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THE UNIT GROUPS OF SEMISIMPLE GROUP ALGEBRAS OF SOME NON-METABELIAN GROUPS OF ORDER 144

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Abstract. We consider all the non-metabelian groups G of order 144 that have exponent either 36 or 72 and deduce the unit group $U(\mathbb{F}_qG)$ of semisimple group algebra $\mathbb{F}_qG$. Here, q denotes the power of a prime, i.e., $q=p^r$ for p prime and a positive integer r. Up to isomorphism, there are 6 groups of order 144 that have exponent either 36 or 72. Additionally, we also discuss how to simply obtain the unit groups of the semisimple group algebras of those non-metabelian groups of order 144 that are a direct product of two nontrivial groups. In all, this paper covers the unit groups of semisimple group algebras of 17 non-metabelian groups.

Keywords: unit group; finite field; Wedderburn decomposition

MSC 2020: 16U60, 20C05

1. Introduction

Let $\mathbb{F}_pG$ denote the semisimple group algebra of a finite group G and a finite field $\mathbb{F}_p$, where the prime p is such that $p\nmid |G|$. The determination of the unit groups of finite group algebras is a well known and extensively studied research problem (cf. [1], [5], [9], [11], [12], [16], [18], [3], [20] and the references therein). This is because the units of group algebras can be utilized in coding theory as well as in cryptography etc., (see [6], [7], [13]). Moreover, for the isomorphism problems and exploration of Lie properties of group algebras, units are very useful (cf. [2]).

It is known that a group is metabelian, provided its derived (or commutator) subgroup is abelian. Bakshi et al. in [1] studied the unit groups of the semisimple group algebras of metabelian groups. Therefore, most of the research in this direction is focused on deducing the unit groups of the semisimple group algebras of non-metabelian groups. In view of this, it is important to know the possible orders of non-metabelian groups. Pazderski in [19] classified the positive integers

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for which there is no non-metabelian group. Consequently, we see that there are non-metabelian groups of order $24k$, $54k$, $60k$ etc., where k is a positive integer. For the non-metabelian groups of order up to 72, unit groups of their corresponding semisimple group algebras are discussed in [14]. The paper [14] covers all the non-metabelian groups of order 24, 48, 54, 60 and 72. Further, the unit groups of the semisimple group algebras of non-metabelian groups of orders 108 and 120 (except S_5 and $SL(2,5)$), are studied in [15], [17]. The unit group of the semisimple group algebras of the symmetric groups S_n can be deduced by determining the degrees of irreducible representations of S_n together with the result that the group S_n splits over every field (cf. [8]). The unit group of the semisimple group algebra of $SL(2,5)$ is determined in [22]. In view of the above discussion, we see that the study of unit groups of the semisimple group algebras of non-metabelian groups up to order 120 is complete, except that of the groups of order 96.

In this paper, we continue in the direction of determining the unit groups of the semisimple group algebras and determine the units groups of the semisimple group algebras of some non-metabelian groups of order 144. To be more precise, we cover the unit groups of group algebras of 17 non-metabelian groups that fulfill one of the following three conditions: (i) exponent 72, (ii) exponent 36, (iii) or that are direct product of their nontrivial subgroups. In order to deduce the unit group $U(\mathbb{F}_qG)$ for $p > 3$, first, we deduce the Wedderburn decomposition of the group algebra $\mathbb{F}_qG$. After obtaining the Wedderburn decomposition, it is straightforward to deduce the unit group. The rest of this paper is structured as follows. The discussion of all the non-metabelian groups of order 144 is given in Section 2. In addition to this, in this section, we discuss a result about the Wedderburn decompositions (or unit groups) of the semsimple group algebras of those groups that are direct product of nontrivial groups. Further, all the basic definitions and results needed in this paper are given in Section 3. Our main results related to the unit groups are discussed in Sections 4 and 5. The last section is concluding in nature.

