A real Shapley value for cooperative games with fuzzy characteristic function

H. Galindo[a], J.M. Gallardo[b], A. Jiménez-Losada[a,*]

[a]*Departamento de Matemática Aplicada II, Escuela Técnica Superior de Ingeniería, Universidad de Sevilla, Camino de los Descubrimientos, 41092 Sevilla, Spain*
[b]*Departamento de Ciencias Integradas, Facultad de Ciencias Experimentales, Universidad de Huelva, Campus de El Carmen , 21007 Huelva, Spain*

Abstract

There are cooperative situations in which the players have only imprecise expectations about the profit that can be obtained by each coalition. In order to model these situations, several families of games have been introduced in the literature. Games with fuzzy characteristic function are among them. The main problem that arises when dealing with one of these games is how to allocate among the players the total profit derived from the cooperation. In this regard it seems reasonable that the vagueness in the payments of the coalitions will cause vagueness in the payoffs of the players. In fact, the values introduced for these games assign a fuzzy payoff to each player in the game. However, in some of these situations it might be necessary to assign a precise payoff to each player. With this purpose, in this paper we use a well known ranking for fuzzy numbers to introduce a real Shapley value for games with fuzzy characteristic function.

Keywords: cooperative game; Shapley value; fuzzy set; fuzzy quantity

1. Introduction

Cooperative games model situations in which a group of players decides to cooperate in order to obtain a profit. A cooperative game is given by the set of all players, called the grand coalition, and a function that determines the payment that each subset of players (coalition) can obtain by cooperating. In this setting it is often assumed that the players know with precision the payment achievable by each coalition. However, in real life this is not always the case. In some situations there are only imprecise expectations about these payments. Game theorists have introduced different types of cooperative games which can be used to model these situations. For this, they have used mathematical tools that allow to deal with the concept of uncertainty. Charnes and Granot [5] made use of Probability Theory and introduced cooperative games in which the coalition payments are random variables with given distribution functions. Various authors have continued this line of research (see [14],

*Corresponding author.
Email address: `ajlosada@us.es (A. Jiménez-Losada)` (A. Jiménez-Losada)

[15]). Branzei et al. [3] used real intervals to model cooperative situations in which the players only know a lower and an upper bound of the profit that can be obtained by each coalition. Cooperative interval games have multiple applications in economics and operations research (see [4]). Mareš and Vlach [10] used another mathematical tool to handle imprecise information: the fuzzy sets introduced by Zadeh [18]. Fuzzy sets had been used before in cooperative game theory by Aubin [1], who considered fuzzy subsets of the coalitions in a cooperative game in order to deal with rates of participation of players within coalitions. The goal of Mareš and Vlach was not to consider fuzzy coalitions but fuzzy payments. Therefore, they made use of fuzzy numbers, which are fuzzy subsets of the set of real numbers. They defined games with fuzzy characteristic function, in which the payment of a coalition is given by a fuzzy number which establishes the grade of feasibility of each possible profit achievable by the coalition. The present paper is focused on this approach.

As with other cooperative games, the main problem that arises when dealing with cooperative games with fuzzy characteristic function is how to share the total profit obtained by the grand coalition. In this regard, it seems reasonable that the imprecision payments of the coalitions should imply imprecision in the players' payoffs. Multiple studies have been carried out in this line of research (see [9], [2], [17]). However, there are situations in which, even if there is vagueness in the profit attainable by the coalitions, precise payoffs for the players are needed. This is the starting point of the present paper. Suppose, for example, a cooperative situation in which the total profit obtained by the grand coalition is known exactly, but there are only expectations about the profit achievable by each proper coalition. Take into account that when a cooperative situation is modeled by a cooperative game, it is supposed that all the players will cooperate and the grand coalition will be formed. This means that the formation of any proper coalition is just a hypothetical scenario and, therefore, it might not be possible to know with precision the profit achievable by each proper coalition. However, the players might need a precise allocation of the total profit. The goal of this paper is to come up with a method for obtaining exact allocations in these situations. We introduce the concept of real value for games with fuzzy characteristic function. By using a function introduced by Yager [16] with the purpose of ranking fuzzy numbers, we obtain a Shapley real value for games with fuzzy characteristic function. We show that this value is characterized by certain axioms which are adaptations of the classic ones used to characterize the Shapley value.

The paper is organized as follows. In section 2 some concepts regarding cooperative games, fuzzy quantities and cooperative games with fuzzy characteristic function are recalled. In section 3 we introduce and characterize the real Shapley value for games with fuzzy characteristic function.

2. Preliminaries

2.1. Cooperative games

A *cooperative game (with transferable utility)* consists of a finite set of players N and a characteristic function $v : 2^N \to \mathbb{R}$ which satisfies $v(\emptyset) = 0$. The elements of N are called players, and the subsets of N coalitions. Given a coalition E, $v(E)$ is the worth of E, and it is interpreted as the collective payment that the players of E would obtain if they cooperate. Frequently, a cooperative game (N, v) is identified with the function v. The family of games with set of players N is denoted by $\mathcal{G}^N$. This set is a $(2^{|N|}-1)$-dimensional real vector space. One basis of $\mathcal{G}^N$ is the set $\left\{\delta_E : E \in 2^N \setminus \{\emptyset\}\right\}$ where for a nonempty coalition E the Dirac game δ_E is defined by

$$\delta_E(F) = \begin{cases} 1 & \text{if } F = E, \\ 0 & \text{otherwise.} \end{cases}$$

Another basis of $\mathcal{G}^N$ is the set $\left\{u_E : E \in 2^N \setminus \{\emptyset\}\right\}$ where for a nonempty coalition E the unanimity game u_E is defined by

$$u_E(F) = \begin{cases} 1 & \text{if } E \subseteq F, \\ 0 & \text{otherwise.} \end{cases}$$

Every game $v \in \mathcal{G}^N$ can be written as

$$v = \sum_{\{E \in 2^N : E \neq \emptyset\}} \triangle_v(E)\ u_E \tag{1}$$

where $\triangle_v(E)$ is called the dividend of the coalition E in the game v and is given by

$$\triangle_v(E) = \sum_{F \subseteq E} (-1)^{|E|-|F|} v(F) \tag{2}$$

for every $E \in 2^N \setminus \{\emptyset\}$.

