain spectrally thin trees in prism graphs Π_n.

Weighted Prism Graph $\Pi_n(a, b)$

We consider a specific weight function where the edges on the cycles C_n^1 and C_n^2 have weight a and edges in the matching M_n have weight b (see Figure 3.8). We denote the weighted prism graph by $\Pi_n(a, b)$, where the corresponding edge weight function is defined as follows.

Definition 3.13. Let $n \in \mathbb{N}$ be a natural number, and let $a, b \in \mathbb{R}_+$. We define the *weighted Prism graph* $\Pi_n(a,b)$ to have edge weight function $w_{a,b} : E \to \mathbb{R}_+$ as

$$(w_{a,b})_{\{i,j\}} = \begin{cases} a, & \text{if } 1 \leq i,j \leq n \\ a, & \text{if } n+1 \leq i,j \leq 2n \\ b, & \text{if } j = i+n. \end{cases}$$

Remark 3.6. For the sake of conciseness, we denote the Laplacian of prism graph $\Pi_n(a,b)$ by

$$\mathcal{L}_{a,b} := \mathcal{L}_{\Pi_n(a,b)}(w_{a,b}).$$

Moreover, to denote the spectral thinness of $\Pi_n(a,b)$, we use

$$\theta_{a,b} := \theta_{\Pi_n, w_{a,b}}.$$

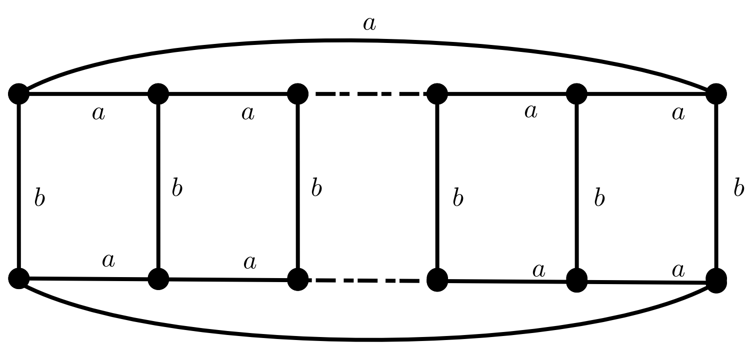


Figure 3.8: Graph Π_n with edge weight function $w_{a,b}$.

A simple general upper bound and lower bound for the thinness parameter of $\Pi_n(a,b)$ can be obtained as follows.

General upper bound for $\theta_{a,b}$

Lemma 3.6. *Let $a, b \in \mathbb{R}_+$ be real numbers, and let $\theta_{a,b}$ be the thinness parameter of $\Pi_n(a,b)$. Then,*

$$\theta_{a,b} \leq \max(\frac{1}{a}, \frac{1}{b}).$$

Proof. Let $\theta \in (0,1]$ and T be a spanning tree of G. Then, for any $x \in \mathbb{R}^{2n}$, we can compute $z := x^\top(\theta\mathcal{L}_{a,b} - \mathcal{L}_T)x$ as

$$\begin{aligned} z &= \theta \sum_{\{i,j\}\in C_n^1\cup C_n^2} a(x_i - x_j)^2 + \theta \sum_{\{i,j\}\in M_n} b(x_i - x_j)^2 - \sum_{\{i,j\}\in E(T)} (x_i - x_j)^2 \\ &\geq \theta \min\{a,b\}\Big(\sum_{\{i,j\}\in E} (x_i - x_j)^2\Big) - \sum_{\{i,j\}\in E(T)} (x_i - x_j)^2. \end{aligned}$$

Now, by replacing $\theta = \max(\frac{1}{a}, \frac{1}{b})$ we have $\theta \min\{a,b\} \geq 1$. Thus, for all $x \in \mathbb{R}^{2n}$ we get

$$\begin{aligned} x^\top(\theta\mathcal{L}_{a,b} - \mathcal{L}_T)x &\geq \sum_{\{i,j\}\in E(\Pi_n)} (x_i - x_j)^2 - \sum_{\{i,j\}\in E(T)} (x_i - x_j)^2 \\ &= x^\top \mathcal{L}_{\Pi_n} x - x^\top \mathcal{L}_T x \\ &\geq 0, \end{aligned}$$

where the last inequality holds since T is a subgraph of unweighted Π_n as well. Therefore,

$$\theta_{a,b} \leq \max(\frac{1}{a}, \frac{1}{b}). \tag{3.19}$$

$\square$

From the above lemma, we can directly obtain the following corollary.

Corollary 3.2. *For any positive real numbers $\varepsilon_a, \varepsilon_b$ the weighted prism graph $\Pi_n(1+\varepsilon_a, 1+\varepsilon_b)$ has a spectrally thin tree.*

General lower bounds for $\theta_{a,b}$

Definition 3.14. Let Π_n be the prism graph on $2n$ vertices. We call two edges $e_1 = \{i,j\} \in C_n^1$ and $e_2 = \{i', j'\} \in C_n^2$ *twin edges* if and only if $i' = n + i$ and $j' = n + j$ (see Figure 3.9).

The following lemma shows how to lower bound spectral thinness of a spanning tree that contains twin edges.

Lemma 3.7. *Let $a, b \in \mathbb{R}_+$ be real numbers and let $T = (V, E(T))$ be a spanning tree of $\Pi_n(a,b)$ with thinness parameter $\theta_{a,b}(T)$. Then, if $E(T)$ contains twin edges we have*

$$\frac{n-1}{na} \leq \theta_{a,b}(T).$$

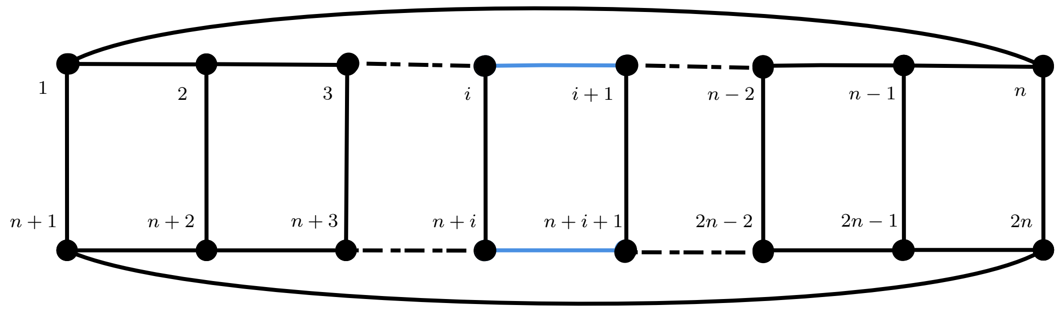


Figure 3.9: The blue edges denote an example of twin edges in the prism graph Π_n.

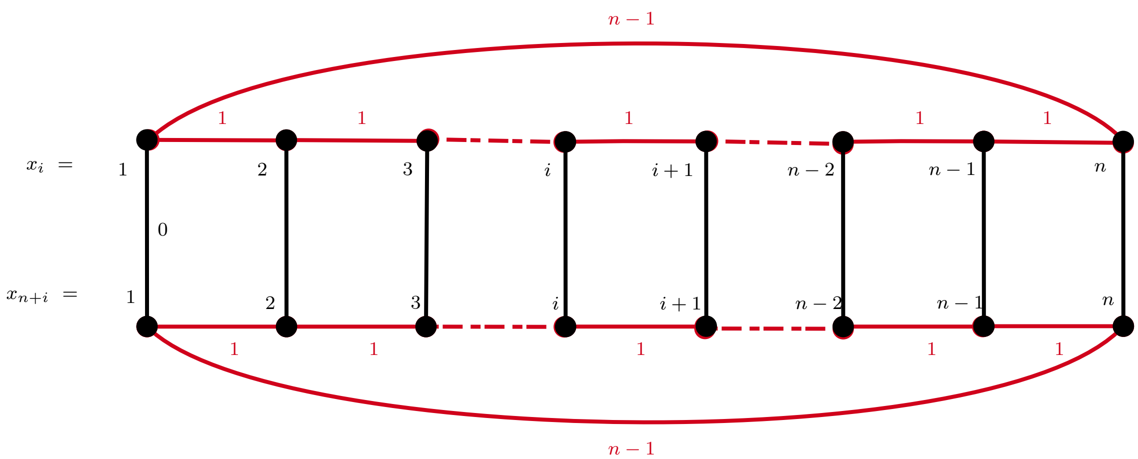


Figure 3.10: Each vertex i is assigned with a value x_i, and the value of $(x_i - x_j)$ for each edge $\{i, j\}$ will be obtained as above.