2. Non-metabelian groups of order 144

In this section, we discuss all the non-metabelian groups of order 144. Up to isomorphism, there are 197 groups of order 144 and 28 of them are non-metabelian. We write all the 28 non-metabelian groups of order 144 in the following list:

(1) $(Q_8 \rtimes C_9) \cdot C_2$, (15) $C_3 \times (A_4 \rtimes C_4)$,

(2) $(Q_8 \rtimes C_9) \rtimes C_2$, (16) $C_3 \rtimes (C_2 \cdot S_4)$,

(3) $((C_2 \times C_2) \rtimes C_9) \rtimes C_4$, (17) $(C_3 \times SL(2,3)) \rtimes C_2$,

(4) $C_2 \times (Q_8 \rtimes C_9)$,

(5) $((C_4 \times C_2) \rtimes C_2) \rtimes C_9$,

(6) $C_2 \times (((C_2 \times C_2) \rtimes C_9) \rtimes C_2)$,

(7) $(C_3 \times C_3) \rtimes ((C_4 \times C_2) \rtimes C_2)$,

(8) $(C_3 \times C_3) \rtimes (C_4 \rtimes C_4)$,

(9) $(C_3 \times C_3) \rtimes D_{16}$,

(10) $(C_3 \times C_3) \rtimes QD_{16}$,

(11) $(C_3 \times C_3) \rtimes Q_{16}$,

(12) $(C_3 \times C_3) \rtimes (C_4 \rtimes C_4)$,

(13) $C_3 \times (C_2 \cdot S_4)$,

(14) $C_3 \times GL(2,3)$,

(18) $(C_3 \times A_4) \rtimes C_4$,

(19) $((C_4 \times S_3) \rtimes C_2) \rtimes C_3$,

(20) $S_3 \times SL(2,3)$,

(21) $C_6 \times SL(2,3)$,

(22) $C_3 \times (((C_4 \times C_2) \rtimes C_2) \rtimes C_3)$,

(23) $(C_3 \times C_3) \rtimes QD_{16}$,

(24) $S_3 \times S_4$,

(25) $C_2 \times ((S_3 \times S_3) \rtimes C_2)$,

(26) $C_2 \times ((C_3 \times C_3) \rtimes Q_8)$,

(27) $C_6 \times S_4$,

(28) $C_2 \times ((C_2 \times A_4) \rtimes C_2)$.

For the following 13 groups in the preceding list that are a direct product of non-trivial groups, the Wedderburn decomposition can be calculated using [14] and the references therein, together with the results that $\mathbb{F}_q(G \times H) \cong \mathbb{F}_q G \bigotimes_{\mathbb{F}_q} \mathbb{F}_q H$ for any finite groups G and H (see also Remark 3.1 in [17]), and

$$\mathbb{F}_{q^a} \bigotimes_{\mathbb{F}_q} \mathbb{F}_{q^b} \cong (\mathbb{F}_{q^l})^t, \quad M_{n_r}(\mathbb{F}_{q^a}) \bigotimes_{\mathbb{F}_q} \mathbb{F}_{q^b} \cong M_{n_r}\Big(\mathbb{F}_{q^a} \bigotimes_{\mathbb{F}_q} \mathbb{F}_{q^b}\Big) \cong \bigoplus_{t\,\text{times}} M_{n_r}(\mathbb{F}_{q^l}),$$

where $a, b \in \mathbb{Z}^+$ and $t = \gcd(a,b)$, $l = \operatorname{lcm}(a,b)$. Here, we keep the same serial number of the groups as in the previous list:

(4) $C_2 \times (Q_8 \rtimes C_9)$,

(13) $C_3 \times (C_2 \cdot S_4)$,

(15) $C_3 \times (A_4 \rtimes C_4)$,

(21) $C_6 \times SL(2,3)$,

(24) $S_3 \times S_4$,

(26) $C_2 \times ((C_3 \times C_3) \rtimes Q_8)$,

(28) $C_2 \times ((C_2 \times A_4) \rtimes C_2)$.

(6) $C_2 \times (((C_2 \times C_2) \rtimes C_9) \rtimes C_2)$,

(14) $C_3 \times GL(2,3)$,

(20) $S_3 \times SL(2,3)$,

(22) $C_3 \times (((C_4 \times C_2) \rtimes C_2) \rtimes C_3)$,

(25) $C_2 \times ((S_3 \times S_3) \rtimes C_2)$,

(27) $C_6 \times S_4$,

For example, we know that $\mathbb{F}_q S_4 \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2$ and $\mathbb{F}_q S_3 \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q)$ for any $p > 3$. This means that for the group (24), i.e., $S_3 \times S_4$, we have $\mathbb{F}_q(S_3 \times S_4) \cong \mathbb{F}_q S_3 \bigotimes_{\mathbb{F}_q} \mathbb{F}_q S_4 \cong (\mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q)) \bigotimes_{\mathbb{F}_q} (\mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2) \cong \Big(\mathbb{F}_q^2 \bigotimes_{\mathbb{F}_q} (\mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2)\Big) \oplus \Big(M_2(\mathbb{F}_q) \bigotimes_{\mathbb{F}_q} (\mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2)\Big) \cong (\mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4) \oplus$

