Games with fuzzy authorization structure: a Shapley value.

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Abstract

A cooperative game consists of a set of players and a characteristic function which determines the maximal gain or minimal cost that every subset of players can achieve when they decide to cooperate, regardless of the actions that the other players take. It is often assumed that the players are free to participate in any coalition, but in some situations there are dependency relationships among the players that restrict their capacity to cooperate within some coalitions. Those relationships must be taken into account if we want to distribute the profits fairly. In this respect, several models have been proposed in literature. In all of them dependency relationships are considered to be complete, in the sense that either a player is allowed to fully cooperate within a coalition or they cannot cooperate at all. Nevertheless, in some situations it is possible to consider another option: that a player has a degree of freedom to cooperate within a coalition. A model for those situations is presented.

Keywords: cooperative game, fuzzy coalition, Shapley value, fuzzy authorization structure

1. Introduction

In a general way, game theory studies cooperation and conflict models, using mathematical methods. This paper is about cooperative game theory. A cooperative game over a finite set of players is defined as a function establishing the worth of each coalition. Given a cooperative game, the main problem that arises is how to assign a payoff to each player in a reasonable way. In this setting, it is often assumed that all of the players are socially identical. In real life, however, political or economic circumstances may impose certain restraints on coalition formation. This idea has led several authors to develop models of games in which relationships among players must be taken into account. Depending on the nature of such relationships, different structures in the set of players have been considered. Myerson (1977) studied games in which communication between players is restricted. He considered graphs to model those restraints. Subsequently, different kinds of limitations on cooperation among

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players have been studied, and various structures have been used for that, like convex geometries (see Bilbao (1998)), matroids (see Bilbao *et al.* (2001)), antimatroids (see Algaba *et al.* (2004)) or augmenting systems (see Bilbao and Ordoñez (2009)). A particularly interesting case of limited cooperation arises when we consider veto relationships between players. In this regard, Gilles *et al.* (1992) modeled situations in which a hierarchical structure imposes some constraints on the behavior of the players in the game. They introduce games with permission structure, that consist of a set of players, a cooperative game and a mapping that assigns to every player a subset of direct subordinates. In this respect, the power of a player over a subordinate can be of different kinds. In the conjunctive approach it is assumed that each player needs the permission of all his superiors, whereas in the disjunctive approach, van den Brink (1997), the permission of any of those superiors will suffice. In each case they consider a new characteristic function, which collects the information given by both the original characteristic function and the permission structure, and that allows them to define a value for games on conjunctive (or respectively disjunctive) permission structures. They provide intuitive characterizations for each case, showing in this way that the values obtained are reasonable. Subsequently, Derks and Peters (1993) generalized those approaches by considering the so-called restrictions. Although their model is more general, the axiomatization given is not as intuitive and straightforward as those given in Gilles *et al.* (1992) and van den Brink (1997) for permission structures.

In all of the models presented so far the dependency relationships are complete, in the sense that either a coalition can veto a player or it does not have any authority over the player. Our aim in this paper will be to provide a new model for games in which players are subject to certain restraints when cooperating within a coalition. We will consider the possibility that such restraints are partial, which will make this model more general than those referenced above.

The paper is organized as follows. In Section 2 we recall some basic definitions and properties about the Shapley value, fuzzy sets and the Choquet integral. In Section 3, we introduce fuzzy authorization structures, that will be used to model situations in which some players depend partially on other players. Then, for each game with fuzzy authorization structure, a new characteristic function, that collects the information from both the game and the structure, is be defined. This characteristic function will allow us to define a Shapley value for games with fuzzy authorization structure. A characterization of this value is given in Section 4. An example is described as well. Finally, in Section 5 some conclusions are given.

2. Preliminaries

2.1. Cooperative TU-games

We recall some concepts regarding cooperative games. A *transferable utility cooperative game* or *TU-game* is a pair (N, v) where N is a finite set and $v : 2^N \to \mathbb{R}$ is a function with $v(\emptyset) = 0$. The elements of $N = \{1, ..., n\}$ are called players, and the subsets of N coalitions. Given a coalition E, $v(E)$ is the worth of E, and it is interpreted as the maximal gain or minimal cost that the players in this coalition can achieve by themselves against the best offensive threat by the complementary coalition. Frequently, a TU-game (N, v) is identified with the function v. A game v is monotone if for every $F \subseteq E \subseteq N$, it holds that $v(F) \leq v(E)$. The family of games with set of players N is denoted by $\mathcal{G}^N$. This set is a $(2^n - 1)$-dimensional real vector space. One basis of this space is the collection $\{u_F : F \subseteq N,\ F \neq \emptyset\}$ where for a nonempty coalition F the unanimity game u_F is defined by

$$u_F(E) = \begin{cases} 1 & \text{if } F \subseteq E, \\ 0 & \text{otherwise.} \end{cases}$$

Every game $v \in \mathcal{G}^N$ can be written as a linear combination of them,

$$v = \sum_{\{E \in 2^N : E \neq \emptyset\}} \triangle_v(E)\, u_E$$

where $\triangle_v(E)$ is the dividend of the coalition E in the game v.

A solution or value on $\mathcal{G}^N$ is a function $\psi : \mathcal{G}^N \to \mathbb{R}^N$ that assigns to each game a vector $(\psi_1(v), \ldots, \psi_n(v))$ where the real number $\psi_i(v)$ is the payoff of the player i in the game (N, v).