A value on $\mathcal{G}^N$ is a mapping $\psi : \mathcal{G}^N \to \mathbb{R}^N$. If $v \in \mathcal{G}^N$ and $i \in N$, the real number $\psi_i(v)$ is the payoff (according to the value ψ) of the player i in the game v. Multiple values have been defined in the literature. The best-known of them is the *Shapley value* [11], which assigns to each player $i \in N$ in a game $v \in \mathcal{G}^N$ a weighted average of the marginal contributions of i to the coalitions. It is formally defined by

$$\phi_i(v) = \sum_{\{E \subseteq N : i \in E\}} p_E\left(v(E) - v(E \setminus \{i\})\right)$$

for every $i \in N$ and every $v \in \mathcal{G}^N$, where

$$p_E = \frac{(|N| - |E|)!\,(|E| - 1)!}{|N|!}$$

for every $E \in 2^N \setminus \{\emptyset\}$.

Some desirable properties for a value $\psi : \mathcal{G}^N \to \mathbb{R}^N$ are the following:

Efficiency: $\sum_{i \in N} \psi_i(v) = v(N)$ for all $v \in \mathcal{G}^N$.

Additivity: $\psi(v_1 + v_2) = \psi(v_1) + \psi(v_2)$ for all $v_1, v_2 \in \mathcal{G}^N$.

Equal treatment: If $v \in \mathcal{G}^N$, $i, j \in N$ and $v(S \cup \{i\}) = v(S \cup \{j\})$ for every $S \subseteq N \setminus \{i, j\}$, then $\psi_i(v) = \psi_j(v)$.

Null player property: A player $i \in N$ is a null player in $v \in \mathcal{G}^N$ if $v(E) = v(E \setminus \{i\})$ for all $E \subseteq N$. If $i \in N$ is a null player in $v \in \mathcal{G}^N$ then $\psi_i(v) = 0$.

These four properties characterize the Shapley value [12].

2.2. Fuzzy quantities

Firstly we recall some definitions regarding fuzzy sets.

Given a set X, a fuzzy subset a of X is defined by its membership function $\mu_a \colon X \to [0, 1]$. For each $x \in X$ the number $\mu_a(x)$ is the degree of membership of x in a. For each $t \in (0, 1]$ the t-cut of a is defined by

$$[a]_t = \{x \in X \colon \mu_a(x) \geqslant t\}$$

Notice that the family of t-cuts determine a. The core of a is defined by

$$core(a) = [a]_1.$$

If a is a fuzzy subset of $\mathbb{R}$, the 0-cut of a is defined by

$$[a]_0 = \overline{\{x \in \mathbb{R} \colon \mu_a(x) > 0\}}.$$

In this paper we will deal with a particular class of fuzzy subsets of $\mathbb{R}$, the class of fuzzy quantities. The term fuzzy quantity has been used in the literature with slightly different meanings. We will use the concept of fuzzy quantity as defined in [13]. A fuzzy subset a of $\mathbb{R}$ is a fuzzy quantity if it satisfies the following conditions:

i) $core(a) \neq \emptyset$.

ii) $[a]_t$ is a closed and bounded interval for every $t \in [0,1]$.

The set of fuzzy quantities will be denoted by $\mathbb{F}$. If $a \in \mathbb{F}$ and $t \in [0,1]$ we denote

$$a_t^+ = \max[a]_t \quad \text{and} \quad a_t^- = \min[a]_t.$$

Notice that given $a, b \in \mathbb{F}$, a is equal to b if and only if $a_t^+ = b_t^+$ and $a_t^- = b_t^-$ for every $t \in [0,1]$.

Let us recall the basics of fuzzy arithmetic (see [6], [7], [8], [13]).

Let $a, b \in \mathbb{F}$.

- The sum $a \oplus b \in \mathbb{F}$ is defined by

$$\mu_{a\oplus b}(x) = \sup\{\min\{\mu_a(y), \mu_b(z)\} \colon y, z \in \mathbb{R},\ y + z = x\}$$

  for every $x \in \mathbb{R}$. Equivalently,

$$[a \oplus b]_t = [a_t^- + b_t^-, a_t^+ + b_t^+]$$

  for every $t \in [0,1]$.

- The difference $a \ominus b \in \mathbb{F}$ is defined by

$$\mu_{a\ominus b}(x) = \sup\{\min\{\mu_a(y), \mu_b(z)\} \colon y, z \in \mathbb{R},\ y - z = x\}$$

  for every $x \in \mathbb{R}$. Equivalently,

$$[a \ominus b]_t = [a_t^- - b_t^+, a_t^+ - b_t^-]$$

  for every $t \in [0,1]$.

- The product $a \odot b \in \mathbb{F}$ is defined by

$$\mu_{a\odot b}(x) = \sup\left\{\min\left\{\mu_a(y), \mu_b\left(z\right)\right\} : y, z \in \mathbb{R},\ yz = x\right\}$$

  for every $x \in \mathbb{R}$. Equivalently,

$$[a \odot b]_t = \left[\min\{a_t^- b_t^-, a_t^- b_t^+, a_t^+ b_t^-, a_t^+ b_t^+\}, \max\{a_t^- b_t^-, a_t^- b_t^+, a_t^+ b_t^-, a_t^+ b_t^+\}\right]$$

  for every $t \in [0,1]$.