$(M_2(\mathbb{F}_q)^2 \oplus M_4(\mathbb{F}_q) \oplus M_6(\mathbb{F}_q)^2)$. In view of this, we conclude that the Wedderburn decompositions of group algebras of those groups that are direct products of two nontrivial groups can be calculated easily. Further, it is straight-forward to compute the unit group from the Wedderburn decomposition. This means one can easily obtain the unit groups of the semisimple group algebras formed by above-mentioned 13 groups. Consequently, in this paper we focus only on those groups that are not a direct product of two smaller nontrivial groups.

3. Preliminaries

Let ε denote the primitive eth root of unity, where e is the exponent of G. As in [4], we define $I_{\mathbb{F}} = \{n\colon \varepsilon \mapsto \varepsilon^n$ is an automorphism of $\mathbb{F}(\varepsilon)$ over $\mathbb{F}\}$, where $\mathbb{F}$ is a finite field. We observe that the Galois group $\mathrm{Gal}\big(\mathbb{F}(\varepsilon), \mathbb{F}\big)$ is a cyclic group. Consequently, for any $\zeta \in \mathrm{Gal}(\mathbb{F}(\varepsilon), \mathbb{F})$, there exists a $s \in \mathbb{Z}_e^*$ such that $\zeta(\varepsilon) = \varepsilon^s$. In other words, $I_{\mathbb{F}}$ is a subgroup of the multiplicative group $\mathbb{Z}_e^*$. Let $\mathsf{g} \in G$ be a p-regular element, so $p \nmid |\mathsf{g}|$. We define γ_{g} as the sum of all conjugates of g for a p-regular element $\mathsf{g} \in G$. Further, let the cyclotomic $\mathbb{F}$-class of γ_{g} be denoted by $S(\gamma_{\mathsf{g}}) = \{\gamma_{\mathsf{g}^n}\colon n \in I_{\mathbb{F}}\}$. To this end, we recall the following two results from [4]. The first result is related to total number of cyclotomic $\mathbb{F}$-classes and the second result is related to the number of elements in any cyclotomic $\mathbb{F}$-class γ_{g}. Let $J(\mathbb{F}G)$ denote the Jacobson radical of group algebra $\mathbb{F}G$.

Theorem 3.1. *The number of simple components of $\mathbb{F}G/J(\mathbb{F}G)$ is equal to the number of cyclotomic $\mathbb{F}$-classes in G.*

Theorem 3.2. *Let z be the number of cyclotomic $\mathbb{F}$-classes in G and ε be same as defined earlier. If K_s and S_s for $1 \leqslant s \leqslant z$ represent the simple components of center of $\mathbb{F}G/J(\mathbb{F}G)$ and the cyclotomic $\mathbb{F}$-classes in G, respectively, then $|S_s| = [K_s : \mathbb{F}]$ for each s, after suitable ordering of the indices.*

Next, we recall the following result with which we already know that $\mathbb{F}$ is one of the simple components of Wedderburn decomposition of $\mathbb{F}G/J(\mathbb{F}G)$ (see [14] for proof).

Lemma 3.1. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ denote the finite dimensional algebras over $\mathbb{F}$. Moreover, let $\mathcal{A}_2$ be semisimple and $\psi\colon \mathcal{A}_1 \to \mathcal{A}_2$ be a surjective homomorphism. Then we have that $\mathcal{A}_1/J(\mathcal{A}_1) \cong \mathcal{A}_3 + \mathcal{A}_2$, where $\mathcal{A}_3$ is a semisimple $\mathbb{F}$-algebra.*

Due to Lemma 3.1, we see that if $J(\mathbb{F}G) = 0$, then $\mathbb{F}$ is always a simple component of $\mathbb{F}G$. To this end, we state a result that characterizes the set $I_{\mathbb{F}}$ (cf. [10]).