Many values have been defined in literature for different families of games. The *Shapley value* (see Shapley (1953)) $\phi(v) \in \mathbb{R}^N$ of a game $v \in \mathcal{G}^N$ is a weighted average of the marginal contributions of each player to the coalitions and formally it is defined by

$$\phi_i(v) = \sum_{\{E \subseteq N : i \in E\}} p_E \left(v(E) - v(E \setminus \{i\}) \right), \quad \text{for all } i \in N,$$

where

$$p_E = \frac{(n - |E|)!\,(|E| - 1)!}{n!}$$

and $|E|$ denotes the cardinality of E.

Some desirable properties for a value $\psi : \mathcal{G}^N \to \mathbb{R}^N$ are the following:

Efficiency: $\sum_{i \in N} \psi_i(v) = v(N)$ for all $v \in \mathcal{G}^N$.

Additivity: $\psi(v_1 + v_2) = \psi(v_1) + \psi(v_2)$ for all $v_1, v_2 \in \mathcal{G}^N$.

Null player property: A player $i \in N$ is a null player in $v \in \mathcal{G}^N$ if $v(E) = v(E \setminus \{i\})$ for all $E \subseteq N$. If $i \in N$ is null player in $v \in \mathcal{G}^N$ then $\psi_i(v) = 0$.

Necessary player property: A player i is a necessary player in $v \in \mathcal{G}^N$ if $v(E) = 0$ for all $E \subseteq N \setminus \{i\}$. If i is a necessary player in a monotone game $v \in \mathcal{G}^N$, then $\psi_i(v) \geq \psi_j(v)$ for all $j \in N$.

These four properties characterize the Shapley value (see van den Brink (1994)).

2.2. Fuzzy sets

Fuzzy subsets of a finite set were described by Zadeh (1965). A *fuzzy subset* of N is a mapping $e : N \longrightarrow [0,1]$ where e assigns to $i \in N$ a degree of membership. A fuzzy subset of N is identified with a vector in $[0,1]^N$. Given $e \in [0,1]^N$ the *support* of e is the set *supp* $(e) = \{i \in N : e_i > 0\}$ and the *image* of e is the set $im(e) = \{e_i : i \in N\}$. If $t \in [0,1]$ the *t-level set* of e is $[e]_t = \{i \in N : e_i \geqslant t\}$. Given $e, f \in [0,1]^N$ standard union and intersection are defined, respectively, by $(e \cap f)_i = \min\{e_i, f_i\}$, $(e \cup f)_i = \max\{e_i, f_i\}$ for all $i = 1, \ldots, n$. The fuzzy sets $e, f \in [0,1]^N$ are called *comonotone* if $(e_i - e_j)(f_i - f_j) \geq 0$ for all $i, j \in N$.

Regarding cooperative game theory, Aubin (1981) defined a *fuzzy coalition* in N as a fuzzy subset e of N where, for all $i \in N$, the number $e_i \in [0,1]$ is regarded as the degree of participation of player i in e. Every coalition $E \subseteq N$ can be identified with the fuzzy coalition $\mathbf{1}^E \in [0,1]^N$ defined by $\mathbf{1}^E_i = 1$ if $i \in E$ and $\mathbf{1}^E_i = 0$ otherwise.

Different Shapley values for games with fuzzy coalitions were studied in Butnariu (1980) and Tsurumi *et al.* (2001).

2.3. The Choquet integral

The Choquet integral was introduced in Choquet (1953). It was originally defined for capacities. Later on, Schmeidler (1986) studied this integral for all set functions. Given $v : 2^N \to \mathbb{R}$ and $e \in [0,1]^N$, the *Choquet integral* of e with respect to v is defined as

$$\int e\, dv = \sum_{p=1}^{q} (s_p - s_{p-1})\, v\left([e]_{s_p}\right), \tag{1}$$

where $im\,(e) \cup \{0\} = \{s_p\}_{p=0}^{q}$ and $0 = s_0 < s_1 < \ldots < s_q$.

It will be useful, when dealing with several fuzzy coalitions, to rewrite the expression above using a superset of $im(e)$, that is,

$$\int e\, dv = \sum_{l=1}^{m} (t_l - t_{l-1})\, v\left([e]_{t_l}\right), \tag{2}$$

where $im(e) \subseteq \{t_l\}_{l=0}^{m}$ and $0 = t_0 < t_1 < \ldots < t_m$.

The following properties of the Choquet integral are known:

(C1) $\int \mathbf{1}^E\, dv = v\,(E)$, for all $E \subseteq N$.

(C2) $\int te\, dv = t \int e\, dv$, for all $t \in [0,1]$.

(C3) $\int e\, dv \leq \int f\, dv$, whenever $e \leq f$ and v is monotone.

(C4) $\int e\, d(cv) = c \int e\, dv$, for $c \in \mathbb{R}$.

(C5) $\int e\, d\,(v_1 + v_2) = \int e\, dv_1 + \int e\, dv_2$.

(C6) $\int (e + f)\; dv = \int e\, dv + \int f\, dv$, when $e + f \leq \mathbf{1}^N$ and e, f are comonotone.

3. Methodology

We aim to present a model of games in which the ability of players to cooperate within a coalition can be limited. To do this, firstly we introduce the structure that will allow us to deal will that kind of dependency relationships. Then we will incorporate the information from the structure with the information from the game. Finally, a value will be proposed.