Notice that the set of real numbers can be embedded into $\mathbb{F}$. Indeed, we can identify $p \in \mathbb{R}$ with the fuzzy quantity determined by the following membership function:

$$\mu_p(x) = \begin{cases} 1 & \text{if } x = p, \\ 0 & \textit{otherwise}. \end{cases}$$

With this identification we have that $\mathbb{R} \subset \mathbb{F}$. Note that the operations $\oplus, \ominus, \odot$ extend, respectively, the sum, subtraction and product of real numbers.

Notice that if $a \in \mathbb{F}$ and $p \in \mathbb{R}$, then

$$\mu_{p \oplus a}(x) = \mu_a(x - p)$$

for every $x \in \mathbb{R}$. Equivalently,

$$[p \oplus a]_t = [p + a_t^-, p + a_t^+]$$

for every $t \in [0, 1]$. And, if $p \in \mathbb{R} \setminus \{0\}$, then

$$\mu_{p \odot a}(x) = \mu_a\left(\frac{x}{p}\right)$$

for every $x \in \mathbb{R}$. Equivalently,

$$[p \odot a]_t = \begin{cases} [pa_t^-, pa_t^+] & \text{if } p > 0, \\ \left[pa_t^+, pa_t^-\right] & \text{if } p < 0. \end{cases}$$

for every $t \in [0, 1]$.

Given $a, b \in \mathbb{F}$, it is said that a is greater than or equal to b, which is denoted by $a \geqslant b$, if $a_t^- \geqslant b_t^-$ and $a_t^+ \geqslant b_t^+$ for every $t \in [0, 1]$.

A fuzzy quantity $a \in \mathbb{F}$ is said to be 0-symmetric if $a_t^- = -a_t^+$ for every $t \in [0, 1]$.

Let us recall some basic properties of the arithmetic operations in $\mathbb{F}$. Let $a, b, c, d \in \mathbb{F}$.

a) $a \oplus b = b \oplus a$.

b) $a \odot b = b \odot a$.

c) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

d) $a \odot (b \odot c) = (a \odot b) \odot c$.

e) $a \oplus 0 = a$.

f) $a \odot 1 = a$.

g) $a \odot 0 = 0$.

h) $a \ominus b = a \oplus ((-1) \odot b)$.

i) If $p \in \mathbb{R}$,

$$\begin{aligned} p \odot (a \oplus b) &= (p \odot a) \oplus (p \odot b), \\ p \odot (a \ominus b) &= (p \odot a) \ominus (p \odot b). \end{aligned}$$

j) If $b, c \geqslant 0$ (or $b, c \leqslant 0$),

$$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c). \tag{3}$$

The best known and most employed metric in $\mathbb{F}$ is the supremum distance. Let us introduce it. Let A and B be nonempty bounded subsets of $\mathbb{R}$. Then,

$$d^*(A, B) = \sup\{\inf\{|x - y| \colon y \in B\} \colon x \in A\}.$$

The Hausdorff distance between A and B is defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}.$$

If $a, b \in \mathbb{F}$ the supremum distance between a and b is defined as

$$d_\infty(a, b) = \sup\{d_H([a]_t, [b]_t) \colon t \in [0, 1]\}.$$

2.3. Cooperative games with fuzzy characteristic function

A *cooperative game with fuzzy characteristic function* consists of a finite and nonempty set N and a characteristic function $v : 2^N \to \mathbb{F}$ that satisfies $v(\emptyset) = 0$. The elements of N are called players, and the subsets of N are called coalitions. For each coalition E, the fuzzy quantity $v(E)$ describes the expectations about the collective payment that can be obtained by the players in E when they cooperate. A cooperative game with fuzzy characteristic function (N, v) will be identified with the mapping v. The class of all cooperative games with fuzzy characteristic function and set of players N is denoted by $\mathcal{FG}^N$. Since $\mathbb{R} \subset \mathbb{F}$, we have that $\mathcal{G}^N \subset \mathcal{FG}^N$. If $v, w \in \mathcal{FG}^N$ and $a \in \mathbb{F}$ the games $v \oplus w, a \odot v \in \mathcal{FG}^N$ are defined by

$$\begin{aligned}(v \oplus w)(E) &= v(E) \oplus w(E),\\ (a \odot v)(E) &= a \odot v(E),\end{aligned}$$

for every $E \in 2^N$.

3. A real value for games with fuzzy expectations

A real value on $\mathcal{FG}^N$ is a mapping $\Psi \colon \mathcal{FG}^N \to \mathbb{R}^N$. If $v \in \mathcal{FG}^N$ and $i \in N$, the real number $\Psi_i(v)$ is the payoff (according to the value Ψ) of the player i in the game v.

We aim to define a real value $\Phi \colon \mathcal{FG}^N \to \mathbb{R}^N$ with nice properties. To this end, we will use the function $M \colon \mathbb{F} \to \mathbb{R}$ introduced by Yager [16] for ordering fuzzy numbers. Given $a \in \mathbb{F}$,

$$M(a) = \frac{1}{2}\int_0^1 a_t^+ \, dt + \frac{1}{2}\int_0^1 a_t^- \, dt.$$

If $v \in \mathcal{FG}^N$ we will denote $v_M = M \circ v$. Notice that $v_M \in \mathcal{G}^N$. Now we can introduce the real value on $\mathcal{FG}^N$ that we propose.