Proof. Suppose e_1 and e_2 are the twin edges of T. Without loss of generality, we can relabel the vertices such that $e_1 = \{1, n\}, e_2 = \{n+1, 2n\}$ be the twin edges in the tree. Now, we define the vector $x \in \mathbb{R}^{2n}$ as follows (see Figure 3.10).

$$x_i = \begin{cases} i, & \text{if } i \leq n \\ i - n, & \text{if } i > n. \end{cases}$$

For $\theta \in (0, 1]$ and x, we can calculate $z = x^\top(\theta \mathcal{L}_{a,b} - \mathcal{L}_T)x$ as

$$
\begin{aligned}
z &= \theta a \sum_{\{i,j\}\in C_n^1 \cup C_n^1} (x_i - x_j)^2 + \theta b \sum_{i,j \in M_n} (x_i - x_j)^2 - \sum_{\{i,j\}\in E(T)} (x_i - x_j)^2 \\
&= \theta a \sum_{\{i,j\}\in C_n^1 \cup C_n^1} (x_i - x_j)^2 - \sum_{\{i,j\}\in E(T)} (x_i - x_j)^2 \\
&= 2\theta a (\sum_{i=1}^{n-1} (i+1-i)^2 + (n-1)^2) - \sum_{\{i,j\}\in E(T)} (x_i - x_j)^2 \\
&\leq 2\theta a((n-1) + (n-1)^2) - 2(n-1)^2.
\end{aligned}
$$

Note that to have $\theta \mathcal{L}_{a,b} - \mathcal{L}_T \succeq 0$, we must have $z \geq 0$. A necessary condition for $z \geq 0$ can be obtained as

$$
z \geq 0 \Rightarrow 2\theta a((n-1) + (n-1)^2) \geq 2(n-1)^2.
$$

Therefore, we get $\frac{n-1}{na} \leq \theta_{a,b}(T)$. □

Lemma 3.8. *Let $a, b \in \mathbb{R}_+$ be real numbers and let T be a spanning tree of $\Pi_n(a,b)$ with thinness parameter $\theta_{a,b}(T)$. Then, if T does not contain any twin edges we have*

$$
\frac{n-1}{nb} \leq \theta_{a,b}(T).
$$

Proof. Suppose T is a spanning tree with no twin edges. Then,

$$
E(T) \cap (C_n^1 \cup C_n^2) \leq n,
$$

and at least $n-1$ edges of T are from the matching edges M_n. Now, consider the incidence vector $x \in \mathbb{R}^{2n}$ defined as

$$
x_i = \begin{cases} 1, & \text{if } i \leq n \\ 0, & \text{if } i > n. \end{cases}
$$

Then, for $\theta \in (0,1]$, $z := x^T(\theta \mathcal{L}_{G_{n,a,b}} - \mathcal{L}_T)x$ is equal to

$$\begin{aligned}
z &= \theta a \sum_{\{i,j\} \in C_n^1 \cup C_n^1} (x_i - x_j)^2 + \theta b \sum_{\{i,j\} \in M_n} (x_i - x_j)^2 - \sum_{\{i,j\} \in E(T)} (x_i - x_j)^2 \\
&= \theta b \sum_{\{i,j\} \in M_n} (x_i - x_j)^2 - \sum_{\{i,j\} \in E(T) \cap M} (x_i - x_j)^2 \\
&= \theta b n - \sum_{\{i,j\} \in E(T) \cap M} (x_i - x_j)^2 \\
&\leq \theta b n - (n-1).
\end{aligned}$$

For $z \geq 0$ we must have

$$\theta b n - (n-1) \geq 0 \ \Leftrightarrow \theta \geq \frac{n-1}{nb}.$$

Hence,

$$\frac{n-1}{nb} \leq \theta_{a,b}(T).$$

□

Combining Lemma 3.6, Lemma 3.7, and Lemma 3.8, we directly get the following proposition.

Proposition 3.26. *Let $a, b \in \mathbb{R}_+$, and let $\theta_{a,b}$ be the thinness parameter of $\Pi_n(a,b)$. Then,*

$$\min\{\frac{n-1}{na}, \frac{n-1}{nb}\} \leq \theta_{a,b} \leq \max(\frac{1}{a}, \frac{1}{b}).$$

Note that the gaps in the above lower bound and upper bound are small if $a = b$. However, for the cases where $a > b$ or $a < b$ the bounds can be very loose. The following lemma simplifies calculation of $\theta_{a,b}$ for the case $a \neq b$.

Lemma 3.9. *Let graph $\Pi_n(a,b)$ be defined as above. Then, we have $\theta_{ka,kb} = \frac{1}{k}\theta_{a,b}$.*

Proof. Let $\theta \in (0,1]$. By definition of Laplacian, the equality

$$\mathcal{L}_{ka,kb} = \frac{1}{k}\mathcal{L}_{a,b}$$

holds. Thus, we have

$$(\theta)\mathcal{L}_{ka,kb} - \mathcal{L}_T \succcurlyeq 0 \iff (k\theta)\mathcal{L}_{a,b} - \mathcal{L}_T \succcurlyeq 0.$$

Therefore, $\frac{\theta_{a,b}}{k}$ satisfies the inequality for $\Pi_n(ka, kb)$ and vice versa. $\square$

Thus, to study the spectral thinness of $\Pi_n(a, b)$, we may consider to study the spectral thinness of $\Pi_n(a, 1)$, with $a > 1$ and $\Pi_n(1, b)$ with $b > 1$, as by Lemma 3.9 the spectral thinness of $\Pi_n(a, b)$ is multiplication of one of the two normalized cases.

Note that even with the above simplification, the weighted graphs $\Pi_n(1, b)$ and $\Pi_n(a, 1)$ are not circulant for any $a, b > 1$. However, in both cases, there is still a lot of symmetry in the graphs that we can benefit from. In the following, we will use a mixture of ideas from Section 3.3 and properties of circulant graphs to bound spectral thinness of $\Pi_n(a, 1)$ and $\Pi_n(1, b)$ (see Table 3.2).

$\Pi_n(a, b)$ for even n and $a, b > 1$	Lower bound	$\leq \theta_{a,b} \leq$	Upper bound
$\Pi_n(1, 1)$	1	$\leq \theta_{1,1} \leq$	1
$\Pi_n(a, 1)$	$\frac{1}{1+2\sqrt{a}}$	$\leq \theta_{a,1} \leq$	1
$\Pi_n(1, b)$	$\frac{1}{2} - \frac{1}{n}$	$\leq \theta_{1,b} \leq$	$\frac{1}{2} + \frac{3}{2b+4}$

Table 3.2: Obtained bounds for prism graph $\Pi_n(a, b)$ where n is an even number and $a, b > 1$ are real numbers.

3.4.5 Weighted Prism Graph $\Pi_n(a, 1)$ for $a > 1$

Harvey and Olver in [20] proved a lower bound on the spectral thinness of weighted prism graphs (see Theorem 4.9 in [20]).

Theorem 3.8. *For given $n, k \in \mathbb{N}$ where n is even and $k \geq n$, there exists a weighted graph with $2n$ vertices and edge connectivity k, such that it does not have $o(\frac{\sqrt{n}}{k})-$spectrally thin spanning tree.*

Specifically, with our notation, they showed that $\theta_{\frac{k}{2},\frac{k}{n}}$ is lower bounded by $\frac{\sqrt{2}}{5}(\frac{\sqrt{n}}{k})$ where n is an even number and $k \geq n$.

While using the same idea, we will improve the result of Harvey and Olver [20] for prism graph to obtain a lower bound on $\theta_{a,1}$.

Theorem 3.9. *Suppose $\Pi_n(a,1)$ is a weighted prism graph where n is an even integer and $a > 1$, and let $\theta_{a,1}$ be the spectral thinness of $\Pi_n(a,1)$. Then,*

$$\frac{1}{1+2\sqrt{a}} \leq \theta_{a,1}.$$

Proof. Let T be any arbitrary spanning tree of $\Pi_n(a,1)$ where $a > 1$. First, note that tree T has at least one edge from the matching. Without loss of generality, we may assume that a matching edge of T is the edge $\{\frac{n}{2}, \frac{3n}{2}\}$, since by rotational relabeling of the vertices we can guarantee the condition holds.

Let $c = 1 - \epsilon$, where

$$\epsilon = \frac{2}{1+\sqrt{a}}.$$

Note that since $a > 1$, we have $0 < \epsilon < 1$. Also, let the certificate of lower bound vector $x \in \mathbb{R}^V$ be defined as follows:

$$x_i = \begin{cases} c^{|\frac{n}{2}-i|}, & \text{if } i \leq n \\ -c^{|\frac{n}{2}-(i-n)|}, & \text{if } i \geq n. \end{cases}$$

We can compute $x^\top \mathcal{L}_T x$ and $x^\top \mathcal{L}_{a,b} x$ as follows.

$$x^\top \mathcal{L}_T x = \sum_{\{i,j\} \in E(T)} (x_i - x_j)^2 \geq (x_{\frac{n}{2}} - x_{\frac{3n}{2}})^2 = 4.$$

On the other hand, we have

$$x^\top \mathcal{L}_{a,1} x = a \sum_{\{i,j\} \in C_n^1 \cup C_n^2} (x_i - x_j)^2 + \sum_{\{i,j\} \in M_n} (x_i - x_j)^2.$$

We analyze each summation separately. Firstly, for the matching edges we have

$$\begin{aligned}
S_M &:= \sum_{i=1}^{n} (x_i - x_{n+i})^2 \\
&= 4 \sum_{i=1}^{n} x_i^2 && \text{(since } x_i = -x_{n+i}\text{)} \\
&\leq 8 \sum_{i=0}^{n} c^i && \text{(since } x_{\frac{n}{2}-i} = x_{\frac{n}{2}+i}\text{)} \\
&< 8 \sum_{i \geq 0} c^i && \text{(since } 0 < c < 1\text{)} \\
&< \frac{8}{1-c}. && \text{(by geometric series)} \\
&= \frac{8}{\epsilon} \\
&= 4(1+\sqrt{a}).
\end{aligned}$$