3.1. Fuzzy authorization structures

The idea is that a set of players may have the power to restrict the ability to cooperate of the rest of the players. So, given a coalition, we will consider the capacity of their players to cooperate within the coalition.

Definition 1. *A fuzzy authorization operator on N is a function $a : 2^N \to [0,1]^N$ that satisfies the following requirements:*

(A1) $a(E) \leqslant \mathbf{1}^E$ *for any* $E \subseteq N$,

(A2) *If* $E \subseteq F$ *then* $a(E) \leqslant a(F)$.

The pair (N, a) will be called a fuzzy authorization structure. The set of fuzzy authorization operators on N will be denoted by $\mathcal{FA}^N$.

Given $a \in \mathcal{FA}^N$, we will denote

$$im(a) = \bigcup_{E \subseteq N} im(a(E)).$$

Suppose that a is a fuzzy authorization operator and v is a game on N. Then, given $E \subseteq N$ and $i \in N$, we will interpret $a_i(E)$ as the proportion of the whole operating capacity of player i that he is allowed to use within coalition E. Or, equivalently, $1 - a_i(E)$ is the fraction of the operating capacity of player i that is under control of coalition $N \setminus E$.

3.2. The restricted game

The restricted game will be the tool used to amalgamate the information from the game and the information from the fuzzy authorization structure.

Definition 2. *Let $v \in \mathcal{G}^N$ and $a \in \mathcal{FA}^N$. The restriction of v on a is the game $v^a \in \mathcal{G}^N$ defined as*

$$v^a(E) = \int a(E)\, dv \quad \text{for all } E \subseteq N.$$

Remark 3. *Using* (2)*, the restriction of* v *on* a *can be written as*

$$v^a(E) = \sum_{l=1}^{m} (t_l - t_{l-1})\, v\left([a(E)]_{t_l}\right) \qquad \textit{for all } E \subseteq N, \tag{3}$$

where $im(a) \subseteq \{t_l\}_{l=0}^{m}$ *and* $0 = t_0 < t_1 < \ldots < t_m$*.*

3.3. A Shapley value for games with fuzzy authorization structure

We apply the Shapley value to the restricted game in order to define a value for games with fuzzy authorization structure.

Definition 4. *The Shapley fuzzy authorization value on the set of players* N *is the allocation rule* $\varphi^N : \mathcal{G}^N \times \mathcal{FA}^N \to \mathbb{R}^N$ *given by*

$$\varphi^N(v, a) = \phi(v^a) \qquad \textit{for all } v \in \mathcal{G}^N \textit{ and } a \in \mathcal{FA}^N.$$

We will write φ (rather than φ^N) and say just *Shapley fuzzy authorization value* as long as there is no possibility of confusion.

4. Results

4.1. A characterization of the Shapley fuzzy authorization value

We aim to prove that the Shapley fuzzy authorization value has good properties with respect to both the game and the fuzzy authorization structure. To do this, we will consider the properties described below.

If $a \in \mathcal{FA}^N$ with $im(a(N)) \subseteq \{0, 1\}$, which means that when the grand coalition is formed each player can use either his full capacity or no capacity at all, the set $supp(a(N))$ can be seen as a carrier (see Shapley (1953)). In that case, we can consider the following efficiency property:

Efficiency. *For every* $v \in \mathcal{G}^N$ *and* $a \in \mathcal{FA}^N$ *with* $im(a(N)) \subseteq \{0, 1\}$ *it holds that*

$$\sum_{i \in N} \psi_i(v, a) = v(supp(a(N))).$$

Additivity is a well-known property of the Shapley value. In our setting, it is as follows:

Additivity. *For every* $v, w \in \mathcal{G}^N$ *and* $a \in \mathcal{FA}^N$ *it holds that*

$$\psi(v+w, a) = \psi(v, a) + \psi(w, a).$$

Given $a \in \mathcal{FA}^N$ and $i, j \in N$, player j depends partially on i according to a if there exists $E \subseteq N$ such that $a_j(E) > a_j(E \setminus \{i\})$. Given $v \in \mathcal{G}^N$ and $a \in \mathcal{FA}^N$, a player $i \in N$ is an irrelevant player in (v, a) if for every $j \in N$ such that j depends partially on i according to a it holds that j is a null player in v. Notice that a null player in v is not necessarily an irrelevant player in (v, a).The null player property is generalized now in the following way:

Irrelevant player. *For every* $v \in \mathcal{G}^N$*,* $a \in \mathcal{FA}^N$ *and* $i \in N$ *such that* i *is an irrelevant player in* (v, a) *it holds that*

$$\psi_i(v, a) = 0.$$

Note that if a is the trivial authorization structure (that is, $a(E) = \mathbf{1}^E$ for every $E \subseteq N$) then the irrelevant players in (v, a) are just the null players in v. From this point of view, the irrelevant player property is a generalization of the null player property.