Definition 1. *The real Shapley value for cooperative games with fuzzy characteristic function is the mapping* $\Phi \colon \mathcal{FG}^N \to \mathbb{R}^N$ *defined as*

$$\Phi(v) = \phi(v_M)$$

for every $v \in \mathcal{FG}^N$*.*

Example 1. Three companies join in a project to make a new product. When the project is finished they have to share the profit obtained from the sale of the product, say 8 million euros. To this end, the situation is modeled by a cooperative game v on the set of players $\{1, 2, 3\}$. Suppose that the profit that can be obtained by each proper coalition cannot be known with precision. This is not strange if we take into account that the only coalition that will actually be formed is N. The formation of any other coalition is just a supposition used to model the situation and obtain a fair allocation of the benefit. Therefore, it seems reasonable that there are only expectations about the profit that each proper coalition could obtain. These expectations are described by some fuzzy quantities, which are indicated below:

$$\mu_{v(\{1\})}(x) = \begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{otherwise} \end{cases}, \quad v(\{2\}) = 0, \quad \mu_{v(\{3\})}(x) = \begin{cases} 1-x, & \text{if } x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_{v(\{1,2\})} = \begin{cases} x & \text{if } x \in [0,1] \\ 1, & \text{if } x \in (1,4] \\ 0, & \text{otherwise} \end{cases}, \quad \mu_{v(\{1,3\})} = \begin{cases} x, & \text{if } x \in [0,1] \\ \frac{4}{3} - \frac{1}{3}x, & \text{if } x \in (1,4] \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_{v(\{2,3\})} = \begin{cases} x, & \text{if } x \in [0,1] \\ \frac{4}{3} - \frac{1}{3}x, & \text{if } x \in (1,4] \\ 0, & \text{otherwise} \end{cases}, \quad v(N) = 8.$$

We obtain the game v_M,

$$v_M(\{1\}) = 0.5,\ v_M(\{2\}) = 0,\ v_M(\{3\}) = 0.25,\ v_M(\{1,2\}) = 2.25,$$

$$v_M(\{1,3\}) = 1.5,\ v_M(\{2,3\}) = 1.5,\ v_M(N) = 8.$$

Finally, we calculate the Shapley value of v_M,

$$\Phi(v) = \phi(v_M) = (2.9166, 2.6666, 2.4166).$$

4. An axiomatization of the real value

In this section our goal will be to characterize this value.

Let us fix a finite and nonempty set N. We introduce some properties that a real value $\Psi\colon \mathcal{FG}^N \to \mathbb{R}^N$ may satisfy:

- EFFICIENCY FOR SYMMETRIC TOTAL PROFIT. If $v \in \mathcal{FG}^N$ and $v(N)$ is a symmetric fuzzy quantity (i.e., there exists $p \in \mathbb{R}$ such that $v(N) \ominus p$ is 0-symmetric) then

$$\sum_{i \in N} \Psi_i(v) = p.$$

  Observe that efficiency for symmetric total profit coincides with the crisp efficiency when $v(N)$ is a crisp number as in Example 1.

- ADDITIVITY. If $v, w \in \mathcal{FG}^N$ then $\Psi(v \oplus w) = \Psi(v) + \Psi(w)$.

- EQUAL TREATMENT. If $v \in \mathcal{FG}^N$, $i, j \in N$ and $v(E \cup \{i\}) = v(E \cup \{j\})$ for every $E \subseteq N \setminus \{i, j\}$, then $\Psi_i(v) = \Psi_j(v)$.

  If $v \in \mathcal{FG}^N$, a player $i \in N$ is said to be a null player in v if $v(E \cup \{i\}) = v(E)$ for every $E \in 2^N$.

- **NULL PLAYER.** If $v \in \mathcal{FG}^N$ and $i \in N$ is a null player in v, then $\Psi_i(v) = 0$.

  Let us consider the topology on $\mathbb{F}$ induced by the metric d_∞. Let us endow $\mathbb{F}^{2^N \setminus \{\emptyset\}}$ with the product topology. Since $\mathcal{FG}^N$ can be identified with the set $\mathbb{F}^{2^N \setminus \{\emptyset\}}$, we have endowed $\mathcal{FG}^N$ with a topology. Now we can state the following property.

- **CONTINUITY.** The value Ψ is a continuous mapping.

- **COMONOTONICITY.** Let $v, w \in \mathcal{FG}^N$ be such that $core(v(E)) \cap core(w(E)) \neq \emptyset$[1] for every $E \in 2^N$. Let $\alpha \in (0,1)$. Consider $h \in \mathcal{FG}^N$ defined by

  $$\mu_{h(E)}(x) = \alpha\mu_{v(E)}(x) + (1-\alpha)\mu_{w(E)}(x)$$

  for every $E \in 2^N$ and every $x \in \mathbb{R}$. Then,

  $$\Psi(h) = \alpha\Psi(v) + (1-\alpha)\Psi(w).$$

Comonotonicity can be understood in the following sense. Given a cooperative situation, we can consider, depending on the analysis of estimations, slightly different games with fuzzy payoffs to represent this situation (so, usually, the cores of the payments assigned to the same coalition will have non empty intersection). Comonotonicity says that the payoff vector of a weighted average of these games is the weighted average of the payoff vectors of the games.

Let us see that Φ satisfies the six properties above.

Theorem 1. *The Shapley value for cooperative games with fuzzy characteristic function satisfies the properties of efficiency for symmetric total profit, additivity, equal treatment, null player, continuity and comonotonicity.*

Proof.

- *Efficiency for symmetric total profit*

  Let $v \in \mathcal{FG}^N$ with $v(N)$ symmetric. Let $p \in \mathbb{R}$ be such that $v(N) \ominus p$ is 0-symmetric. It is clear that

  $$\frac{1}{2}v(N)_t^+ + \frac{1}{2}v(N)_t^- = p \quad \text{for every } t \in [0,1]. \tag{4}$$

[1]So $\mu_{v(E)}$ and $\mu_{w(E)}$ are comonotonic functions.

Notice that, from the definition of Φ and the efficiency property of the Shapley value, it follows that

$$\begin{aligned}\sum_{i\in N}\Phi_i(v) &= \sum_{i\in N}\phi_i(v_M)=v_M(N)=M(v(N)) \\ &= \frac{1}{2}\int_0^1 v(N)_t^+dt+\frac{1}{2}\int_0^1 v(N)_t^-dt, \end{aligned} \tag{5}$$

which, by (4), is equal to p.