Similarly, for the cycle edges we get

$$\begin{aligned}
S_C &:= 2a \sum_{i=1}^{n-1} (x_i - x_{i+1})^2 + a(x_n - x_1) \\
&\leq 4a \sum_{i=0}^{n/2-1} (c^i - c^{i+1})^2 && \text{(since } x_i = -x_{n+i}\text{)} \\
&\leq 4a \sum_{i \geq 0} (c^i - c^{i+1})^2 && \text{(since } 0 < c^i - c^{i+1} < 1\text{)} \\
&= 4a(1-c)^2 \sum_{i \geq 0} c^{2i} \\
&= \frac{4a(1-c)^2}{1-c^2} && \text{(by geometric series)} \\
&= \frac{4a(1-c)}{1+c} \\
&= \frac{4a\epsilon}{2-\epsilon} \\
&= 4\sqrt{a}.
\end{aligned}$$

As a result, we have the following

$$\frac{x^\top \mathcal{L}_T x}{x^\top \mathcal{L}_{a,b} x} \geq \frac{4}{4+8\sqrt{a}} = \frac{1}{1+2\sqrt{a}}. \qquad \square$$

In order to compare this lower bound with the lower bound of Harvey and Olver [20] for prism graph we use the fact given in Lemma 3.9. More specifically, for $a > b$ the prism graph $\Pi_n(a,b)$ has

$$\theta_{a,b} = \frac{1}{b}\theta_{\frac{a}{b},1} \geq (\frac{1}{b})\frac{1}{1+2\sqrt{\frac{a}{b}}} = \frac{1}{b+2\sqrt{ab}}.$$

Therefore, for the choice of $\hat{a} = \frac{k}{2}$ and $\hat{b} = \frac{k}{n}$ we have

$$\begin{aligned}
\theta_{\hat{a},\hat{b}} &\geq \frac{1}{\frac{k}{n} + 2\sqrt{\frac{k}{2}\frac{k}{n}}} \\
&= \frac{\sqrt{n}}{\frac{k}{\sqrt{n}} + \sqrt{2}k} \\
&= \frac{\sqrt{n}}{k(\frac{1}{\sqrt{n}} + \sqrt{2})} \\
&\geq \frac{\sqrt{n}}{k(\frac{1}{\sqrt{2}} + \sqrt{2})} \\
&= \frac{3}{\sqrt{2}}\frac{\sqrt{n}}{k},
\end{aligned}$$

where the third inequality holds for $n \geq 2$. Therefore, we obtain a better lower bound compared to $\frac{\sqrt{2}}{5}\frac{\sqrt{n}}{k}$ in [20].

3.4.6 Weighted Prism Graph $\Pi_n(1,b)$ for $b > 1$

Proposition 3.27. *Let $n \in \mathbb{N}$ be an even number, and let $b > 1$ be a real number. Then, for the prism graph $\Pi_n(1,b)$, and any spanning tree T of $\Pi_n(1,b)$ the spectral thinness is* *at least*

$$\theta_{1,b}(T) \geq \frac{1}{2} - \frac{1}{2n}. \tag{3.20}$$

Proof. By Lemma 3.7, any spanning tree T of $\Pi_n(1,b)$ that has twin edges will have spectral thinness at least $\frac{n-1}{n}$. Therefore, to lower bound $\theta_{1,b}$ we only need to consider a tree that has at least $n-1$ matching edges in M_n.

We consider the general cases where $|E(T) \cap M_n| = t$, where $t \in \{n-1, n\}$. If $|E(T) \cap M_n| = t$, then $E(T)$ has exactly $2n - t - 1$ edges from the cycle edges C_n^1 and C_n^2. Let $x \in \mathbb{R}^{2n}$ be defined as follows:

$$x_i := \begin{cases} 1, & \text{if } i \text{ is odd} \\ 0, & \text{if } i \text{ is even.} \end{cases}$$

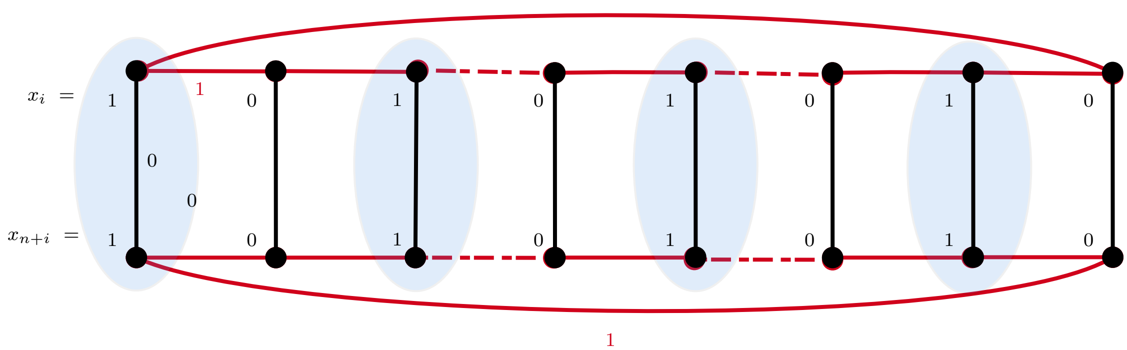


Figure 3.11: Combinatorial cut for prism graph in $\Pi_n(1,b)$. The vertices in blue circles are on one shore of the cut and other vertices are on the other shore of the cut.

Then, as shown in Figure 3.11, the edges in the cut, defined by x, are the cycle edges $C_n^1 \cup C_n^2$, and we have:

$$x^\top \mathcal{L}_{1,b} x = 2n$$
$$x^\top \mathcal{L}_T x = 2n - t - 1.$$

Thus, we get

$$\begin{aligned}\theta_{1,b}(T) &\geq \frac{x^\top \mathcal{L}_T x}{x^\top \mathcal{L}_{1,b} x} \\ &\geq \frac{2n-t-1}{2n} \\ &\geq \frac{n-1}{2n} && (\text{ since } t \in \{n-1, n\}) \\ &= \frac{1}{2} - \frac{1}{2n}.\end{aligned}$$

Thus, in general, we will have

$$\theta_{1,b} \geq \min\{\frac{1}{2} - \frac{1}{2n}, \frac{n-1}{n}\} = \frac{1}{2} - \frac{1}{2n}.$$

□

Upper bound on $\theta_{1,b}$

As shown in the proof of the previous proposition, to obtain a spectrally thin spanning tree of $\Pi_n(1,b)$, we need to consider graphs that do not have twin edges.

In this section, we will assume that number of vertices in each cycle n is an even number $n = 2k$. Before, we proceed further in this section, we will present a relabeling of $\Pi_n(1,b)$, such that the corresponding matrix $\mathcal{L}_{1,b}$ has interesting properties. Let, relabelled vertices in graph be as

$$\begin{aligned}V_1 &= \{2i-1 : i \in [k]\} \cup \{n+2i : i \in [k]\} \\ V_2 &= \{2i : i \in [k]\} \cup \{n+(2i-1) : i \in [k]\},\end{aligned}$$

where V_1 and V_2 are the upper and lower cycle vertices with edge set of each cycle equal to

$$\begin{aligned}C_n^1 &= \{\{n+2i,\ (2i \pm 1 \mod 2n)\} : i \in [k]\} \\ C_n^2 &= \{2i, n+(2i \pm 1 \mod 2n)\} : i \in [k-1]\} \\ M_n &= \{\{i, n+i\} : i \in [n]\}.\end{aligned}$$

Note that with this relabeling (see Figure 3.12), vertices $\{1, \ldots n\}$ and $\{n+1, \ldots 2n\}$

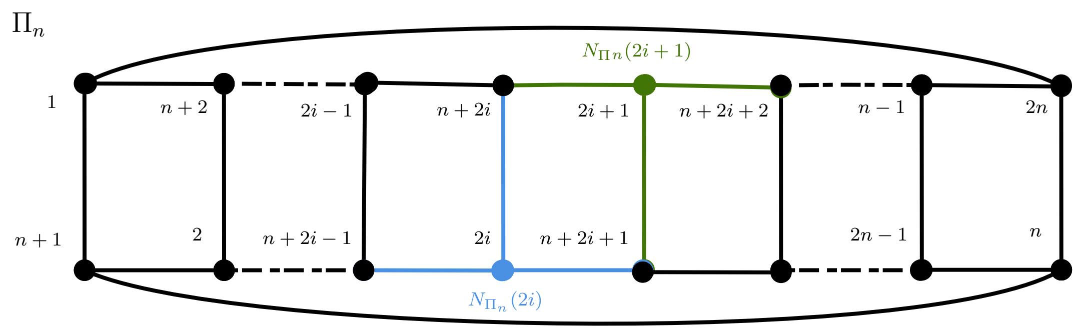


Figure 3.12: Relabelled vertices of prism graph.

form a bipartite graph. Also, the i-th vertex for $i \leq n$ is connected to vertices $n+i$ with edge weight b and vertices $n+i+1$ and $n+i-1$ with weights 1.