Theorem 3.3. *Let $\mathbb{F}$ be a finite field with order $q = p^r$ for some prime p and $r \in \mathbb{Z}^+$. Let e fulfill $\gcd(e, q) = 1$, ε be the primitive eth root of unity. Moreover, let $|q|$ denote the order of q modulo e, then we have $I_{\mathbb{F}} = \{1, q, q^2, \ldots, q^{|q|-1}\} \bmod e$.*

Finally, we end this subsection by stating two results from [21]. The first result is related to the commutative simple components of the group algebra $\mathbb{F}_q G$ and the second one is related to the relationship between the Wedderburn decomposition of $\mathbb{F}_q G$ and Wedderburn decomposition of $\mathbb{F}_q(G/H)$, where H is a normal subgroup of G. Let G' denote the commutator subgroup of G.

Theorem 3.4. *Let the group algebra $\mathcal{R}G$ be semisimple, where $\mathcal{R}$ denotes a commutative ring. Then, we have $\mathcal{R}G \cong \mathcal{R}\big(G/G'\big) \oplus \Delta(G, G')$, where $\mathcal{R}\big(G/G'\big)$ is the sum of all commutative simple components of $\mathcal{R}G$, and $\Delta(G, G')$ is the sum of all non-commutative simple components.*

Theorem 3.5. *Let H be a normal subgroup of G and $\mathcal{R}G$ be a semisimple group algebra. Then $\mathcal{R}G \cong \mathcal{R}(G/H) \oplus \Delta(G, H)$, where $\Delta(G, H)$ denotes the ideal (left) of $\mathcal{R}G$ generated by the set $\{h - 1\colon\ h \in H\}$.*

4. Non-metabelian groups of order 144 having exponent 72

In this section, we consider all the non-metabelian groups of order 144 that have exponent 72 and deduce the unit groups of their semisimple group algebras. The non-metabelian groups of order 144 that have exponent 72 are (up to isomorphism) $G_1 := (Q_8 \rtimes C_9).C_2$, and $G_2 := (Q_8 \rtimes C_9) \rtimes C_2$.

4.1. The group $G_1 = (Q_8 \rtimes C_9) \cdot C_2$. The presentation of G_1 is the following:

$$\begin{aligned}\langle x, y, z, w, t, u\colon\ & x^2u^{-1}, [y,x]z^{-1}y^{-1}, [z,x]z^{-1}, [w,x]u^{-1}t^{-1}w^{-1}, [t,x]t^{-1}w^{-1},\\ & [u,x], y^3z^{-2}, [z,y], [w,y]t^{-1}w^{-1}, [t,y]u^{-1}w^{-1}, [u,y], z^3, [w,z],\\ & [t,z], [u,z], w^2u^{-1}, [t,w]u^{-1}, [u,w], t^2u^{-1}, [u,t], u^2\rangle,\end{aligned}$$

where $[x, y] = x^{-1}y^{-1}xy$. Also G_1 has 15 conjugacy classes as shown in the table below.

R	e	x	y	z	w	u	xw	y^2	yt	zw	zu	xyw	y^2w	yz^2	yz^2t
S	1	36	8	2	6	1	18	8	8	12	2	18	8	8	8
O	1	4	9	3	4	2	8	9	18	12	6	8	18	9	18

where R, S and O, respectively, denote the representative, size and order (of the representative) of a conjugacy class. From the above description of G_1, it is clear that G_1 has exponent 72. Also $G_1' \cong Q_8 \rtimes C_9$. Next, we give the unit group of $\mathbb{F}_q G_1$ when $p > 3$.

Theorem 4.1. *The unit group U_1 of $\mathbb{F}_q G_1$, for $q = p^k$ and $p > 3$, where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements, is as follows:*

(1) *for* $p^k \in \{1, 17, 55, 71\} \bmod 72$, $U_1 \cong (\mathbb{F}_q^*)^2 \oplus GL_2(\mathbb{F}_q)^6 \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_4(\mathbb{F}_q)^4 \oplus GL_6(\mathbb{F}_q)$;

(2) *for* $p^k \in \{5, 11, 13, 29, 43, 59, 61, 67\} \bmod 72$, $U_1 \cong (\mathbb{F}_q^*)^2 \oplus GL_2(\mathbb{F}_q) \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_4(\mathbb{F}_q) \oplus GL_6(\mathbb{F}_q) \oplus GL_2(\mathbb{F}_{q^2}) \oplus GL_2(\mathbb{F}_{q^3}) \oplus GL_4(\mathbb{F}_{q^3})$;

