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Instability of axion inflation in the regime of strong backreaction

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Abstract

In axion-U(1) inflation models, in the limit of strong coupling between the inflaton and the gauge field, one helicity mode of the gauge field undergoes exponential enhancement. This amplification, governed by the inflaton velocity, leads to a delayed backreaction of the gauge field on the inflaton motion, resulting in oscillations in the inflaton velocity. In this thesis, we conduct an analytical examination of the evolution equation, relaxing certain assumptions typically found in the literature. Additionally, we explore the numerical effects of a time-delayed friction term within a single-field inflation model.

Contents

Introduction

Primordial inflation was introduced for the first time [1] to explain the homogeneity and spatial flatness of the universe on large scales. It assumes that in the early universe, the vacuum energy of a scalar field ϕ (inflaton) dominated the energy density of the universe, acting as a source with an equation of state $P \simeq -\rho$ [2, 3], providing an accelerated expansion. Alongside giving the right initial condition to a flat and homogeneous universe, the quantum fluctuations of the inflaton provide the seeds for energy density perturbations responsible for the cosmic microwave background (CMB) anisotropies and subsequent structure formation in the universe [4–8]. Over the years, several workable models of inflation have been proposed, but a definitive particle physics model of inflation has not yet been determined. A crucial issue is the UV sensitivity of inflationary models, namely, the presence of physics beyond the Standard Model (SM) that can spoil the required flatness of the potential via radiative corrections. A possible way to preserve the model from such corrections is to postulate the presence of a shift symmetry in the inflaton action. We refer to scalar fields that enjoy this symmetry $\phi \to \phi + \mathit{constant}$ as axions, and the first model of inflation involving axions was introduced by Ref. [9], and named *natural inflation*. In natural inflation models, instanton effects [10] cause the shift symmetry to break into a discrete one $\phi \to \phi + 2\pi f$, where f is the axion decay constant, giving rise to a periodic potential of the form

$$V(\phi) = \Lambda^4 \left[1 - \cos\left(\frac{\phi}{f}\right)\right] \ . \tag{1}$$

However, the initial formulation of natural inflation necessitates $f \sim M_p$ ($M_p \simeq 2.4 \cdot 10^{18}$ GeV) to establish a suitably prolonged inflationary period, but this seems to be problematic in the context of super gravity and string theory [11, 12]. Moreover, cosmological observations [13] ruled out the simplest realization of natural inflation.

To realize a model of natural inflation compatible with cosmological observations, and for $f \ll M_p$, Amber and Sorbo introduced a coupling of the axion with a $U(1)$ gauge group. They showed in [14] that as ϕ rolls down the potential, it produces gauge field quanta at the expense of its kinetic energy, and if the coupling is sufficiently strong, the dissipation allows to realize slow roll and a sufficiently long period of inflation, even for a steep potential. These mechanisms in which slow roll is achieved through dissipation effects due to particle production can also be seen in models of warm inflation [15] and trapped inflation [16]. In the first study of this model [14], it was discovered that due to coupling, a helicity state of the gauge field is amplified by a factor proportional to the exponential of the inflaton velocity $\dot{\phi}$, and consequently, the gauge field can strongly backreacts on the inflaton evolution. The initial studies on this model assume that the inflaton is homogeneous. Among these, there is the original work [14], which is purely analytical and assumes a steady-state evolution of the inflaton velocity. This perspective contrasts with numerical results, also achieved within the limit of homogeneous inflaton, but precisely solving the evolution equations of the gauge field and backreaction (though exact in the limit of numerical inflaton) [17–22]. Oscillations of $\dot{\phi}$ are observed around the mean value of AS. Among them, the reference [20] attributes the effect to a time delay. There is also the analytical counterpart by Peloso-Sorbo, performed again in the homogeneous backreaction regime, obtaining oscillations and time delay analytically, within the limit of a small departure from AS. Finally, there are lattice studies that solve the system exactly [23–26]. All of these follow an oscillation and confirm previous results. However, the latest work [26] has a larger dynamical range, manages to track the evolution a bit longer, and observes an initial oscillation followed by damping. This motivates an analytical study beyond the homogeneous backreaction regime. This thesis represents the first step in this direction.

In this framework, the following work further investigates the nature of the oscillations of $\dot{\phi}$. This is done both analytically studying the equation of motion of the two fields, relaxing some assumptions always present in the literature, and numerically analyzing the consequences of a delay in the backreaction term.

Outline

The thesis is structured as follows

Chapter 1: Introduction to the standard model of cosmology and horizons.

Chapter 2: Introduction to inflation, addressing the problem that brought its introduction, the simplest realizations of inflation, the background dynamics, and the properties of the quantum fluctuations of the inflaton.

Chapter 3: Review of the AS model and of the analytical and numerical studies carried out.

Chapter 4: In this chapter, we present the first original contribution of this work, where we compute the equations of motion for the two fields ϕ and A_μ, including terms proportional to A_0, and terms related to the gradients of the inflaton $\partial_i\phi$. These new terms are treated as small perturbations.

Chapter 5: In the final chapter, we investigate the aftermath of a delayed friction term in the equation of motion of the inflaton. Specifically, we consider the inflationary model $\lambda\phi^4$ and add a delayed term $\Gamma\dot{\phi}$ to the equation of motion, taking Γ constant. Even in the presence of friction, the model remains ruled out by observations. However, our interest is not in the specific model per se, but in the study of the effect of delayed backreactions, in a different and simpler setup than the AS model.

Conclusions: Finally, we summarize the results obtained in this work. We first review the analytical analysis carried out in Chapter 4, which led us to write a master equation for the perturbations of the inflaton field that was not already present in the literature. Then we summarize the results obtained in Chapter 5, where we performed a numerical study on the effects of the delayed term in the equation of motion of the inflaton, showing that also in a different context from the AS one, oscillations of the inflaton velocity $\dot{\phi}$ arise.

Chapter 1

The FLRW Universe

The aim of the following chapter is to introduce the Hot Big Bang (HBB) model, specifically the geometrical environment of our universe, its dynamics, and the shortcomings of the HBB model that brought to the formulation of the inflationary paradigm. This presentation follows from [27–30]

1.1 The geometry of the universe

The theory of general relativity states that spacetime is a four dimensional manifold on which is defined a metric tensor, $g_{\mu\nu}$, and that it evolves according to the Einstein field equations (EFE)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{M_p^2} T_{\mu\nu} \ , \tag{1.1}$$

being $R_{\mu\nu}$ the Ricci tensor, R the Ricci scalar, $M_p = 1/\sqrt{8\pi G}$ the reduced Planck mass, G the universal gravitational constant, that in the international system is $G = 6.674310^{-11}\ \mathrm{m^3 kg^{-1} s^{-2}}$, and $T_{\mu\nu}$ the stress-energy tensor. Moreover, the Ricci tensor is

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\lambda\mu} \ , \tag{1.2}$$

a combination of derivatives and products of Christoffel symbols $\Gamma^\lambda_{\mu\nu}$, which is a connection that gives a way of relating vectors in the tangent spaces of nearby points, and is defined as

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right) \ . \tag{1.3}$$

Soon after Einstein theory had been formulated a question arose spontaneously: Which solution of the field equation corresponds to our universe? Although mathematics lays the foundation, it alone fails to yield a solution to the equation; hence, we must augment our approach with assumptions based on observational evidence. Modern cosmological observations, first among them the CMB spectrum, seem to support the assertions of the cosmological principle, namely of our universe to be isotropic and homogeneous on large scales. Another key observation is that the universe is not static but it expands, currently at an accelerated pace. This goes under the name of Hubble law and it is supported both by astronomical evidence, like the analyses of the redshifts of standard candles (type Ia Supernovae), and also by cosmological observations, again CMB. These two unconnected events give us an estimate of the expansion rate of the Universe today H_0, also known as the Hubble rate.

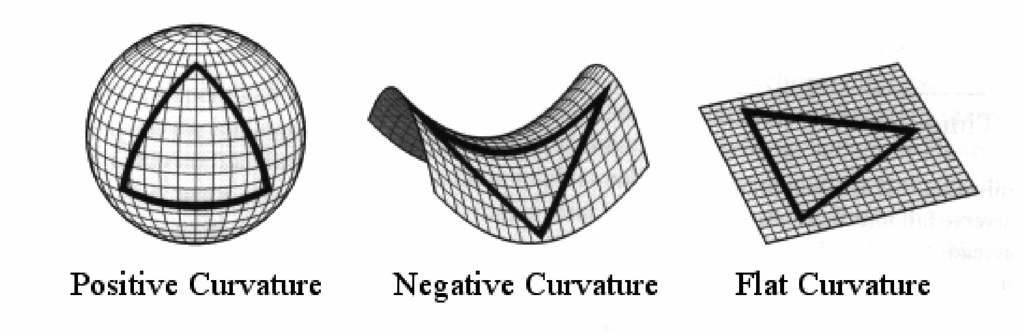

Figure 1.1: Graphic representation of two dimensional spacetimes curvature.[1]

Under the assumptions of homogeneity, isotropy, and dynamism, by the end of the '20s Alexander Friedmann and Georges Lemaitre independently obtained an exact solution of the EFE, while in the '30s Howard Robertson and Arthur Walker studied the properties of such metric. Finally, this metric now goes under the name of Friedmann-Lemaitre-Robertson-Walker (FLRW) metric and in $c = 1$ units reads

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + sin^2\theta d\phi^2\right)\right] \ , \tag{1.4}$$

where $a(t)$ is the scale factor that parametrizes how the distance between two fixed points of the universe changes in time, while k is a real constant related to the spatial curvature of the universe, conventionally normalized such that it assumes values $k \in \{1, -1, 0\}$ depending on whether the universe has a positive (open), negative (closed) or zero (flat) curvature, fig. (1.1).

This specific form of the metric (1.4) implies the choice of a specific coordinate system t, r, θ, ϕ which is the one of a comoving observer with respect to (wrt) the expansion of the universe. Comoving and proper length are related through the relation

$$\lambda_p(t) = a(t)\lambda_c \ . \tag{1.5}$$

In the same way, time coordinates satisfy the same relation, defining conformal time as

$$d\tau = \frac{dt}{a(t)} \ . \tag{1.6}$$

Furthermore, performing a change of coordinates

$$r = f_k(\chi) = \begin{cases} sin\chi & \text{if} \quad k = 1 \\ \chi & \text{if} \quad k = 0 \\ sinh\chi & \text{if} \quad k = -1 \end{cases} \ , \tag{1.7}$$

employing (1.6) and (1.7) is possible to rewrite the metric (1.4) in the form

$$ds^2 = a^2(\tau)\left[-d\tau^2 + d\chi^2 + f_k^2(\chi)\left(d\theta^2 + sin^2\theta d\phi^2\right)\right] \ . \tag{1.8}$$

This metric is the conformally flat FLRW metric, and its peculiarity is that considering constant angular coordinates $d\theta = 0$ and $d\phi = 0$, the null geodesics $ds^2 = 0$ are the same we find in Minkowski spacetime ($\tau = \pm\chi$).

[1] Source: https://pages.uoregon.edu/jschombe/cosmo/lectures/lec15.html

1.2 The dynamics of the universe

To study the evolution of the scale factor, it is necessary to substitute the metric (1.4) into the EFE (1.1) for a specific form of the stress-energy tensor $T_{\mu\nu}$. However, the choice of the stress-energy tensor is definitely not a trivial one, in principle, one could consider the general relativistic definition taking

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_p}{\delta g^{\mu\nu}} \ , \tag{1.9}$$

where S_p is some action describing the particle content of the underlying quantum field theory (QFT), considering every field and its interactions. However, this would require a complicated computation and complete trust in the effectiveness of the present QFT description, which could fail at very large energy scales, like those in the early universe. The alternative is to focus on the global properties, modeling the content of the universe with a perfect fluid, that in its rest frame is fully characterized by its energy density and the isotropic pressure field. The related stress-energy tensor is

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu} \ , \tag{1.10}$$

where u^μ is the four-velocity of the fluid. Remembering that T^{0i} is related to the momentum density in the i-direction, the assumptions made require $T^{0i} = 0$ otherwise it would identify a privileged direction spoiling isotropy. Moreover, $T^{0i} = 0$ means $(\rho + P)u^0u^i = 0$ and since the four-velocity is defined

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{d\vec{x}}{d\tau}\right) \ , \tag{1.11}$$

being τ the affine parameter chosen to characterize the motion of the fluid, this is only possible if $u^i = 0$. This means that the assumptions of isotropy and homogeneity hold only in a specific frame, the one in which the cosmic fluid is at rest. This could be expected since an isotropic universe in one frame, is not isotropic in a frame obtained performing a boost on the former one. Turning now to Einstein equations, plugging the perfect fluid stress-energy tensor (1.10), and the FLRW metric (1.4) into (1.1), two independent equations are obtained, one from the 00-component

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \ , \tag{1.12}$$

being H the Hubble rate and $\dot{a}$ the time derivative of the scale factor, and the other from the trace of the ij-component

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \ . \tag{1.13}$$

These two equations relate the derivatives of the scale factor to the fluid energy density and pressure. A third equation is obtained by exploiting the 0-component of the conservation equations $\nabla_\mu T^{\mu\nu} = 0$, and it reads

$$\dot{\rho} = -\frac{3\dot{a}}{a}(\rho + P) \ . \tag{1.14}$$

The collection of these three equations (1.12 - 1.14) is referred to as the Friedmann equations. However, even though it consists of three variables (a, ρ, P) in three equations, it does not constitute a complete set of equations. It can be demonstrated that only two of these equations are independent, since (1.14) follows from (1.12) and (1.13) through Bianchi identity. To complete the set of equations it is customary to introduce an equation of state. This type of equation can assume numerous forms and typically relates the isotropic pressure field of the fluid to various quantities, like the energy density, the temperature, the entropy, and so on. A common choice is to consider a barotropic equation of state, namely a relation between the pressure field and the energy density. Such an equation of state has the form

$$P = \omega\rho \ , \tag{1.15}$$

where ω is the equation of state parameter and his numerical value depends on the specific component. The pressure field of a perfect fluid exhibits the property of additivity, in other words, the total pressure of the fluid is the sum of the pressure of its components, thus the equation of state (1.15) can be explicitly written

$$P = \sum_i P_i = \sum_i \omega_i \rho_i \ . \tag{1.16}$$

Conceptually this form is equivalent to the former one, however, this expression stresses the idea of the cosmic fluid being made up of different components each characterized by different equations of state. For what concerns the global evolution of the universe, different components are specified by different equations of state parameters ω_i and an ordinary classification is: $\omega = 0$ to indicate non-relativistic particles (also referred to as matter), $\omega = 1/3$ to indicate ultra-relativistic particles (also referred to as radiation), while everything that falls under $\omega < -1/3$ indicates what is commonly called dark energy. To solve the system it is convenient to substitute the equation of state (1.15) into the third Friedmann equation (1.14), which now reads

$$\dot{\rho} + \frac{3\dot{a}}{a}\rho\,(1+\omega) = 0 \ . \tag{1.17}$$

Solving this equation permits to find the evolution of the energy density of a given component, in terms of the scale factor. The relation one finds is

$$\rho = \rho_i \left(\frac{a_i}{a}\right)^{3(1+\omega)} \ , \tag{1.18}$$

in this case, the pedices i indicates quantities evaluated at some initial time. This is because we are able to describe the dynamics of the ratio of energy densities and scale factor because the equation (1.12) is invariant under $a(t) \to c\ a(t)$ for any c constant[2]. Plugging relation (1.18) into (1.12), it is possible to find the time evolution of the scale factor. Assuming that only one component dominates the energy budget of the universe, which holds true for nearly the entirety of the universe evolution because the energy density of different components scales differently, the solutions are

$$a = \begin{cases} a_i \left(\frac{t}{t_i}\right)^{\frac{2}{3(1+\omega)}} & \text{if} \ \ \omega \neq -1 \\ a_i e^{H(t-t_i)} & \text{if} \ \ \omega = -1 \end{cases} \tag{1.19}$$

The special case $\omega = -1$ represents a specific kind of dark energy, the Cosmological constant (Λ). This component is characterized by a constant energy density, as can be seen from eq. (1.17), since replacing $\omega = -1$ it reads $\dot{\rho} = 0$. Considering other aforementioned components, hence fixing $\omega = 0$ and $\omega = 1/3$ in the solution (1.18), one obtains the evolution of the energy densities of matter and radiation respectively

$$\rho_m = \rho_{mi} \left(\frac{a_i}{a}\right)^3 \ , \tag{1.20}$$

$$\rho_r = \rho_{ri} \left(\frac{a_i}{a}\right)^4 \ . \tag{1.21}$$

The dependency $\rho_m \propto a^{-3}$ was expected since the energy density of matter dilutes solely due to the expansion of the universe in all three spatial directions. In contrast, the additional factor a^{-1} in the solution for the energy density of radiation arises from the further effect that the expansion of the universe has on the energy of relativistic particles, that is stretching their wavelength (redshift) resulting in an overall dependency $\rho_r \propto a^{-4}$. Regarding the energy density

[2] This is true only for a flat universe, but as we are going to see this is an adequate approximation for our universe.

of radiation, it is useful also to relate it to the temperature of the photons field, this can be done employing the properties of the black body radiation for which $\rho_r \propto T^4$, consequently this relation implies that during a radiation dominated epoch $a \propto T^{-1}$. These results concerning the evolution of the energy densities provide significant insights. They show that the energy budget of the universe was initially dominated by radiation, followed by a matter dominated epoch, while just recently the main contributor to the total energy density became a form of dark energy. The succession of the dominion of the various components is qualitatively shown in Figure (1.2). This gives us information about the evolution of the scale factor at different stages of the evolution of the universe, in fact from (1.19) we find that initially, the scale factor grew like $a \propto t^{1/2}$ during radiation domination, then while matter dominated it scaled like $a \propto t^{2/3}$ and recently things changed as the scale factor started to evolve like $a \propto e^{Ht}$.

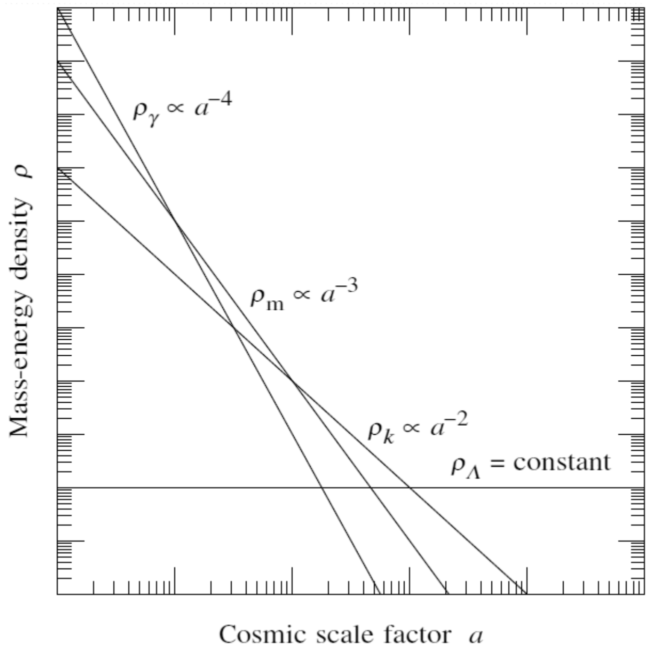

Figure 1.2: Comparison between the different dependencies of the energy densities wrt the scale factor. A detailed discussion regarding the energy density associated to curvature, ρ_k, will be carried out in the next sections.[3]

Regarding energy densities, there is still an important definition to give. The first Friedmann equation (1.12) shows the interplay between the total energy density and the intrinsic curvature in their contribution to the Hubble rate. Imposing in this equation $k = 0$ defines a value of the energy density that flattens the universe, such a value is called critical density, ρ_c, and is defined as

$$\rho_c \equiv \frac{3H^2}{8\pi G} . \tag{1.22}$$

To quantify whether this value is close to the actual energy density of the universe it is convenient to introduce the density parameter, defined as the ratio between the observed energy density and the critical energy density

$$\Omega \equiv \frac{\rho}{\rho_c} = \frac{3H^2\rho}{8\pi G} , \tag{1.23}$$

[3] Source: https://jila.colorado.edu/~ajsh/courses/astr3740_14/evol.html

from the meaning of the critical density, it is evident that measuring $\Omega = 1$ would reflect in a flat universe, while Ω either smaller or greater than 1 would reflect an open or closed universe respectively. However, it is worth stressing that it will always be impossible to assert that the universe is flat. No matter how precise measurements become, every result is always associated with an error, and measuring $\Omega = 1$ with an error, both open and closed universes would not be excluded.

Before introducing the motivations of the introduction of the inflationary model there are still a few concepts that need to be introduced, related to the concept of horizon.

1.3 Cosmological horizon

Horizon is a frequent concept in cosmology, and it represents a boundary beyond which information cannot arrive to a given observer. The cosmological horizon (or particle horizon)is a physical distance indicating the radius of a region causally connected to a central observer, and it is defined as

$$d_H(t) = a(t) \int_0^t \frac{dt'}{a(t')} \; . \tag{1.24}$$

Substituting in (1.24) the solution previously found for $a(t)$ (1.19), it is possible to solve the integral finding

$$d_H(t) = \frac{3(1+\omega)}{1+3\omega} t \; , \qquad \omega > -\frac{1}{3} \; . \tag{1.25}$$

This relation shows how the cosmological horizon is finite and meaningful only for $\omega > -1/3$, which translates in components with an equation of state $P > -1/3\rho$, and if substituted this latter inequality into eq. (1.13) gives $\ddot{a} < 0$. The particle horizon exists and it is finite if and only if the expansion of the universe is decelerated.

1.4 Hubble radius

Another important length in cosmology is the Hubble radius, which is defined as

$$R_H(t) \equiv \frac{1}{H(t)} = t_H \; , \tag{1.26}$$

where t_H is the Hubble time, the inverse of the Hubble rate, and represents a characteristic expansion time. Making use of the definition of the Hubble rate, and relation (1.19), one finds that for $\omega > -1/3$

$$R_H(t) = \frac{3}{2}(1+\omega)ct = \frac{1+3\omega}{2} d_H(t) \; . \tag{1.27}$$

Considering again standard components, and order of magnitudes only, the latter relation reads $R_H(t) \sim d_H(t)$.