Then, we can write the Laplacian $\mathcal{L}_{a,b}$ as a block matrix

$$\mathcal{L}_{a,b} := \begin{bmatrix} (b+2)I_n & A_1 \\ A_1^\top & (b+2)I_n \end{bmatrix},$$

where A_1 is a circulant matrix with essential row $c_1 = [-b, -1, 0, \ldots, 0, -1]$.

Next, we will define an example a spanning tree with no twin edges, which is also a Hamiltonian path in $\Pi_n(1, b)$.

Definition 3.15. Let $n \in \mathbb{N}$ be even, $b > 1$ be a real number, and $\Pi_n(1, b)$ be the prism graph. We define the *zigzag path* of $\Pi_n(1, b)$ denoted by P_Z to be the Hamiltonian path that contains all of the matching edges in M_n. I.e., P_Z

$$P_Z = 1, n+1, 2, n+2, 3, n+3 \ldots, n, 2n.$$

Note that by adding the edge $e = \{1, 2n\}$ the zigzag path becomes a cycle which we call the zigzag cycle and denote with C_Z (see Figure 3.13). It is not hard to see that with the relabelling mentioned in the above (see Figure 3.14) we have

$$\mathcal{L}_{C_Z} = \begin{bmatrix} 2I_n & A_2 \\ A_2^\top & 2I_n \end{bmatrix},$$

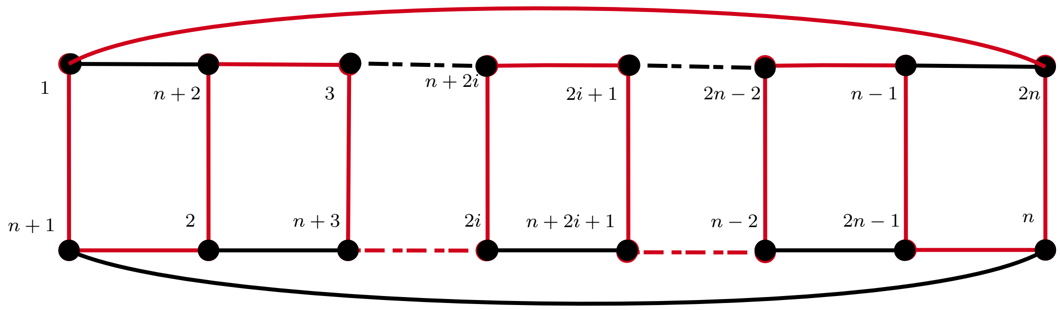


Figure 3.13: Hamiltonian cycle C_Z in the prism graph $\Pi_n(1, b)$.

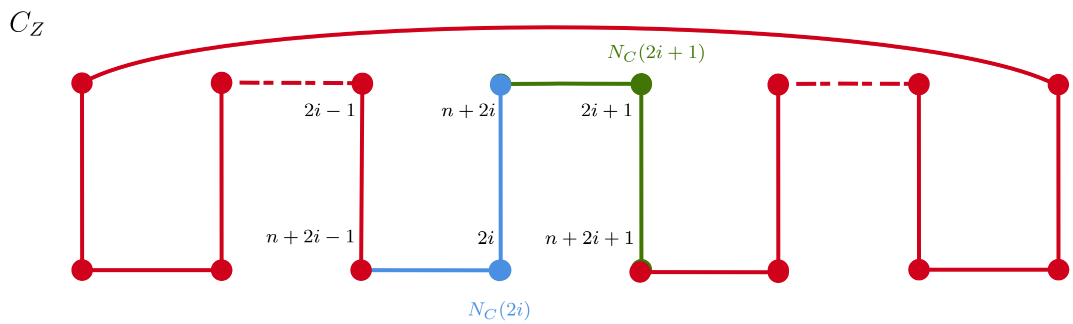


Figure 3.14: The figure shows the corresponding vertices of the Zigzag cycle C_Z in the new labelling of vertices.

where A_2 is a circulant matrix with essential row

$$c_2 = [-1, 0, 0, \ldots, 0, -1].$$

Now for a fixed $\theta \in (0, 1]$, We define $\mathcal{D}_\theta := \theta\mathcal{L}_{a,b} - \mathcal{L}_{C_Z}$ to obtain

$$\mathcal{D}_\theta = \begin{bmatrix} (\theta(b+2)-2)I_n & \theta A_1 - A_2 \\ \theta A_1^\top - A_2^\top & (\theta(b+2)-2)I_n \end{bmatrix},$$

where $A = \theta A_1 - A_2$ is also circulant (see Figure 3.14) with essential row

$$c = [1 - b\theta, -\theta, 0, \ldots, 0, 1 - \theta].$$

To show that $\mathcal{D}_\theta$ is a PSD matrix, we only need to confirm that the smallest eigenvalue of $\mathcal{D}_\theta$, is non-negative. Notice that

$$\mathcal{D}_\theta = \Big(\theta(b+2) - 2\Big) I_{2n} + B, \tag{3.21}$$

where

$$B = \begin{bmatrix} 0 & A \\ A^\top & 0 \end{bmatrix}.$$

To find the eigenvalues of $\mathcal{D}_\theta$, we will use the following lemma.

Lemma 3.10. *For $\theta \in (0,1]$, let B be the matrix defined as*

$$B = \begin{bmatrix} 0 & A \\ A^\top & 0 \end{bmatrix},$$

where $A = \theta A_1 - A_2$ is a circulant matrix with essential row

$$c = [1 - b\theta, -\theta, 0, \dots, 0, 1-\theta].$$

Then, the set of eigenvalues of B are exactly $\pm\beta_j$, for $0 \le j \le n-1$, where

$$\beta_j := \sqrt{\frac{(b\theta-1)^2}{4\theta(1-\theta)} - 4\theta(1-\theta)\Big(\cos\frac{2\pi j}{n} - \frac{(b\theta-1)(2\theta-1)}{4\theta(1-\theta)}\Big)^2}.$$

Additionally, if $\frac{(b\theta-1)(2\theta-1)}{4\theta(1-\theta)} \ge 1$, then the maximum value $\hat{\beta} := \max_j \beta_j$ equals to

$$\hat{\beta} = |\theta(2b+2) - 2|.$$

We will postpone the proof of Lemma 3.10 to the end of this section, and utilize the lemma to upper bound the thinness of cycle C_Z.

Proposition 3.28. *Let $\Pi_n(1,b)$ with $b > 1$ be the weighted prism graph on $2n$ vertices such that $n = 2k$ is an even number. Then, for the Hamiltonian cycle C_Z in $\Pi_n(1,b)$ we have*

$$\theta_{a,b}(C_Z) \le \frac{1}{2} + \frac{3}{2b+4}. \tag{3.22}$$

Proof of Proposition 3.28. Let prism graph Π_n with the relabeling described earlier be given (similar to Figure 3.13). Recall that as in (3.21), we have

$$\mathcal{D}_\theta = \Big(\theta(b+2) - 2\Big) I_{2n} + B.$$

Thus, we may assume $\theta(b+2) - 2 \geq 0$ as otherwise $\mathcal{D}_\theta$ has negative diagonal entries and it is not PSD. Next, by Lemma 3.10, the set of unordered eigenvalues of $\mathcal{D}_\theta$ can be computed as

$$\{\theta(b+2) - 2 \pm \beta_j : 0 \leq j \leq n-1\}.$$

Thus, the minimum eigenvalue of $\mathcal{D}_\theta$ is equal to

$$\begin{aligned}\lambda_{2n}(\mathcal{D}_\theta) &= \theta(b+2) - 2 - \max_{0 \leq j \leq n-1} \beta_j \\ &= \theta(b+2) - 2 - \hat{\beta}.\end{aligned}$$

Additionally, if we have $\frac{(b\theta-1)(2\theta-1)}{4\theta(1-\theta)} \geq 1$, then $\hat{\beta} = |\theta(b+2) - 2|$ and the minimum eigenvalue of $\mathcal{D}_\theta$ is equal to

$$\begin{aligned}\lambda_{2n}(\mathcal{D}_\theta) &= \theta(b+2) - 2 - |\theta(b+2) - 2| \\ &= 0,\end{aligned}$$

where the second equality holds since we assumed $\theta(b+2) - 2 \geq 0$. Therefore, a sufficient condition for $\mathcal{D}_\theta$ to be a positive semidefinite matrix is to have

$$\begin{aligned}\frac{(b\theta-1)(2\theta-1)}{4\theta(1-\theta)} \geq 1 &\iff (2b+4)\theta^2 - (b+6)\theta + 1 \geq 0 \\ &\iff \theta \leq \theta_1 \text{ or } \theta \geq \theta_2,\end{aligned}$$

where $\theta_1 < \theta_2$ are the roots of the quadratic function

$$f(\theta) = (2b+4)\theta^2 - (b+6)\theta + 1.$$

We can now compute θ_1, θ_2 as

$$\theta_1 = \frac{b+6-\sqrt{b^2+4b+20}}{4b+8}, \qquad \theta_2 = \frac{b+6+\sqrt{b^2+4b+20}}{4b+8}.$$