(3) *for* $p^k \in \{7, 23, 25, 31, 41, 47, 49, 65\} \bmod 72$, $U_1 \cong (\mathbb{F}_q^*)^2 \oplus GL_2(\mathbb{F}_q)^3 \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_4(\mathbb{F}_q) \oplus GL_6(\mathbb{F}_q) \oplus GL_2(\mathbb{F}_{q^3}) \oplus GL_4(\mathbb{F}_{q^3})$;

(4) *for* $p^k \in \{19, 35, 37, 53\} \bmod 72$, $U_1 \cong (\mathbb{F}_q^*)^2 \oplus GL_2(\mathbb{F}_q)^4 \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_4(\mathbb{F}_q)^4 \oplus GL_6(\mathbb{F}_q) \oplus GL_2(\mathbb{F}_{q^2})$.

P r o o f. Since $\mathbb{F}_q G_1$ is semisimple, using Lemma 3.1 we get

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q \bigoplus_{r=1}^{m-1} M_{n_r}(\mathbb{F}_r) \quad \text{for some } m \in \mathbb{Z}. \tag{4.1}$$

First assume that k is a multiple of 6. Then for any prime $p > 3$ we have $p^k \equiv 1 \bmod 8$ and $p^k \equiv 1 \bmod 9$, which means $p^k \equiv 1 \bmod 72$. Therefore, as $I_{\mathbb{F}} = \{1\}$, $|S(\gamma_g)| = 1$ for each $g \in G_1$. Hence, (4.1), Theorems 3.1 and 3.2 provide that $\mathbb{F}_q G_1 \cong \mathbb{F}_q \bigoplus_{r=1}^{14} M_{n_r}(\mathbb{F}_q)$. Utilizing Theorem 3.4 with $G_1/G_1' \cong C_2$ to obtain $\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \bigoplus_{r=1}^{13} M_{n_r}(\mathbb{F}_q)$, where $n_r \geqslant 2$ with $142 = \sum_{r=1}^{13} n_r^2$. To this end, consider the normal subgroup $H_1 := \langle u \rangle$ of G_1. Note that $G_1/H_1 \cong ((C_2 \times C_2) \rtimes C_9) \rtimes C_2)$ and $\mathbb{F}_q(G_1/H_1) \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q)$ (cf. [14]). This with Theorem 3.5 implies that $\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q) \bigoplus_{r=1}^{6} M_{n_r}(\mathbb{F}_q)$, where $n_r \geqslant 2$ with $72 = \sum_{r=1}^{6} n_r^2$. This equation gives the following 2 possibilities for n_r's, $(2^2, 4^4)$ and $(2, 3^3, 4, 5)$ where y^z means $(y, y, \ldots, z\,\text{times})$. Since the Wedderburn decomposition is unique for the group algebra $\mathbb{F}_q G_1$, suppose, if possible, that the second possibility is true, i.e., $(2, 3^3, 4, 5)$ is the required choice for n_r's. We will arrive at some contradiction to show that $(2^2, 4^4)$ is the actual value of n_r's. By assumption, we have $\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q)^5 \oplus M_3(\mathbb{F}_q)^5 \oplus M_4(\mathbb{F}_q) \oplus M_5(\mathbb{F}_q) \oplus M_6(\mathbb{F}_q)$. Since this holds for any prime $p > 3$, we let $p = 5$. Due to Proposition 1 from [3], we know that if $\mathbb{F}_p G \cong \bigoplus_{t=1}^{j} M_{n_t}(\mathbb{F}_{q_t})$, then p does not divide any of the n_t. Consequently,

$(2, 3^3, 4, 5)$ can not be the choice for n_r's. Therefore, we have

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q)^6 \oplus M_3(\mathbb{F}_q)^2 \oplus M_4(\mathbb{F}_q)^4 \oplus M_6(\mathbb{F}_q). \tag{4.2}$$