In the context of inflation, as will be discussed in the next section, one is interested in comparing the extension of a certain region to the horizon. For such analysis, it is convenient to work with comoving quantities, so that the dimension of a region is constant during the expansion of the universe. The comoving counterpart of the Hubble radius can be found employing the relation between comoving and physical quantities (1.5)

$$r_H(t) = \frac{R_H(t)}{a(t)} = \frac{1}{a(t)H(t)} = \frac{1}{\dot{a}(t)} \; . \tag{1.28}$$

Valuable information about the comoving Hubble radius comes from its derivative

$$\dot{r}_H(t) = \left(\frac{1}{\dot{a}(t)}\right)^{\cdot} = -\frac{\ddot{a}(t)}{\dot{a}^2(t)} \ . \tag{1.29}$$

Together with the second Friedmann equation (1.13), relation (1.29) shows that for ordinary components ($\omega_m = 0$ and $\omega_r = 1/3$), for which the expansion is always decelerated, the comoving Hubble radius grows monotonically. This characteristic behaviour, will be crucial while presenting the flaws of the HBB model in the next section.

[3] Source: https://www.slac.stanford.edu/econf/C070730/talks/dodelson_073107a.pdf

Chapter 2

The Inflationary Universe

The standard Big Bang model provided accurate predictions of the evolution of the universe at various stages, managing to fully explain most phenomena we observe. CMB spectrum, the abundance of light nuclei, and Hubble law are some of the most notable remarks of the HBB model, however, there are good reasons to believe that the model is incomplete as it is flawed by shortcomings. Such shortcomings are not related to intrinsic problems of the model, instead, they arise in the form of very tuned initial conditions, that must be imposed to explain features of the universe we observe today. Despite the name, the Big Bang theory is not the theory of a Bang, as common sense would suggest, it is the theory of the aftermath of a bang. What is missing is an adequate theory giving the HBB model the right initial conditions.

2.1 The horizon problem

Up to this point, the homogeneity of the universe is always been used as a strength of the cosmological model, however, even if it simplifies the mathematical treatment, it represents a problem when trying to explain how it has been reached in the HBB framework.

The horizon problem arises from the fact that in a standard FLRW universe, $r_H(t)$ is monotonically increasing. This implies that over time, the area within the causal reach of any observer expands, leading to larger scales entering the horizon. However, the issue stems from an observational point of view. Looking for example at two points on the CMB, separated by an angular distance larger than 2 degrees, we are looking at photons coming from regions on the last scattering surface, that were not in causal connection at the time of decoupling. How come the photons coming from these two regions, that never had the chance to exchange information prior to their emission, exhibit the same temperature up deviations of the order 10^{-5}?

At this point, there are two possible approaches to solve this issue. The first one is to assume that the universe was already born homogeneous with such a high degree of precision, however, this solution is not satisfactory. A second approach consists in finding a dynamic mechanism, that starting from a broad range of initial conditions, gives back a homogeneous universe. This is what inflation does introducing a period of accelerated expansion in the early universe. To understand how an accelerated expansion cures the horizon problem, it is sufficient to notice that a positive $\ddot{a}(t)$ causes $r_H(t)$ to shrink, recall eq. (1.29). If this regime is sustained for long enough, then the scales that now enter the horizon, in the early universe had the chance to connect with each other, in the far past of this accelerated stage. This would explain the extreme homogeneity we observe in the temperature distribution of CMB photons, Figure (2.2).

[1] Source: `https://www.ctc.cam.ac.uk/outreach/origins/inflation_zero.php`

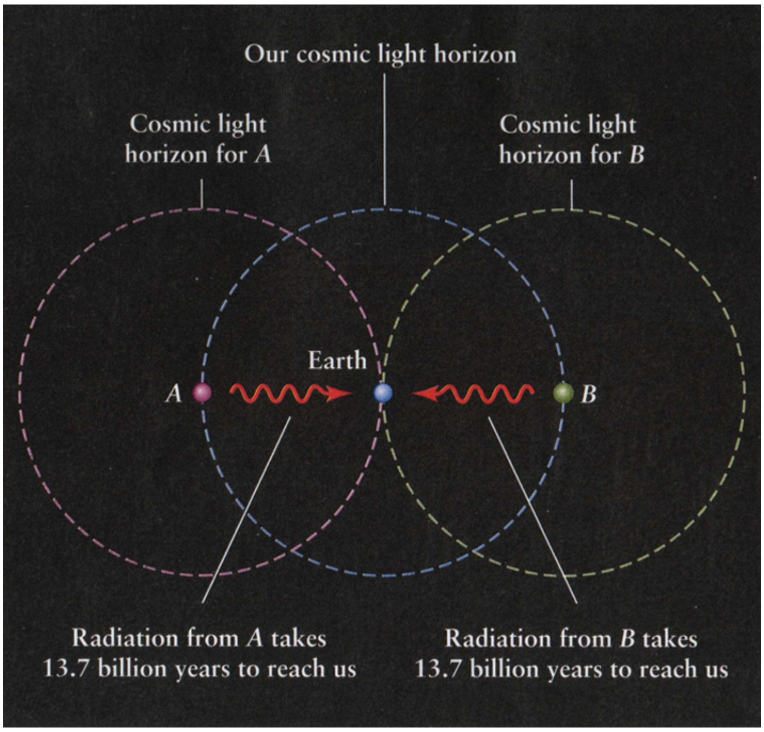

Figure 2.1: Graphic representation of the horizon problem, in its most extreme problematic case of two CMB photons reaching us from opposite directions.[1]

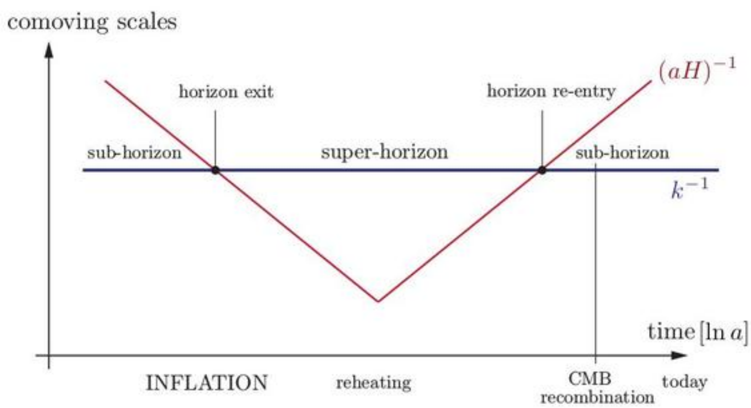

Figure 2.2: Graphic representation of the solution to the horizon problem. Red line: evolution of the comoving Hubble radius from the inflationary epoch, through the end of inflation (reheating) and until today. Black line, the evolution of the comoving length of dimension $\lambda \sim k^{-1}$, that exits the horizon during inflation and re-entered the horizon before atom recombination, and subsequent CMB emission. Source: [31].

While the discussion about what caused inflation is postponed to section 2.4 below, it is important now to study how much inflation should have lasted in order to cure the horizon problem. To do so it is necessary to define the number of e-folds, the quantity used to characterize the progression of inflation

$$N \equiv \int_{t_i}^{t_f} H(t)dt = ln\left(\frac{a_f}{a_i}\right) \ , \tag{2.1}$$

namely, N quantifies how much the universe grew during inflation, from a given moment t_i to the final moment t_f

To solve the horizon problem it is sufficient to require that the comoving Hubble radius at the beginning of inflation was bigger (or equal to find the minimum amount of inflation we need) than its value today, this would imply that the largest scale observable today was in causal connection before inflation started. This translates into

$$r_H(t_i) \geq r_H(t_0) \ , \tag{2.2}$$

from the definition of r_H (1.28)

$$\frac{1}{a_i H_i} \geq \frac{1}{a_0 H_0} \ , \tag{2.3}$$

which implies

$$\frac{a_f}{a_i} \geq \frac{a_f H_i}{a_0 H_0} \ . \tag{2.4}$$

To give a qualitative result, it is convenient to employ the relation $a \propto T^{-1}$ on the right-hand side

$$\frac{a_f}{a_i} \geq \frac{T_0}{H_0}\frac{H_i}{T_{Reh}} \ , \tag{2.5}$$

where it has been assumed for simplicity, that inflation ends with an instantaneous decay e thermalization, that restored the energy density diluted during inflation, and T_{Reh} represents the temperature of the universe right after this ending phase of inflation called reheating. Which translates into a minimum number of e-folds

$$N \geq ln\left(\frac{T_0}{H_0}\right) + ln\left(\frac{H_i}{T_{Reh}}\right) \ . \tag{2.6}$$

Taking now $T_0 = 2.7$ K $\simeq 10^{-13}$ GeV and $H_0 \simeq 10^{-42}$ GeV, (2.6) becomes

$$N \geq 67 + ln\left(\frac{H_i}{T_{Reh}}\right) \ . \tag{2.7}$$

The second term is model dependent, nonetheless, it is possible to give a rough estimate. Assuming that inflation was a quasi de-Sitter epoch, namely that H remains almost constant during inflation $H_i \simeq H_{Reh}$, and assuming that inflation ends leaving the universe in a radiation dominated state (initial condition for the HBB model to take over the description of the evolution of the universe), one can derive an estimate of the final Hubble rate (H_{Reh}) using the first Friedmann equation

$$\begin{aligned} H^2_{Reh} &= \frac{8}{3}\pi G \rho_{r\,Reh} \\ &= \frac{8}{3}\pi G \frac{\pi^2}{30} g_* T^4_{Reh} \\ &\propto \frac{T^4_{Reh}}{M_p^2} \ . \end{aligned} \tag{2.8}$$

Plugging this result into (2.7), the last term becomes $ln\,(T_{Reh}/M_p)$. To get an estimate is necessary to fix a value of T_{Reh}, however, only bounds on its value have been fixed so far. From the lack of observation of a gravitational wave background, and in the context of single field models of slow-roll inflation (concepts that will be properly introduced in Section 2.4), have been fixed [32] an upper bound of the reheating temperature of $T_{Reh} \lesssim 1.7 \cdot 10^{13}$ GeV. Inserting now a reheating temperature of this order of magnitude, one finds $T_{Reh}/M_p \sim 10^{-5}$, which turns into a minimum number of e-folds

$$N \gtrsim 56 \ . \tag{2.9}$$

2.1.1 Minimum number of e-folds

In the previous section, we derived the minimum number of e-folds necessary to solve the horizon problem. However, the discussion about the minimum duration of inflation is delicate and can be deepened further. Current observations are telling us that the universe is homogeneous on the largest scales we can probe today (CMB). This means that we must require the modes corresponding to the CMB today, to exit the horizon during inflation, so that before that moment, they were in causal connection. Following the computation of Ref. [33], the relation employed to determine when a given mode k, exits the horizon during inflation is given by

$$k = a_k H_I \ , \tag{2.10}$$

where H_I is the Hubble rate during inflation, while a_k is the scale factor at horizon crossing. The epoch of horizon exit is therefore related to the present magnitude of the scale factor a_0, in Hubble units by

$$\frac{a_0 H_0}{k} = \frac{a_0 H_0}{a_k H_I} \ . \tag{2.11}$$

Let us assume now that between the end of inflation and the reheating epoch, there has been a matter dominated epoch (due to the oscillatory phase of the inflaton around the minimum of the potential), and that after reheating the universe was in a radiation dominated epoch that lasted until the matter-radiation equivalence. Exploiting these assumptions, and recalling the relations of the energy density as a function of the scale factor (1.2), and that the total energy density is proportional to H^2 through (1.12), it is possible to recast (2.11) into

$$\frac{k}{a_0 H_0} = \frac{a_k}{a_{end}} \frac{a_{end}}{a_{reh}} \frac{a_{reh}}{a_0} \frac{H_I}{H_0} \ . \tag{2.12}$$

Employing the definition of the number of e-folds (3.23), the first term on the rhs is related to the number of e-folds elapsed from the moment the mode k crossed out the horizon, to the end of inflation, as

$$\frac{a_k}{a_{end}} = e^{-N_k} \ . \tag{2.13}$$

Taking now the logarithm of (2.12), the number of e-folds N_k can be written as [33]

$$N_k = 62 - \ln \frac{k}{a_0 H_0} - \ln \frac{10^{16}\ \text{GeV}}{V_k^{1/4}} + \ln \frac{V_k^{1/4}}{V_{end}^{1/4}} - \frac{1}{3} \ln \frac{V_{end}^{1/4}}{\rho_{reh}^{1/4}} \ . \tag{2.14}$$

This relation depends on three energy scales (V_k, V_{end} and ρ_{reh}), and hence depends on the specific inflationary model. Considering now the largest scales we can probe (CMB), COBE constraint of the inflaton energy at the moment this mode exited the horizon $V_{CMB}^{1/4} \lesssim 10^{16}$ GeV [34], taking $V_{CMB}^{1/4} \sim V_{end}^{1/4}$ and assuming reheating to be prompt, (2.14) tell us that the current observable universe exited the horizon about 62 e-folds before the end of inflation.

2.2 The flatness problem

The latest results from the Planck collaboration [13] show that the density parameter (1.23) today is very close to unity

$$|1 - \Omega(t_0)| < 0.4 \times 10^{-2} \quad [95\%\text{CL}] \ . \tag{2.15}$$

To understand what this result means, one has to rearrange the first Friedmann equation (dividing by H^2 and employing the definition of the density parameter) in the following form

$$\Omega(t) - 1 = \frac{k}{a^2(t)H^2(t)} = \frac{k}{\dot{a}^2(t)} = k r_H^2(t) \ . \tag{2.16}$$

From the latter equality, it is clear that the flat universe condition is unstable since the absolute value of the last term of (2.16) is always growing. This means that curvature becomes more and more dominant over time unless the universe is not precisely flat ($k = 0$). If one computes the value that $|1 - \Omega(t)|$ should have had in the early universe to explain its current value, one finds at the Planck time ($t_p \simeq 10^{-44}$ s) an upper bound of

$$|1 - \Omega(t_p)| < 10^{-62} \ . \tag{2.17}$$

Again requiring a fine tuning of the curvature parameter with such precision is not a satisfactory approach. However it is intuitive to see how inflation dynamically solves this issue, again a period of accelerated expansion would make r_H shrink, and if it lasts long enough then for (2.16) Ω would be pushed towards one sufficiently to explain the value we observe today, see figure (2.3).

The way inflation solves this problem exhibits the attractor nature of the inflationary model, namely that the universe could start with whatever curvature, but if inflation manages to start, and if it lasts enough, it is able to bring any initial condition to the one required to explain the observations we make today. Finally, it is possible to show that a rough estimation of how much inflation should last in order to solve the flatness problem, would again give a minimum number of e-folding $N \gtrsim [60, 70]$.

2.3 Unwanted relics

Assuming that the universe started in a high temperature state, in the context of grand unified theories (GUT), massive topological defects X should have been produced, and, if they survived annihilation, they could contribute to the present energy density overclosing the universe ($\Omega_X \gg 1$). Such relics should be the by-product of the spontaneous symmetry breaking (SSB) of some gauge symmetry that occurred around the GUT scale, $T_{GUT} \sim 10^{14} \div 10^{16}$ GeV. There are different kinds of topological defects, depending on the symmetry group that gets broken, and the various possible defects can be characterized by their dimensions. Listed in growing order of their spatial dimensions there are: magnetic monopoles, point-like (0D) massive particles. Cosmic strings (1D), typically related to the breaking of a $U(1)$ group. Domain walls (2D), related to the breaking of a discrete symmetry. Cosmic texture (3D), typically associated with the breaking of a $SU(2)$ group. More recently other possibilities came up, like gravitinos and moduli respectively in the contexts of supergravity theories and supersymmetry theories.

To solve this problem, relics must be produced before inflation starts, such that as inflation proceeds and the universe expands the relics get diluted enough to be still today rare to prevent

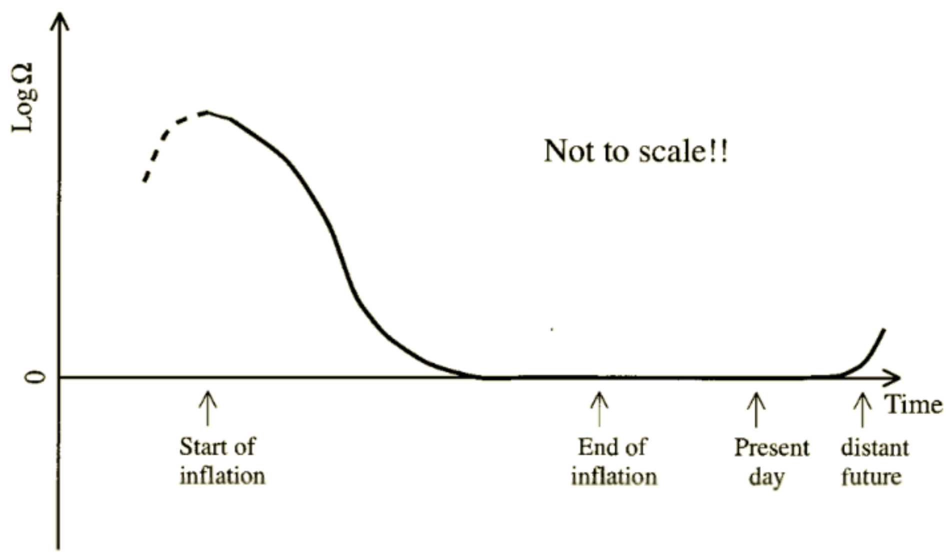

Figure 2.3: Graphic representation of the inflationary solution to the horizon problem. Source [30].

them from overclosing the universe. However, this dilution affects every kind of particle, hence it is necessary that after inflation takes care of the problems affecting the HBB model, an ending mechanism that reheats the universe giving the right initial conditions to have a radiation dominated epoch in which nuclei form, then atoms combine and photons decouple from electrons, and lastly large scale structure are able to form.

2.4 Simplest realization of inflation

Inflation has been defined as a period of accelerated expansion that occurred in the early universe, introduced to address the shortcomings of the HBB model. However, nothing has been said yet about how it can be realized dynamically. To introduce this topic it is functional to recall the effects of the cosmological constant. This component, characterized by an equation of state $P_\Lambda = -\rho_\Lambda$, and that causes an accelerated expansion of the universe, recall eq. (1.13), can be associated with a stress-energy tensor of a perfect fluid

$$T^\Lambda_{\mu\nu} = (P_\Lambda + \rho_\Lambda) u_\mu u_\nu + P_\Lambda g_{\mu\nu} = -\rho_\Lambda g_{\mu\nu} \ . \tag{2.18}$$

According to the latter relation, it would be possible to mimic the effects of the cosmological constant, in the context of particle physics, if a given particle species would have a stress-energy tensor $T_{\mu\nu}$ characterized by a vacuum expectation value (VEV) such that

$$\langle T_{\mu\nu} \rangle \equiv -\langle \rho \rangle g_{\mu\nu} \ . \tag{2.19}$$

This can be easily realized. Considering for example a scalar field ϕ, and a lagrangian density

$$\mathcal{L}_\phi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \ , \tag{2.20}$$

the associated stress-energy tensor (1.9) is then

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \mathcal{L} \ . \tag{2.21}$$

Assuming now that we have a ground state such that $\langle\phi_0\rangle$ is constant in time and space, all the derivatives in (2.21) vanish, leaving

$$\langle T_{\mu\nu}\rangle = -V(\langle\phi_0\rangle)g_{\mu\nu} \ , \tag{2.22}$$

this would act as an effective Λ providing an accelerated expansion.

The choice of a scalar field is not the only possibility, but it is the simplest one. In fact, when choosing the candidate particle species, one must be careful not to spoil any symmetry present in the universe, since this component must dominate the energy budget of the universe to provide inflation. One clear example of a choice that is not viable is a gauge field A_μ, in this case, a non-vanishing VEV in this case would identify a preferred direction spoiling rotational invariance (isotropy). Although the above example of a constant scalar field in the minimum of its potential would provide an accelerated expansion of the universe, it does not work as an inflationary model because inflation eventually ends, and a perfectly constant VEV of the scalar field does not allow any dynamics that regulate the duration of inflation.

To overcome this issue, it is possible to consider a dynamic scalar field, hence one that has not yet reached the minimum of its potential. The background of this field $\phi = \phi_0(t)$, is only time-dependent and produces a homogeneous and isotropic background, hence the nonvanishing components of the associated stress-energy tensor (2.21) are

$$\begin{cases} T^0_0 = -\left(\frac{1}{2}\dot{\phi}_0 + V(\phi_0)\right) = -\rho_\phi \\ T^i_j = \left(\frac{1}{2}\dot{\phi}_0 - V(\phi_0)\right)\delta^i_j = P_\phi\delta^i_j \end{cases} . \tag{2.23}$$

These two equations show that in case $\dot{\phi}_0 \ll V(\phi)$, indeed the background $\phi_0(t)$ behaves almost as an effective Λ, leading to an inflating background. Furthermore, in QFT a field can not be perfectly homogeneous, so the full inflaton field is given by

$$\phi(t,\vec{x}) = \phi_0(t) + \delta\phi(t,\vec{x}) \ , \tag{2.24}$$

where the additional term represents quantum fluctuations, which play a pivotal role in cosmology, giving the seeds to primordial fluctuations that eventually evolve into CMB anisotropies and large scale structures, due to gravitational clustering, during the matter dominated epoch.

2.4.1 Background dynamics of inflation

The dynamics of the inflaton field follows from the variational principle

$$\frac{\delta S_\phi}{\delta\phi} = 0 \ , \tag{2.25}$$

of an action describing a scalar field minimally coupled to gravity

$$S_\phi = \int d^4x\sqrt{-g}\mathcal{L}_\phi \ . \tag{2.26}$$

The result is an equation of the type $\Box\phi_0 = \frac{\partial V}{\partial\phi_0}$, where the d'Alambertian operator $\Box\phi_0$ in curved spacetime is defined as

$$\Box\phi_0 = \phi_0{}^{;\mu}{}_{;\mu} = \frac{1}{\sqrt{-g}}\left(g^{\mu\nu}\sqrt{-g}\phi_{0,\mu}\right)_{,\nu} \ , \tag{2.27}$$

where the semicolon indicates the covariant derivative, while the comma the normal one. Carrying out the computation in FLRW the equation of motion for a scalar field becomes

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 = -\frac{\partial V}{\partial\phi_0} \ , \tag{2.28}$$

its main difference from the usual Klein-Gordon equation is the presence of a friction term $3H\dot{\phi}_0$, arising from the universe's expansion. Together with this equation, one should also consider the Friedmann equation, neglecting the curvature term (that anyway during inflation would be washed away quickly). The system describing the dynamics of the scalar field is

$$\begin{cases} H^2 = \frac{8\pi G}{3}\rho_{\phi_0} = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}_0 + V(\phi_0)\right) \\ \ddot{\phi}_0 + 3H\dot{\phi}_0 = -\frac{\partial V}{\partial\phi_0} \ . \end{cases} \tag{2.29}$$

2.5 Slow roll inflation

A simple realization of inflation, on which many early models of inflation were based [2, 35], relies on a scalar field that for a certain amount of time during its evolution provides a negative pressure, causing an accelerated expansion of the universe. The stress-energy tensor (2.23) indicates that this can be the case if $V(\phi_0) \gg \frac{1}{2}\dot{\phi}^2$, thus $P_\phi \simeq -V(\phi_0) \simeq -\rho_\phi$, and the simplest way to achieve this condition is to require the potential to be sufficiently flat.