Given $a \in \mathcal{FA}^N$ and $i, j \in N$, we say that i has veto power over j according to a if $a_j(N \setminus \{i\}) = 0$. Players who have veto power over a necessary player will expect to be treated as another necessary player. This leads us to consider the following property:

Veto power over a necessary player. *For every monotone game* $v \in \mathcal{G}^N$*,* $a \in \mathcal{FA}^N$ *and* $i, j \in N$ *such that* j *is a necessary player in* v *and* i *has veto power over* j *according to* a *it holds that, for all* $k \in N$*,*

$$\psi_i(v, a) \geqslant \psi_k(v, a).$$

Note that if a is the trivial authorization structure then the players with veto power over a necessary player according to a are just the necessary players for the game. So the property of veto power over a necessary player is a generalization of the necessary player property.

Let $a \in \mathcal{FA}^N$, $\emptyset \neq T \subseteq N$ and $i \in T$. The fuzzy authorization operator a describes a situation in which some players may need the permission from other players in order to use a fraction of their operating capacity. In such situation, if coalition T is formed, player i will be allowed to use a proportion of his capacity equal to $a_i(T)$. Now suppose that somehow the players in T acquire the power to authorize player i to use a bigger proportion of his capacity, say $s \in (a_i(T), 1]$. The new situation would be described by the fuzzy authorization operator $a^{T,i,s}$ defined as

$$a^{T,i,s}(E) = \begin{cases} a(E) \cup (s \cdot \mathbf{1}^{\{i\}}) & \text{if } T \subseteq E, \\ a(E) & \text{otherwise.} \end{cases}$$

In this case, it would be reasonable to expect that all the players in T will benefit equally from the change. This is what the following property states:

Fairness. *For every $v \in \mathcal{G}^N$, $a \in \mathcal{FA}^N$, $T \in 2^N \setminus \{\emptyset\}$, $i \in T$ and $s \in [0, 1]$ it holds that*

$$\psi_j\left(v, a^{T,i,s}\right) - \psi_j(v, a) = \psi_i\left(v, a^{T,i,s}\right) - \psi_i(v, a) \quad \text{for all } j \in T.$$

Notice that if $s \leqslant a_i(T)$ then $a^{T,i,s} = a$. Therefore, the expression above is non trivial only if $s \in (a_i(T), 1]$.

Two fuzzy authorization operators a and a' are called *comonotone* if $a(E)$ and $a'(E)$ are comonotone for every $E \subseteq N$.

If we suppose that each player has an amount of a certain resource and that the profit that can be made from those resources is proportional to the quantities, we could consider a property like the following, that establishes, in a way, linearity between the authorization operator and the payoff:

Comonotonicity. *For every $v \in \mathcal{G}^N$, $a, a' \in \mathcal{FA}^N$ comonotone and $t \in [0, 1]$ it holds that*

$$\psi(v, ta + (1-t)a') = t\,\psi(v, a) + (1-t)\,\psi(v, a').$$

Theorem 5. *An allocation rule $\psi : \mathcal{G}^N \times \mathcal{FA}^N \to \mathbb{R}^N$ is equal to the Shapley fuzzy authorization value if and only if it satisfies the properties of efficiency, additivity, irrelevant player, veto power over a necessary player, fairness and comonotonicity.*

Proof. Firstly it will be proved that φ satisfies the properties in the theorem.

EFFICIENCY. Let $v \in \mathcal{G}^N$ and $a \in \mathcal{FA}^N$ with $im(a(N)) \subseteq \{0,1\}$. It holds that

$$\sum_{i\in N} \varphi_i(v,a) = \sum_{i\in N} \phi_i(v^a) = v^a(N) = \int a(N)\,dv = \int \mathbf{1}^{supp(a(N))}\,dv = v(supp(a(N))),$$

where we have used the efficiency of the Shapley value and property (C1).

ADDITIVITY. Let $v, w \in \mathcal{G}^N$ and $a \in \mathcal{FA}^N$. From (C5) it follows that for every coalition E it holds that

$$(v+w)^a(E) = \int a(E)\,d(v+w) = \int a(E)\,dv + \int a(E)\,dw = v^a(E) + w^a(E).$$

Therefore, $(v+w)^a = v^a + w^a$. From this fact and the additivity of the Shapley value we get

$$\begin{aligned}\varphi(v+w,a) &= \phi\left((v+w)^a\right) = \phi(v^a + w^a)\\ &= \phi(v^a) + \phi(w^a) = \varphi(v,a) + \varphi(w,a).\end{aligned}$$

IRRELEVANT PLAYER. Let $v \in \mathcal{G}^N$, $a \in \mathcal{FA}^N$ and $i \in N$ an irrelevant player in (v,a). We must show that $\varphi_i(v,a) = 0$. Taking into consideration that $\varphi_i(v,a) = \phi_i(v^a)$ and that the Shapley value satisfies the null player property, it will suffice to prove that i is a null player in v^a. Let $E \subseteq N$. Take $\{t_l\}_{l=0}^m \supseteq im(a)$ with $0 < t_0 < t_1 < \ldots < t_m$. From (3) we obtain that

$$v^a(E) - v^a(E \setminus \{i\}) = \sum_{l=1}^{m} (t_l - t_{l-1}) \left(v\left([a(E)]_{t_l}\right) - v\left([a(E\setminus\{i\})]_{t_l}\right)\right). \tag{4}$$

For any $l = 1,\ldots,m$, it is clear from (A2) that $[a(E\setminus\{i\})]_{t_l} \subseteq [a(E)]_{t_l}$. Besides, since i is an irrelevant player in (v,a), it follows that the players in $[a(E)]_{t_l} \setminus [a(E\setminus\{i\})]_{t_l}$ are null players in v. Hence, it holds that

$$v\left([a(E)]_{t_l}\right) = v\left([a(E\setminus\{i\})]_{t_l}\right) \quad \text{for all } l = 1,\ldots,m. \tag{5}$$

From (4) and (5) we obtain $v^a(E) = v^a(E \setminus \{i\})$. Therefore, i is a null player in v^a.