Now, we consider the other remaining possibilities for k. We can easily see that for any k, $p^k \in \{1, 3, 5, 7\} \bmod 8$ and $p^k \in \{1, 2, 4, 5, 7, 8\} \bmod 9$. Due to this, we have 24 possible values of p^k modulo 72. Note that for $p^k \in \{1, 17, 55, 71\} \bmod 72$, $|S(\gamma_g)| = 1$ for all $g \in G_1$ as $I_{\mathbb{F}} = \{1\}$. Consequently, the Wedderburn decomposition is given by (4.2). Further, for $p^k \in \{5, 11, 13, 29, 43, 59, 61, 67\} \bmod 72$, we have that $S(\gamma_y) = \{\gamma_y, \gamma_{y^2}, \gamma_{yz^2}\}$, $S(\gamma_{xw}) = \{\gamma_{xw}, \gamma_{xyw}\}$, $S(\gamma_{yt}) = \{\gamma_{yt}, \gamma_{yz^2t}, \gamma_{y^2w}\}$, and for the other representatives g, $S(\gamma_g) = \{\gamma_g\}$. Therefore, (4.1) and Theorems 3.1, 3.2 imply that $\mathbb{F}_q G_1 \cong \mathbb{F}_q \bigoplus_{r=1}^{6} M_{n_r}(\mathbb{F}_q) \oplus M_{n_7}(\mathbb{F}_{q^2}) \bigoplus_{r=8}^{9} M_{n_r}(\mathbb{F}_{q^3})$. Since $G_1/G_1' \cong C_2$, Theorem 3.4 yields $\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \bigoplus_{r=1}^{5} M_{n_r}(\mathbb{F}_q) \oplus M_{n_6}(\mathbb{F}_{q^2}) \bigoplus_{r=7}^{8} M_{n_r}(\mathbb{F}_{q^2})$. Further, again consider the normal subgroup $H_1 = \langle u \rangle$ of G_1. Using [14], we know that $\mathbb{F}_q(G_1/H_1) \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^3})$. This with Theorem 3.5 provides that $\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q) \oplus M_{n_1}(\mathbb{F}_q) \oplus M_{n_2}(\mathbb{F}_{q^2}) \oplus M_2(\mathbb{F}_{q^3}) \oplus M_{n_3}(\mathbb{F}_{q^3})$ with $72 = n_1^2 + 2n_2^2 + 3n_3^2$, $n_r \geqslant 2$ for all r. The last equation has a unique solution given by $(4, 2, 4)$. Consequently, the Wedderburn decomposition is given by

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2 \oplus M_4(\mathbb{F}_q) \oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^2}) \oplus M_2(\mathbb{F}_{q^3}) \oplus M_4(\mathbb{F}_{q^3}). \tag{4.3}$$

Next, for $p^k \in \{7, 23, 25, 31, 41, 47, 49, 65\} \bmod 72$, we observe that $S(\gamma_y) = \{\gamma_y, \gamma_{y^2}, \gamma_{yz^2}\}$, $S(\gamma_{yt}) = \{\gamma_{yt}, \gamma_{yz^2t}, \gamma_{y^2w}\}$, and for the other representatives g, $S(\gamma_g) = \{\gamma_g\}$. Therefore, (4.1), Theorems 3.1, 3.2 and 3.4 with $G_1/G_1' \cong C_2$ deduce that $\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \bigoplus_{r=1}^{7} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=8}^{9} M_{n_r}(\mathbb{F}_{q^3})$. To this end, again consider H_1 and use $\mathbb{F}_q(G_1/H_1) \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^3})$, along with Theorem 3.5, to see that $\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q) \bigoplus_{r=1}^{3} M_{n_r}(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^3}) \oplus M_{n_4}(\mathbb{F}_{q^3})$ with $72 = \sum_{r=1}^{3} n_r^2 + 3n_4^2$, $n_r \geqslant 2$ for all r. The last equation has two solutions given by $(2^2, 4^2)$, and $(2, 4, 5, 3)$. Consequently, with the same logic as considered in the case when $I_{\mathbb{F}} = \{1\}$, we conclude that the required Wedderburn decomposition is

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q)^3 \oplus M_3(\mathbb{F}_q)^2 \oplus M_4(\mathbb{F}_q) \oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^3}) \oplus M_4(\mathbb{F}_{q^3}). \tag{4.4}$$