- *Additivity*

  Let $v,w\in\mathcal{FG}^N$. Let $E\in 2^N$. Notice that

$$\begin{aligned}(v\oplus w)_M(E) &= M(v(E)\oplus w(E)) \\ &= \frac{1}{2}\int_0^1 (v(E)\oplus w(E))_t^+dt+\frac{1}{2}\int_0^1 (v(E)\oplus w(E))_t^-dt \\ &= \frac{1}{2}\int_0^1 (v(E)_t^+ + w(E)_t^+)dt+\frac{1}{2}\int_0^1 (v(E)_t^- + w(E)_t^-)dt \\ &= \frac{1}{2}\int_0^1 v(E)_t^+dt+\frac{1}{2}\int_0^1 v(E)_t^-dt+\frac{1}{2}\int_0^1 w(E)_t^+dt+\frac{1}{2}\int_0^1 w(E)_t^-dt \\ &= v_M(E)+w_M(E).\end{aligned}$$

  We conclude that $(v\oplus w)_M=v_M+w_M$. From this fact and the additivity of the Shapley value we have that

$$\Phi(v\oplus w)=\phi((v\oplus w)_M)=\phi(v_M+w_M)=\phi(v_M)+\phi(w_M)=\Phi(v)+\Phi(w).$$

- *Equal treatment*

  Let $v\in\mathcal{FG}^N$ and $i,j\in N$ be such that $v(E\cup\{i\})=v(E\cup\{j\})$ for every $E\subseteq N\setminus\{i,j\}$. It is clear that $v_M(E\cup\{i\})=v_M(E\cup\{j\})$ for every $E\subseteq N\setminus\{i,j\}$. By the property of equal treatment of the Shapley value we obtain that $\phi_i(v_M)=\phi_j(v_M)$. It follows that $\Phi_i(v)=\Phi_j(v)$.

- *Null player*

  Let $v\in\mathcal{FG}^N$, $i\in N$ be such that i is a null player in v. It is clear that i is a null player in v_M. By the property of null player of the Shapley value it follows that $\phi_i(v_M)=0$. We conclude that $\Phi_i(v)=0$.

- *Continuity*

Notice that the Shapley value on $\mathcal{G}^N$ is a linear mapping from $\mathbb{R}^{2^N\setminus\{\emptyset\}}$ into $\mathbb{R}^N$. This implies that $\phi\colon \mathcal{G}^N \to \mathbb{R}^N$ is continuous. Therefore, it is clear that in order to prove that Φ is continuous, it suffices to show that the function $M\colon \mathbb{F} \to \mathbb{R}$ is continuous. Let $M^+, M^-\colon \mathbb{F} \to \mathbb{R}$ defined by

$$M^+(a) = \int_0^1 a_t^+ dt,$$

$$M^-(a) = \int_0^1 a_t^- dt,$$

for every $a \in \mathbb{F}$. Since $M = \frac{1}{2}(M^+ + M^-)$ it is enough to prove that M^+ and M^- are continuous. Let us prove that M^+ is continuous (the reasoning for M^- is analogous). Let $a \in \mathbb{F}$ and let $\epsilon > 0$. Let $b \in \mathbb{F}^N$ be such that $d_\infty(a,b) < \epsilon$. We have that

$$\left|M^+(a) - M^+(b)\right| \leqslant \int_0^1 \left|a_t^+ - b_t^+\right| dt. \tag{6}$$

Take $t_0 \in [0,1]$. Let us prove that $\left|a_{t_0}^+ - b_{t_0}^+\right| \leqslant d_H([a]_{t_0}, [b]_{t_0})$. Suppose that $a_{t_0}^+ \geqslant b_{t_0}^+$ (the case $a_{t_0}^+ < b_{t_0}^+$) is analogous). We have that

$$\begin{aligned}
\left|a_{t_0}^+ - b_{t_0}^+\right| &= a_{t_0}^+ - b_{t_0}^+ = \min\{|a_{t_0}^+ - y|\colon y \in [b]_{t_0}\} \\
&\leqslant \max\{\min\{|x-y|\colon y \in [b]_{t_0}\}\colon x \in [a]_{t_0}\} \\
&= d^*([a]_{t_0}, [b]_{t_0}) \leqslant d_H([a]_{t_0}, [b]_{t_0}).
\end{aligned}$$

We have proved that

$$\left|a_t^+ - b_t^+\right| \leqslant d_H([a]_t, [b]_t) \tag{7}$$

for every $t \in [0,1]$.

By (6) and (7),

$$\begin{aligned}
\left|M^+(a) - M^+(b)\right| &\leqslant \int_0^1 d_H([a]_t, [b]_t) dt \\
&\leqslant \int_0^1 d_\infty(a,b) dt = d_\infty(a,b) < \epsilon
\end{aligned}$$

- *Comonotonicity*

Let $v, w \in \mathcal{FG}^N$ be such that $core(v(F)) \cap core(w(F)) \neq \emptyset$ for every $F \in 2^N$. Let

$\alpha \in (0,1)$. Consider $h \in \mathcal{FG}^N$ defined by

$$\mu_{h(F)}(x) = \alpha\mu_{v(F)}(x) + (1-\alpha)\mu_{w(F)}(x) \tag{8}$$

for every $F \in 2^N$ and every $x \in \mathbb{R}$. We aim to prove that $\Phi(h) = \alpha\Phi(v) + (1-\alpha)\Phi(w)$. To this end, it suffices to prove that

$$h_M = \alpha v_M + (1-\alpha)w_M. \tag{9}$$

Let $E \in 2^N$. Take $p \in core(v(E)) \cap core(w(E))$. Let λ denote the Lebesgue measure. If $K \subseteq \mathbb{R}^2$ then $\mathbf{1}_K \colon \mathbb{R}^2 \to \{0,1\}$ is the indicator function of K, which is defined as

$$\mathbf{1}_K(x,t) = \begin{cases} 1 & \text{if } (x,t) \in K, \\ 0 & \text{if } (x,t) \notin K. \end{cases}$$