VETO POWER OVER A NECESSARY PLAYER. Let $v \in \mathcal{G}^N$, $a \in \mathcal{FA}^N$ and $i, j \in N$ be such that v is monotone, j is a necessary player in v and i has veto power over j according to a. We want to prove that $\varphi_i(v,a) \geqslant \varphi_k(v,a)$ for all $k \in N$. Firstly, from the monotonicity of v and (A2) it is easy to derive that v^a is monotone. Then, since $\varphi(v,a) = \phi(v^a)$, v^a is monotone and the Shapley value satisfies the necessary player property, it will be enough to show that i is a necessary player in v^a. Given $E \subseteq N \setminus \{i\}$ we get

$$v^a(E) = \int a(E)\, dv = \sum_{l=1}^{m} (t_l - t_{l-1})\, v\left([a(E)]_{t_l}\right),$$

where $im(a(E)) \cup \{0\} = \{t_l\}_{l=0}^m$ and $0 = t_0 < t_1 < \ldots < t_m$. Using (A2) and the fact that i has veto power over j according to a we can write $a_j(E) \leq a_j\left(N \setminus \{i\}\right) = 0$. Therefore, $j \notin [a(E)]_{t_l}$ for all $l = 1, ..., m$. Since j is a necessary player in v it holds that $v\left([a(E)]_{t_l}\right) = 0$ for all $l = 1, ..., m$. Consequently, $v^a(E) = 0$.

Fairness. Let $v \in \mathcal{G}^N$, $a \in \mathcal{FA}^N$, $T \in 2^N \setminus \{\emptyset\}$, $i \in T$ and $s \in [0,1]$. Take $j \in T$. We must prove that

$$\varphi_j\left(v, a^{T,i,s}\right) - \varphi_j(v,a) = \varphi_i\left(v, a^{T,i,s}\right) - \varphi_i(v,a). \tag{6}$$

Using the definition of the Shapley value, we can write

$$\begin{aligned}\varphi_j\left(v, a^{T,i,s}\right) &= \phi_j\left(v^{a^{T,i,s}}\right) = \sum_{\{E \subseteq N : j \in E\}} p_E \left[v^{a^{T,i,s}}(E) - v^{a^{T,i,s}}\left(E \setminus \{j\}\right)\right] \\ &= \sum_{\{E \subseteq N : j \in E\}} p_E \left[\int a^{T,i,s}(E)\, dv - \int a^{T,i,s}\left(E \setminus \{j\}\right)\, dv\right]. \end{aligned} \tag{7}$$

Similarly,

$$\begin{aligned}\varphi_j\left(v, a\right) &= \phi_j\left(v^{a}\right) = \sum_{\{E \subseteq N : j \in E\}} p_E \left[v^{a}(E) - v^{a}\left(E \setminus \{j\}\right)\right] \\ &= \sum_{\{E \subseteq N : j \in E\}} p_E \left[\int a(E)\, dv - \int a\left(E \setminus \{j\}\right)\, dv\right]. \end{aligned} \tag{8}$$

Taking into account that $a^{T,i,s}(F) = a(F)$ if $T \nsubseteq F$, we obtain, subtracting (8) from (7), that

$$\varphi_j\left(v, a^{T,i,s}\right) - \varphi_j\left(v,a\right) = \sum_{\{E \subseteq N : T \subseteq E\}} p_E \left[\int a^{T,i,s}(E)\, dv - \int a(E)\, dv\right].$$

Finally, (6) follows from the fact that the last expression does not depend on the player $j \in T$ chosen.

COMONOTONICITY. Let $v \in \mathcal{G}^N$, $a, a' \in \mathcal{FA}^N$ comonotone and $t \in [0,1]$. It is clear that $ta, (1-t)a' \in \mathcal{FA}^N$ and are also comonotone. Given a coalition E, we obtain, using (C6) and (C2), that

$$\begin{aligned} v^{ta+(1-t)a'}(E) &= \int t\, a(E) + (1-t)\, a'(E)\, dv \\ &= t \int a(E)\, dv + (1-t) \int a'(E)\, dv = tv^a(E) + (1-t)v^{a'}(E). \end{aligned}$$

Hence, $v^{ta+(1-t)a'} = tv^a + (1-t)v^{a'}$. From this equality and the linearity of the Shapley value we get

$$\begin{aligned} \varphi(v, ta + (1-t)a') &= \phi\left(v^{ta+(1-t)a'}\right) = \phi\left(tv^a + (1-t)v^{a'}\right) \\ &= t\phi\left(v^a\right) + (1-t)\phi\left(v^{a'}\right) \\ &= t\varphi(v,a) + (1-t)\varphi(v,a'). \end{aligned}$$

We have proved that the Shapley fuzzy authorization value satisfies all of the properties mentioned in the theorem. Now we will show that such properties uniquely determine the Shapley fuzzy authorization value.