Finally, we are remaining with the choices $p^k \in \{19, 35, 37, 53\} \bmod 72$. For these choices, we observe that $S(\gamma_{xw}) = \{\gamma_{xw}, \gamma_{xyw}\}$, and for the other representatives g, $S(\gamma_g) = \{\gamma_g\}$. Therefore, (4.1), Theorems 3.1, 3.2 and 3.4 deduce that $\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \bigoplus_{r=1}^{11} M_{n_r}(\mathbb{F}_q) \oplus M_{n_{12}}(\mathbb{F}_{q^2})$, $n_r \geqslant 2$. To this end, again consider H_1 and utilize $\mathbb{F}_q(G_1/H_1) \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q)$, along with Theorem 3.5, to reach at $\mathbb{F}_qG_1 \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q) \bigoplus_{r=1}^{4} M_{n_r}(\mathbb{F}_q) \oplus M_{n_5}(\mathbb{F}_{q^2})$ with $72 = \sum_{r=1}^{4} n_r^2 + 2n_5^2$, $n_r \geqslant 2$ for all r. The last equation has three solutions given by $(2^2, 4^3)$, $(2, 3, 4, 5, 3)$ and $(4^4, 2)$. To this end, we consider the normal subgroup $H_2 := \langle z \rangle$ with the corresponding factor group $G_1/H_2 \cong C_2.S_4$. From [14], we know that $\mathbb{F}_q(G_1/H_2) \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2 \oplus M_4(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^2})$. Consequently, we conclude that $M_2(\mathbb{F}_{q^2})$ must be the Wedderburn component of $\mathbb{F}_qG$ which implicitly implies that $(4^4, 2)$ is the possible choice for n_r's. Therefore, the Wedderburn decomposition is

$$\mathbb{F}_qG_1 \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^2 \oplus M_4(\mathbb{F}_q)^4 \oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^2}). \tag{4.5}$$

It is straight-forward to deduce the unit group from the Wedderburn decomposition. □

4.2. The group $G_2 = (Q_8 \rtimes C_9) \rtimes C_2$. The presentation of G_2 is the following:

$$\langle x, y, z, w, t, u \colon x^2, [y,x]z^{-1}y^{-1}, [z,x]z^{-1}, [w,x]u^{-1}t^{-1}w^{-1}, [t,x]t^{-1}w^{-1}, [u,x], y^3z^{-2}, [z,y], [w,y]t^{-1}w^{-1}, [t,y]u^{-1}w^{-1}, [u,y], z^3, [w,z], [t,z], [u,z], w^2u^{-1}, [t,w]u^{-1}, [u,w], t^2u^{-1}, [u,t], u^2\rangle.$$

Also G_2 has 15 conjugacy classes, as shown in the table below.

R	e	x	y	z	w	u	xw	y^2	yt	zw	zu	xyw	y^2w	yz^2	yz^2t
S	1	36	8	2	6	1	18	8	8	12	2	18	8	8	8
O	1	2	9	3	4	2	8	9	18	12	6	8	18	9	18

Note that G_2 has exponent 72 and $G_2' \cong Q_8 \rtimes C_9$. Next, we give the unit group of $\mathbb{F}_qG_2$.

Theorem 4.2. *The unit group U_2 of $\mathbb{F}_qG_2$, for $q = p^k$ and $p > 3$, where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements, is as follows:*

(1) *for* $p^k \in \{1, 17, 19, 35\} \bmod 72$, $U_2 \cong (\mathbb{F}_q^*)^2 \oplus GL_2(\mathbb{F}_q)^6 \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_4(\mathbb{F}_q)^4 \oplus GL_6(\mathbb{F}_q)$;
(2) *for* $p^k \in \{5, 7, 13, 23, 29, 31, 47, 61\} \bmod 72$, $U_2 \cong (\mathbb{F}_q^*)^2 \oplus GL_2(\mathbb{F}_q) \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_4(\mathbb{F}_q) \oplus GL_6(\mathbb{F}_q) \oplus GL_2(\mathbb{F}_{q^2}) \oplus GL_2(\mathbb{F}_{q^3}) \oplus GL_4(\mathbb{F}_{q^3})$;
(3) *for* $p^k \in \{11, 25, 41, 43, 49, 59, 65, 67\} \bmod 72$, $U_2 \cong (\mathbb{F}_q^*)^2 \oplus GL_2(\mathbb{F}_q)^3 \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_4(\mathbb{F}_q) \oplus GL_6(\mathbb{F}_q) \oplus GL_2(\mathbb{F}_{q^3}) \oplus GL_4(\mathbb{F}_{q^3})$;
(4) *for* $p^k \in \{37, 53, 55, 71\} \bmod 72$, $U_2 \cong (\mathbb{F}_q^*)^2 \oplus GL_2(\mathbb{F}_q)^4 \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_4(\mathbb{F}_q)^4 \oplus GL_6(\mathbb{F}_q) \oplus GL_2(\mathbb{F}_{q^2})$.