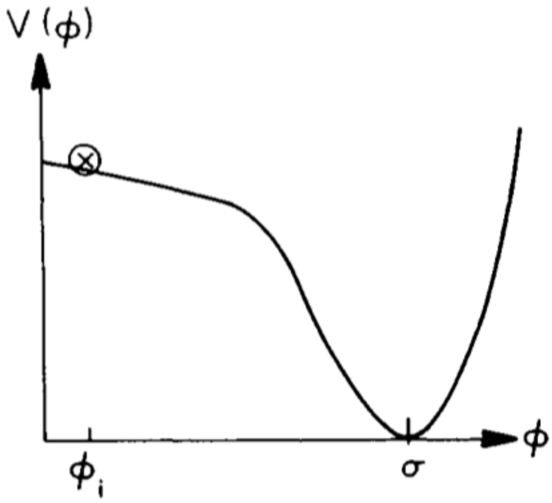

Figure 2.4: Example of slow roll inflation potential. Source [29]

To show why this would indeed provide inflation, suppose initially conditions are unsuitable for inflation[2], and the potential energy is negligible wrt the kinetic one, $\frac{1}{2}\dot{\phi}_0 \gg V(\phi_0)$, therefore

[2] The issue of the inflaton initial conditions is more subtle than how it is presented in the following example, whose only purpose is to show the attractor feature of slow roll inflation. Together with large kinetic energy, one should also be concerned about inhomogeneities that give rise to a nonvanishing gradient term, however, in this section we neglect this term, assuming a perfectly homogeneous background. For a more detailed discussion about the initial condition problem of the inflaton field, see [36].

$P_\phi \simeq \frac{1}{2}\dot{\phi}_0^2 \simeq \rho_\phi$ which would mean an equation of state parameter (1.15) $\omega_\phi \simeq 1$. This would reflect, for (1.18) in an energy density of the inflation field scaling as

$$\rho_\phi \sim \rho_{kin} \propto a^{-3(1+\omega_\phi)} \simeq a^{-6} \; . \tag{2.30}$$

This strong decrease factor, combined with a flat potential leads to the conclusion that sooner or later the potential energy will dominate over the kinetic one, and a negative pressure is achieved. Once again, this example makes it stand out the attractor nature of the inflationary model.

To study the dynamics of slow roll inflation, it is necessary to substitute in (2.29) the condition $V(\phi_0) \gg \frac{1}{2}\dot{\phi}_0^2$, already seen, and also a second condition $3H\dot{\phi}_0 \gg \ddot{\phi}_0$ that controls the duration of inflation (because if $\dot{\phi}_0$ is small enough at some time t_*, but $\ddot{\phi}_0$ is not small, then it will generally lead to a large $\dot{\phi}_0$ soon after t_*). Under such assumptions, the equations read

$$\begin{cases} H^2 \simeq \frac{8\pi G}{3} V(\phi_0) \\ 3H\dot{\phi}_0 \simeq -\frac{\partial V}{\partial \phi} \; . \end{cases} \tag{2.31}$$

To quantify the slow roll regime it is customary to introduce the so called slow roll parameters ε and η. The first slow roll parameter is defined as

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} \; . \tag{2.32}$$

The relation between this parameter and inflation becomes evident when computing $\ddot{a}$

$$\ddot{a} = (\dot{a})^{\cdot} = (aH)^{\cdot} = (\dot{a}H + a\dot{H}) = (aH^2 + a\dot{H}) = aH^2\left(1 + \frac{\dot{H}}{H^2}\right) = aH^2(1-\varepsilon) \; , \tag{2.33}$$

this shows that $\varepsilon < 1$ is required to have accelerated expansion. In order to understand how ε is related to the slow roll conditions, first it is necessary to derive the first equation of (2.29) wrt time

$$2H\dot{H} = \frac{8\pi G}{3}\left(\dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi}\right) \; , \tag{2.34}$$

where prime denotes the derivative wrt the inflaton field. Then use the second equation of (2.29) for the second derivative of the field, $\ddot{\phi} = -V' - 3H\dot{\phi}$

$$2H\dot{H} = -8\pi G H\dot{\phi}^2 \; , \tag{2.35}$$

that finally gives

$$\dot{H} = -4\pi G\dot{\phi}^2 \; . \tag{2.36}$$

This relation is always true for the simplest models of inflation given by the lagrangian density (2.20). At this point, the slow roll parameter ε can be recast into

$$\varepsilon = -\frac{\dot{H}}{H^2} = \frac{4\pi G\dot{\phi}^2}{H^2} \simeq \frac{4\pi G\dot{\phi}^2}{\frac{8\pi G}{3}V(\phi)} = \frac{3}{2}\frac{\dot{\phi}^2}{V(\phi)} \; , \tag{2.37}$$

where the third relation holds only under the slow roll regime. From the last equality of (2.37) ε can be interpreted as the ratio between the kinetic and potential energy, therefore to have slow roll the condition that ε must satisfy is

$$\varepsilon \ll 1 \ . \tag{2.38}$$

Furthermore, another important relation for ε comes from exploiting $\dot{\phi} \simeq -\frac{V'}{3H}$

$$\varepsilon \simeq \frac{3\dot{\phi}^2}{2V(\phi)} \simeq \frac{3V'^2}{18H^2V} = \frac{1}{6}\left(\frac{V'}{V}\right)^2 \frac{1}{\frac{8}{3}\pi G} = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \ . \tag{2.39}$$

Now, condition (2.38) translates into a condition on the potential

$$\frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \ll 1 \ , \tag{2.40}$$

before, the request for flat potential was qualitative, now it becomes quantitative.

The second slow roll parameter is defined starting from the condition $|\ddot{\phi}| \ll |3H\dot{\phi}|$

$$\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} \ , \tag{2.41}$$

and consequently must satisfy

$$|\eta| \ll 1 \tag{2.42}$$

To relate this parameter to the potential it is necessary first to derive the slow roll relation $\dot{\phi} \simeq -\frac{V'}{3H}$ wrt time to get

$$\ddot{\phi} = -\frac{V''\dot{\phi}}{3H} + \frac{\dot{H}}{3H^2}V' \ , \tag{2.43}$$

plugging this into the definition of η one obtains

$$\eta \simeq \frac{V''}{3H^2} - \frac{\dot{H}}{H^2}\frac{V'}{3H\dot{\phi}} \simeq \eta_V - \varepsilon \ , \tag{2.44}$$

where in the last relation it has been used $3H\dot{\phi} \simeq -V'$, and defined

$$\eta_V \equiv \frac{V''}{3H^2} \simeq \frac{1}{8\pi G}\frac{V''}{V} \ . \tag{2.45}$$

The requirements $\varepsilon \ll 1$ and $|\eta| \ll 1$ imply that also

$$|\eta_V| = \frac{|V''|}{3H^2} \ll 1 \ , \tag{2.46}$$

must hold during inflation, setting a second condition on the potential

$$\frac{1}{8\pi G}\left|\frac{V''}{V}\right| \ll 1 \ . \tag{2.47}$$

Although the first condition on V (2.40) was necessary for inflation to start, the latter is needed to ensure that inflation lasts a sufficient amount of time.

2.6 Properties of primordial perturbations

The previous sections showed how the background of the inflaton field $\phi_0(t)$ provided an accelerated expansion that made the universe nearly homogeneous. However, the universe today presents structures on many scales, from planets to clusters of galaxies. The seeds of these inhomogeneities are the fluctuations of the inflaton field $\delta\phi(t, \vec{x})$, (recall (2.24)), that during inflation, the wavelength λ of these perturbations got stretched to cosmological scales, and during the matter dominated epoch, thanks to a phenomenon known as gravitational instability, started to grow. The aforementioned perturbations are referred to as scalar perturbations; however, they do not constitute the sole type of perturbations involved in the early universe. When considering also metric perturbations we need to take into account also other perturbations that can be characterized by the way they transform under rotations. Using this classification, we distinguish scalar, vector, and tensor perturbations. These different modes are not coupled at linear order and hence evolve independently. Moreover, in scalar field models of inflation, vector perturbations are not supported, and hence we are going to neglect them. In the following section, relations between primordial perturbations and observable quantities are defined, for a comprehensive discussion, see [37, 38].

Primordial perturbations are described by a random field $\delta\phi(t, \vec{x})$ characterized by a vanishing vacuum expectation value and are almost Gaussian distributed. The deviation from Gaussian distribution is encoded in the nonlinear parameter f_{NL} and in a shape function. The nonlinear parameter is related to their amplitude and is proportional to the slow roll parameters, and observations so far did not show any evidence of primordial non-Gaussianities (see the Planck constraint on f_{NL} [39]).

2.6.1 Amplitude of the perturbations

To study the amplitude of scalar perturbations, let us start by considering perturbations of a massive scalar field χ in a quasi de Sitter stage. For this analysis, it is customary to expand $\delta\chi(t, \vec{x})$ into Fourier modes as

$$\delta\chi(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \delta\chi_k(t) \ . \tag{2.48}$$

The equation of motion for such perturbations is

$$\delta\ddot{\chi}_k + 3H\delta\dot{\chi}_k + \frac{k^2}{a^2}\delta\chi_k + m_\chi^2 \delta\chi_k = 0 \ . \tag{2.49}$$

Following the computation of [37], we redefine

$$\delta\chi_k = \frac{\delta\sigma_k}{a(t)} \ . \tag{2.50}$$

Using conformal time (1.6), in a quasi de Sitter expansion, one can show that for small values of ϵ the scale factor becomes

$$a(\tau) = -\frac{1}{H\tau^{1+\epsilon}} \ . \tag{2.51}$$

We can now rewrite equation (2.49)

$$\delta\sigma_k'' + \left(k^2 M^2(\tau)\right)\delta\sigma_k = 0\ , \tag{2.52}$$

where the squared mass term is given by

$$M^2(\tau) = m_\chi^2 a^2 - \frac{a''}{a}\ , \tag{2.53}$$

and where in quasi de Sitter, at first order in ϵ we can write

$$\frac{a''}{a} \simeq \frac{1}{\tau^2}(2+3\epsilon)\ . \tag{2.54}$$

Taking now $m_\chi^2/H^2 = 3\eta_\chi \ll 1$ in analogy to the slow roll parameters, and expanding for small values of ϵ and η_χ, we can recast equation (2.52) into

$$\delta\sigma_k'' + \left[k^2 - \frac{1}{\tau^2}\left(\nu_\chi^2 - \frac{1}{4}\right)\right]\delta\sigma_k = 0\ , \tag{2.55}$$

where ν_χ at first order in ϵ and η_χ is given by

$$\nu_\chi \simeq \frac{3}{2} + \epsilon - \eta_\chi\ . \tag{2.56}$$

Assuming ν_χ to be real, the solution to equation (2.55) reads

$$\delta\sigma_k = \sqrt{-\tau}\left[c_1(k)H_{\nu_\chi}^{(1)}(-k\tau) + c_2(k)H_{\nu_\chi}^{(2)}(-k\tau)\right]\ , \tag{2.57}$$

where $H_{\nu_\chi}^{(1)}$ and $H_{\nu_\chi}^{(2)}$ are the Hankel functions of the first and second kind. This solution exhibits two different behaviours of the perturbations of a scalar field on sub-horizon and super-horizon scales. Imposing the sub-horizon condition ($k \gg aH$), the solution describes a plane wave that oscillates with a decreasing amplitude. On the other hand, on super-horizon scales ($k \ll aH$), the fluctuations of a scalar field, with a nonvanishing mass, have a tiny dependence upon time. Returning to the original field χ, the amplitude of its fluctuations on super-horizon scales is

$$|\delta\chi_k| \simeq \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu_\chi}\ . \tag{2.58}$$

Let us stress that the above discussion holds for a massive scalar field, in a quasi de Sitter stage, and it is not adequate to fully describe inflaton perturbations. To do so, one should also take into account perturbations of the metric that for simplicity we disregarded. However, it is possible to show that a proper computation of the amplitude of inflaton fluctuations gives a similar result to (2.58), with a different parameter ν, which in this case is related to the slow roll parameters via

$$\nu \simeq \frac{3}{2} - \eta + 3\epsilon\ . \tag{2.59}$$

For a comprehensive discussion, see Ref. [37].

2.6.2 Power spectrum

As mentioned above, the vacuum expectation value of the fluctuations of the inflaton field vanishes. However, the fluctuations are correlated through different points, which means that the vacuum expectation values of products of the field could be nonvanishing. One example is the two point correlation function

$$\xi(r) = \langle \delta\phi(t, \vec{x} + \vec{r})\delta\phi(t, \vec{x})\rangle \ , \tag{2.60}$$

where ξ depends only on the modulus of $\vec{r}$ under the assumption of statistical homogeneity and isotropy. Starting from the Fourier transform of the two point correlation function is possible to get the power spectrum

$$\langle \delta\phi_k(t)\delta\phi_{k'}(t)\rangle = (2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}')P(k) \ , \tag{2.61}$$

where the power spectrum is related to the inflaton fluctuation as

$$P(k) = |\delta\phi_k|^2 \ . \tag{2.62}$$

An additional frequently employed definition of the power spectrum is

$$\Delta_{\delta\phi}(k) \equiv \frac{k^3}{2\pi}P(k) = \frac{k^3}{2\pi}|\delta\phi_k|^2 = \left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu} \ . \tag{2.63}$$

Let us define now the comoving curvature perturbation as

$$\mathcal{R} = H\frac{\delta\phi_k}{\dot{\phi}} \quad , \quad \delta g_{ij,scalar} = 0 \ , \tag{2.64}$$

this is a gauge invariant quantity, evaluated in the gauge where scalar perturbations of the spatial components of the metric vanish.

It is then possible to compute the power spectrum of $\mathcal{R}_k$ on super-horizon scales as

$$\Delta_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2}\frac{H^2}{\dot{\phi}^2}|\delta\phi_k|^2 = \frac{k^3}{4M_p^2\epsilon\phi^2}|\delta\phi_k|^2 = \frac{1}{2M_p^2\epsilon}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu} \ . \tag{2.65}$$

The amplitude of the power spectrum of scalar perturbations, computed at the pivot scale $k_* = 0.002$ MpC^{-1}, is found to be [40] $(2.445 \pm 0.096) \cdot 10^{-9}$ by the COBE collaboration. From (2.65) we get the spectral index n_s as

$$n_s - 1 = \frac{d\ln\Delta_{\mathcal{R}}(k)}{d\ln k} \ , \tag{2.66}$$

and from (2.65) it is

$$n_s - 1 = 3 - 2\nu = 2\eta_V - 6\epsilon \ . \tag{2.67}$$

In the context of slow-roll inflation, the conditions $|\eta_V|, \epsilon \ll 1$ give, according to (2.67), $n_s \simeq 1$. This, in turn, for (2.66), implies an almost scale-invariant power spectrum, as the derivative of the power spectrum wrt the wave number k vanishes.

2.6.3 Tensor perturbations

Gravitational waves are produced during inflation. Tensor perturbations of the metric induced by inflation constitute a stochastic background of gravitational waves (GW), and are described by a symmetric $h_{ij} = h_{ji}$, traceless ($h^i_i = 0$) and transverse ($\partial^i h_{ij} = 0$) tensor, with only two degrees of freedom, the two possible polarization states $\lambda = +, \times$. These perturbations are related to the spatial components of the metric, as before in a gauge where there are no scalar perturbations in the spatial components of the perturbed FLRW metric, via

$$g_{ij} = a^2(t)(\delta_{ij} + h_{ij}) \ . \tag{2.68}$$

It is possible to expand the tensor perturbations into Fourier modes as

$$h_{ij}(t, \vec{x}) = \sum_{\lambda=+,\times} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} h_\lambda(t, \vec{k}) \varepsilon^\lambda_{ij}(\vec{k}) \ , \tag{2.69}$$

where $\varepsilon^\lambda_{ij}(\vec{k})$ are the symmetric, traceless and transverse polarisation tensors. Rescaling the modes as

$$v_k = \frac{a}{2} M_p h_k \ , \tag{2.70}$$

the modes v_k satisfy equation

$$v_k'' + \left(k^2 - \frac{a''}{a}\right) v_k = 0 \ . \tag{2.71}$$

On super-horizon scales, this is solved by

$$|v_k| = \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu_T} \ , \tag{2.72}$$

where

$$\nu_T \simeq \frac{3}{2} - \epsilon \ . \tag{2.73}$$

Similarly to the case of scalar perturbations, the power spectrum associated with tensor perturbations is given by

$$\Delta_T(k) = \frac{8}{M_p^2} \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{n_T} \ , \tag{2.74}$$

and hence, a corresponding spectral index

$$n_T = \frac{d \ln \Delta_T(k)}{d \ln k} = 3 - 2\nu_T = -2\epsilon. \tag{2.75}$$

Having now both the power spectrum associated with scalar and tensor perturbation, we define the tensor-to-scalar ratio, as the ratio of the two power spectra, evaluated at horizon crossing

$$r \equiv \frac{\Delta_T(k)}{\Delta_{\mathcal{R}}(k)}\bigg|_{k=aH} = \frac{8\left(\frac{H}{2\pi M_p}\right)^2}{(2\epsilon)^{-1}\left(\frac{H}{2\pi M_p}\right)^2} = 16\epsilon = -8n_T \ . \tag{2.76}$$

The relation $r = 8n_T$ is also known as the consistency relation and holds for every single scalar field model of inflation.

2.6.4 Observable prediction

The power spectra of scalar and tensor perturbations introduced four degrees of freedom, two amplitudes, and two spectral tilt. However, of these degrees of freedom one can be fixed employing the consistency relation, while one amplitude can be fixed employing the normalization of CMB anisotropies[3], set from COBE [41] to be of the order of 10^{-5}. Thus, only two degrees of freedom are necessary to build the parameter space, and consequently to classify inflationary models. The parameters used are (r, n_s), it is then possible to plot in this plane the prediction of different inflationary models, and then compare them with the Planck observations.

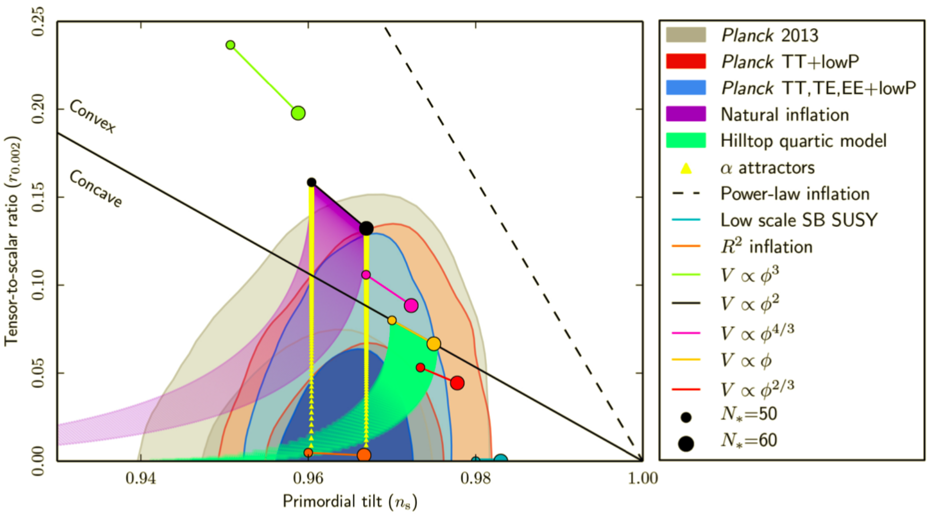

Figure 2.5: Marginalized joint 68% and 95% CL regions for n_s and r at $k = 0.002$ Mpc^{-1} from Planck compared to the theoretical prediction of selected inflationary models. It has been assumed $dn_s/d\ln k = 0$. Source [42].

The predictions to lowest order in the slow roll approximation for (n_s, r) at $k = 0.002$ Mpc^{-1} of a few inflationary models, assuming an uncertainty for the total number of e-folds $(50 < N < 60)$ are shown in figure 2.5.

[3] In fact, being $\mathcal{R}$ constant on super-horizon scales, it can be used to relate the scalar perturbations $\delta\phi$ that crossed out the horizon during inflation, to those that reentered the horizon during a radiation dominated epoch, that we observe as temperature anisotropies in the CMB spectrum.

Chapter 3

Amber-Sorbo Model

The inflationary universe was first introduced in [1] to solve some inconsistencies of the HBB model. The theory of inflation became quickly popular not only because it provides appropriate initial conditions for a homogeneous and isotropic universe, but also because it naturally generates primordial fluctuations that eventually evolve into CMB anisotropies and later on into large scale structures. In its simplest realization, inflation is driven by a single scalar field, slowly rolling down its potential, acting as a source that dominates the energy budget of the universe, with an equation of state $\rho \approx -P$. As we have seen, this requires a very flat inflaton potential. From a model building point of view, inflationary models typically suffer from being sensitive to unknown ultraviolet (UV) particle physics. The Standard Model (SM) of particle physics is in fact an Effective Field Theory (EFT) whose cut-off scale can be pushed at most up to the Planck scale, where quantum gravity is expected to become relevant. It is also possible that new physics arises before the Planck scale, possibly explaining the field content and the value assumed by some parameters of the Standard Model. Examples of such physics are supersymmetry and grand unified theories (GUT). New physics can spoil the required flatness of the potential via radiative corrections. One way to protect the potential and hence the inflaton dynamics is to assume some symmetry that prevents interactions between the inflationary sector and other particles. In 1990, a model of natural inflation was introduced, featuring a pseudo-Nambu-Goldstone boson (pNGB), an axion-like particle, as inflaton. The coupling of the axion ϕ to matter enjoy a shift symmetry $\phi \to \phi + \text{constant}$, which prevent them from generating (at the loop level) a potential term for the inflaton. This shift symmetry is broken (typically, by instanton effects [10]) to generate a potential that, to leading order acquires the form,

$$V(\phi) = \Lambda^4 \left[1 - \cos\left(\frac{\phi}{f}\right)\right] \ , \tag{3.1}$$

where f is the so called axion decay constant, while Λ^4 is the scale of the potential, and hence of the energy density that drove inflation.