Let $\psi : \mathcal{G}^N \times \mathcal{FA}^N \to \mathbb{R}^N$ be such that it satisfies the properties of efficiency, additivity, irrelevant player, veto power over a necessary player, fairness and comonotonicity. We must prove that $\psi = \varphi$.

Firstly we will show that

$$\psi(c\, u_E, a) = \varphi(c\, u_E, a) \text{ for all } E \in 2^N \setminus \{\emptyset\},\ c > 0 \text{ and } a \in \mathcal{FA}^N \text{ with } im(a) \subseteq \{0,1\}. \quad (9)$$

Given $a \in \mathcal{FA}^N$ we define

$$m(a) = \sum_{F \subseteq N} |supp\,(a(F))|\,.$$

We will prove (9) by induction on $m(a)$.

BASE CASE. Let $a \in \mathcal{FA}^N$ be such that $m(a) = 0$. It holds that $a(F) = 0$ for all $F \subseteq N$. It is clear that all of the players are irrelevant players in $(c\,u_E, a)$. Since ψ satisfies the irrelevant player property, it follows that $\psi(cu_E, a) = 0$. We conclude that $\psi(cu_E, a) = \varphi(cu_E, a)$.

INDUCTIVE STEP. Let $a \in \mathcal{FA}^N$ be such that $im(a) \subseteq \{0, 1\}$ and $m(a) > 0$. We consider the three following sets:

$$\begin{aligned} H_1 &= \{i \in N : i \text{ is an irrelevant player in } (c\,u_E, a)\}\,, \\ H_2 &= \{i \in N : \exists j \in E \text{ such that } i \text{ has veto power over } j \text{ according to } a\}\,, \\ H_3 &= N \setminus (H_1 \cup H_2). \end{aligned}$$

Note that $E \subseteq H_2$, since any player has veto power over himself.

Since φ and ψ satisfy the irrelevant player property it holds that

$$\varphi_k(c\,u_E, a) = 0 \quad \text{for all } k \in H_1, \tag{10}$$

$$\psi_k(c\,u_E, a) = 0 \quad \text{for all } k \in H_1. \tag{11}$$

From the property of veto power over a necessary player we can derive that there exist $b, b' \in \mathbb{R}$ such that

$$\varphi_k(c\,u_E, a) = b \quad \text{for all } k \in H_2. \tag{12}$$

$$\psi_k(c\,u_E, a) = b' \quad \text{for all } k \in H_2. \tag{13}$$

Now suppose that $i \in H_3$. Since $i \notin H_1$ there must exist $j \in E$ such that j depends partially on i according to a. This means that there exists $F \subseteq N$ such that $a_j(F) > a_j(F \setminus \{i\})$. Since $im(a) \subseteq \{0, 1\}$ it holds that $a_j(F \setminus \{i\}) = 0$. Notice that $F \neq N$, since otherwise i would have veto power over j and this would contradict $i \notin H_2$. Take T minimal such that $T \subseteq F$ and $j \in supp(a(T))$. It is clear that $i \in T$. We define the fuzzy authorization structure $\tilde{a}$ given by

$$\tilde{a}(S) = \begin{cases} a(T) - \mathbf{1}^{\{j\}} & \text{if } S = T, \\ a(S) & \text{otherwise.} \end{cases}$$

It is straightforward to check that $\tilde{a} \in \mathcal{FA}^N$ and $\tilde{a}^{T,j,1} = a$. Since φ and ψ satisfy the property of fairness, we obtain that

$$\begin{aligned} \varphi_i\,(c\,u_E, a) - \varphi_i(c\,u_E, \tilde{a}) &= \varphi_j\,(c\,u_E, a) - \varphi_j(c\,u_E, \tilde{a}), \\ \psi_i\,(c\,u_E, a) - \psi_i(c\,u_E, \tilde{a}) &= \psi_j\,(c\,u_E, a) - \psi_j(c\,u_E, \tilde{a}). \end{aligned}$$

Taking into account that $E \subseteq H_2$ we get from (12) and (13) that $\varphi_j(c\,u_E, a) = b$ and $\psi_j(c\,u_E, a) = b'$. If we substitute these values into the equalities above we have

$$\varphi_i\,(c\,u_E, a) = b + \varphi_i(c\,u_E, \tilde{a}) - \varphi_j(c\,u_E, \tilde{a}), \tag{14}$$

$$\psi_i\,(c\,u_E, a) = b' + \psi_i(c\,u_E, \tilde{a}) - \psi_j(c\,u_E, \tilde{a}). \tag{15}$$

As $m(\tilde{a}) = m(a) - 1$, it follows by induction hypothesis that $\varphi(c\,u_E, \tilde{a}) = \psi(c\,u_E, \tilde{a})$. From this equality, (14) and (15) it follows that

$$\varphi_i\,(c\,u_E, a) - \psi_i\,(c\,u_E, a) = b - b'.$$

So we have proved that

$$\varphi_k\,(c\,u_E, a) - \psi_k\,(c\,u_E, a) = b - b' \quad \text{for all } k \in H_3. \tag{16}$$

Now, on the one hand, from (10), (11), (12), (13) and (16) we get that

$$\sum_{k \in N} \varphi_k(c\,u_E, a) - \sum_{k \in N} \psi_k(c\,u_E, a) = (b - b')|H_2 \cup H_3|,$$

and, on the other hand, as φ and ψ are efficient and $im(a(N)) \subseteq \{0,1\}$, we know that

$$\sum_{k\in N} \varphi_k(c\,u_E, a) = \sum_{k\in N} \psi_k(c\,u_E, a).$$

Therefore, it follows that $(b-b')|H_2 \cup H_3| = 0$. Since $\emptyset \neq E \subseteq H_2$ it holds that $b = b'$, which leads to $\psi(c\,u_E, a) = \varphi(c\,u_E, a)$.