To upper bound θ_1 note that $b^2 + 4b + 20 = (b+2)^2 + 16$. Therefore, we have

$$\begin{aligned}
\theta_1 &= \frac{b + 6 - \sqrt{(b+2)^2 + 16}}{4b + 8} \\
&\leq \frac{b + 6 - \sqrt{(b+2)^2}}{4b + 8} \\
&= \frac{4}{4b + 8} \\
&= \frac{1}{b + 2}.
\end{aligned}$$

As a result, for any value $\theta \leq \theta_1$ we do not satisfy $\theta(b+2) - 2 \geq 0$ and $\mathcal{D}_\theta$ is not a PSD matrix. On the other hand, note that $\sqrt{(b+2)^2 + 16} \geq b + 4$ as $b \geq 1$. Therefore, we get

$$\begin{aligned}
\theta_2 &= \frac{b + 6 + \sqrt{(b+2)^2 + 16}}{4b + 8} \\
&\leq \frac{b + 6 + (b + 4)}{4b + 8} \\
&= \frac{1}{2} + \frac{3}{2b + 4}.
\end{aligned}$$

Therefore, for any value $\theta \geq \frac{1}{2} + \frac{3}{2b+4}$ both conditions

$$\theta(b+2) - 2 \geq 0$$

and

$$\frac{(b\theta - 1)(2\theta - 1)}{4\theta(1 - \theta)} \geq 1$$

are satisfied, and we have $\mathcal{D}_\theta \succeq 0$. This proves that

$$\theta_{1,b}(C_Z) \leq \frac{1}{2} + \frac{3}{2b + 4}.$$

□

Thus, we can conclude that for even $n \in \mathbb{N}$, the spectral thinness of the prism graph $\Pi_n(1, b)$ can be bounded as follows:

Corollary 3.3. *For even value $n = 2k$ and $b > 1$, the spectral thinness of $\Pi_n(1, b)$ is* *bounded by*

$$\frac{1}{2} - \frac{1}{2n} \leq \theta_{1,b} \leq \frac{1}{2} + \frac{3}{2b+4}. \tag{3.23}$$

Proof.

$$\begin{aligned}
\frac{1}{2} - \frac{1}{2n} &\leq \theta_{1,b} && (\text{ by Proposition 3.27}) \\
&\leq \theta_{1,b}(P_Z) \\
&\leq \theta_{1,b}(C_Z) && (\text{ by Proposition 3.18}) \\
&\leq \frac{1}{2} + \frac{3}{2b+4}. && (\text{ by Proposition 3.28})
\end{aligned}$$

□

Finally, we will finish this case, by proving Lemma 3.10.

Proof of Lemma 3.10. To compute eigenvalues of B we consider the square matrix B^2, where

$$B^2 = \begin{bmatrix} AA^\top & 0 \\ 0 & A^\top A \end{bmatrix}.$$

Note that $A^\top$ is also a circulant matrix with essential row

$$c' = [1 - b\theta, 1 - \theta, 0, \dots, 0, -\theta].$$

Since circulant matrices commute (Proposition 3.21 (iii)), we have $AA^\top = A^\top A$. Moreover, since $A, A^\top$ are circulant, both share a same set of eigenvectors

$$v_j = \frac{1}{\sqrt{n}}[\omega_n^{0j}, \omega_n^{1j}, \dots, \omega_n^{(n-1)j}]^\top$$

with eigenvalues

$$\lambda_A(v_j) := \sqrt{n} c v_j = \sum_{k=0}^{n-1} c_k \omega_n^{kj},$$
$$\lambda_{A^\top}(v_j) := \sqrt{n} c' v_j = \sum_{k=0}^{n-1} c'_k \omega_n^{kj}.$$

Also, the vectors $V_j = \begin{bmatrix} v_j \\ v_j \end{bmatrix}$ and $U_j = \begin{bmatrix} v_j \\ -v_j \end{bmatrix}$ form a set of eigenvectors for B^2 with eigenvalues $\lambda_{B^2}(V_j)$ and $\lambda_{B^2}(U_j)$ is equal to

$$\beta_j^2 = \lambda_A(v_j)\lambda_{A^\top}(v_j).$$

Notice that if β_j^2 is an eigenvalue of B^2, then either β_j or $-\beta_j$ is an eigenvalue of B. Additionally, since B is a bipartite matrix, by Proposition 3.8, β_j is eigenvalue of B if and only if $-\beta_j$ is an eigenvalue of B. Therefore, the set of eigenvalues of B is equal to:

$$\{\pm\beta_j : 0 \le j \le n-1\}.$$

By calculation, we have that

$$\begin{aligned}
\beta_j^2 &= \lambda_A(v_j)\lambda_{A^\top}(v_j) \\
&= \Big((1-b\theta) - \theta\omega^j + (1-\theta)\omega^{(n-1)j}\Big) \times \Big((1-b\theta) + (1-\theta)\omega^j - \theta\omega^{(n-1)j}\Big) \\
&= (1-b\theta)^2 + \theta^2 + (1-\theta)^2 - 2\theta(1-\theta)\cos\frac{4\pi j}{n} + 2(1-b\theta)(1-2\theta)\cos\frac{2\pi j}{n}.
\end{aligned}$$

Then, by simplifying the above formulation using $\cos 2x = 2\cos^2 x - 1$ we get,

$$\begin{aligned}
\beta_j^2 &= 2\theta(1-\theta) + (1-b\theta)^2 + \theta^2 + (1-\theta)^2 - 4\theta(1-\theta)\cos^2\frac{2\pi j}{n} + 2(1-b\theta)(1-2\theta)\cos\frac{2\pi j}{n} \\
&= 1 + (1-b\theta)^2 - 4\theta(1-\theta)\Big(\cos\frac{2\pi j}{n} - \frac{(1-b\theta)(1-2\theta)}{4\theta(1-\theta)}\Big)^2 + \frac{(1-b\theta)^2(1-2\theta)^2}{4\theta(1-\theta)} \\
&= 1 + \frac{(b\theta-1)^2}{4\theta(1-\theta)} - 4\theta(1-\theta)\Big(\cos\frac{2\pi j}{n} - \frac{(b\theta-1)(2\theta-1)}{4\theta(1-\theta)}\Big)^2.
\end{aligned}$$

Next, we will argue that if

$$\frac{(b\theta-1)(2\theta-1)}{4\theta(1-\theta)} \geq 1 \tag{3.24}$$

then we have

$$\hat{\beta} = |\theta(2b+2)-2|.$$

Note that since $\theta \in (0,1)$, we have $4\theta(1-\theta) \geq 0$. Then, to maximize β_j, we equivalently can minimize the value of

$$\Big(\cos\frac{2\pi j}{n} - \frac{(b\theta-1)(2\theta-1)}{4\theta(1-\theta)}\Big)^2.$$

Now, if (3.24) holds, then to minimize above, the best choice is set $j=0$ so that $\cos\frac{2\pi j}{n}=1$. Then, we get

$$\begin{aligned}
\beta_0^2 &= 1 + \frac{(b\theta-1)^2}{4\theta(1-\theta)} - 4\theta(1-\theta)\Big(1 - \frac{(b\theta-1)(2\theta-1)}{4\theta(1-\theta)}\Big)^2 \\
&= 1 + \frac{(b\theta-1)^2}{4\theta(1-\theta)}\Big(4\theta(1-\theta)\Big) - \frac{(b\theta-1)^2(2\theta-1)^2}{4\theta(1-\theta)} + 2(b\theta-1)(2\theta-1) \\
&= 1 + \frac{(b\theta-1)^2}{4\theta(1-\theta)}\Bigg[1-(2\theta-1)^2\Bigg] - 4\theta(1-\theta) + 2(b\theta-1)(2\theta-1) \\
&= 1 + (b\theta-1)^2 - 4\theta(1-\theta) + 2(b\theta-1)(2\theta-1) \\
&= 1 + (b\theta-1)\Big((b\theta-1)+(4\theta-2)\Big) - 4\theta(1-\theta) \\
&= (b^2+4b+4)\theta^2 - (4\theta+8)\theta + 4 \\
&= (b+2)^2\theta^2 - 2(b+2)\theta + 4 \\
&= \Big((b+2)\theta-2\Big)^2.
\end{aligned}$$

Therefore, we get

$$\hat{\beta} = |(b+2)\theta-2|.$$

□

The computational calculation showed that roughly speaking, the value of $\theta_{1,b}(P_{2n})$ is closer to the upper bound $\frac{1}{2}+\frac{1}{b}$ (see Figure 3.15).