We have that

$$\begin{aligned} h_M(E) &= \frac{1}{2}\int_0^1 h(E)_t^+ dt + \frac{1}{2}\int_0^1 h(E)_t^- dt \\ &= \frac{1}{2}\int_0^1 (p + \lambda(\{x \geqslant p \colon \mu_{h(E)}(x) \geqslant t\}))dt + \frac{1}{2}\int_0^1 (p - \lambda(\{x \leqslant p \colon \mu_{h(E)}(x) \geqslant t\}))dt \\ &= \frac{p}{2} + \frac{1}{2}\int_0^1 \left(\int_p^{+\infty} \mathbf{1}_{\{(x,t)\in[p,+\infty)\times[0,1] \colon \mu_{h(E)}(x)\geqslant t\}}(x,t)dx\right)dt \qquad (10) \\ &\quad + \frac{p}{2} - \frac{1}{2}\int_0^1 \left(\int_{-\infty}^p \mathbf{1}_{\{(x,t)\in(-\infty,p]\times[0,1] \colon \mu_{h(E)}(x)\geqslant t\}}(x,t)dx\right)dt \end{aligned}$$

which, by Fubini's Theorem, is equal to

$$\begin{aligned} & p + \frac{1}{2}\int_p^{+\infty} \left(\int_0^1 \mathbf{1}_{\{(x,t)\in[p,+\infty)\times[0,1] \colon \mu_{h(E)}(x)\geqslant t\}}(x,t)dt\right)dx \qquad (11) \\ & -\frac{1}{2}\int_{-\infty}^p \left(\int_0^1 \mathbf{1}_{\{(x,t)\in(-\infty,p]\times[0,1] \colon \mu_{h(E)}(x)\geqslant t\}}(x,t)dt\right)dx \\ = \; & p + \frac{1}{2}\int_p^{+\infty} \lambda(\{t \in [0,1] \colon \mu_{h(E)}(x) \geqslant t\})dx - \frac{1}{2}\int_{-\infty}^p \lambda(\{t \in [0,1] \colon \mu_{h(E)}(x) \geqslant t\})dx \\ = \; & p + \frac{1}{2}\int_p^{+\infty} \mu_{h(E)}(x)dx - \frac{1}{2}\int_{-\infty}^p \mu_{h(E)}(x)dx. \end{aligned}$$

We have proved that

$$h_M(E) = p + \frac{1}{2}\int_p^{+\infty} \mu_{h(E)}(x)dx - \frac{1}{2}\int_{-\infty}^p \mu_{h(E)}(x)dx \tag{12}$$

Similarly, we can prove that

$$v_M(E) = p + \frac{1}{2}\int_p^{+\infty} \mu_{v(E)}(x)dx - \frac{1}{2}\int_{-\infty}^{p} \mu_{v(E)}(x)dx \tag{13}$$

and

$$w_M(E) = p + \frac{1}{2}\int_p^{+\infty} \mu_{w(E)}(x)dx - \frac{1}{2}\int_{-\infty}^{p} \mu_{w(E)}(x)dx. \tag{14}$$

From (8), (12), (13) and (14) we obtain (9).

□

Now we aim to prove that if a value on $\mathcal{FG}^N$ satisfies the six properties stated in the previous theorem then this value is equal to the real Shapley value for cooperative games with fuzzy characteristic function.

Theorem 2. *If a real value Ψ on $\mathcal{FG}^N$ satisfies the properties of efficiency for symmetric total profit, additivity, equal treatment, null player, continuity and comonotonicity, then Ψ is equal to the real Shapley value for cooperative games with fuzzy characteristic function.*

Proof. Suppose that $\Psi\colon \mathcal{FG}^N \to \mathbb{R}^N$ satisfies the properties stated in the theorem. Our goal is to prove that $\Psi = \Phi$. The proof will be done in several steps. In each step it will be shown that $\Psi(v) = \Phi(v)$ for every v in a certain class of games in $\mathcal{G}^N$.

Step 1 Let $E \in 2^N \setminus \{\emptyset\}$ and let $a \in \mathbb{F}$ be such that $|\{\mu_a(z)\colon z \in \mathbb{R}\}| = 2$. Our goal is to prove that

$$\Psi(a \odot u_E) = \Phi(a \odot u_E). \tag{15}$$

It is clear that there exist $x, y \in \mathbb{R}$ with $x \leqslant y$ such that

$$\mu_a(z) = \begin{cases} 1 & \text{if } z \in [x, y], \\ 0 & \text{if } z \in \mathbb{R} \setminus [x, y]. \end{cases}$$

The players in $N \setminus E$ are null players in $a \odot u_E$. By the property of null player,

$$\Psi_i(a \odot u_E) = 0 \quad \text{for every } i \in N \setminus E. \tag{16}$$

By the property of equal treatment, there exists $p \in \mathbb{R}$ such that

$$\Psi_i(a \odot u_E) = p \quad \text{for every } i \in E. \tag{17}$$

Notice that a is symmetric, since $a \ominus \frac{x+y}{2}$ is 0-symmetric. From (16), (17) and the property of efficiency for symmetric total profit it follows that

$$\Psi_i(a \odot u_E) = \begin{cases} \frac{x+y}{2|E|} & \text{if } i \in E, \\ 0 & \text{if } i \in N \setminus E. \end{cases}$$

Since we have used only the properties in the theorem, we conclude (15).