We have proved (9). Now we will show that

$$\psi(c\,u_E, a) = \varphi(c\,u_E, a) \text{ for all } E \in 2^N \setminus \{\emptyset\},\ c > 0 \text{ and } a \in \mathcal{FA}^N. \tag{17}$$

For every fuzzy authorization operator $a \in \mathcal{FA}^N$ we consider

$$z(a) = |im(a) \setminus \{0,1\}|\,.$$

We will prove (17) by strong induction on $z(a)$.

Base case. It has already been proved.

Inductive step. Let $a \in \mathcal{FA}^N$ be such that $z(a) > 0$. Take $t \in im(a) \setminus \{0,1\}$. Consider the fuzzy authorization operators $a^{[0,t]}$ and $a^{[t,1]}$ defined by

$$a_i^{[0,t]}(F) = \min\left(1, \frac{a_i(F)}{t}\right),$$
$$a_i^{[t,1]}(F) = \max\left(0, \frac{a_i(F)-t}{1-t}\right),$$

for every $F \subseteq N$ and $i \in N$. Moreover it holds that

$$ta^{[0,t]} + (1-t)a^{[t,1]} = a.$$

Let us see that $a^{[0,t]}$ and $a^{[t,1]}$ are comonotone. We have to prove that for every $F \subseteq N$ and $i, j \in N$ it holds that

$$\left[a_i^{[0,t]}(F) - a_j^{[0,t]}(F)\right]\left[a_i^{[t,1]}(F) - a_j^{[t,1]}(F)\right] \geq 0.$$

It suffices to consider these cases:

1) If $a_i(F), a_j(F) \geq t$ then $a_i^{[0,t]}(F) = a_j^{[0,t]}(F) = 1$.

2) If $a_i(F), a_j(F) \leq t$ then $a_i^{[t,1]}(F) = a_j^{[t,1]}(F) = 0$.

3) If $a_i(F) \geq t > a_j(F)$ then $a_i^{[0,t]}(F) = 1 > a_j^{[0,t]}(F)$ and also $a_i^{[t,1]}(F) \geq 0 = a_j^{[t,1]}(F)$.

Now, since φ and ψ satisfy comonotonicity it holds that

$$\psi(c\, u_E, a) = t\, \psi(c\, u_E, a^{[0,t]}) + (1-t)\, \psi(c\, u_E, a^{[t,1]}).$$

$$\varphi(c\, u_E, a) = t\, \varphi(c\, u_E, a^{[0,t]}) + (1-t)\, \varphi(c\, u_E, a^{[t,1]}),$$

Since $z(a^{[0,t]}), z(a^{[t,1]}) < z(a)$ it follows by induction hypothesis that $\psi(c\, u_E, a^{[0,t]}) = \varphi(c\, u_E, a^{[0,t]})$ and $\psi(c\, u_E, a^{[t,1]}) = \varphi(c\, u_E, a^{[t,1]})$. Hence

$$\psi(c\, u_E, a) = t\, \varphi(c\, u_E, a^{[0,t]}) + (1-t)\, \varphi(c\, u_E, a^{[t,1]}) = \varphi(c\, u_E, a).$$

So we have proved (17). Now, take $E \in 2^N \setminus \{\emptyset\}$, $a \in \mathcal{FA}^N$ and $c < 0$. Using additivity and the irrelevant player property we have that

$$\psi(c\, u_E, a) + \psi(-c\, u_E, a) = 0,$$

$$\varphi(c\, u_E, a) + \varphi(-c\, u_E, a) = 0,$$

and, hence,

$$\psi(c\, u_E, a) = -\psi(-c\, u_E, a) = -\varphi(-c\, u_E, a) = \varphi(c\, u_E, a).$$

We have seen that $\psi(c\, u_E, a) = \varphi(c\, u_E, a)$ for all $c \in \mathbb{R}$, $E \in 2^N \setminus \{\emptyset\}$ and $a \in \mathcal{FA}^N$. Finally, take $v \in \mathcal{G}^N$ and $a \in \mathcal{FA}^N$. It holds that

$$\begin{aligned}
\psi(v, a) &= \psi\left(\sum_{\{E \subseteq N : E \neq \emptyset\}} \Delta_v(E)\, u_E, a\right) = \sum_{\{E \subseteq N : E \neq \emptyset\}} \psi\left(\Delta_v(E)\, u_E, a\right) \\
&= \sum_{\{E \subseteq N : E \neq \emptyset\}} \varphi\left(\Delta_v(E)\, u_E, a\right) = \varphi\left(\sum_{\{E \subseteq N : E \neq \emptyset\}} \Delta_v(E)\, u_E, a\right) = \varphi(v, a).
\end{aligned}$$

□

4.2. Example

Imagine the following situation. A consumer electronics company wants to make a new product. To do this, the company needs several components from various suppliers. We will focus on three of those suppliers. For $i = 1, 2, 3$ supplier i produces component i. The company has signed an agreement with the three suppliers that establishes the following:

- The company will pay i dollars to supplier i for every unit of component i delivered before the deadline.
- The company will pay a total of $2(i+j)$ dollars to suppliers i and j for every pair made up of a unit of component i and a unit of component j delivered before the deadline.
- The company will pay a total of 20 dollars to the three suppliers for every set made up of a unit of each component delivered before the deadline.