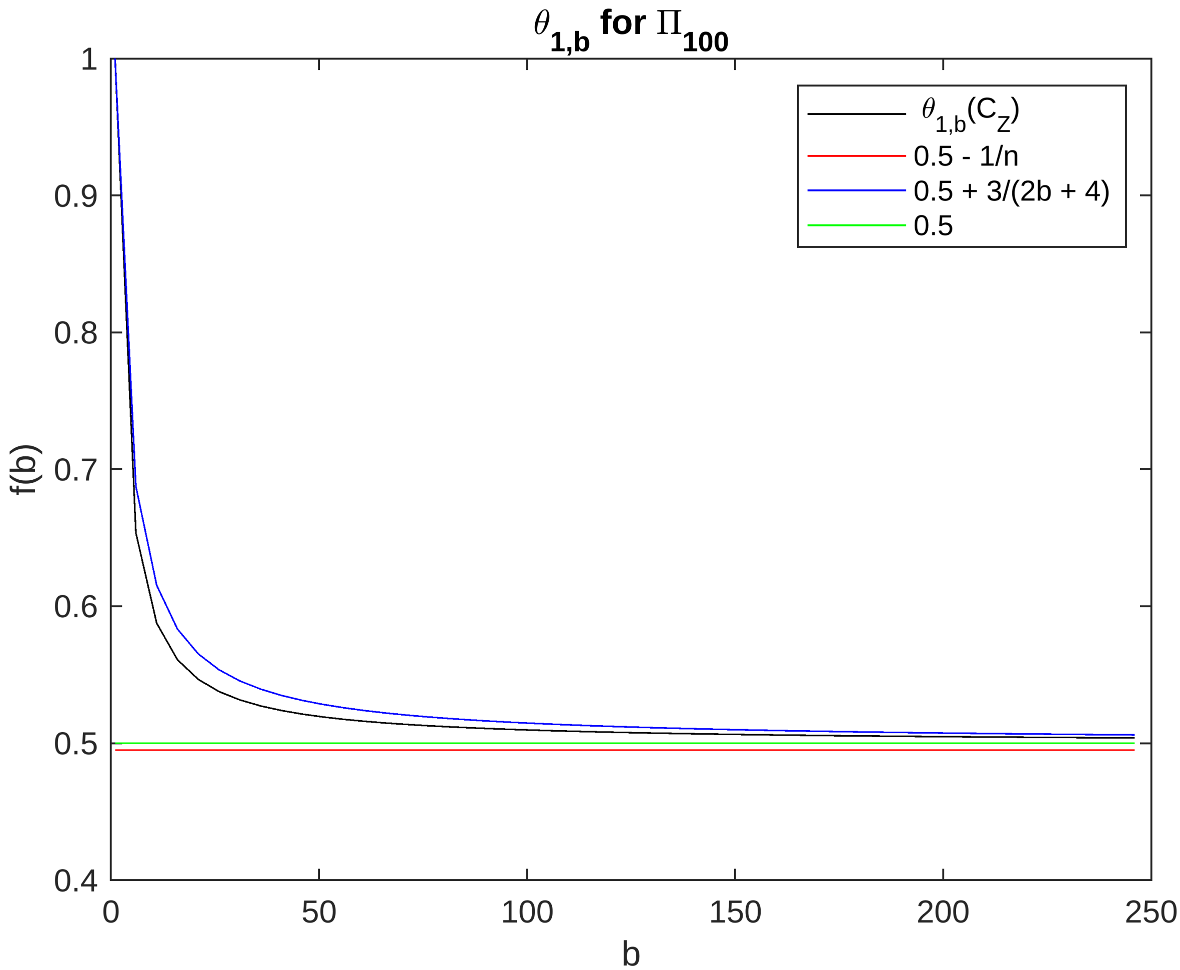


Figure 3.15: a=1, n=100, enumerate on b. The black curve shows the experimental computational of $\theta_{1,b}$, and the blue curve and the red line show the obtained upper and lower bounds for $\theta_{1,b}$.

Chapter 4

Conclusion and Future Research

In this thesis, we discussed notions of combinatorially and spectrally thin trees which are two related families of spanning trees with interesting properties and applications. However, as we saw, existence of such spanning trees is still unknown for many families of graphs, and there are many questions to investigate that could be an intermediate step towards resolution of thin tree conjecture.

Edge disjoint spanning trees To begin with, we showed that graphs without two spanning trees do not have combinatorially thin trees. Additionally, we also covered a result of Merker and Postle [29], which shows families of 4-regular graphs with two edge-disjoint spanning trees that do not have combinatorially thin trees. Regarding this result, an interesting open question is that whether having three or higher number of edge-disjoint spanning trees is enough for a graph to be combinatorially thin (see Goemans [19]).

Combinatorial thinness with respect to specific cuts Note that a combinatorially α-thin tree has to have at most α fraction of each cut-set $\delta(U) \subseteq E$ in the graph G. However, for some applications we might be able to relax this condition by requiring the spanning tree to be only α-thin for specific cut-sets of the graph. In a recent work, Klien and Olver [25] considered a laminar family of cuts $\mathcal{L} \subseteq 2^V$ such that for any two cuts $U_1, U_2 \in \mathcal{L}$ we have either U_1 and U_2 are disjoint or they are subset of each other; i.e., $U_1 \subset U_2$ or $U_2 \subset U_1$. In fact, they proved that given a $k-$edge connected graph and a laminar family of cuts $\mathcal{L} \subseteq 2^V$, there exists a spanning tree which is $O(\frac{1}{k})-$thin for the cuts in $\mathcal{L}$. This new result may also be considered as another piece of evidence toward correctness of strong thin tree conjecture. Therefore, another direction for research on thin trees is to investigate the existence of thin trees with respect to specific cut families.

Combinatorial thinness for structured families of graphs As we mentioned in Chapter 2, Oveis-Gharan and Saberi [16] proved existence of combinatorially thin tree in families of graphs with bounded genus. Additionally, for planar graph G, we showed how to translate their algorithm to an algorithm working on the original planar graph G. More specifically, we used the fact that in k-connected planar graphs the underlying graph is sparse, and one can maintain an ordering of edges throughout the algorithm while selecting proper edges and contracting them.

An interesting approach to further understanding this result is to see exactly what properties should be satisfied by G for this algorithm to work and whether we can extend their result to other family of graphs with sprase underlying graph.

Relations between combinatorial and spectral thinness Even though calculating $\alpha_G(T)$ is believed to be a hard problem for an arbitrary graph G and spanning tree T, computing $\theta_G(T)$ is possible in polynomial time. Since spectrally thin trees are a subset of combinatorially thin trees, we can obtain an upper bound on α_G whenever spectral thin trees exist.

Although strong thin tree conjecture suggests existence of combinatorially $O(\frac{1}{k})$- thin trees in k-connected graphs, Harvey and Olver [20], and later, Anari and Oveis-Gharan [2] showed instances of k-connected (weighted) graphs without spectrally thin trees. More specifically, one can construct k-connected (weighted) graphs with n vertices such that all edges in a specific cut have effective resistance $1 - O(k^2/n)$. Therefore, if strong thin tree conjecture is correct, we have family of graphs where the gap $\theta_G - \alpha_G$ becomes arbitrary close to one. On the other hand, we know that $\theta_G = O(\mathrm{Reff}_G)$ [27, 20]. From this result, we further know that in a k-connected edge-transitive graph G we have $\theta_G = O(\frac{1}{k})$. As a result, if strong thin tree conjecture holds, for families of k-connected edge-transitive graphs we have $\theta_G/\alpha_G = O(1)$.

Nevertheless, the exact relations between α_G, θ_G, k, and Reff_G are still unknown. Therefore, another direction of research in this area is to study the gaps such as $\theta_G - \alpha_G$ and ratios such as θ_G/α_G for different families of graphs which may be useful in further understanding of the strong thin tree conjecture.

Hamiltonian paths As another observation in Chapter 3, we investigated Hamiltonian paths as a good candidate for both spectrally and combinatorially thin trees. Hamiltonian graphs are not limited to the examples we covered and many families of distance regular graphs have Hamiltonian properties. In fact, in 1969 Lovász conjectured that every connected vertex-transitive graph has a Hamiltonian path, and in a recent work, Merino et al. [28] proved this conjecture for the case of Kneser graphs. Investigating the difference and similarities of spectral and combinatorial thinness of Hamiltonian paths, where they exist, is an interesting research direction which might shed light on the relation of θ_G and α_G is some graphs.

Computation complexity of θ_G It is known that verifying the exact value of α_G is a NP-hard problem in the general setting. However, even though we know how to efficiently compute $\theta_G(T)$ for a specific tree T, we currently do not have a complete understanding of the computational complexity of θ_G in the general setting. Finding polynomial time algorithms to compute θ_G for arbitrary graphs or proving NP-hardness of this problem can also be an interesting research direction.

Semidefinite programming Finally, we can consider the problem of finding spectral thinness of a given graph G as an optimization problem and consider semidefinite relaxation of it. In fact, $\theta_{G,w}$ is equal to the optimal value of the following optimization problem.

$$\begin{array}{ll} \theta_{G,w} = \min & \theta \\ \text{subject to} & \theta.\mathcal{L}_G(w) \succeq \mathcal{L}_G(x), \\ & x \in SP(G) \cap \{0,1\}^E. \end{array} \tag{P1}$$

Therefore, we can obtain a lower bound on θ_G by considering a semidefinite relaxation of (P1) as

$$\begin{array}{ll} \min & \theta \\ \text{subject to} & \theta\mathcal{L}_G(w) \succeq \mathcal{L}_G(x) \\ & Ax \geq b, \end{array} \tag{SP1}$$

where $A \in \mathbb{R}^{V \times E}$ and $b \in \mathbb{R}^E$ are the constraint identifiers of spanning tree polytope $SP(G)$. Note that well-posed semidefinite programming problems can be solved in polynomial time to obtain an approximate optimal solution. Therefore, the above proposition gives a very easy algorithmic way of obtaining a lower bound on spectral thinness $\theta_{G,w}$. However, this lower bound can be loose.