Step 2 Let $E \in 2^N \setminus \{\emptyset\}$ and let $a \in \mathbb{F}$ be such that $|\{\mu_a(z) \colon z \in \mathbb{R}\}| = 3$. We aim to prove that

$$\Psi(a \odot u_E) = \Phi(a \odot u_E). \tag{18}$$

It is clear that there exist $l \in (0, 1)$ and $x, y, r, s \in \mathbb{R}$ with $x \leqslant r \leqslant s \leqslant y$ and $s - r < y - x$ such that

$$\mu_a(z) = \begin{cases} 1 & \text{if } z \in [r, s], \\ l & \text{if } z \in [x, y] \setminus [r, s], \\ 0 & \text{if } z \in \mathbb{R} \setminus [x, y]. \end{cases}$$

Let $b, c \in \mathbb{F}$ defined by

$$\mu_b(z) = \begin{cases} 1 & \text{if } z \in [x, y], \\ 0 & \text{if } z \in \mathbb{R} \setminus [x, y], \end{cases}$$

$$\mu_c(z) = \begin{cases} 1 & \text{if } z \in [r, s], \\ 0 & \text{if } z \in \mathbb{R} \setminus [r, s]. \end{cases}$$

Notice that

$$\mu_a(z) = l\mu_b(z) + (1 - l)\mu_c(z)$$

for every $z \in \mathbb{R}$. Therefore,

$$\mu_{(a \odot u_E)(F)}(z) = l\mu_{(b \odot u_E)(F)}(z) + (1 - l)\mu_{(c \odot u_E)(F)}(z)$$

for every $F \in 2^N$ and every $z \in \mathbb{R}$. Moreover,

$$core((b \odot u_E)(F)) \cap core((c \odot u_E)(F)) \neq \emptyset$$

for every $F \in 2^N$. By the property of comonotonicity and (15),

$$\begin{aligned}\Psi(a \odot u_E) &= l\Psi(b \odot u_E) + (1-l)\Psi(c \odot u_E) \\ &= l\Phi(b \odot u_E) + (1-l)\Phi(c \odot u_E) = \Phi(a \odot u_E).\end{aligned}$$

Step 3 Let $E \in 2^N \setminus \{\emptyset\}$ and let $a \in \mathbb{F}$ be such that $\{\mu_a(z) \colon z \in \mathbb{R}\}$ is a finite set. We aim to prove that

$$\Psi(a \odot u_E) = \Phi(a \odot u_E). \tag{19}$$

It is clear that there exist $l_1, \ldots, l_{n-1} \in (0,1)$ with $l_1 < \ldots < l_{n-1} = 1$ and $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$ with $x_1 \leqslant \ldots \leqslant x_n \leqslant y_n \leqslant \ldots y_1$ such that

$$\mu_a(z) = \begin{cases} 1 & \text{if } z \in [x_n, y_n] \\ l_i & \text{if } z \in [x_i, y_i] \setminus [x_{i+1}, y_{i+1}], \\ 0 & \text{if } z \in \mathbb{R} \setminus [x_1, y_1]. \end{cases}$$

Consider $b_1, \ldots, b_n \in \mathbb{F}$ defined as

$$\mu_{b_n}(z) = \begin{cases} 1 & \text{if } z \in [x_n, y_n], \\ 0 & \text{if } z \in \mathbb{R} \setminus [x_n, y_n], \end{cases}$$

$$\mu_{b_i}(z) = \begin{cases} 1 & \text{if } z = 0, \\ l_i & \text{if } z \in [x_i - x_{i+1}, y_i - y_{i+1}] \setminus \{0\}, \\ 0 & \text{if } z \in \mathbb{R} \setminus [x_i - x_{i+1}, y_i - y_{i+1}]. \end{cases}$$

for $i = 1, \ldots, n-2$. It can be easily verified that

$$a = \bigoplus_{i=1}^{n} b_i.$$

Therefore, it is clear that

$$a \odot u_E = \bigoplus_{i=1}^{n} b_i \odot u_E.$$

By additivity, (18) and (15),

$$\Psi(a \odot u_E) = \Psi(\bigoplus_{i=1}^{n} b_i \odot u_E) = \sum_{i=1}^{n} \Psi(b_i \odot u_E) = \sum_{i=1}^{n} \Phi(b_i \odot u_E) = \Phi(\bigoplus_{i=1}^{n} b_i \odot u_E) = \Phi(a \odot u_E).$$

Step 4 Our goal is to prove that

$$\Psi(a \odot u_E) = \Phi(a \odot u_E) \tag{20}$$

for every $E \in 2^N \setminus \{\emptyset\}$ and for every $a \in \mathbb{F}$.

Let $E \in 2^N \setminus \{\emptyset\}$ and let $a \in \mathbb{F}$. Since we have already proved (19) we can suppose that $\{\mu_a(z) \colon z \in \mathbb{R}\}$ is not finite. By continuity and (19), in order to prove (20) it is enough to show that there are games with the form $b \odot u_E$, where $\{\mu_b(z) \colon z \in \mathbb{R}\}$ is finite, arbitrarily close to the game $a \odot u_E$. To this end, it suffices to see that we can find fuzzy quantities b, with $\{\mu_b(z) \colon z \in \mathbb{R}\}$ finite, arbitrarily close to a.

Let $\epsilon > 0$. Let $[a]_0 = [r, s]$ and let $x_1, \ldots, x_n \in \mathbb{R}$ such that $r = x_1 < \ldots < x_n = s$, $x_i - x_{i-1} < \epsilon$ for every $i = 2, \ldots, n$ and $x_k \in core(a)$ for some $k \in \{1, \ldots, n\}$. Consider $b \in \mathbb{F}$ defined by

$$\mu_b(z) = \begin{cases} \mu_a(x_i) & \text{if } z \in [x_i, x_{i+1}) \text{ with } i < k, \\ 1 & \text{if } z = x_k, \\ \mu_a(x_i) & \text{if } z \in (x_{i-1}, x_i] \text{ with } i > k, \\ 0 & \text{if } z \in \mathbb{R} \setminus [r, s]. \end{cases}$$