In this context, in 2010 Amber and Sorbo [14] explored the idea of introducing a coupling between the axion and some U(1) gauge group, leading to a backreaction term in the equation of motion of the inflaton that provides an additional friction term. This constitutes a realization of the idea of warm inflation [15]. The following chapter will present the dynamics and the phenomenology of this model of inflation.

3.1 Natural inflation

The slow roll prescription permits to realize inflation from a broad range of initial conditions. However, not all models of slow roll inflation are devoid of fine tuning issues, indeed, as one deepens the particle physics structure of inflationary models, one finds that radiative corrections arising from self-interactions, or interactions with other particles, usually spoil the flatness of the potential making inflation end too soon. This is also known as $\eta-$problem. Natural inflation models avoid this problem [9], by using an axionic field as inflaton candidate, that initially enjoys shift symmetry $\phi \to \phi + c$, giving rise naturally to a flat potential $V(\phi)$, that eventually gets explicitly broken, and the axion acquires a potential of the form (3.1) becoming a pNGB. In models with a large global symmetry breaking scale f, PNGBs are very weakly interacting, since their couplings are suppressed by inverse powers of f. As a result of the symmetry breaking the potential becomes periodic (3.1) and an inflationary epoch can be achieved for $f \sim M_p$ and $\Lambda \sim M_{GUT}$, mass scales that arise naturally in particle physics models [9].

In the previous chapter (subsection 2.6.4), it was discussed that the tensor-to-scalar ratio r and the spectral tilt n_s are the two observable parameters of the primordial power spectrum used to evaluate the validity of inflationary models. Observations of these two parameters [42], entirely rule out standard natural inflation, which as shown in figure 2.5, lies completely outside the 95% CL region. Examples of possible approaches to construct natural inflation models that align with the present constraints on the values of r and n_s are multi-axions models [43, 44], or models that modify the inflaton dynamics by including additional friction term coming from coupling with other particle species [14].

3.2 The AS action

Soon after its introduction, natural inflation became popular as it seems to cure the UV problem of inflationary models because axions are the simplest spin-zero degree of freedom with a non-trivial radiatively stable potential, and are largely produced in beyond standard model theories. However there is a flaw, it was shown in [9] that to sustain a sufficient amount of inflation it is necessary to have $f \sim M_p$. While this is not an issue for QFT, it represents a problem for quantum gravity, which is expected to destroy every global symmetry (hence the shift symmetry in the context of natural inflation) on super-Planckian scales [11]. Moreover, in string theory explicit realization, it always holds $f < M_p$ [12].

To overcome these problems, Amber and Sorbo proposed a realization of natural inflation [14] for which inflation can occur for steep potential, even if $f \ll M_p$, if one includes an interacting term between the axion and some $U(1)$ gauge field A_μ, via a dimension-5 operator $\Phi F_{\mu\nu}\tilde{F}^{\mu\nu}$. Thus the full action of a minimally coupled to gravity inflation, in the AS model reads

$$S = \int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}R - \frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi - V(\Phi) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\alpha}{4f}\Phi F_{\mu\nu}\tilde{F}^{\mu\nu}\right] , \tag{3.2}$$

where Φ is the inflaton field, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the field strength of the gauge field A^μ, $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is its Hodge dual, with the Levi-Civita tensor defined such that $\varepsilon^{0123} = 1/\sqrt{-g}$. The other quantities appearing in (3.2) are α a dimensionless constant related to the coupling strength, while f is the mass dimension one axion decay constant, and the potential the one already seen for natural inflation (3.1).

3.2.1 Background solution

To study the dynamics of the background, perturbations of the metric and of the inflaton are going to be neglected, thus, in conformal time (1.6), the inflaton is described by a homogeneous field $\Phi(\tau)$, while the gauge field $A_\mu(\tau, \vec{x})$ is evaluated in the gauge $A_0 = \vec{\nabla} \cdot \vec{A} = 0$. Moreover the classical field $\vec{A}(\tau, \vec{x})$ is promoted to a quantum operator $\hat{\vec{A}}(\tau, \vec{x})$, whose decomposition into creation and annihilation operators, in Fourier space read

$$\hat{\vec{A}}(\tau, \vec{x}) = \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^{3/2}} \left[\vec{\epsilon}_\lambda(\vec{k}) A_\lambda(\tau, k) a_\lambda^{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + \text{h.c.}\right] , \tag{3.3}$$

where $\lambda = \pm$ indicate the two helicity states, while $\vec{\epsilon}_\pm$ are the helicity vectors, defined such that

$$\begin{aligned} &\vec{k} \cdot \vec{\epsilon}_\pm = 0 \\ &\vec{k} \times \vec{\epsilon}_\pm = \mp i |\vec{k}| \vec{\epsilon}_\pm \ , \end{aligned} \tag{3.4}$$

and the annihilation and creation operators obey

$$\left[a_\lambda\left(\vec{k}\right), a_{\lambda'}^\dagger\left(\vec{k}'\right)\right] = \delta_{\lambda\lambda'} \delta(\vec{k} - \vec{k}') \ . \tag{3.5}$$

Under these assumptions, the equations of motion for the inflaton Φ and the mode functions A_λ are

$$\begin{aligned} &\Phi'' + 2aH\Phi' + a^2 \frac{\partial V(\Phi)}{\partial \Phi} = -\frac{\alpha^2}{4\pi^2 a^3 f} \int dk\ k^2 \frac{\partial}{\partial \tau} \left[|A_+|^2 - |A_-|^2\right] \ , \\ &A_\lambda'' + \left[k^2 \pm 2k\frac{\xi}{\tau}\right] A_\lambda = 0 \ , \end{aligned} \tag{3.6}$$

where prime denotes the derivative wrt conformal time. Furthermore, it has been assumed the limit of quasi de-Sitter space-time $a(\tau) = -1/(H\tau)$, and defined

$$\xi \equiv \frac{\alpha \Phi'}{2faH} \ , \tag{3.7}$$

the parameter controlling the production of gauge degrees of freedom, assumed to be constant during inflation. The standard procedure involves solving the second equation for A_λ initially, resulting in a solution that is a functional of the inflaton field, which is then replaced into the inflaton equation of motion, obtaining an integro-differential equation for the homogeneous field Φ. Solving the equation for the gauge field, one finds that, depending on the sign of the inflaton velocity, and hence of ξ, one of the two helicity modes is strongly amplified near horizon crossing. Given the symmetry of the system, it is possible to assume, without loss of generality, a positive velocity for the inflaton, $\dot{\phi} > 0$, such that the enhanced term is A_+ while the other mode A_- can be disregarded. For this reason, from now on we set $A = A_+$, disregarding the negligible polarization. The exact solution to the second equation of the system (3.6) involves the regular and irregular Coulomb wave functions, F_0 and G_0, and it reads

$$A(\tau, k) = \frac{1}{\sqrt{2k}} [iF_0(\xi, -k\tau) + G_0(\xi, -k\tau)] \ . \tag{3.8}$$

Defining $x = -k\tau$ an approximate expression for the solutions on large scales, is given by [14]

$$\begin{aligned} A_1(\tau, k)\Big|_{x \ll 2\xi} &\simeq \frac{1}{\sqrt{2k}} \left[\left(\frac{x}{2\xi}\right)^{1/4} e^{\pi\xi - 2\sqrt{2\xi x}} + \frac{i}{2}\left(\frac{x}{2\xi}\right)^{1/4} e^{2\sqrt{2\xi x} - \pi\xi} \right] , \\ A_2(\tau, k) &= A_1^*(\tau, k) \ , \end{aligned} \tag{3.9}$$

normalized by the relation $A_1 A_2' - A_2 A_1' = i$. In the regime of unstable growth, the real part dominates and it is possible to further approximate the solution to

$$A(\tau, k) \simeq \frac{1}{\sqrt{2k}} \left(\frac{k}{2\xi a H}\right)^{1/4} e^{\pi\xi - 2\sqrt{\frac{2\xi k}{aH}}} \ . \tag{3.10}$$

By inserting this expression in (3.6) and performing the integration, we obtain

$$\Phi'' + 2aH\Phi' + a^2 \frac{\partial V(\Phi)}{\partial \Phi} \simeq -\frac{a^2 \alpha}{f} \frac{315 H^4}{2^{17}\pi^2} \frac{e^{2\pi\xi}}{\xi^4} \ . \tag{3.11}$$

Employing the slow roll conditions, and assuming that the dominant friction term is the one related to the gauge field production, the first two terms at the left hand side of (3.11) can be neglected, so that this relation simplifies to

$$\frac{\alpha}{f} \frac{315 H^4}{2^{17}\pi^2} \frac{e^{2\pi\xi}}{\xi^4} \simeq -\frac{\partial V(\Phi)}{\partial \Phi} \ . \tag{3.12}$$

3.2.2 Perturbations

An analytical study of perturbations on the AS solution has been done for the first time in [45]. Denoting from now on quantities related to the background with $\bar{\Phi}$ and $\bar{A}$, the fields are now decomposed into

$$\begin{aligned} \Phi &= \bar{\Phi} + \delta\Phi \ , \\ A &= \bar{A} + \delta A \ . \end{aligned} \tag{3.13}$$

The goal is to solve equations (3.6) to first order in the perturbations $\delta\Phi$ and δA. However, it is important to stress that this does not represent a complete perturbative study of the AS solution as metric perturbations, possible inhomogeneities of the inflation, and perturbations of H will be disregarded. However, this is still an interesting case study since it analyses from an analytical point of view the systems studied numerically in previous works [17–21, 23–25], where the instability of the AS solution (3.12) has been observed.

Perturbing the equations of motion of the two fields (3.6), the resulting system is

$$\begin{aligned} &\delta\Phi'' + 2aH\delta\Phi' + a^2 \frac{\partial^2 V(\Phi)}{\partial \Phi^2}\delta\Phi = -\frac{\alpha}{f a^2} \int \frac{d^3k}{(2\pi)^3} \frac{k}{2} \frac{\partial}{\partial \tau}[\bar{A}\delta A^* + \bar{A}^*\delta A] \ , \\ &\delta A'' + \left(k^2 - \frac{k\bar{\Phi}'}{f}\right)\delta A = \frac{\alpha \bar{A}}{f}\delta\Phi' \ . \end{aligned} \tag{3.14}$$

This time, the second equation can be solved by employing the Green function method. To do so, it is necessary to first find the Green function that satisfies

$$\left[\frac{\partial^2}{\partial\tau^2} + k^2 - \frac{k\bar{\Phi}'}{f}\right] G_k(\tau,\tau') = \delta(\tau-\tau') \ . \tag{3.15}$$

The Green function is built from the solutions of the homogeneous equation associated to the second equation in (3.14) (namely, this equation with the right hand side set to zero). Such solutions have been already given in eq. (3.9) above. Given those solutions $\bar{A}_1$ and $\bar{A}_2$, a Green function that solves equation (3.15) is given by the combination

$$G_k(\tau,\tau') = \frac{\bar{A}_1(\tau)\bar{A}_2(\tau') - \bar{A}_1(\tau')\bar{A}_2(\tau)}{W(\tau')}\theta(\tau-\tau') \ , \tag{3.16}$$

where θ is a Heaviside function, with the dependence on τ and τ' chosen so to respect causality, and where $W(\tau') = \bar{A}_1'(\tau')\bar{A}_2(\tau') - \bar{A}_1(\tau')\bar{A}_2'(\tau')$ the Wronskian. This leads to

$$G_k(\tau,\tau') \simeq -\frac{(xx')^{1/4}}{\sqrt{2\xi}k}\sinh\left[2\sqrt{2\xi}\left(\sqrt{x}-\sqrt{x'}\right)\right]\theta(x'-x)\theta(2\xi\gamma^2-x') \ , \tag{3.17}$$

where the last factor has been added to account for the fact that the form (3.9) of the solutions that we employ is only valid for x and $x' > 2\xi$. This term effectively serves for the UV regularization of the mode functions. In this expression, we introduced an order one constant γ, as a measure of the uncertainty of the regularization [45].

Having obtained the Green function that satisfies equation (3.15), we can now employ it in the solution of the second equation of (3.14), which formally reads

$$\delta A(\tau,k) = \frac{\alpha k}{f}\int^{\tau} d\tau' G_k(\tau,\tau')\bar{A}(\tau',k)\delta\Phi'(\tau') \ . \tag{3.18}$$

Inserting this solution into the first equation of (3.14), one finds

$$\begin{aligned}\delta\Phi'' + 2aH\delta\Phi' + a^2V''\delta\Phi = &-\frac{\alpha^2}{f^2a^2}\int^{\tau} d\tau'\delta\Phi(\tau') \\ &\times\frac{\partial}{\partial\tau}\int\frac{d^3k}{(2\pi)^3}\frac{k^2}{2}\left[G_k^*(\tau,\tau')\bar{A}(\tau)\bar{A}^*(\tau') + \text{c.c.}\right] \ .\end{aligned} \tag{3.19}$$

3.2.3 Analytical solutions

Solutions of (3.19) have been studied in ref. [45]. To do so, it is necessary to insert the form of the Green function (3.17), and the dominant term of the gauge field mode (3.9), into equation (3.19) and defining a new integration variable as $y \equiv -2\xi k\tau'$, it reads [45]

$$\begin{aligned}&\delta\Phi'' + 2aH\delta\Phi' + a^2V''\delta\Phi = \\ &= \frac{\alpha^2}{f^2a^2}\frac{e^{2\pi\xi}}{2^8\pi^2\xi^5}\int^{\tau}\frac{d\tau'}{(-\tau')^4}\delta\Phi'(\tau')\frac{\partial}{\partial\tau}\int_0^{4\xi_\gamma^2} dy\, y^3\sqrt{\tau\tau'}\left[e^{-4\sqrt{y}} - e^{-4\sqrt{y}\sqrt{\frac{-\tau}{-\tau'}}}\right] \ ,\end{aligned} \tag{3.20}$$

where it has been defined $\xi_\gamma \equiv \xi\gamma$.

Noticing that, under the working assumption of disregarding slow roll variations, every term of (3.20) evolves in time as $\delta\Phi/\tau^2$, it is reasonable to look for "power law" solutions of the type

$$\delta\Phi = C(-\tau)^{-\frac{1+\zeta}{2}} , \tag{3.21}$$

where C and ζ are constants, however, due to the linearity of the equation, the constant $C \neq 0$ can be neglected.

Since equation (3.20) has real coefficients, the solution can be complex[1], allowing ζ to have an imaginary part. In this case, we must impose Re $\zeta > -8$ otherwise the integral (3.20) would diverge at $\tau' = -\inf$. Linear combinations of these solutions [45] are of the form

$$\delta\Phi \propto (-H\tau)^{-\frac{1+\text{Re}\zeta}{2}} \cos\left(\frac{\text{Im}\zeta}{2}\ln(-H\tau) + \varphi\right) , \tag{3.22}$$

where φ is an arbitrary phase. This result shows that a complex ξ corresponds to an oscillatory solution about the AS one. Moreover, the oscillations converge for Re $\xi < -1$, while they depart from the AS solution for Re $\xi > -1$.

3.3 Time lag of the backreaction

Having shown that the AS solution (3.12) is unstable analytically, we still need to provide the physical and more intuitive reason for this instability. When the system was solved analytically, various assumptions were employed, with one of the key assumptions being that the velocity of the inflaton field to be constant, motivated by the slow-roll approximation. However, this approximation breaks down in the regime of substantial gauge field production because the backreaction term in the inflaton equation will affect severely the evolution of the inflaton, reflecting in variations in the inflaton velocity when the backreaction dominates. Different numerical analyses of the AS model have been conducted [17–21, 23–25], showing that gauge field production, backreacting on this non-linear system, results in an oscillatory behaviour in the inflaton velocity. This is precisely the result that was obtained analytically in [45] and that was reviewed in the previous subsection. Among the works that obtained the instability numerically, ref. [20] also explained that this oscillatory behaviour could be traced back to a time delay in the backreacting term wrt a variation in the inflaton velocity. To get to this result, it is necessary to run through the equation of motion of the background fields once again.

The system of equations studied in [20] was slightly different from the one presented in the previous section (3.6), as the time variable used is the number of e-folds N

$$N \equiv \int aH d\tau . \tag{3.23}$$

We can use this relation to "trade" derivatives wrt to conformal time to derivatives wrt number of e-folds

$$\begin{aligned} \frac{d}{d\tau} &= aH\frac{d}{dN} \\ \frac{d^2}{d\tau^2} &= a^2H^2\left(\frac{d^2}{dN^2} + (1-\varepsilon)\frac{d}{dN}\right) , \end{aligned} \tag{3.24}$$

where $\varepsilon = -H'/H \ll 1$ is the slow-roll parameter. With this new re-parametrization, the equations of motion assume the form

[1] In this case the complex solution is always accompanied by its conjugate.

$$\begin{aligned}
&\Phi'' - \varepsilon\Phi' + 3\Phi' + \frac{1}{H^2}\frac{\partial V}{\partial \Phi} - \frac{1}{fH^2}\langle \vec{E}\cdot\vec{B}\rangle = 0 \\
&A''_\lambda(\vec{k}) + (1-\varepsilon)A'_\lambda(\vec{k}) + \frac{k}{aH}\left(\frac{k}{aH} + 2\lambda\xi\right)A_\lambda(\vec{k}) = 0 \ ,
\end{aligned} \tag{3.25}$$

where prime denotes the derivative wrt N, and where the backreaction term, assuming the velocity of the inflaton field to be negative, and hence considering only the enhanced mode A_-[2], is given by

$$\langle \vec{E}\cdot\vec{B}\rangle = -\frac{1}{a^4}\int \frac{dk}{4\pi}k^3\frac{d}{d\tau}|A_-(\tau,\vec{k})|^2 \ . \tag{3.26}$$

Inserting the solution (3.10) and computing the integral, equation (3.26) is approximated by

$$\langle \vec{E}\cdot\vec{B}\rangle \simeq -\frac{e^{2\pi\xi}}{2^{21}\pi^2\xi^4}H^4\int_0^{x_{uv}} x^7 e^{-x} \simeq -2.4\cdot 10^{-4}\frac{H^4}{\xi^4}e^{2\pi\xi} \ , \tag{3.27}$$

being $x_{uv} \sim 8\xi$ a cut-off scale preventing UV divergence, while $x \equiv 4\sqrt{2\xi k/(aH)}$.
To close the system of equations, it is necessary to also consider Friedmann equation

$$3H^2M_P^2 = V(\Phi) + \frac{1}{2}H\Phi'^2 + \left\langle \frac{E^2+B^2}{2}\right\rangle \ , \tag{3.28}$$

where the energy density associated with the gauge field, considering only the enhanced mode, is given by

$$\left\langle \frac{E^2+B^2}{2}\right\rangle = \frac{1}{a^4}\int \frac{dk}{4\pi^2}k^2\left(\left|\frac{dA_-(\tau,\vec{k})}{d\tau}\right|^2 + k^2\left|A_-(\tau,\vec{k})\right|^2\right) \ , \tag{3.29}$$

which, adopting the same line of reasoning as for (3.27), becomes

$$\left\langle \frac{E^2+B^2}{2}\right\rangle \simeq \frac{e^{2\pi\xi}}{2^{19}\pi^2\xi^3}H^4\left[\int_0^{x_{uv}} x^6 e^{-x}dx + \frac{1}{2^6\xi^2}\int_0^{x_{uv}} x^8 e^{-x}dx\right] \simeq 1.3\cdot 10^{-4}\frac{H^4}{\xi^3}e^{2\pi\xi} \ . \tag{3.30}$$

When the production of gauge field becomes efficient, which means $\xi \gtrsim 1$ (actually it is necessary to require $\xi \gtrsim 3$ since smaller values require a more careful regularization scheme [46, 47]), the equation for the enhanced mode A_- (3.25) becomes tachyonic when

$$m^2_{\lambda=-1} = \frac{k}{aH}\left(\frac{k}{aH} - 2\xi\right) < 0 \ , \tag{3.31}$$

that occurs for $\frac{k}{aH} < 2\xi$.

At $\frac{k}{aH} = \xi$, $m^2_{\lambda=-1}$ reaches its minimum $-\xi^2$, this corresponds to the peak growth of the mode A_-. However, due to the derivative wrt τ and the k^4 prefactor, the integrand of the backreaction

[2] We perform this choice here, which is opposite to the one we made above, to match the notation of ref. [20].

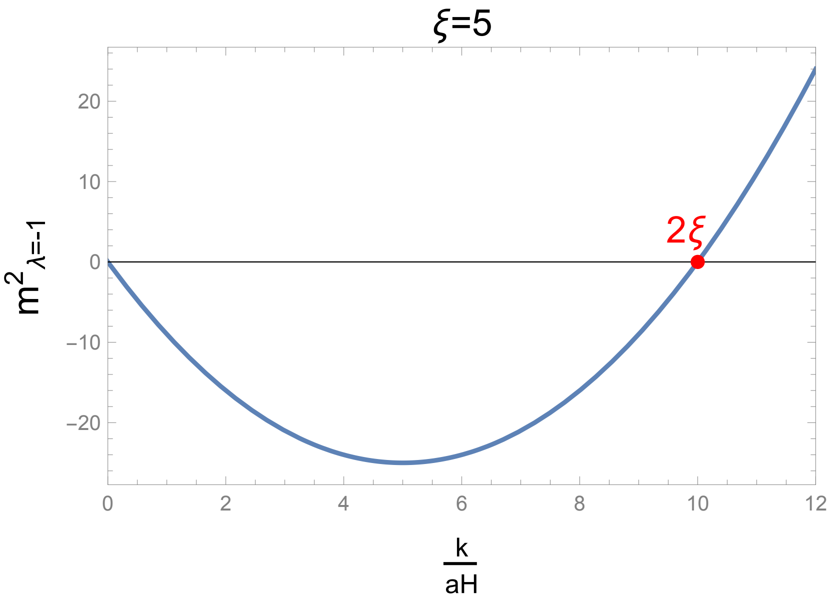

Figure 3.1: Mass term of equation (3.25) for $\lambda = -1$ and $\xi = 3$.

takes its maximum value at approximately $\frac{k}{aH} = \frac{2}{\xi}$ (see fig. 3.2), this means that there is a delay in the response of the backreaction of about

$$\Delta N_\xi \simeq -\ln\frac{2}{\xi} + \ln\xi = \ln\frac{\xi^2}{2} \ , \tag{3.32}$$

e-folds. Finally, the instability freezes out when the friction term $\sim A'_-$ takes over, at $\frac{k}{aH} < \frac{1}{2\xi}$ when the squared "mass term" becomes larger than -1.