Example 4.1. For the complete graph K_n with weight function $w = \mathbb{1}$, the thinness parameter θ_{K_n} is lower bounded by $\frac{4(n-1)}{n^2}$ as described in Theorem 3.6. However, for

$x = \frac{2}{n}.\mathbb{1} \in \mathbb{R}^E$, we have $x \in SP(K_n)$. Moreover, if we consider the SDP (P1) relaxation we get

$$\theta \mathcal{L}_{K_n}(\mathbb{1}) \succeq \frac{2}{n}\mathcal{L}_{K_n}(\mathbb{1}).$$

Therefore, $(2/n, x)$ is a pair of feasible solution for the SDP relaxation. Thus, the optimal value of (P1) $\tilde{\theta} < \frac{2}{n}$ which has a considerable gap with $\frac{4(n-1)}{n^2}$.

As another research direction one can investigate the spectral thinness problem by finding better SDP relaxations which may lead to tighter lower bounds on θ_G.

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APPENDICES

Appendix A

Missing Proofs

Proof of Claim 1.1

Proof. Firstly, note that since $\bar{E}$ corresponds to set of all arcs $a \in A$ of G with non-zero x^*_a, we have

$$\begin{aligned}
x^*(A) &= \sum_{\{u,v\}\in\bar{E}} (x^*_{uv} + x^*_{vu}) \\
&= \sum_{\{u,v\}\in\bar{E}} (\frac{n}{n-1})\bar{x}_{uv} \\
&= (\frac{n}{n-1})\bar{x}(\bar{E})
\end{aligned}$$

Thus, we can show the first constraint in (1.7) holds as,

$$\begin{aligned}
\bar{x}(\bar{E}) &= (\frac{n-1}{n})x^*(A) \\
&= (\frac{n-1}{n})\Big(\sum_{v\in V}(\sum_{(v,u)\in A} x^*_{vu})\Big) \\
&= (\frac{n-1}{n})\Big(\sum_{v\in V} x^*(\delta^+_G(v))\Big) \\
&= (\frac{n-1}{n})|V| = n-1,
\end{aligned}$$

where the last equality holds since x^* is feasible in (1.8).

Secondly, for each subset of vertices $U \subsetneq V$ let

$$\bar{x}(\bar{E}[U]) = \sum_{\{u,v\} \in \bar{E}[U]} \bar{x}_{uv}.$$

Then, we show $\bar{x}(\bar{E}[U]) < |U| - 1$. By definition,

$$\begin{aligned}
\sum_{v \in U} \bar{x}(\delta_{\bar{G}}(v)) &= 2\bar{x}(\bar{E}[U]) + \bar{x}(\delta_{\bar{G}}(U)) \\
&= 2\bar{x}(\bar{E}[U]) + \frac{n-1}{n}(x^*(\delta^+_G(U)) + x^*(\delta^-_G(U))) \\
&\geq 2\bar{x}(\bar{E}[U]) + 2(\frac{n-1}{n}),
\end{aligned}$$

where the last inequality holds since we have $x^*(\delta^-_G(U)) = x^*(\delta^+_G(U)) \geq 1$.

On the other hand, we have

$$\begin{aligned}
&\sum_{v \in U} \bar{x}(\delta_{\bar{G}}(v)) = \sum_{v \in U} \frac{n-1}{n}(x^*(\delta^+_G(v)) + x^*(\delta^-_G(v))) \\
&= \sum_{v \in U} 2\frac{n-1}{n} \\
&= 2\frac{n-1}{n}|U|.
\end{aligned}$$

Combining both, we get that

$$\bar{x}(\bar{E}[U]) \leq \frac{n-1}{n}(|U| - 1) < |U| - 1. \tag{A.1}$$

Since the last inequality is strict, $\bar{x}$ is in the relative interior of $\mathrm{SP}(\bar{G})$. $\square$

Proof of Lemma 2.2

Proof. Let $G = (V, E)$ be a graph with $\Delta(G) = D$, and let $T = (V, E(T))$ be a spanning tree in G. For each edge $e = \{u, v\} \in E(T)$, suppose W_e, U_e be the vertex sets of $T \setminus e$ two connected components. Define $f(e) := ||W_e| - |U_e||$, and let edge $e^* = \{u, v\}$ be the

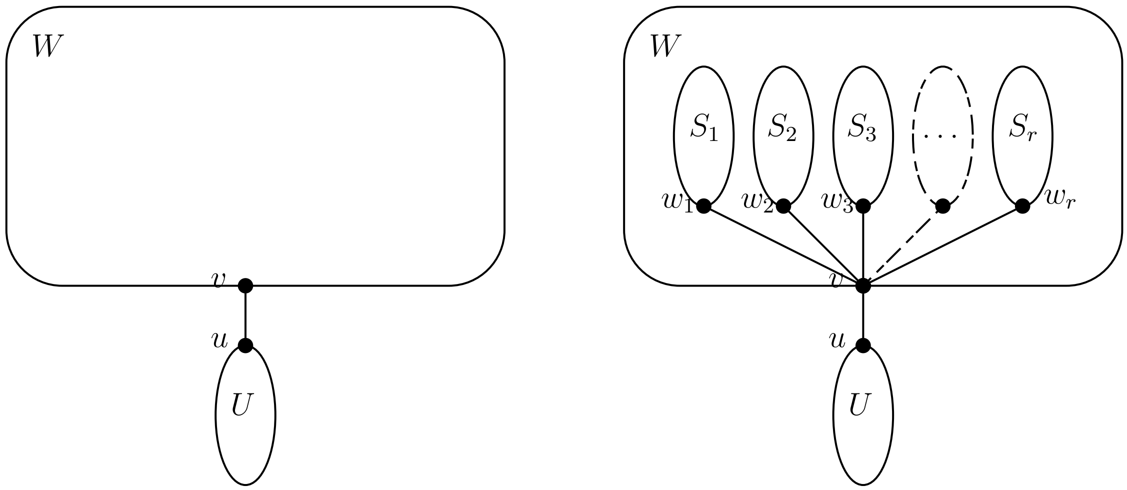

Figure A.1: The components U and W corresponding to cut-edge $e = \{u, v\}$.

edge that minimizes the value of function f. Without loss of generality, suppose U_{e^*} be the smaller size component, and suppose $u \in U_{e^*}$.

Now, let $\{w_1, \dots w_r\}$ be the neighbours of v in W_{e^*} for some $r \leq D - 1$. Moreover, for each $i \in [r]$ let $S_i \subset V$ be the vertex set of $T \setminus \{w_i, v\}$ connected component which is fully contained in W_{e^*}. Note that by our definition, we have

$$W_{e^*} = \{v\} \cup S_1 \cup \dots \cup S_r.$$

Now we will prove two claims in the following.

Claim A.1. $|S_i| \leq \frac{n}{2}$, *for all* $i \in [r]$.

Proof. Without loss of generality, suppose we have $|S_1| > \frac{n}{2}$. Then, consider edge $e = \{w_1, v\}$, and note that

$$\begin{aligned} f(e) &= ||S_1| - |(W_{e^*} \setminus S_1) \cup U_{e^*}|| \\ &= |S_1| - (|U_{e^*}| + |S_2| + \dots |S_r| + 1) && (\text{since } |S_1| > \frac{n}{2}) \\ &< |S_1| - |U_{e^*}| \\ &< |W_{e^*}| - |U_{e^*}| && (\text{since } S_1 \subsetneq W_{e^*}) \\ &= f(e^*), \end{aligned}$$

which is a contradiction since we assumed e^* minimizes function f. $\square$

Claim A.2. *$|S_j| \leq |U_{e^*}|$ for all $j \in [r]$.*

Proof. Without loss of generality, suppose $|S_1| > |U_{e^*}|$, then consider edge $e = \{w_1, v\}$. Since by Claim A.1 we know that $|S_1| \leq \frac{n}{2}$, we have

$$\begin{aligned} f(e) &= ||S_1| - |V \setminus S_1|| \\ &= |V \setminus S_1| - |S_1| \\ &= (|W_{e^*}| - |S_1| + |U_{e^*}|) - |S_1| \\ &= |W_{e^*}| + |U_{e^*}| - 2|S_1|. \end{aligned}$$

On the other hand, $f(e^*) = |W_{e^*}| - |U_{e^*}|$. Therefore,

$$\begin{aligned} f(e) - f(e^*) &= (|W_{e^*}| + |U_{e^*}| - 2|S_1|) - (|W_{e^*}| - |U_{e^*}|) \\ &= 2|U_{e^*}| - 2|S_1| \\ &< 0. \end{aligned}$$

Hence, $f(e) < f(e^*)$ which is a contradiction. □

Now, in order to show $\min\{|U_{e^*}|, |W_{e^*}|\} = |U_{e^*}| \geq \frac{n}{d+1}$, note that we have

$$|W_{e^*}| = \sum_{i=1}^{r} |S_i| + 1,$$

and also,

$$|W_{e^*}| + |U_{e^*}| = \sum_{i=1}^{r} |S_i| + |U_{e^*}| + 1 = n.$$

Since by Claim A we have$|S_i| \leq |U_{e^*}|$, the following inequalities holds:

$$n - 1 = \sum_{i=1}^{r} |S_i| + |U_{e^*}| \leq (r+1)|U_{e^*}| \leq D|U_{e^*}|.$$