Let us see that $d_\infty(a, b) \leqslant \epsilon$. To this end, it suffices to prove that $d_H([a]_t, [b]_t) < \epsilon$ for every $t \in (0, 1]$. Let $t \in (0, 1]$. Let $[a]_t = [p, q]$. It is clear that $p \leqslant x_k \leqslant q$. Let $h \in \{1, \ldots, k-1\}$ and $l \in \{k+1, \ldots, n\}$ be such that $p \in (x_h, x_{h+1}]$ and $q \in [x_{l-1}, x_l)$. We have that

$$[x_{h+1}, x_{l-1}] \subseteq [a]_t \subset (x_h, x_l) \tag{21}$$

It can be easily verified that

$$\mu_b(x_h) = \mu_a(x_h) < t,$$

$$\mu_b(x_{h+1}) = \mu_a(x_{h+1}) \geqslant t,$$

$$\mu_b(x_{l-1}) = \mu_a(x_{l-1}) \geqslant t,$$

$$\mu_b(x_l) = \mu_a(x_l) < t.$$

Hence, $[x_{h+1}, x_{l-1}] \subseteq [b]_t \subset (x_h, x_l)$. From these inclusions, (21) and the inequalities $x_{h+1} - x_h < \epsilon$ and $x_l - x_{l-1} < \epsilon$ it easily follows that $d_H([a]_t, [b]_t) < \epsilon$.

Step 5 Our goal is to prove that

$$\Psi(a \odot \delta_E) = \Phi(a \odot \delta_E) \tag{22}$$

for every $E \in 2^N \setminus \{\emptyset\}$ and for every $a \in \mathbb{F}$.

Let $E \in 2^N \setminus \{\emptyset\}$ and let $a \in \mathbb{F}$. By (1) and (2),

$$\delta_E = \sum_{\{F \in 2^N \setminus \{\emptyset\}:\, E \subseteq F\}} (-1)^{|F|-|E|} u_F,$$

whence

$$\delta_E + \sum_{\{F \in 2^N \setminus \{\emptyset\}:\, E \subseteq F\}} u_F = \sum_{\{F \in 2^N \setminus \{\emptyset\}:\, E \subseteq F, |F|-|E| \in 2\mathbb{Z}\}} 2u_F,$$

that is,

$$\delta_E(H) + \sum_{\{F \in 2^N \setminus \{\emptyset\}:\, E \subseteq F\}} u_F(H) = \sum_{\{F \in 2^N \setminus \{\emptyset\}:\, E \subseteq F, |F|-|E| \in 2\mathbb{Z}\}} 2u_F(H),$$

for every $H \subseteq N$. If we multiply by a and apply (3) we obtain

$$(a \odot \delta_E)(H) \oplus \bigoplus_{\{F \in 2^N \setminus \{\emptyset\}:\, E \subseteq F\}} (a \odot u_F)(H) = \bigoplus_{\{F \in 2^N \setminus \{\emptyset\}:\, E \subseteq F, |F|-|E| \in 2\mathbb{Z}\}} ((2 \odot a) \odot u_F)(H),$$

for every $H \subseteq N$. Hence,

$$(a \odot \delta_E) \oplus \bigoplus_{\{F \in 2^N \setminus \{\emptyset\}:\, E \subseteq F\}} (a \odot u_F) = \bigoplus_{\{F \in 2^N \setminus \{\emptyset\}:\, E \subseteq F, |F|-|E| \in 2\mathbb{Z}\}} ((2 \odot a) \odot u_F).$$

which, by additivity, leads to

$$\Psi_i(a \odot \delta_E) + \sum_{\{F \in 2^N \setminus \{\emptyset\}:\, E \subseteq F\}} \Psi_i(a \odot u_F) = \sum_{\{F \in 2^N \setminus \{\emptyset\}:\, E \subseteq F, |F|-|E| \in 2\mathbb{Z}\}} \Psi_i((2 \odot a) \odot u_F) \tag{23}$$

and

$$\Phi_i(a \odot \delta_E) + \sum_{\{F \in 2^N \setminus \{\emptyset\}:\, E \subseteq F\}} \Phi_i(a \odot u_F) = \sum_{\{F \in 2^N \setminus \{\emptyset\}:\, E \subseteq F, |F|-|E| \in 2\mathbb{Z}\}} \Phi_i((2 \odot a) \odot u_F) \tag{24}$$

for every $i \in N$. From (20), (23) and (24), it is concluded that $\Psi_i(a \odot \delta_E) = \Phi_i(a \odot \delta_E)$ for every $i \in N$. We have proved (22).

Step 6 We aim to prove that

$$\Psi(v) = \Phi(v)$$

for every $v \in \mathcal{FG}^N$.

Let $v \in \mathcal{FG}^N$. Notice that

$$v = \bigoplus_{E \in 2^N \setminus \{\emptyset\}} (v(E) \odot \delta_E).$$

By additivity and (22),

$$\Psi(v) = \sum_{E \in 2^N \setminus \{\emptyset\}} \Psi(v(E) \odot \delta_E) = \sum_{E \in 2^N \setminus \{\emptyset\}} \Phi(v(E) \odot \delta_E) = \Phi(v),$$

which completes the proof.

$\square$

5. Conclusions

In this paper, a new type of solution for games with fuzzy characteristic function is proposed. Unlike the solutions for these games that have so far been introduced in the literature, in which the players' payoffs are described by fuzzy numbers, we define a value in which the players' payoffs are given by real numbers. This value is obtained from the classic Shapley value and from the index for fuzzy numbers introduced by Yager [16].

Given a game with fuzzy characteristic function, there is a nice relation between the real payoffs given by our solution and the fuzzy payoffs given by the Hukuhara-Shapley value [17], since our value can be obtained by applying the Yager index to the Hukuhara-Shapley value. But, whereas the Hukuhara-Shapley value is defined only for some partificular games with characteristic function, the solution that we present is defined for any game with fuzzy characteristic function.

Future research could explore the use of different indices for fuzzy numbers in the study of cooperative situations with imprecise information.

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