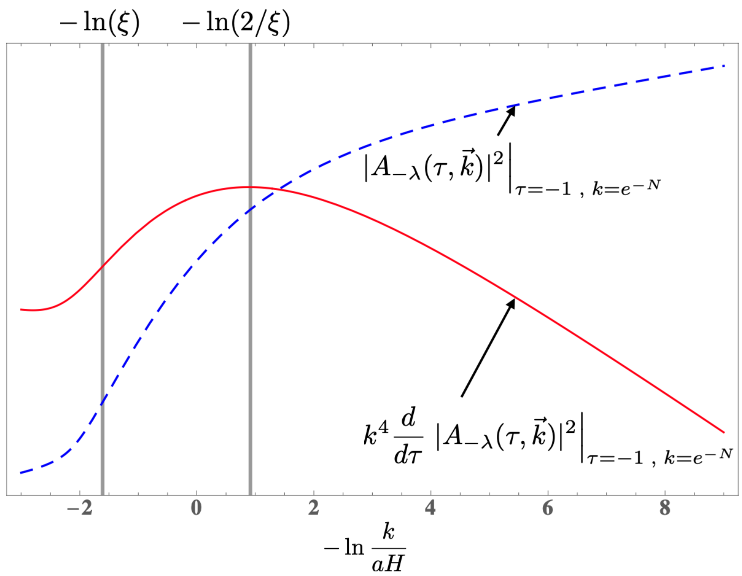

Figure 3.2: Blue dash: the square of the mode $|A_-(\tau,\vec{k})|^2$. Red solid: integrand of the backreaction term $\langle\vec{E}\cdot\vec{B}\rangle$. The curves are displayed as a function of k such that $\ln\left(\frac{k}{aH}\right) = 0$ correspond to horizon-sized modes. Here ξ is assumed constant and it has been fixed to $\xi = 5$. Both curves are evaluated at $\tau = -1$ and the wavenumber k is expressed as the number of e-folds after horizon crossing $-\ln\frac{k}{aH}$. Source [20].

Due to the delayed response, at the moment of maximal growth $\frac{k}{aH} = \xi > 1 \Rightarrow -\ln\xi < 0$ the gauge mode still sits inside the horizon, while the largest contribution to the backreaction occurs when they are super-horizon $\frac{k}{aH} = \frac{2}{\xi} < 1 \Rightarrow -\ln\frac{2}{\xi} > 0$.

Ref. [20] showed that, due to the delay, the backreaction prevents an adiabatic evolution of the inflaton velocity, in contrast to the assumption made to arrive to the AS solution (3.12). Instead, it gives a non-monotonic evolution. To see this, they studied how $\langle \vec{E} \cdot \vec{B} \rangle$ changes in response to a sudden variation of ξ. Starting from an initial condition $\xi = \xi_0$, they assumed that at a certain $N = N_0$, the parameter ξ suddenly changes to $\xi = \xi_1 = \xi_0 + \Delta\xi > \xi_0$. This variation has been modeled as

$$\xi(N) = \xi_0 + \frac{1}{2}\Delta\xi(1 + \tanh(\mu_\xi(N - N_0))) \ . \tag{3.33}$$

For $N \leq N_0$, the major contribution to the backreaction (3.26) comes from modes $A_-(\vec{k})$ with $\frac{k}{aH} \simeq \frac{2}{\xi_0}$, and hence from the solution (3.27)

$$|\langle \vec{E} \cdot \vec{B} \rangle_{N_0}| \simeq 2.4 \cdot 10^{-4} \frac{H^4}{\xi_0^4} e^{2\pi\xi_0} \ , \tag{3.34}$$

Conversely, the integral gets dominated by modes $A_-(\vec{k})$ with $\frac{k}{aH} \simeq \frac{2}{\xi_1}$ at latter e-folds $N = N_0 + \Delta N_\xi$, and the amplitude of the backreaction becomes

$$|\langle \vec{E} \cdot \vec{B} \rangle_{N_0+\Delta N_\xi}| \simeq 2.4 \cdot 10^{-4} \frac{H^4}{\xi_1^4} e^{2\pi\xi_1} \ > \ |\langle \vec{E} \cdot \vec{B} \rangle_{N_0}| \ . \tag{3.35}$$

Figure (3.3) shows the effects of the delayed response of the backreaction to sudden variations in the inflaton velocity, that reflect in a transition from $\xi = 5$ to $\xi = 6$, described by (3.33), and it also shows how the transition occurs smoothly.

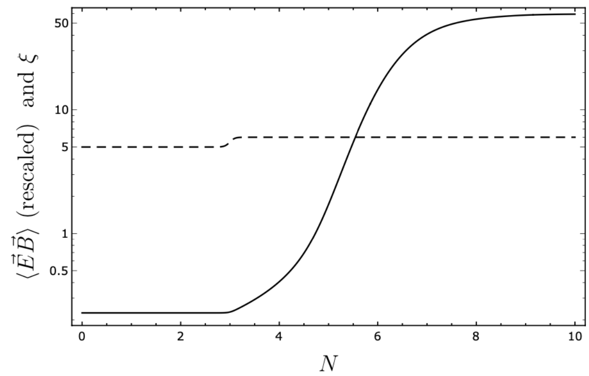

Figure 3.3: Black solid: numerically computed response of $\langle \vec{E} \cdot \vec{B} \rangle$ to a change of ξ. (Black dash) obtained setting: $N_0 = 3$, $\xi_0 = 5$, amplitude $\Delta\xi = 1$, and steepness $\mu_\xi = 10$, in eq. (3.33). Source [20].

The asymptotic behaviours of the backreaction are: for small N given by (3.34) where $\langle \vec{E} \cdot \vec{B} \rangle \simeq 0.25$ GeV^4. Conversely, after the rapid variation of ξ, the backreaction at large N asymptotically reaches the value given by (3.35) of $\langle \vec{E} \cdot \vec{B} \rangle \simeq 51$ GeV^4.

3.4 Numerical result

The numerical simulations carried out in [20] were aimed at solving the system made by equations (3.25) and (3.26), and have been done taking values of the axion decay constant such that

$M_P/f = 20, 25$, and potential of the form $V(\Phi) = m^2\Phi^2/2$ with $m = 6 \cdot 10^{-6} M_P$. The evolution equations were solved iteratively, in the sense that we now explain. The first step of the iteration is to gather an estimate on the backreaction and the energy density of the gauge field, assuming ξ constant solving

$$\begin{aligned}\langle \vec{E} \cdot \vec{B} \rangle_0 &= \frac{1}{2^{21}\pi^2} \frac{H_0^4}{\xi^4} e^{2\pi\xi} \int_0^{8\xi} x^7 e^{-x} dx \\ \left\langle \frac{E^2 + B^2}{2} \right\rangle_0 &= \frac{6!}{2^{19}\pi^2} \frac{H_0^4}{\xi^3} e^{2\pi\xi} ,\end{aligned} \tag{3.36}$$

where H_0 is calculated via the Friedmann equation (3.28) neglecting the backreaction. At this point, the new estimate of the backreaction is plugged into the equation of the inflaton field (3.25), which is then solved. The second step consists of solving the equation for the gauge field in (3.25) for an appropriate array of modes k, focusing on the modes experiencing the tachyonic instability, solving equation (3.28) taking into account the energy density coming from (3.36). One then plugs the gauge mode function obtained in this way into the discretized integral (3.36), so to obtain an improved backreaction term. This cycle is then repeated, starting from the improved backreaction. The iterations stop when there is no appreciable difference between two consecutive solutions. Although not proven in ref. [20] that the process consistently converges at a reasonable rate, once convergence occurs, as in the analysis shown above, the result is a self-consistent solution of the system of equations (3.25) and (3.26).

The initial conditions for the inflaton field have been chosen at the CMB scale in accordance with the slow-roll solution in the absence of coupling to the gauge field. The results of the numerical simulation exhibit the oscillatory behaviour of the inflaton velocity. Figure (3.4) displays the results obtained in [20] where they plot the parameter ξ as a function of the number of e-folds, and for two different values of f, showing how oscillations get more pronounced for stronger backreaction. This oscillatory behaviour from the AS solution was then confirmed analytically by [45], as we studied above.

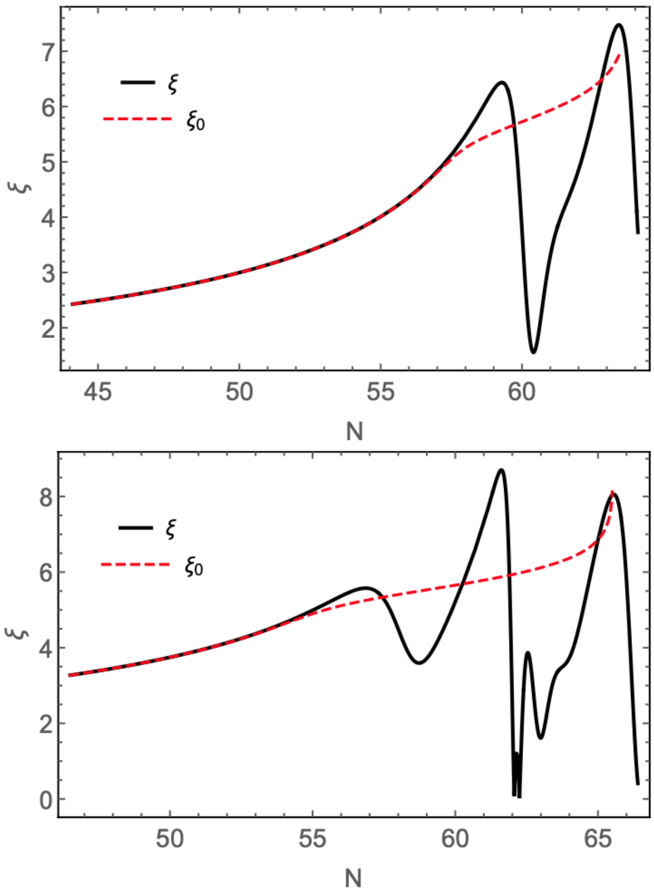

Figure 3.4: Top: $1/f = 20$. Bottom: $1/f = 25$. Comparison between the numerical simulation (Solid Black) and the analytical result of the inflaton equation of motion assuming the backreaction fixed by (3.27) (Red Dashed). Source [20].

Chapter 4

EOM in a generic gauge

The analytical study of the AS model is a complicated task. To simplify things, it is unavoidable to exploit some assumptions that simplify the equations. The main assumption made in both analytical [45] and numerical [17–21, 23–25] studies was the homogeneity of the inflaton $\partial_i \Phi = 0$. This allows to specify the gauge $A_0 = \partial_i A^i = 0$. Among the numerical studies mentioned above, only [23] studied the backreaction, including inflaton inhomogeneities. However, they studied only the last few (~ 5) e-folds of inflation, and mainly focused on the preheating phase of inflation. Moreover, their results exhibit a strong sensitivity on the parameters involved. For these reasons, it seems important to make progress on the analytical side, and this chapter represents a first step.

In the following chapter, we are going to derive the master equation for the inflaton-gauge field system, including inflaton inhomogeneities. This complete equation has never been computed in the literature. We will use conformal time τ, hence our background metric will be $g_{\mu\nu} = a^2(\tau)\text{diag}(-,+,+,+)$. We will indicate with $' = \frac{d}{d\tau}$ while with $\dot{} = \frac{d}{dt}$, moreover, the two derivative operators are related by $\dot{\phi} = \frac{1}{a}\phi'$. Lastly, we define $\mathcal{H} = \frac{a'}{a}$.

4.1 Equations of motion in real space

Taking into account also spatial inhomogeneities of the inflaton field, we can fix the gauge $\partial_i A^i = 0$, so that the spatial components A_i can still be written as the sum of the two helicity states as expressed in equation (3.3). However, for a non homogeneous inflaton, this gauge condition is no longer compatible with $A_0 = 0$, so that the temporal component of the gauge field needs to be taken into account.

Let us start by computing the equations of motion for the gauge field A_μ, considering how the action (3.2) varies at linear order under $A^\mu \to A^\mu + \delta A^\mu$. Under this transformation, the two terms of the action proportional to the gauge field become

$$\frac{1}{2} F^{\mu\nu} \delta F_{\mu\nu} = -\partial_\nu \left(F^{\mu\nu} \delta A_\mu \right) + \partial_\nu F^{\mu\nu} \delta A_\mu \tag{4.1}$$

$$\frac{\alpha}{2f} \phi \tilde{F}^{\mu\nu} \delta F_{\mu\nu} = -\partial_\nu \left(\frac{\alpha}{f} \phi \tilde{F}^{\mu\nu} \delta A_\mu \right) + \frac{\alpha}{f} \partial_\nu \phi \tilde{F}^{\mu\nu} \delta A_\mu + \frac{\alpha}{f} \phi \partial_\nu \tilde{F}^{\mu\nu} \delta A_\mu \ . \tag{4.2}$$

At this point, we apply the least action principle and, by eliminating the surface terms, we obtain the covariant expression

$$F^{\mu\nu}_{\ ;\nu} + \frac{\alpha}{f}\phi_{,\nu}\tilde{F}^{\mu\nu} + \frac{\alpha}{f}\phi\tilde{F}^{\mu\nu}_{\ ;\nu} = 0 \ , \tag{4.3}$$

where for the inflaton the covariant derivative reduces the normal one since it is a scalar field. However, for the dual $\tilde{F}^{\mu\nu}$ the covariant derivative is always vanishing, so we are left with

$$F^{\mu\nu}_{\ ;\nu} + \frac{\alpha}{f}\phi_{,\nu}\tilde{F}^{\mu\nu} = 0 \ . \tag{4.4}$$

From these equations, we derive the scalar and vectorial equations for the gauge field A_μ

$$\partial_i\partial_i A_0 + \frac{\alpha}{f}\phi_{,i}\epsilon^{0ijk}\partial_j A_k = 0 \ . \tag{4.5}$$

$$\partial_0\partial_0 A_i - \partial_j\partial_j A_i - \partial_0\partial_i A_0 - \frac{\alpha}{f}\phi_{,0}\epsilon^{0ijk}\partial_j A_k + \frac{\alpha}{f}\phi_{,j}\epsilon^{0ijk}\left(\partial_0 A_k - \partial_k A_0\right) = 0 \ , \tag{4.6}$$

where from now on, repeated lower indices are summed using a Kronecker delta (for the full computation, see Appendix A). Equation (4.5) shows that we can fix $A_0 = 0$ only in the case of homogeneous inflaton. Now, as anticipated at the beginning of this section, we are not allowed to do it anymore. However, this equation also shows that A_0 is not a dynamic degree of freedom, but it is determined in terms of the spatial components, and therefore can be easily integrated out, as we do in the following.

The equation of motion for the inflaton, which comes from the action (3.2) in the $\partial^i A_i$ gauge, is

$$\phi'' + 2\mathcal{H}\phi' - \nabla^2\phi + a^2\frac{\partial V}{\partial\phi} = -\frac{\alpha}{4f}a^2 F_{\mu\nu}\tilde{F}^{\mu\nu} \ . \tag{4.7}$$

Writing the right-hand side in terms of the latter relation in terms of A, we get

$$\phi'' + 2\mathcal{H}\phi' - \nabla^2\phi + a^2\frac{\partial V}{\partial\phi} = -\frac{\alpha a^{-2}}{f}\epsilon^{ijk}(\partial_0 A_i - \partial_i A_0)\partial_j A_k \ . \tag{4.8}$$

For the full derivation, see [Appendix B].

4.2 Equations of motion in Fourier space

The purpose of the following section is to write a linearized system of equations for the perturbations of the two fields, on the AS solutions that we take as background, in Fourier space. From now on, background and perturbations will be separated according to

$$\begin{aligned} \phi(\tau,\vec{x}) &= \bar{\phi}(\tau) + \delta\phi(\tau,\vec{x}) \\ A_i(\tau,\vec{x}) &= \bar{A}_i(\tau,\vec{x}) + \delta A_i(\tau,\vec{x}) \ , \\ A_0(\tau,\vec{x}) &= A_0(\tau,\vec{x}) \end{aligned} \tag{4.9}$$

being A_0 already a perturbation, not present in the AS model. For what concerns the inflaton and the spatial components of the gauge field, in momentum space, they read

$$
\begin{aligned}
\phi(\tau,\vec{x}) &= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} e^{i\vec{k}\cdot\vec{x}} \left[(2\pi)^{\frac{3}{2}}\delta^{(3)}(\vec{k})\bar{\phi}(\tau) + \delta\phi(\tau,\vec{k})\right] \\
A_i(\tau,\vec{x}) &= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} e^{i\vec{k}\cdot\vec{x}} \left[\bar{A}_i(\tau,k) + \delta A_i(\tau,k)\right] \ ,
\end{aligned}
\tag{4.10}
$$

where $\delta A_i(\tau,\vec{x})$ is also promoted to a quantum operator via

$$
\delta A_i(\tau,\vec{x}) = \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \left[\varepsilon_{i\lambda}\left(\vec{k}\right)\delta A_\lambda(\tau,k)\hat{a}_\lambda\left(\vec{k}\right)e^{i\vec{k}\cdot\vec{x}} + h.c.\right]
\tag{4.11}
$$

The relation for A_0 in momentum space can be obtained by replacing relations (4.10) into (4.5), (see Appendix C)

$$
A_0(\tau,\vec{k}) = i\frac{\alpha}{f}\frac{k_i}{k^2}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}|\vec{k}-\vec{p}|\bar{A}_i(\tau,\vec{k}-\vec{p})\delta\phi(\tau,\vec{p}) \ .
\tag{4.12}
$$

Employing relations (4.12) and (4.10) in equations (4.6) and (4.8), we finally find the equation of the perturbed gauge field

$$
\delta A_i''(\tau,\vec{k}) + k^2\delta A_i(\tau,\vec{k}) - i\frac{\alpha}{f}\bar{\phi}'(\tau)\epsilon^{ijk}k_j\delta A_k(\tau,\vec{k}) = -i\frac{\alpha}{f}\int \frac{d^3p}{(2\pi)^{\frac{2}{3}}}\delta\phi(\tau,\vec{p})\epsilon^{ijk}k_j\bar{A}_k'(\tau,\vec{k}-\vec{p}) \ ,
\tag{4.13}
$$

and the inflaton field, in Fourier space

$$
\begin{aligned}
&\delta\phi''(\tau,\vec{k}) + 2\mathcal{H}\delta\phi'(\tau,\vec{k}) + k^2\delta\phi(\tau,\vec{k}) + a^2V''\delta\phi(\tau,\vec{k}) + \\
&+ \frac{\alpha^2}{a^2f^2}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\frac{d^3q}{(2\pi)^{\frac{3}{2}}}\left|\vec{k}-\vec{p}\right||\vec{p}-\vec{q}|\bar{A}_i(\tau,\vec{k}-\vec{p})\bar{A}_i(\tau,\vec{p}-\vec{q})\delta\phi(\tau,\vec{q}) = \\
= &-\frac{\alpha}{a^2f}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\left|\vec{k}-\vec{p}\right|\bar{A}_i(\tau,\vec{k}-\vec{p})\delta A_i'(\tau,\vec{p}) - \frac{\alpha}{a^2f}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\bar{A}_i'(\tau,\vec{k}-\vec{p})|\vec{p}|\delta A_i(\tau,\vec{p}) \ .
\end{aligned}
\tag{4.14}
$$

For a comprehensive computation, see Appendix C.

4.3 Master equation for $\delta\phi$

To obtain one master equation for the perturbations of the inflaton field, we first solve equation (4.13) formally, and then we substitute the solution, which will be a functional of $\delta\phi$, into equation (4.14). We note that the homogeneous part of equation (4.13) is the same as the one obtained in [45] in the case $A_0 = 0$ and $\partial_i\phi = 0$, which means that the two equations share the same Green function $G_k(\tau,\tau')$ that we already presented in eq. (3.16), employing the same background solution (3.9). However, the solution is different because of the presence of a different forcing term on the right-hand side. The solution we find is

$$
\delta A_i(\tau,\vec{k}) = -i\frac{\alpha}{f}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}d\tau' G_k(\tau,\tau')\epsilon^{ijk}k_j\bar{A}_k'(\tau',\vec{k}-\vec{p})\delta\phi(\tau',\vec{p})
\tag{4.15}
$$

Plugging this solution into the integrals on the right-hand side of equation (4.14), we get for the first one

$$
\begin{aligned}
&-\frac{\alpha}{a^2 f}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\left|\vec{k}-\vec{p}\right| \bar{A}_i(\tau,\vec{k}-\vec{p})\delta A_i'(\tau,\vec{p}) = \\
=&\frac{i\alpha^2}{a^2 f^2}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\left|\vec{k}-\vec{p}\right| \bar{A}_i(\tau,\vec{k}-\vec{p})\int \frac{d^3q}{(2\pi)^{\frac{3}{2}}}\int^{\tau} d\tau' \,\frac{\partial G_p(\tau,\tau')}{\partial\tau}\epsilon^{ijk}p_j\bar{A}_k'(\tau',\vec{p}-\vec{q})\delta\phi(\tau',\vec{q}) \ ,
\end{aligned}
\tag{4.16}
$$

while for the second one

$$
\begin{aligned}
&-\frac{\alpha}{a^2 f}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\bar{A}_i'(\tau,\vec{k}-\vec{p})|\vec{p}|\delta A_i(\tau,\vec{p}) = \\
=&-\frac{\alpha^2}{a^2 f^2}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\bar{A}_i'(\tau,\vec{k}-\vec{p})|\vec{p}|\int \frac{d^3q}{(2\pi)^{\frac{3}{2}}} d\tau' G_p(\tau,\tau')\epsilon^{klm}(p_l)\bar{A}_m'(\tau',\vec{p}-\vec{q})\delta\phi(\tau',\vec{q}) = \\
=&-\frac{\alpha^2}{a^2 f^2}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\frac{d^3q}{(2\pi)^{\frac{3}{2}}} d\tau' \bar{A}_i'(\tau,\vec{k}-\vec{p})|\vec{p}|\epsilon^{klm}p_l\bar{A}_m'(\tau',\vec{p}-\vec{q})G_p(\tau,\tau')\delta\phi(\tau',\vec{q}) \ .
\end{aligned}
\tag{4.17}
$$

At this point, we just have to put all the pieces together, and we end up with

$$
\begin{aligned}
&\delta\phi''(\tau,\vec{k}) + 2\mathcal{H}\delta\phi'(\tau,\vec{k}) + k^2\delta\phi(\tau,\vec{k}) + a^2V''\delta\phi(\tau,\vec{k}) + \\
&+\frac{\alpha^2}{a^2 f^2}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\frac{d^3q}{(2\pi)^{\frac{3}{2}}}\left|\vec{k}-\vec{p}\right| |\vec{p}-\vec{q}|\, \bar{A}_i(\tau,\vec{k}-\vec{p})\bar{A}_i(\tau,\vec{p}-\vec{q})\delta\phi(\tau,\vec{q}) = \\
=&\frac{i\alpha^2}{a^2 f^2}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\frac{d^3q}{(2\pi)^{\frac{3}{2}}}\left|\vec{k}-\vec{p}\right| \bar{A}_i(\tau,\vec{k}-\vec{p})\int^{\tau} d\tau' \,\frac{\partial G_p(\tau,\tau')}{\partial\tau}\epsilon^{ijk}p_j\bar{A}_k'(\tau',\vec{p}-\vec{q})\delta\phi(\tau',\vec{q}) + \\
&-\frac{\alpha^2}{a^2 f^2}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\frac{d^3q}{(2\pi)^{\frac{3}{2}}} d\tau' \bar{A}_i'(\tau,\vec{k}-\vec{p})|\vec{p}|\epsilon^{klm}p_l\bar{A}_m'(\tau',\vec{p}-\vec{q})G_p(\tau,\tau')\delta\phi(\tau',\vec{q}) \ .
\end{aligned}
\tag{4.18}
$$

Equation (4.18) represents the evolution equation for inflaton field perturbations in the context of the AS model. Comparing this result with (3.19) reveals how relaxing the assumptions of perfectly homogeneous inflaton field, and vanishing 0-component of A_μ, adds a degree of complexity in the equation describing the perturbations of the inflaton field. This equation has not been given in the existing literature. Solving this equation is beyond the scope of the present thesis.