Therefore, we have $|U_{e^*}| \geq \frac{n-1}{D}$ and both shores of $T \setminus e^*$ has vertex set of size at least $\frac{n}{D+1}$. □

Proof of Theorem 3.2

Proof. Note that $\lambda_n = 0$ with corresponding eigenvector $v_n = \mathbb{1}_E$. By Theorem 3.1, we have

$$\begin{aligned}
\lambda_{n-1} &= \min_{x \perp v_n = \mathbb{1}} \frac{x^\top \mathcal{L}_G x}{x^\top x} \\
&= \min_{x \perp v_n = \mathbb{1}} \frac{\sum_{\{i,j\} \in E} (x_i - x_j)^2}{\sum_i x_i^2}.
\end{aligned}$$

Now, for an arbitrary but fixed cut $U \subset V$ with $|U| \leq \frac{n}{2}$, define vector $x \in \mathbb{R}^n$ as

$$x_i := \begin{cases} \frac{1}{|U|} & \text{if } v_i \in U \\ -\frac{1}{|V|-|U|} & \text{if } v_i \notin U. \end{cases}$$

Notice that we have

$$\langle x, \mathbb{1} \rangle = \sum_{i=1}^{n} x_i = |U|.\frac{1}{|U|} - (|V| - |U|).\frac{1}{|V| - |U|} = 0.$$

Therefore, $x \perp \mathbb{1}$ and we can show that

$$\begin{aligned}
\lambda_{n-1} &\leq \frac{x^\top \mathcal{L}_G x}{x^\top x} \\
&= \frac{\sum_{\{i,j\} \in E} (x_i - x_j)^2}{\sum_i x_i^2} \\
&= \frac{|\delta(U)|(\frac{1}{|U|} + \frac{1}{|V|-|U|})^2}{|U|(\frac{1}{|U|})^2 + (|V| - |U|).(\frac{1}{|V|-|U|})^2} \\
&= |\delta(U)|(\frac{1}{|U|} + \frac{1}{|V| - |U|}) \\
&= \frac{|\delta(U)|.|V|}{|U|.(|V| - |U|)} \\
&= \frac{|\delta(U)|}{|U|} \times \frac{|V|}{|V| - |U|} \\
&\leq 2\Phi(U),
\end{aligned}$$

where the last equality holds as $|V| - |U| \geq \frac{|V|}{2}$. Since the inequality holds for all $U \subset V$, we have

$$\frac{\lambda_{n-1}}{2} \leq \Phi(G). \tag{A.2}$$

□

Proof of Lemma 3.5

Proof. Note that if $n < 4$, then $\frac{4}{n} > 1$ is a trivial upper bound on the thinness of any tree of K_n. Therefore, suppose $n \geq 5$ and let T be a non-path tree of K_n. Therefore, T has a vertex of degree at least three. Therefore, we only need to consider the following cases.

i. T has a vertex v_1 of degree $\deg_T(v_1) \geq 4$.

ii. T has two adjacent vertices v_1, v_2 of degree $\deg_T(v_1) = \deg_T(v_2) = 3$.

iii. For every vertex v_1 of T with degree $\deg_T(v_1) = 3$, all of its neighbours v_2, v_3, v_4 have degree at most 2.
$$2 \geq \deg_T(v_2) \geq \deg_T(v_3) \geq \deg_T(v_4).$$

The first two cases can be proven similar to proof of Proposotion 3.17.

For the third case, let $N = \Big(N_T(v_2) \cup N_T(v_3) \cup N_T(v_4)\Big) \setminus v_1$ be the vertices at distance two from v_1 in the spanning tree T, which are connected to $v_2, \ldots, v_4$. Without loss of generality, let $v_1 = 1$, $v_2 = 2$, $v_3 = 3$, $v_4 = 4$ be the first vertices in the vertex labeling (see Figure A.2). We define $x \in \mathbb{R}^V$ to be

$$x_i = \begin{cases} 4, & \text{if } i = 1 \\ -2, & \text{if } i = 2, 3, 4 \\ 1, & \text{if } i \in N \\ 0, & \text{otherwise.} \end{cases}$$

Now, we claim that in each case, we have

$$\theta_{K_n}(T) \geq \frac{x^\top \mathcal{L}_T x}{x^\top \mathcal{L}_{K_n} x} \geq \frac{4}{n}.$$

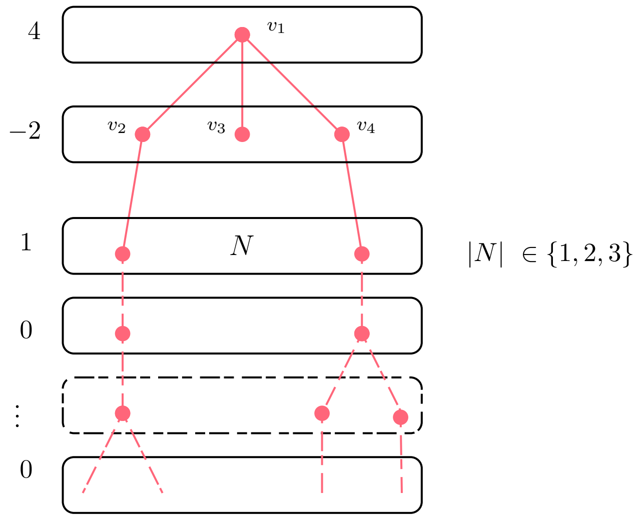

Figure A.2: Spanning tree with no two adjacent degree three vertices and the corresponding values of x_i.

We first consider $x^\top \mathcal{L}_T x$. Note that since $n > 5$, at least one of the vertices in N has to be connected to a vertex outside of $\{1,2,3,4\} \cup N$. Additionally, each vertex in N is connected to exactly one of the vertices $\{2,3,4\}$. Therefore,

$$
\begin{aligned}
x^\top \mathcal{L}_T x &= \sum_{\{i,j\}\in E(T)} (x_i - x_j)^2 \\
&\geq= \sum_{j\in\{2,3,4\}} 6^2 + \sum_{\{i,j\}\in E(T): i\in\{2,3,4\}, j\in N} 3^2 + \sum_{\{i,j\}\in E(T): i\notin\{1,2,3,4\}, j\in N} 1 \\
&\geq 109 + 9|N|.
\end{aligned}
$$

To compute $x^\top \mathcal{L}_{K_n} x$ we proceed as follows. For vertex 1 we have,

$$
\begin{aligned}
\sum_{1<j\leq n} (x_1 - x_j)^2 &= \sum_{j\in\{2,3,4\}} 6^2 + \sum_{j\in N} 3^2 + \sum_{j\notin N\cup\{2,3,4\}} 4^2 \\
&= 108 + 9|N| + (n - 1 - (|N| + 3))16 \\
&= 16n - 7|N| + 44.
\end{aligned}
$$

Also, for the other edges $\{i, j\}$ with $i \in \{2, 3, 4\}$ and $j > 1$ we have,

$$\begin{aligned}
\sum_{i \in \{2,3,4\}} \sum_{i<j} (x_i - x_j)^2 &= \sum_{i \in \{2,3,4\}} \sum_{j \in N} 3^2 + \sum_{i \in \{2,3,4\}} \sum_{j \notin N \cup \{1,2,3,4\}} 4 \\
&= 3 \times 9 \times |N| + 3 \times 4 \times (n - 1 - (3 + |N|)) \\
&= 12n + 15|N| - 48.
\end{aligned}$$

Finally, all the edges $\{i, j\}$ with $i \in N$ and $j \notin \{1, 2, 3, 4\} \cup N$ we have,

$$\begin{aligned}
\sum_{i \in N} \sum_{i<j} &= \sum_{i \in N} \sum_{j \notin \{1,2,3,4\} \cup N} (x_i - x_j)^2 \\
&= |N|((n - 1) - 4 - (|N| - 1)) \\
&= |N|n - |N| - |N|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
x^\top \mathcal{L}_{K_n} x &= \sum_{1 \leq i < j \leq n} (x_i - x_k)^2 \\
&= \sum_{1 < j \leq n} (x_1 - x_j)^2 + \sum_{i \in \{2,3,4\}} \sum_{i<j} (x_i - x_j)^2 + \sum_{i \in N} \sum_{j \notin \{i<j} (x_i - x_j)^2 \\
&= (28 + |N|)n - (|N| - 2)^2.
\end{aligned}$$

Therefore, we can lower bound the thinness parameter $\theta_{K_n}(T)$ with

$$\begin{aligned}
\theta_{K_n}(T) &\geq \frac{x^\top \mathcal{L}_T x}{x^\top \mathcal{L}_G x} \\
&\geq \frac{16n - 7|N| + 44}{(28 + |N|)n - (|N| - 2)^2} \\
&\geq \begin{cases} \frac{118}{23n-1}, & \text{if } |N| = 1 \\ \frac{127}{30n}, & \text{if } |N| = 2 \\ \frac{136}{31n-3}, & \text{if } |N| = 3. \end{cases} \\
&\geq \frac{4}{n}.
\end{aligned}$$

□