Each supplier has calculated that it would be able to produce one million units of the corresponding component before the deadline. This situation can be modeled with a cooperative game $(\{1, 2, 3\}, v)$, where, for every $E \subseteq \{1, 2, 3\}$, $v(E)$ is the revenue (in millions) obtained by coalition E.

$$v(\{1\}) = 1, \qquad v(\{2\}) = 2, \qquad v(\{3\}) = 3,$$
$$v(\{1,2\}) = 6, \qquad v(\{1,3\}) = 8, \qquad v(\{2,3\}) = 10, \quad v(\{1,2,3\}) = 20.$$

Imagine now the following. In order to produce component 3, supplier 3 makes use of a technology developed and patented by supplier 1. Supplier 3 has calculated that if they decided to produce component 3 without using that technology, they would only be able to produce seven hundred thousand units before the deadline. They also use a technology patented by supplier 2, and the speed of production would drop 50 percent if they did without that technology. Finally, it they decided to do without both technologies, they could only produce four hundred thousand units of component 3 before the deadline.

For every $E \subseteq \{1, 2, 3\}$ and $i \in \{1, 2, 3\}$, $a_i(E)$ indicates the fraction of its maximal productive capacity that player i can reach if it does not have the authorization of the players in $\{1,2,3\} \setminus E$.

E	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$a(E)$	$(1,0,0)$	$(0,1,0)$	$(0,0,0.4)$	$(1,1,0)$	$(1,0,0.5)$	$(0,1,0.7)$	$(1,1,1)$

We calculate the restricted game:

$$v^a(\{1\}) = v(\{1\}) = 1, \qquad v^a(\{2\}) = v(\{2\}) = 2, \qquad v^a(\{3\}) = 0.4\, v(\{3\}) = 1.2,$$
$$v^a(\{1,2\}) = v(\{1,2\}) = 6, \qquad v^a(\{1,3\}) = 0.5\, v(\{1,3\}) + 0.5 v(\{1\}) = 4.5,$$
$$v^a(\{2,3\}) = 0.7\, v(\{2,3\}) + 0.3\, v(\{2\}) = 7.6, \qquad v^a(\{1,2,3\}) = v(\{1,2,3\}) = 20.$$

Finally, a payoff vector for the suppliers is

$$\varphi(v,a) = (5.6833, 7.7333, 6.5833).$$

5. Conclusions

We have defined and characterized a value for games with fuzzy authorization structure. The model presented is more general than those introduced in previous papers (Gilles *et al.* (1992), Derks and Peters (1993), van den Brink (1997), Algaba *et al.* (2004)), since it allows us to deal with partial dependency relationships. The value introduced is applicable to situations in which we have a cooperative game and a collection of restrictions on coalition formation.

Other solutions for games with fuzzy authorization structure remain to be studied.

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References

Algaba E., Bilbao J.M., van den Brink R. and Jiménez-Losada A. (2004). Cooperative games on antimatroids. *Discrete mathematics* 282: 1-15.

Aubin J.P. (1981). Cooperative fuzzy games, *Mathematics of Operations Research* 6: 1-13.

Bilbao J.M. (1998). Axioms for the Shapley value on convex geometries. *European Journal of Operational Research* 110: 368-376.

Bilbao J.M., Driessen T.S.H., Jiménez-Losada A. and Lebrón E.A. (2001). The Shapley value for games on matroids: the static model. *Mathematical methods of Operations Research* 53: 333-348.

Bilbao J.M. and Ordóñez M. (2009). Axiomatizations of the Shapley value for games on augmenting systems. *European Journal of Operational Research* 196: 1008-1014.

Brink R. van den (1994). Relational Power in Hierarchical Organizations. Ph. D. Thesis, 1994.

Brink R. van den (1997). An axiomatization of the disjunctive permission value for games with a permission structure. *International Journal of Game Theory* 26(1): 27-43.

Butnariu D. (1980). Stability and Shapley value for an n-person fuzzy game, *Fuzzy Sets and Systems* 4: 63-72.

Choquet G. (1953). Theory of Capacities, *Annales de l'Institut Fourier* 5: 131-295.

Derks J. and Peters H. (1993). A Shapley value for games with restricted coalitions, *International Journal of Game Theory* 21: 351-366.

Gilles R.P., Owen G. and van den Brink R. (1992). Games with permission structures: the conjunctive approach, *International Journal of Game Theory* 20: 277-293.

Myerson R.B. (1977). Graphs and cooperation in games, *Mathematics of Operations Research* 2(3): 225-229.

Shapley L.S. (1953). A value for n-person games, *Annals of Mathematics Studies* 28: 307-317.

Schmeidler D. (1986). Integral representation without additivity, *Proceedings of the American Mathematical Society* 97: 255-261.

Tsurumi M., Tanino T., Inuiguchi M. (2001). A Shapley function on a class of cooperative fuzzy games, *European Journal of Operational Research* 129: 596-618.

Zadeh L.A. (1965). Fuzzy sets, *Information and Control* 8: 338-353.