Chapter 5

Numerical analysis of inflation with dissipation and delayed backreaction

The analysis carried out in Chapter 3, showed that the instability of the AS solution (3.12), can be explained in terms of a delayed backreaction. Practically, gauge field modes produced at a time t backreact on the motion of the inflaton at a time $t' = t + \delta t$, thus the backreaction term induces in a differential equation, where every term is evaluated at t', a term that depends on the history of $d\phi/dt$ at times earlier than t'. This was first argued in [20], and then analytically proved in [45] through the delayed kernel we find in equation (3.20).

In the literature are not present many other examples of delayed backreaction. The aim of this chapter is to study the consequences of this effect on a toy model of inflation with dissipation. For definiteness, we consider the simplest model of warm inflation, in which dissipation introduces a term $\Gamma\dot{\phi}$ in the inflaton equation, where Γ is the decay rate of the inflaton in other species, which we take constant for simplicity. We assume a delay in this term, namely that the inflaton derivative is evaluated at some earlier time wrt the other terms in the equation. Our purpose is not to present a fully working model of inflation, but rather to study this effect in the simplest context, and to see whether it can result in oscillations of the inflaton velocity, analogously to those encountered in the AS model, and visible in Figure 3.4 of this thesis.

Under these assumptions, the evolution of the inflaton is controlled by

$$\ddot{\phi}(t) + 3H(t)\dot{\phi}(t) + \Gamma\dot{\phi}(t - \delta t) + \frac{\partial V(\phi)}{\partial \phi} = 0 \ . \tag{5.1}$$

The analysis has been carried out considering the "Power-Law" potential of the form

$$V(\phi) = \frac{\lambda}{4}\phi^4 \ . \tag{5.2}$$

This specific model of inflation was ruled out almost a decade ago by the Planck collaboration [42]. However, let us emphasize that the main objective of the forthcoming investigation is the study of the effects of time delay on the inflaton dynamics.

5.1 System of equations

The starting point of the following analysis is the system of equations

$$\ddot{\phi}(t) + 3H(t)\dot{\phi}(t) + \Gamma\dot{\phi}(t) + \lambda\phi^3(t) = 0 \ , \tag{5.3}$$

$$H^2(t) = \frac{1}{3M_p^2}\left(\frac{1}{2}\dot{\phi}^2(t) + \frac{\lambda}{4}\phi^4(t) + \rho_\gamma(t)\right) \ , \tag{5.4}$$

where for the time being, we consider $\Gamma\dot{\phi}(t)$ without the time delay, but we will introduce it shortly, and $\rho_\gamma(t)$ is the energy density of the products of the inflaton decay, which we consider to be a component characterized by an equation of state $\omega = 1/3w$. It is convenient to reparameterize ϕ, Γ, and the time variable t in order to make these equations dimensionless. This is done to make the problem more general, losing the dependency on scales or parameters we are unable to fix (e.g. λ). We derive the adimensional variables φ, τ, γ, and h through the definitions

$$\begin{aligned} \phi &\equiv M_p\varphi \ , \\ t &\equiv \frac{\tau}{M_p\sqrt{\lambda}} \ , \\ \Gamma &\equiv M_p\sqrt{\lambda}\gamma \ , \\ H &\equiv M_p\sqrt{\lambda}h \ . \end{aligned} \tag{5.5}$$

In this new set of variables, equations (5.3) and (5.4), became

$$\ddot{\varphi}(\tau) + 3h(\tau)\dot{\varphi}(\tau) + \gamma\dot{\varphi}(\tau) + \varphi^3(\tau) = 0 \ , \tag{5.6}$$

$$h^2(\tau) = \frac{1}{3}\left(\frac{1}{2}\dot{\varphi}^2(\tau) + \frac{1}{4}\varphi^4(\tau) + \tilde{\rho}_\gamma(\tau)\right) \ , \tag{5.7}$$

where now dot denotes the derivative wrt the dimensionless time variable τ, and $\tilde{\rho}_\gamma(\tau)$ is the dimensionless energy density of the decay products. From equations (5.6) and (5.7) we derive the following evolution equation for $h(\tau)$

$$\dot{h}(\tau) + 2h^2(\tau) + \frac{1}{6}\dot{\varphi}^2(\tau) - \frac{1}{6}\varphi^4(\tau) = 0 \ . \tag{5.8}$$

In the AS model, it has been observed a delay of a fixed number of e-folds ($\delta n \sim 5$) rather than a fixed temporal delay δt [20, 45]. For this reason we will write the equations in terms of number of e-folds, instead of time, and then introduce the delay δn. Furthermore, it has been decided to use the number of e-folds as a parameter to indicate the evolution of the system. To rewrite the derivatives wrt τ, as derivatives wrt n[1] we employ the relation

$$dn \equiv h \ d\tau \ , \tag{5.9}$$

from which follow the relations

[1] In the previous chapters the number of e-folds was indicated with N, now we use n to conform to the notation adopted in this chapter for the dimensionless variables but notice that the number of e-folds is always dimensionless.

$$\frac{d}{d\tau} = h\frac{d}{dn} \ , \tag{5.10}$$
$$\frac{d^2}{d\tau^2} = h^2\frac{d^2}{dn^2} + hh'\frac{d}{dn} \ , \tag{5.11}$$

where, from now on, prime will be used to indicate the derivative wrt n.

Finally, the system of dimensionless equations that we are going to study in the next sections read

$$\varphi''(n) + \left(1 + \frac{1}{6h^2(n)} - \frac{1}{6}\varphi'^2(n)\right)\varphi'(n) + \frac{\gamma}{h^2(n)}h(n-\delta n)\varphi'(n-\delta n) + \frac{1}{h^2(n)}\varphi^3(n) = 0 \ , \tag{5.12}$$
$$h'(n) + 2h(n) - \frac{1}{6h(n)}\varphi^4(n) + \frac{1}{6}h(n)\varphi'^2(n) = 0 \ . \tag{5.13}$$

For the full derivation, see Appendix D. In the following sections, we are going to present the numerical solutions of equations (5.12) and (5.13), carried out using *Mathematica*, first in the case of absence of extra friction ($\gamma = 0$), then introduce the friction term ($\gamma \neq 0$), and in the end consider a delay (δn) in the friction term. Each time we will use the previous results to set the initial conditions necessary to solve the differential equations.

5.2 Numerical solutions without friction

To numerically solve equations (5.12) and (5.13), disregarding the additional friction term, we first need to find the slow roll relations for $\varphi'(n)$, $\varphi(n)$, and $h(n)$, which we use to set the initial conditions. Employing the slow roll conditions, the system of equations reads

$$3h^2(n)\varphi'(n) \simeq -\varphi^3(n) \ , \tag{5.14}$$
$$h^2(n) \simeq \frac{1}{12}\varphi^4(n) \ . \tag{5.15}$$

From the slow roll equations, we find the relations

$$\varphi(n) \simeq \sqrt{-8n} \ , \tag{5.16}$$
$$\varphi'(n) \simeq -\sqrt{-\frac{2}{n}} \ , \tag{5.17}$$
$$h(n) \simeq -\frac{1}{2\sqrt{3}}8n \ , \tag{5.18}$$

where in this notation n is negative and goes from a chosen value of $n_i = -63$ at the beginning of inflation and then increases towards zero. The full computation can be found in Appendix E.

Employing relations (5.16,5.17,5.18) as initial conditions at n_i, we numerically solve equations (5.12) and (5.13) for $\gamma = 0$. The results for this configuration are shown in figures 5.1 and 5.2, where for representation purposes the number of e-folds has been rescaled in order to start a

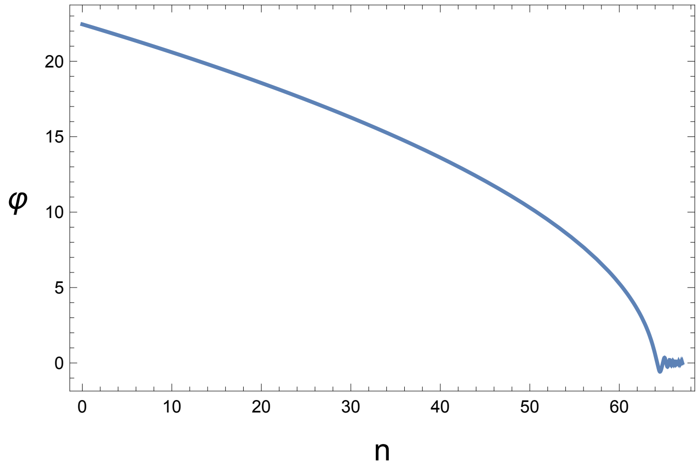

Figure 5.1: Numerical evolution of the inflaton field $\varphi(n)$.

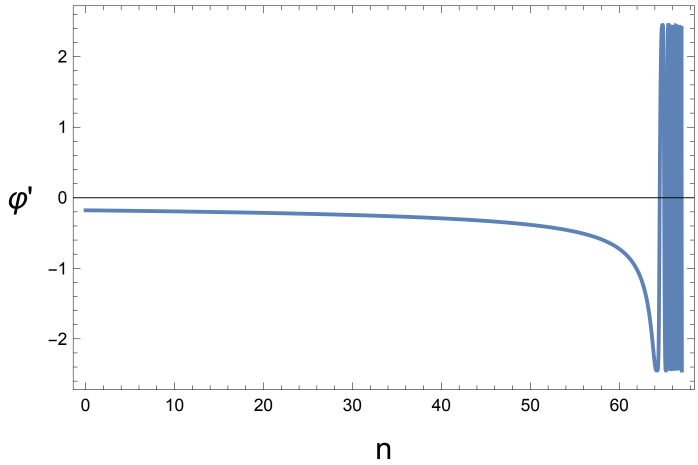

Figure 5.2: Numerical evolution of the inflaton velocity $\varphi'(n)$.

zero when inflation begins and to be positive. Notice in particular how, for low n, the inflaton velocity is small, in accordance with the slow roll assumption, and that after inflation ends, the inflaton oscillates about the minimum of the potential.

To find the moment at which inflation ends, we compute $\frac{a''}{a}$ and then identify the end of inflation as the first time it becomes negative. To do so, we consider

$$\frac{\ddot{a}}{a}(\tau) = h^2(\tau) + \dot{h}(\tau) = \frac{1}{12}\varphi^4(\tau) - \frac{1}{3}\dot{\varphi}^2(\tau) \ , \tag{5.19}$$

where the second equality follows from equations (5.8) and (5.7). Exploiting now relation (D.3) it is possible to rewrite the latter equation as

$$\frac{a''}{a}(n) = \frac{1}{12}\varphi^4(n) - \frac{1}{3}h^2(n)\varphi'^2(n) \ . \tag{5.20}$$

Figure 5.3 shows the evolution of a''/a near the first zero, which is reached after $n_{end} = 63.04$ e-folds, notice that this result is in close accordance with the slow roll assumption of 63 e-folds of inflation.

Once we find the end of inflation, we also determine the value of the inflaton at that moment $\varphi_{end} = 2.34$. We do not want the additional friction term to significantly alter the total number

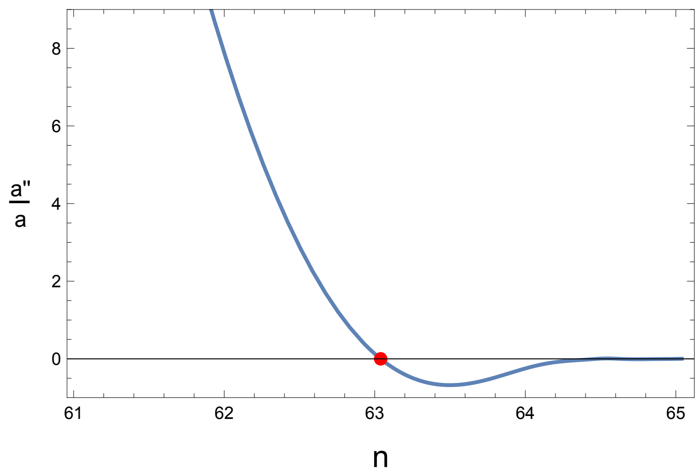

Figure 5.3: Evolution of $a''/a < 0$ (blue line) near the end of inflation (red dot) identified as the moment at which $a''/a < 0$.

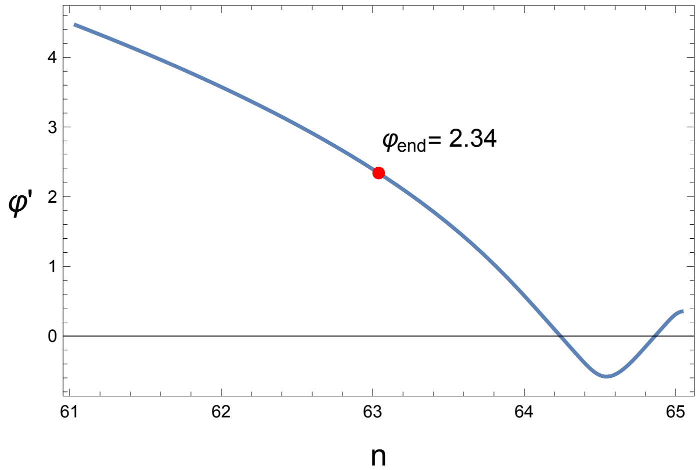

Figure 5.4: Evolution of $\varphi(n)$ (blue line) near the end of inflation. Value of the inflaton at the end of inflation (red dot).

of e-folds of inflation because we aim to examine the behavior of φ' around $N \sim 60$ e-folds before the end of inflation. However, we have verified that the addition of the term $\gamma\varphi'$ induces an overdamping regime, which significantly prolongs the duration of inflation. To counteract this, we are forced to decrease gamma to a level where the backreaction becomes entirely negligible at $N \sim 60$ e-folds from the end of inflation. Since this is a toy model and it is expected that in a realistic model the physics governing the backreaction around ~ 60 e-folds differs from the physics causing the inflation to end, we separate the two aspects in the toy model. The simplest way to implement this is to always assume that inflation ends at $\phi_{end} = 2.34$, with the idea that at this point another mechanism (such as a change in the inflaton potential) terminates inflation when that value is reached.

5.3 Numerical solutions in the case $\gamma \neq 0$ and $\delta n = 0$

Let us now consider the backreaction from dissipation, with no time delay. To do so, we need to fix the value of γ. Since gamma is constant, while the Hubble rate decreases with time, we can conveniently work in a regime in which gamma is significantly smaller than $3h$ at the beginning of our evolution, so that the backreaction is initially irrelevant and the evolution obtained in the previous subsection can be used as an initial condition, and then gamma becomes relevant at some point before the end of inflation. We define γ as

$$\gamma \equiv \frac{3h_{60}}{\alpha} \; , \tag{5.21}$$

where h_{60} indicates the value that the Hubble rate assumes 60 e-folds before the end of inflation, as computed in the previous section, while α will be fixed at $\alpha = \{25, 20, 15, 10\}$. These values of α have been chosen in order to get a friction $\gamma\varphi'$ that becomes important only in the final stage of inflation. We can check that these are the right choices considering again the Hubble rate computed in the absence of extra friction term and computing the ratio $\gamma/(3h_n)$, where the subscript n indicates the Hubble rate computed n e-folds before the end of inflation.

	$\frac{\gamma}{3h_{60}}$	$\frac{\gamma}{3h_{40}}$	$\frac{\gamma}{3h_{10}}$	$\frac{\gamma}{3h_{1}}$
$\alpha = 25$	0.04	0.06	0.23	1.40
$\alpha = 20$	0.05	0.07	0.29	1.75
$\alpha = 15$	0.07	0.10	0.38	2.33
$\alpha = 10$	0.10	0.15	0.58	3.49

Table 5.1: Ratio between γ and the Hubble friction computed using the Hubble rate found in the previous section, for different values of α.

As shown in Table 5.1, the term $\gamma\varphi'$ in all cases starts at least one order of magnitude smaller than $3h\varphi'$. It becomes more important as inflation continues, and it is of the same order as the Hubble friction around the end of inflation. Although Table 5.1 compares γ with the Hubble rate computed in case $\gamma = 0$, it represents the only way to confront the two frictions involved before solving the new system of equations. Once h is computed in the scenario where $\gamma \neq 0$, we revisit the same table for further validation.

Solving numerically equations (5.12) and (5.13), imposing as initial conditions at $n = 0$ the solutions $\varphi(0)$, $\varphi(0)$, and $h(0)$ obtained in the previous section, we find that inflation ends ($\varphi(n) = \varphi_{end}$) after $n_{end} = \{73.88, 76.61, 81.16, 90.25\}$ e-folds for $\alpha = \{25, 20, 15, 10\}$, respectively.

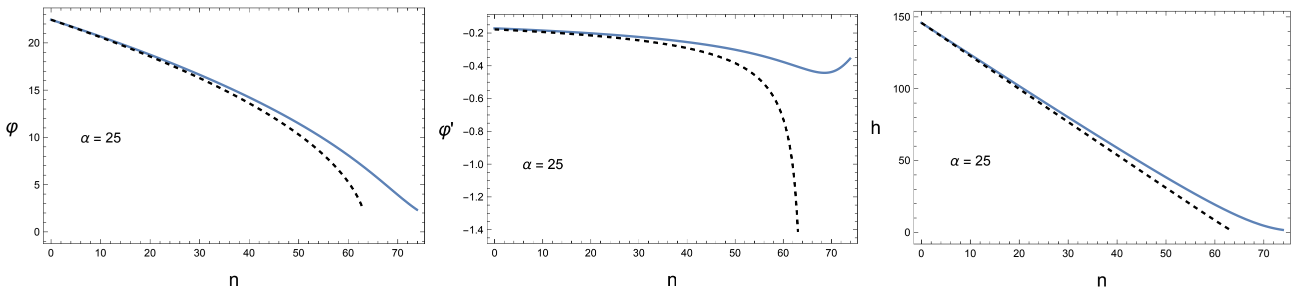

Figure 5.5: (Blue solid) Numerical evolution of $\varphi(n)$, $\varphi'(n)$, and $h(n)$, for $\alpha = 25$ without delay δn. (Black dashed) Numerical evolution of $\varphi(n)$, $\varphi'(n)$, and $h(n)$, in absence of extra friction term.

In Figures 5.5 and 5.6 we present the numerical solutions for the evolution of the inflaton $\varphi(n)$, the inflaton velocity $\varphi'(n)$, and the Hubble rate $h(n)$, compared to their evolution in case $\gamma = 0$. These figures show clearly how adding a friction term makes inflation last more e-fold, as could be expected. Moreover, we see how the evolution of the inflaton velocity is visibly affected in the final stages of inflation, when the extra friction term becomes larger than the Hubble friction, slowing down the motion of the inflaton. As mentioned above, we again compare γ with the Hubble rate, computed for $\gamma \neq 0$, to confirm that the ratios described in Table 5.1 are accurate.

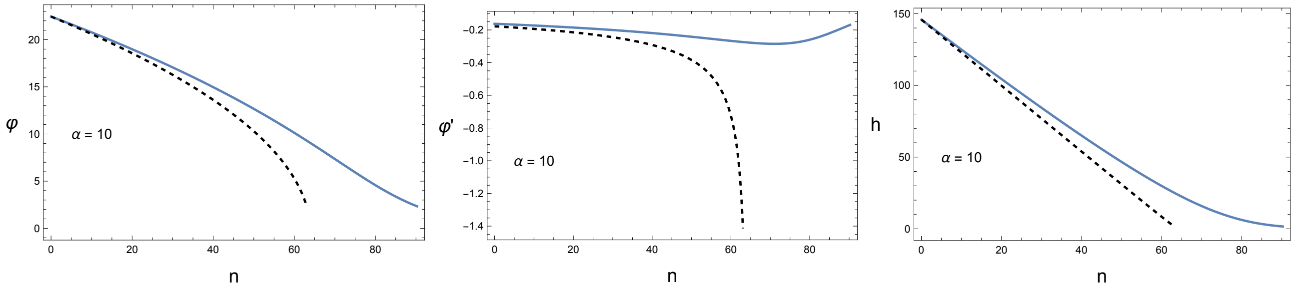

Figure 5.6: (Blue solid) Numerical evolution of $\varphi(n)$, $\varphi'(n)$, and $h(n)$, for $\alpha = 10$ without delay δn. (Black dashed) Numerical evolution of $\varphi(n)$, $\varphi'(n)$, and $h(n)$, in absence of extra friction term.

	$\frac{\gamma}{3h_{60}}$	$\frac{\gamma}{3h_{40}}$	$\frac{\gamma}{3h_{10}}$	$\frac{\gamma}{3h_{1}}$
$\alpha = 25$	0.05	0.08	0.44	2.44
$\alpha = 20$	0.06	0.10	0.63	3.22
$\alpha = 15$	0.09	0.16	1.07	4.57
$\alpha = 10$	0.17	0.30	2.31	7.35

Table 5.2: Ratio between γ and the Hubble friction computed considering the extra friction term, for different values of α.

Comparing the data presented in Tables 5.1 and 5.2, we notice that in the latter the values of the ratios are larger, but the general sense of negligible extra friction at the beginning and similar frictions near the end is preserved. Notice finally, that the sole inclusion of the extra friction term is not sufficient for inducing oscillations in the inflaton field velocity (Figures **??** and **??**), as expected.

5.4 Numerical solutions in the case $\gamma \neq 0$ and $\delta n \neq 0$

To numerically integrate equations (5.12) and (5.13), in the case of a non vanishing delay δn, it is necessary to specify the past history of $\varphi(n)$, $\varphi'(n)$, $h(n)$, and not simply their values at $n = 0$. To do so, our ansatz is to use the solutions obtained in the previous section, as the history of the functions in the interval $n \in [0, \delta n]$, and, with these initial conditions, solve equations (5.12) and (5.13) for $n > \delta n$.

α	δn	n_{end}
25	5	81.57
	10	90.65
	15	100.19
20	5	84.81
	10	93.87
	15	103.57
15	5	91.0
	10	98.75
	15	108.03
10	5	102.12
	10	113.47
	15	128.31

Table 5.3: Total number of e-folds from the fixed initial value $\varphi_{in} = 22.45$ to the fixed final value $\varphi_{end} = 2.34$, for different cases of friction α, and different delay δn.

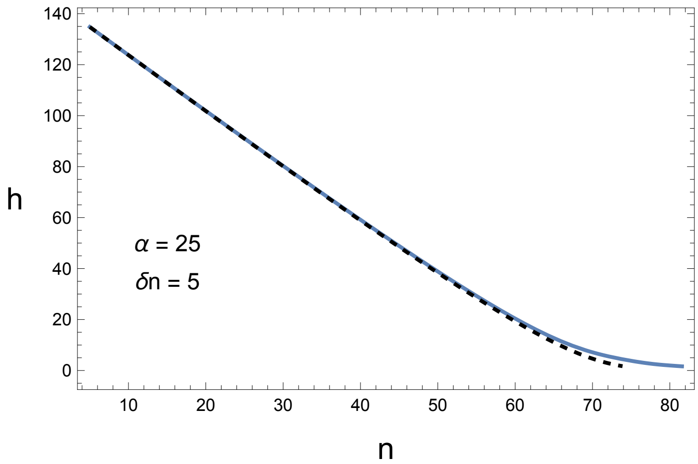

Figure 5.7: (Blue solid) Evolution of the Hubble rate h, for $\alpha = 25$ and $\delta n = 5$. (Black dashed) Evolution of the Hubble rate, for $\alpha = 25$ and $\delta n = 0$

Solving the equations and identifying again the end of inflation as the moment when $\varphi(n) = \phi_{end}$, we find that for the different values of α, increasing the delay δn reflects into a longer duration of inflation (according to Table 5.3). This relation between the total number of e-folds and the delay was to be expected, due to the presence of $h(n - \delta n)$ in the delayed friction term in equation (5.13). In fact, h during inflation decreases monotonically (Figure 5.7), therefore increasing the delay δn corresponds to considering larger values of h at a given n, which results in a stronger friction.

Introducing the delay δn does not appear to have a great visual impact on the evolution of the inflaton field φ, apart from the total duration of inflation, as we see in Figures 5.8.

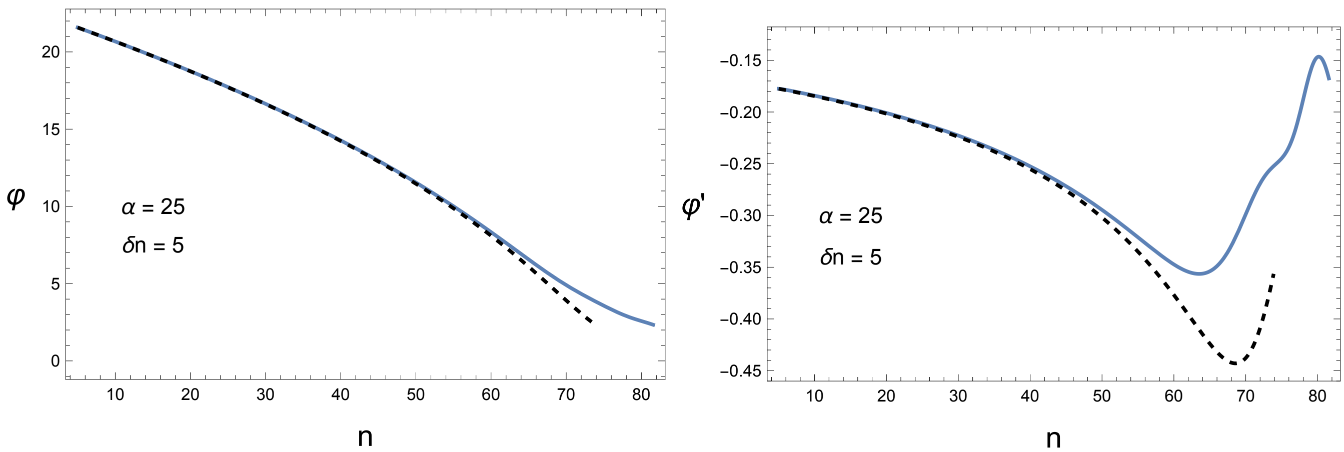

Figure 5.8: (Blue solid) Numerical evolution of $\varphi(n)$, $\varphi'(n)$, and $h(n)$, for $\alpha = 25$ and a delay $\delta n = 5$. (Black dashed) Numerical evolution of $\varphi(n)$, $\varphi'(n)$, for $\alpha = 25$ and without delay δn.

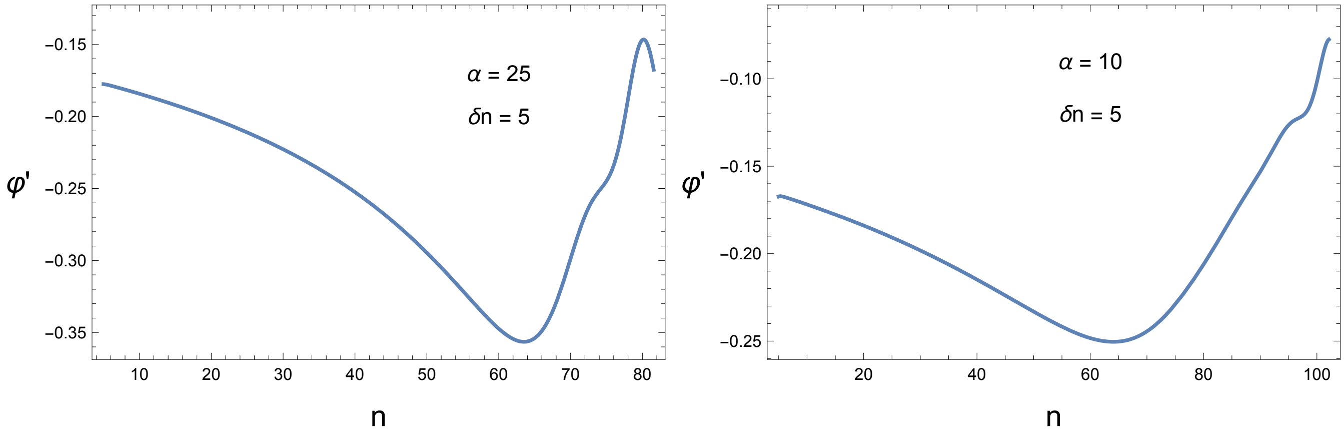

Figure 5.9: Comparison of the evolution of the inflaton velocity for a fixed delay $\delta n = 5$ and for $\alpha = 25$ and $\alpha = 10$ respectively.

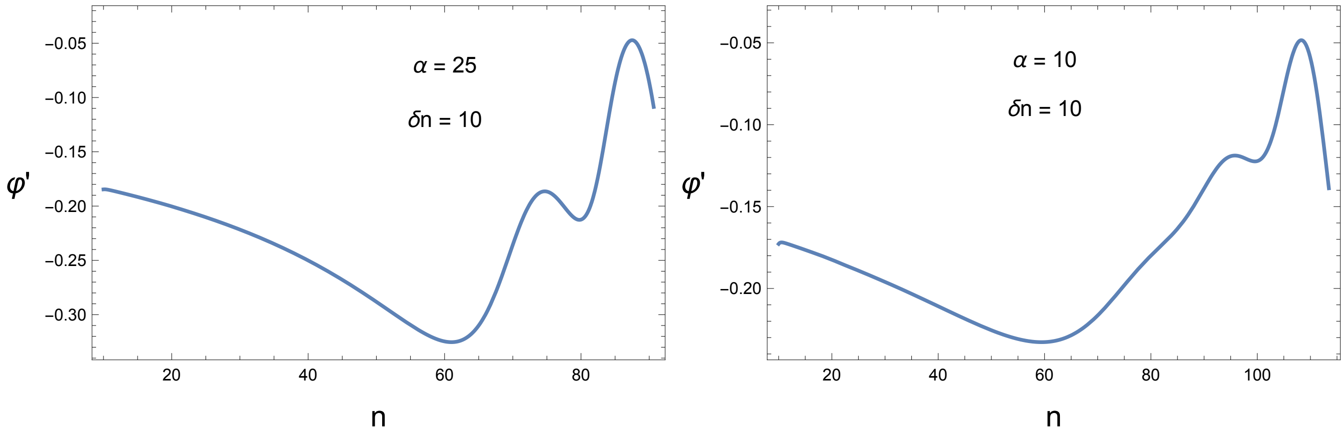

Figure 5.10: Comparison of the evolution of the inflaton velocity for a fixed delay $\delta n = 10$ and for $\alpha = 25$ and $\alpha = 10$ respectively.

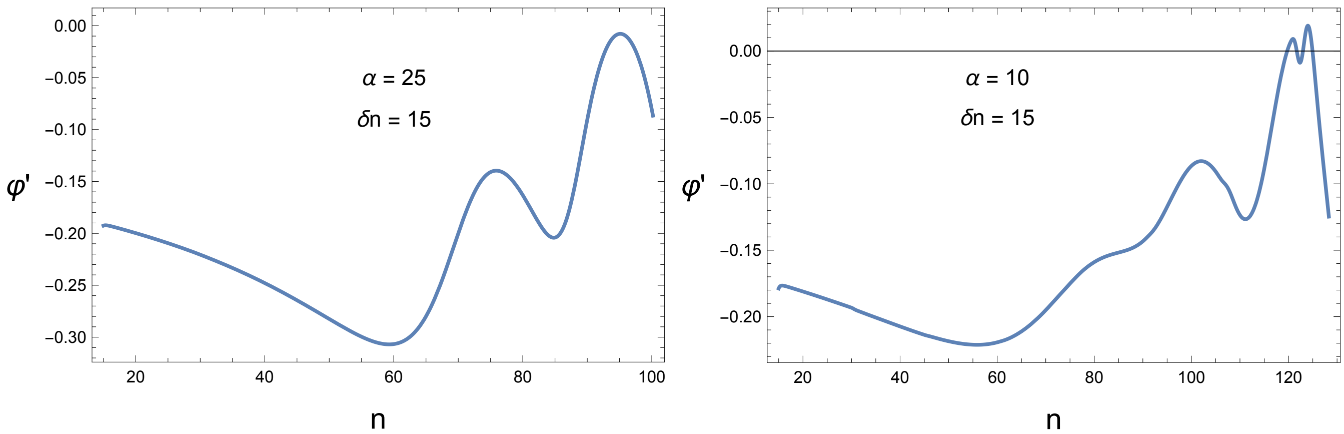

Figure 5.11: Comparison of the evolution of the inflaton velocity for a fixed delay $\delta n = 15$ and for $\alpha = 25$ and $\alpha = 10$ respectively.

On the other hand, the impact of the delay on the inflaton velocity φ' is immediately visible, as the motion is slowed even more and oscillations arise. Figures 5.9 to 5.11 illustrate that as the delay δn increases, the oscillations become more pronounced. Conversely, when augmenting the friction strength γ (or decreasing α), the oscillations appear to lose their periodicity. We notice in Figure 5.11 that for a long delay $\delta n = 15$ and a large γ ($\alpha = 10$), at around $n \simeq 120$

the velocity changes sign. This has also been observed in [20], and it has been claimed that this occurs because as φ' drops, so does the friction term, but due to the time lag, there is a moment when the velocity is so small, compared to the friction, that it temporarily changes sign.

This numerical analysis showed that the oscillatory behaviour of the inflaton velocity is not a phenomenon strictly related to the AS model, but it extends to other models of inflation with a delayed backreaction. Specifically, we studied the case of $V(\phi) = \lambda/4\phi^4$ inflation for a backreaction modeled as $\Gamma\phi'$ with Γ constant. A further study could involve a similar numerical analysis, for inflation models that fit well the latest observations [13], and for a more realistic backreaction term.

Conclusions

In this thesis, we investigated instabilities within the Amber-Sorbo (AS) model of inflation [14] in the regime of strong backreaction of the gauge field on the inflaton.

In Chapter 1 we introduced the FLRW universe, focusing specifically on the geometry of the universe, its dynamics, and the concept of horizon. This is done to establish the context in which the inflationary paradigm was introduced.

In Chapter 2 we present the inflationary universe, stressing the shortcomings of the Hot Big Bang (HBB) model, that motivated the introduction of inflation. In this chapter, we also discuss the simplest realization of inflation, the dynamics of the inflaton background, and the properties of the perturbations of the inflaton field.

In Chapter 3 we set up the theoretical framework, introducing the topic of axion inflation and discussing both the theoretical and observational limitations of the former model [9]. This sets the stage in which AS inflation was introduced [14]. This is a natural inflation model, based on a mechanism in spirit similar to that at work in warm [15] and trapped [16] inflation, with the main difference that the inflaton in the AS model is coupled to a derivative of the produced field, which allows one to obtain a dissipative process, without invoking more complicated structures as in the other models. We start from the original work [14], that analytically studied this system under the assumptions of homogeneous inflaton and steady-state evolution of the inflaton velocity. This evolution was then found to be in contrast with numerical studies that solved the evolution equations of the gauge field and backreaction [17–22], where oscillations of the inflaton velocity around the mean value of AS. Among the cited works, reference [20] was the first to attribute the origin of the oscillations to a time delay. Further confirming this phenomenon, there are lattice simulations that have solved the system exactly [23–26]. In this chapter, we also focus on reviewing the analysis performed in [45] and [20], which are essential for the studies that we present in the subsequent chapters.

In Chapter 4 we computed the equations of motion for the inflaton and gauge field, relaxing some assumptions commonly made in the literature. For what concerns the gauge field A_μ, typically the gauge chosen to perform the computation is the one identified by the condition $\partial^i A_i = 0$. Furthermore, in the literature, it is customary to make the assumption of homogeneous inflaton $\partial_i \phi = 0$, this reflects on a constrain equation for the temporal component of the gauge field such that $A_0 = 0$. In this chapter, instead, we decided to analytically study the system, including inflaton inhomogeneities $\partial_i \phi \neq 0$, and as a consequence we also had to account for a different constraint equation $A_0 \neq 0$. According to these choices, we computed the equations of motion for the two fields (equations (4.5), (4.6) and (4.8)), obtaining an extra scalar equation (4.5) for the gauge field, which is not present in the gauge $A_0 = 0$. Subsequently, we perturbed the system and computed the equations in Fourier space (equations (4.12), (4.13) and (4.14)). This system of equations is eventually solved to obtain a master equation for the perturbed inflaton field (4.18), analogous to the one computed in [45] neglecting the gradient terms of the inflaton and taking $A_0 = 0$, reported in equation (3.14). Comparing these two equations makes evident the degree of complexity brought by the relaxations of the previous assumptions, in the resulting integro-differential equation (3.19). This equation might be solved analytically and perturbatively (in the limit of small inhomogeneities) expanding around the analytic solution of [45], that was obtained by solving the corresponding master equation in the absence of inflaton inhomogeneities. However, solving this equation is beyond the scope of the present thesis.

In Chapter 5 we conducted a numerical investigation on the consequences of a time-delayed friction term, in a system of equations of a typical inflationary model. The model we studied is one of a single scalar field in a potential $V(\phi) = \lambda/4\phi^4$. This specific model was chosen for the excursion of the Hubble rate H during inflation. Having a weak or strong backreaction means having a dissipation term $\Gamma\dot{\phi}$ negligible or comparable to the standard friction term $3H\dot{\phi}$, respectively. Therefore, we have considered a model characterized by a significant variation of H during inflation, to set the initial stage as a standard slow-roll inflation with negligible dissipation. Then, there is a dynamic transition into a regime of significant backreaction. In this model, there is no agreement with observations, but our goal is to study the consequences of a delayed backreaction. This is essentially an unexplored aspect in the context of inflation, except for the AS model, and our objective was to see if the emergence of oscillations in $\dot{\phi}$ was a peculiarity of AS or if it could be reproduced in other models. We find that this also occurs in the $\lambda\phi^4$ model (see figures 5.9 - 5.11).

From the results obtained in this thesis, two possible continuations are spontaneous. The first consists in solving the master equation (4.18), either numerically on a lattice, or pertubatively to obtain an analytical solution. The second extension of this work could be to carry out the same numerical study that we performed in Chapter 5, for different potentials and for different forms of the extra friction term.

Appendices

Appendix A

Computation of the equations for A_0 and A_i

To compute the equations for the gauge field, let us consider the two terms of

$$F^{\mu\nu}_{;\nu} + \frac{\alpha}{f}\phi_{,\nu}\tilde{F}^{\mu\nu} = 0 \ . \tag{A.1}$$

separately. The first term can be recast as

$$F^{\mu\nu}_{;\nu} = \partial_\nu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + \Gamma^\mu_{\nu\lambda}F^{\lambda\nu} + \Gamma^\nu_{\nu\lambda}F^{\mu\lambda} = \partial_\nu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + \Gamma^\nu_{\nu\lambda}F^{\mu\lambda}, \tag{A.2}$$

while the second term of (A.1)

$$\begin{aligned}\phi_{,\nu}\tilde{F}^{\mu\nu} &= \phi_{,0}\tilde{F}^{\mu 0} + \phi_{,j}\tilde{F}^{\mu j} \\ &= \phi_{,0}\left(\partial^\mu A^0 - \partial^0 A^\mu\right) + \phi_{,j}\left(\partial^\mu A^j + \partial^j A^\mu\right) \ .\end{aligned} \tag{A.3}$$

At this point, we need to separate the two cases $\mu = 0$ and $\mu = i$. Let us start from (A.2)

$\mu = 0:$

$$\partial_\nu\left(\partial^0 A^\nu - \partial^\nu A^0\right) + \Gamma^\nu_{\nu\lambda}F^{0\lambda} = \partial_0\left(\partial^0 A^0 - \partial^0 A^0\right) + \partial_i\left(\partial^0 A^i - \partial^i A^0\right) \tag{A.4}$$

where we used the fact that λ must be spatial in order for $F^{0\lambda}$ to be non-vanishing, but then $\Gamma^\nu_{\nu i}$ is zero for every choice of ν.

$$\begin{aligned}&\partial_i\left(g^{00}g^{ij}\partial_0 A_j - g^{ij}g^{00}\partial_j A_0\right) = \partial_i a^{-4}\left(\partial_i A_0 - \partial_0 A_i\right) = \\ &= a^{-4}\partial_i\partial_i A_0 - a^{-4}\partial_0\partial_i A_i = a^{-4}\partial_i\partial_i A_0.\end{aligned} \tag{A.5}$$

$\mu = i:$

$$\begin{aligned}&\partial_\nu\left(\partial^i A^\nu - \partial^\nu A^i\right) + \Gamma^\nu_{\nu\lambda}F^{i\lambda} = \partial_0\left(\partial^i A^0 - \partial^0 A^i\right) + \partial_k\left(\partial^i A^k - \partial^k A^i\right) + \\ &+ \Gamma^0_{00}F^{i0} + \Gamma^j_{j0}F^{i0} = \partial_0\left(g^{ij}g^{00}\partial_j A_0 - g^{ij}g^{00}\partial_0 A_j\right) + \\ &+ \partial_k\left(g^{ij}g^{kl}\partial_j A_l - g^{ij}g^{kl}\partial_l A_j\right) + 4\mathcal{H}\left(g^{ij}g^{00}\partial_j A_0 - g^{ij}g^{00}\partial_0 A_j\right) = \\ &= 4a^{-5}a'\partial_i A_0 - a^{-4}\partial_0\partial_i A_0 - 4a^{-5}a'\partial_0 A_i + a^{-4}\partial_0\partial_0 A_i + a^{-4}\partial_i\partial_j A_j + \\ &- a^{-4}\partial_k\partial_k A_i - 4a^{-4}\mathcal{H}\partial_i A_0 + 4a^{-4}\mathcal{H}\partial_0 A_i = \\ &= a^{-4}\partial_0\partial_0 A_i - a^{-4}\partial_k\partial_k A_i - a^{-4}\partial_0\partial_i A_0.\end{aligned} \tag{A.6}$$

Instead, for (A.3) we have

$\mu = 0$:

$$\frac{\alpha}{f}\phi_{,i}\tilde{F}^{0i} = \frac{\alpha}{f}\phi_{,i}\frac{\epsilon^{0ijk}}{2\sqrt{-g}}F_{jk} = \frac{\alpha}{f}\phi_{,i}\frac{\epsilon^{0ijk}}{2\sqrt{-g}}\left(\partial_j A_k - \partial_k A_j\right) = a^{-4}\frac{\alpha}{f}\phi_{,i}\epsilon^{0ijk}\partial_j A_k. \tag{A.7}$$

$\mu = i$:

$$\begin{aligned}\phi_{,0}\tilde{F}^{i0} + \phi_{,j}\tilde{F}^{ij} &= -a^{-4}\phi_{,0}\epsilon^{0ijk}\partial_j A_k + \frac{a^{-4}}{2}\phi_{,j}\epsilon^{ij\rho\sigma}\left(\partial_\rho A_\sigma - \partial_\sigma A_\rho\right) = \\ &= -a^{-4}\phi_{,0}\epsilon^{0ijk}\partial_j A_k + a^{-4}\phi_{,j}\epsilon^{0ijk}\left(\partial_0 A_k - \partial_k A_0\right).\end{aligned} \tag{A.8}$$

Summing the various contributions together we find the equations for A_0 and A_i

$$\partial_i\partial_i A_0 + \frac{\alpha}{f}\phi_{,i}\epsilon^{0ijk}\partial_j A_k = 0\ . \tag{A.9}$$

$$\partial_0\partial_0 A_i - \partial_j\partial_j A_i - \partial_0\partial_i A_0 - \frac{\alpha}{f}\phi_{,0}\epsilon^{0ijk}\partial_j A_k + \frac{\alpha}{f}\phi_{,j}\epsilon^{0ijk}\left(\partial_0 A_k - \partial_k A_0\right) = 0\ , \tag{A.10}$$

Appendix B

Expansion of $F_{\mu\nu}\tilde{F}^{\mu\nu}$

To express the right-hand side of equation (4.7) in terms of the gauge field A_μ, we start by rewriting the contraction of the two tensor strengths as

$$\begin{aligned} F_{\mu\nu}\tilde{F}^{\mu\nu} &= \frac{\epsilon^{\mu\nu\rho\sigma}}{2\sqrt{-g}}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho) \\ &= \frac{\epsilon^{\mu\nu\rho\sigma}}{2\sqrt{-g}}(\partial_\mu A_\nu \partial_\rho A_\sigma - \partial_\mu A_\nu \partial_\sigma A_\rho - \partial_\nu A_\mu \partial_\rho A_\sigma + \partial_\nu A_\mu \partial_\sigma A_\rho) \\ &= \frac{\epsilon^{\mu\nu\rho\sigma}}{2\sqrt{-g}}(2\partial_\mu A_\nu \partial_\rho A_\sigma - 2\partial_\nu A_\mu \partial_\rho A_\sigma) \\ &= \frac{2\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\partial_\mu A_\nu \partial_\rho A_\sigma \ , \end{aligned} \tag{B.1}$$

then we explicitly write it in terms of A_0 and A_i, noting that we can either have both ν and σ spatial, or one spatial and the other temporal.

$\nu \wedge \sigma \neq 0$:

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu \partial_\rho A_\sigma &= \epsilon^{0ijk}\partial_0 A_i \partial_j A_k + \epsilon^{ij0k}\partial_i A_j \partial_0 A_k \\ &= \epsilon^{0ijk}\partial_0 A_i \partial_j A_k + \epsilon^{0ijk}\partial_i A_j \partial_0 A_k \\ &= \epsilon^{0ijk}\partial_0 A_i \partial_j A_k + \epsilon^{0jki}\partial_j A_k \partial_0 A_i \\ &= \epsilon^{0ijk}\partial_0 A_i \partial_j A_k + \epsilon^{0ijk}\partial_j A_k \partial_0 A_i \\ &= 2\epsilon^{0ijk}\partial_0 A_i \partial_j A_k \ . \end{aligned} \tag{B.2}$$

$\nu \oplus \sigma = 0$:

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu \partial_\rho A_\sigma &= \epsilon^{i0jk}\partial_i A_0 \partial_j A_k + \epsilon^{ijk0}\partial_i A_j \partial_k A_0 \\ &= \epsilon^{i0jk}\partial_i A_0 \partial_j A_k + \epsilon^{i0jk}\partial_i A_j \partial_k A_0 \\ &= \epsilon^{i0jk}\partial_i A_0 \partial_j A_k + \epsilon^{j0ki}\partial_j A_k \partial_i A_0 \\ &= \epsilon^{i0jk}\partial_i A_0 \partial_j A_k + \epsilon^{i0jk}\partial_j A_k \partial_i A_0 \\ &= 2\epsilon^{i0jk}\partial_i A_0 \partial_j A_k \ . \end{aligned} \tag{B.3}$$

Summing these two results together, considering that $\epsilon^{i0jk} = -\epsilon^{0ijk}$ and that fixing the first index to be zero $\epsilon^{0ijk} = \epsilon^{ijk}$, we get

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = 4\epsilon^{ijk}(\partial_0 A_i - \partial_i A_0)\partial_j A_k \ . \tag{B.4}$$

Appendix C

Full derivation of the equations for $A_0(\tau, \vec{k})$, $\delta A_i(\tau, \vec{k})$, and $\delta\phi(\tau, \vec{k})$

C.1 $A_0(\tau, \vec{k})$

Let us start by considering equation (4.5) in Fourier space

$$\int \frac{d^3k'}{(2\pi)^{\frac{3}{2}}} e^{i\vec{k}'\cdot\vec{x}}(-k'^2)A_0(\tau, \vec{k}') + \frac{\alpha}{f}\int \frac{d^3p\ d^3q}{(2\pi)^3} e^{i(\vec{q}+\vec{p})\cdot\vec{x}} ip_i \delta\phi(\tau, \vec{p}) iq_j \bar{A}_k(\tau, \vec{q})\epsilon^{ijk} = 0 \ , \tag{C.1}$$

at this point we multiply everything by $e^{-i\vec{k}\cdot\vec{x}}$ and integrate in $\frac{d^3x}{(2\pi)^{2/3}}$

$$\begin{aligned} &\int \frac{d^3k' d^3x}{(2\pi)^3} e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}(-k'^2)A_0(\tau, \vec{k}') + \\ &+ \frac{\alpha}{f}\int \frac{d^3p\ d^3q\ d^3x}{(2\pi)^{\frac{9}{2}}} e^{-i(\vec{k}-\vec{q}-\vec{p})\cdot\vec{x}} ip_i \delta\phi(\tau, \vec{p}) iq_j \bar{A}_k(\tau, \vec{q})\epsilon^{ijk} = 0 \ , \end{aligned} \tag{C.2}$$

the integral in $\frac{d^3x}{(2\pi)^{2/3}}$ of the exponential gives us a delta

$$\int \frac{d^3k'}{(2\pi)^{\frac{3}{2}}} \delta^{(3)}(\vec{k} - \vec{k}')(-k'^2)A_0(\tau, \vec{k}') - \frac{\alpha}{f}\int \frac{d^3p\ d^3q}{(2\pi)^3} \delta^{(3)}(\vec{k} - \vec{p} - \vec{q}) p_i \delta\phi(\tau, \vec{p}) q_j \bar{A}_k(\tau, \vec{q})\epsilon^{ijk} = 0 \ , \tag{C.3}$$

and finally, we can make use of the deltas to solve one integral in each term.

$$-k^2 A_0(\tau, \vec{k}) = \frac{\alpha}{f}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} p_i \delta\phi(\tau, \vec{p})(k_j - p_j)\bar{A}_k(\tau, \vec{k} - \vec{p})\epsilon^{ijk} \ , \tag{C.4}$$

before exploiting the properties of the helicity vectors let us rewrite the right-hand side as

$$
\begin{aligned}
-k^2 A_0(\tau,\vec{k}) &= \frac{\alpha}{f}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\delta\phi(\tau,\vec{p})(p_i-k_i+k_i)(k_j-p_j)\bar{A}_k(\tau,\vec{k}-\vec{p})\epsilon^{ijk} \\
&= \frac{\alpha}{f}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\delta\phi(\tau,\vec{p})k_i(k_j-p_j)\bar{A}_k(\tau,\vec{k}-\vec{p})\epsilon^{ijk} \\
&= -i\frac{\alpha}{f}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\delta\phi(\tau,\vec{p})k_i\left|\vec{k}-\vec{p}\right|\bar{A}_i(\tau,\vec{k}-\vec{p})\ ,
\end{aligned}
\tag{C.5}
$$

where we have used $(p_i-k_i)(p_j-k_j)\epsilon^{ijk}=0$ and $(p_j-k_j)\bar{A}_k(\tau,\vec{k}-\vec{p})\epsilon^{ijk}=-i\left|\vec{k}-\vec{p}\right|\bar{A}_i(\tau,\vec{k}-\vec{p})$ neglecting the mode that does not experience the unstable enhancement. We now have the relation for A_0 in Fourier space

$$
A_0(\tau,\vec{k}) = i\frac{\alpha}{f}\frac{k_i}{k^2}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\left|\vec{k}-\vec{p}\right|\bar{A}_i(\tau,\vec{k}-\vec{p})\delta\phi(\tau,\vec{p})\ . \tag{C.6}
$$

C.2 $\delta A_i(\tau,\vec{k})$

For $\delta A_i(\tau,\vec{k})$ let us start from (4.6) in Fourier space. Using the same trick used in the previous section, of multiplying everything by $e^{-i\vec{k}\cdot\vec{x}}$, integrate $\frac{d^3x}{(2\pi)^{2/3}}$, and exploiting the Dirac delta, we can write it

$$
\begin{aligned}
&\delta A_i''(\tau,\vec{k}) + k^2\delta A_i(\tau,\vec{k}) - ik_iA_0'(\tau,\vec{k}) - i\frac{\alpha}{f}\int\frac{d^3p}{(2\pi)^{\frac{2}{3}}}\delta^{(3)}(\vec{p})\bar{\phi}'(\tau)\epsilon^{ijk}(k_j-p_j)\delta A_k(\tau,\vec{k}-\vec{p})+ \\
&-i\frac{\alpha}{f}\int\frac{d^3p}{(2\pi)^{\frac{2}{3}}}\delta\phi'(\tau,\vec{p})\epsilon^{ijk}(k_j-p_j)\bar{A}_k(\tau,\vec{k}-\vec{p}) + i\frac{\alpha}{f}\int\frac{d^3p}{(2\pi)^{\frac{2}{3}}}\epsilon^{ijk}p_j\delta\phi(\tau,\vec{p})\bar{A}_k'(\tau,\vec{k}-\vec{p}) = 0\ ,
\end{aligned}
\tag{C.7}
$$

where we neglected the term coming from the vector product between the gradient of ϕ and the gradient of A_0 because it would be of second order. Replacing the result obtained for A_0 (C.6) we end up with

$$
\begin{aligned}
&\delta A_i''(\tau,\vec{k}) + k^2\delta A_i(\tau,\vec{k}) + \frac{\alpha}{f}\int\frac{d^3p}{(2\pi)^{\frac{2}{3}}}\left|\vec{k}-\vec{p}\right|\left[\delta\phi'(\tau,\vec{p})\bar{A}_i(\tau,\vec{k}-\vec{p}) + \delta\phi(\tau,\vec{p})\bar{A}_i'(\tau,\vec{k}-\vec{p})\right]+ \\
&-\frac{\alpha}{f}\int\frac{d^3p}{(2\pi)^{\frac{2}{3}}}\left|\vec{k}-\vec{p}\right|\delta\phi'(\tau,\vec{p})\bar{A}_i(\tau,\vec{k}-\vec{p}) - i\frac{\alpha}{f}\int\frac{d^3p}{(2\pi)^{\frac{2}{3}}}\delta^{(3)}(\vec{p})\phi'(\tau)\epsilon^{ijk}(k_j-p_j)\delta A_k(\tau,\vec{k}-\vec{p})+ \\
&-\frac{\alpha}{f}\int\frac{d^3p}{(2\pi)^{\frac{3}{2}}}\left|\vec{k}-\vec{p}\right|\bar{A}_i'(\tau,\vec{k}-\vec{p})\delta\phi(\tau,\vec{p}) + i\frac{\alpha}{f}\int\frac{d^3p}{(2\pi)^{\frac{3}{2}}}\epsilon^{ijk}k_j\bar{A}_k'(\tau,\vec{k}-\vec{p})\delta\phi(\tau,\vec{p}) = 0\ ,
\end{aligned}
\tag{C.8}
$$

to get to these last two terms we replaced p_j in the last integral of (C.7) with $-(k_j-p_j-k_j)$, exploited the usual properties of the helicity vectors. Finally after deleting all the terms that cancel we get

$$
\delta A_i''(\tau,\vec{k}) + k^2\delta A_i(\tau,\vec{k}) - \frac{\alpha}{f}\bar{\phi}'(\tau)\left|\vec{k}\right|\delta A_i(\tau,\vec{k}) = -i\frac{\alpha}{f}\int\frac{d^3p}{(2\pi)^{\frac{2}{3}}}\delta\phi(\tau,\vec{p})\epsilon^{ijk}k_j\bar{A}_k'(\tau,\vec{k}-\vec{p}) \tag{C.9}
$$

C.3 $\delta\phi(\tau,\vec{k})$

We now want to perturb equation (4.8) and write it in momentum space. For this purpose, let us consider separately the two sides

LHS: The left-hand side of equation (4.8) can be perturbed and written in Fourier space as

$$\delta\phi''(\vec{k}) + 2\mathcal{H}\delta\phi'(\vec{k}) + k^2\delta\phi(\vec{k}) + a^2V''\delta\phi(\vec{k}) \ . \tag{C.10}$$

RHS: The right-hand side requires more work, first let us perturb it

$$-\frac{\alpha}{fa^2}\epsilon^{ijk}\left[\partial_0\delta A_i(\tau,\vec{x})\partial_k\bar{A}_k(\tau,\vec{x}) + \partial_0\bar{A}_i(\tau,\vec{x})\partial_j\delta A_k(\tau,\vec{x}) - \partial_i A_0(\tau,\vec{x})\partial_j\bar{A}_k(\tau,\vec{x})\right] \ . \tag{C.11}$$

At this point, we apply the same procedure adopted to go from (C.1) to (C.4), and we obtain

$$\begin{aligned}
&-\frac{\alpha}{fa^2}\int\frac{d^3p}{(2\pi)^{\frac{3}{2}}}\delta A_i'(\tau,\vec{p})i(k_j-p_j)\bar{A}_k(\tau,\vec{k}-\vec{p})\epsilon^{ijk}+\\
&-\frac{\alpha}{fa^2}\int\frac{d^3p}{(2\pi)^{\frac{3}{2}}}\bar{A}_i'(\tau,\vec{p})i(k_j-p_j)\delta A_k(\tau,\vec{k}-\vec{p})\epsilon^{ijk}+\\
&+\frac{\alpha}{fa^2}\int\frac{d^3p}{(2\pi)^{\frac{3}{2}}}ip_iA_0(\tau,\vec{p})i(k_j-p_j)\bar{A}_k(\tau,\vec{k}-\vec{p})\epsilon^{ijk} \ .
\end{aligned} \tag{C.12}$$

Then we exploit the helicity vector properties

$$\begin{aligned}
&-\frac{\alpha}{fa^2}\int\frac{d^3p}{(2\pi)^{\frac{3}{2}}}\delta A_i'(\tau,\vec{p})|\vec{k}-\vec{p}|\bar{A}_i(\tau,\vec{k}-\vec{p})+\\
&-\frac{\alpha}{fa^2}\int\frac{d^3p}{(2\pi)^{\frac{3}{2}}}\bar{A}_i'(\tau,\vec{p})|\vec{k}-\vec{p}|\delta A_i(\tau,\vec{k}-\vec{p})+\\
&+\frac{\alpha}{fa^2}\int\frac{d^3p}{(2\pi)^{\frac{3}{2}}}ip_iA_0(\tau,\vec{p})|\vec{k}-\vec{p}|\bar{A}_i(\tau,\vec{k}-\vec{p}) \ ,
\end{aligned} \tag{C.13}$$

and we substitute the expression for A_0 (C.6)

$$\begin{aligned}
&-\frac{\alpha}{fa^2}\int\frac{d^3p}{(2\pi)^{\frac{3}{2}}}\delta A_i'(\tau,\vec{p})|\vec{k}-\vec{p}|\bar{A}_i(\tau,\vec{k}-\vec{p})+\\
&-\frac{\alpha}{fa^2}\int\frac{d^3p}{(2\pi)^{\frac{3}{2}}}\bar{A}_i'(\tau,\vec{p})|\vec{k}-\vec{p}|\delta A_i(\tau,\vec{k}-\vec{p})+\\
&-\frac{\alpha^2}{f^2a^2}\int\frac{d^3p}{(2\pi)^{\frac{3}{2}}}\frac{d^3q}{(2\pi)^{\frac{3}{2}}}|\vec{k}-\vec{p}||\vec{p}-\vec{q}|\bar{A}_i(\tau,\vec{k}-\vec{p})\bar{A}_i(\tau,\vec{p}-\vec{q})\delta\phi(\vec{q}) \ .
\end{aligned} \tag{C.14}$$

Now that we have worked out both sides of the equation we can join them, and we get

$$
\begin{aligned}
&\delta\phi''(\tau,\vec{k}) + 2\mathcal{H}\delta\phi'(\tau,\vec{k}) + k^2\delta\phi(\tau,\vec{k}) + a^2V''\delta\phi(\tau,\vec{k})+\\
&+\frac{\alpha^2}{a^2f^2}\int\frac{d^3p}{(2\pi)^{\frac{3}{2}}}\frac{d^3q}{(2\pi)^{\frac{3}{2}}}|\vec{k}-\vec{p}||\vec{p}-\vec{q}|\bar{A}_i(\tau,\vec{k}-\vec{p})\bar{A}_i(\tau,\vec{p}-\vec{q})\delta\phi(\tau,\vec{q}) =\\
=&-\frac{\alpha}{a^2f}\int\frac{d^3p}{(2\pi)^{\frac{3}{2}}}|\vec{k}-\vec{p}|\bar{A}_i(\tau,\vec{k}-\vec{p})\delta A_i'(\tau,\vec{p}) - \frac{\alpha}{a^2f}\int\frac{d^3p}{(2\pi)^{\frac{3}{2}}}\bar{A}_i'(\tau,\vec{k}-\vec{p})|\vec{p}|\delta A_i(\tau,\vec{p})\ .
\end{aligned}
\tag{C.15}
$$

Appendix D

Evolution equations for $\varphi(n)$ and $h(n)$

To find the evolution equation for $\varphi(n)$ and $h(n)$, let us start from equations for $\varphi(\tau)$ and $h(\tau)$

$$\ddot{\varphi}(\tau) + 3h(\tau)\dot{\varphi}(\tau) + \gamma\dot{\varphi}(\tau) + \varphi^3(\tau) = 0 \ , \tag{D.1}$$

$$\dot{h}(\tau) + 2h^2(\tau) + \frac{1}{6}\dot{\varphi}^2(\tau) - \frac{1}{6}\varphi^4(\tau) = 0 \ . \tag{D.2}$$

Employing the relations between the derivative wrt τ, and the one wrt n

$$\frac{d}{d\tau} = h\frac{d}{dn} \ , \tag{D.3}$$

$$\frac{d^2}{d\tau^2} = h^2\frac{d^2}{dn^2} + hh'\frac{d}{dn} \ , \tag{D.4}$$

equations (D.1) and (D.2) can be written as

$$h^2(n)\varphi''(n) + h(n)h'(n)\varphi'(n) + 3h^2(n)\varphi'(n) + \gamma h(n)\varphi'(n) + \frac{1}{h^2(n)}\varphi^3(n) = 0 \ , \tag{D.5}$$

$$h(n)h'(n) + 2h^2(n) + \frac{1}{6}h^2(n)\varphi'^2(n) - \frac{1}{6}\varphi^4(n) = 0 \ , \tag{D.6}$$

and then we can recast them into

$$\varphi''(n) + \frac{h'(n)}{h(n)}\varphi'(n) + 3\varphi'(n) + \frac{\gamma}{h(n)}\varphi'(n) + \varphi^3(n) = 0 \ , \tag{D.7}$$

$$h'(n) + 2h(n) + \frac{1}{6}h(n)\varphi'^2(n) - \frac{1}{6h(n)}\varphi^4(n) = 0 \ . \tag{D.8}$$

Isolating now $h'(n)$ in (D.8) and substituting it into the second term of (D.7), we finally obtain the evolution equations for the two quantities in the variable n, that without considering the delay for the moment, reads

$$\varphi''(n) + \left(1 + \frac{1}{6h^2(n)} - \frac{1}{6}\varphi'^2(n)\right)\varphi'(n) + \frac{\gamma}{h^2(n)}h(n)\varphi'(n) + \frac{1}{h^2(n)}\varphi^3(n) = 0 \; , \tag{D.9}$$

$$h'(n) + 2h(n) - \frac{1}{6h(n)}\varphi^4(n) + \frac{1}{6}h(n)\varphi'^2(n) = 0 \; . \tag{D.10}$$

Appendix E

Computation of the slow roll initial conditions

In the absence of extra friction, the equations describing the system read

$$\ddot{\varphi}(\tau) + 3h(\tau)\dot{\varphi}(\tau) + \varphi^3(\tau) = 0 \ , \tag{E.1}$$

$$h^2(\tau) = \frac{1}{3}\left(\frac{1}{2}\dot{\varphi}^2(\tau) + \frac{1}{4}\varphi^4(\tau)\right) \ . \tag{E.2}$$

Assuming slow roll, these relations become

$$3h(\tau)\dot{\varphi}(\tau) + \varphi^3(\tau) \simeq 0 \ , \tag{E.3}$$

$$h^2(\tau) \simeq \frac{1}{12}\varphi^4(\tau) \ , \tag{E.4}$$

while in terms of number of e-folds, the slow roll relations become

$$3h^2(n)\varphi'(n) + \varphi^3(n) \simeq 0 \ , \tag{E.5}$$

$$h^2(n) \simeq \frac{1}{12}\varphi^4(n) \ . \tag{E.6}$$

Substituting the second equation into the first one we obtain

$$\frac{1}{4}\varphi^4(n)\varphi'(n) \simeq -\varphi^3(n) \tag{E.7}$$

$$\varphi(n)\varphi'(n) \simeq -4 \tag{E.8}$$

$$\varphi(n)d\varphi(n) \simeq -4dn \ , \tag{E.9}$$

and integrating this gives

$$\int_{\varphi_i}^{\varphi} \tilde{\varphi} d\tilde{\varphi} \simeq -4\int_{n_i}^{n} dn \tag{E.10}$$

$$\varphi(n) \simeq \sqrt{-8n} \ , \tag{E.11}$$

where we set $\varphi_i^2/2 = -4n_i$, and we are considering negative number of e-folds. Replacing now, the result (E.11) into equations (E.8) and (E.6), we obtain the slow roll relation for $\varphi'(n)$ and $h(n)$

$$\sqrt{-8n}\varphi'(n) \simeq -4 \; , \tag{E.12}$$

$$h(n) \simeq \frac{1}{\sqrt{12}} - 8n \; , \tag{E.13}$$

and finally, we obtain

$$\varphi'(n) \simeq -\sqrt{-\frac{2}{n}} \; , \tag{E.14}$$

$$h(n) \simeq -\frac{4}{\sqrt{3}}n \; . \tag{E.15}$$

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