Proof. On the similar lines of the proof of Theorem 4.1, we have $\mathbb{F}_qG_2 \cong \mathbb{F}_q \bigoplus_{r=1}^{m-1} M_{n_r}(\mathbb{F}_r)$, for some $m \in \mathbb{Z}$. Note that the group G_2 also has a normal subgroup $H_1 := \langle u\rangle$ such that $G_2/H_1 \cong ((C_2 \times C_2) \rtimes C_9) \rtimes C_2)$. Therefore, as in Theorem 4.1, for $p^k \in \{1, 17, 19, 35\} \bmod 72$, we note that $|S(\gamma_g)| = 1$ for all $g \in G_1$ as $I_{\mathbb{F}} = \{1\}$. Consequently, the Wedderburn decomposition is given by (4.2). Next, for $p^k \in \{5, 7, 13, 23, 29, 31, 47, 61\} \bmod 72$, we have that $S(\gamma_y) = \{\gamma_y, \gamma_{y^2}, \gamma_{yz^2}\}$, $S(\gamma_{xw}) = \{\gamma_{xw}, \gamma_{xyw}\}$, $S(\gamma_{yt}) = \{\gamma_{yt}, \gamma_{yz^2t}, \gamma_{y^2w}\}$, and for the other representatives g, $S(\gamma_g) = \{\gamma_g\}$. We note that this case is also similar to that in the group G_1. Therefore, the Wedderburn decomposition is given by (4.3). Next, for $p^k \in \{11, 25, 41, 43, 49, 59, 65, 67\} \bmod 72$, we observe that $S(\gamma_y) = \{\gamma_y, \gamma_{y^2}, \gamma_{yz^2}\}$, $S(\gamma_{yt}) = \{\gamma_{yt}, \gamma_{yz^2t}, \gamma_{y^2w}\}$, and for the other representatives g, $S(\gamma_g) = \{\gamma_g\}$. In this case, the Wedderburn decomposition is given by (4.4), since this case is again similar to that in the group G_1. Finally, we are remaining with the choices $p^k \in \{37, 53, 55, 71\} \bmod 72$ and for these choices we observe that $S(\gamma_{xw}) = \{\gamma_{xw}, \gamma_{xyw}\}$, and for the other representatives g, $S(\gamma_g) = \{\gamma_g\}$. Note that the group G_2 has no normal subgroup H_2 such that G_2/H_2 is isomorphic to $C_2 \cdot S_4$ as in the previous theorem. Therefore, the results of previous theorem are not applicable in this case. So, (4.1) (with G_1 replaced by G_2), Theorems 3.1, 3.2 and 3.4 deduce that $\mathbb{F}_qG_2 \cong \mathbb{F}_q^2 \bigoplus_{r=1}^{11} M_{n_r}(\mathbb{F}_q) \oplus M_{n_{12}}(\mathbb{F}_{q^2})$, $n_r \geqslant 2$. Consider the normal subgroup H_1 and utilize $\mathbb{F}_q(G_2/H_1) \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q)$ along with Theorem 3.5 to arrive at $\mathbb{F}_qG_2 \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q) \bigoplus_{r=1}^{4} M_{n_r}(\mathbb{F}_q) \oplus M_{n_5}(\mathbb{F}_{q^2})$ with $72 = \sum_{r=1}^{4} n_r^2 + 2n_5^2$, $n_r \geqslant 2$ for all r. This equation has three choices for n_r's $(2^2, 4^3)$, $(2, 3, 4, 5, 3)$ and $(4^4, 2)$. At this point, we consider the normal subgroup $H_2 := \langle z\rangle$ with the corresponding factor group $G_2/H_2 \cong GL(2, 3)$. From [14], we know that $\mathbb{F}_q(G_2/H_2) \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2 \oplus M_4(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^2})$. Consequently, we conclude that $M_2(\mathbb{F}_{q^2})$ must be the Wedderburn component of $\mathbb{F}_qG$, which implicitly implies that $(4^4, 2)$ is the possible choice for n_r's. Therefore, the Wedderburn decomposition is again given by (4.5). □

5. Non-metabelian groups of order 144 having exponent 36

In this section, we consider all the non-metabelian groups of order 144 that have exponent 36 and deduce the unit groups of their semisimple group algebras. Up to isomorphism, there are 4 non-metabelian groups of order 144 that have exponent 36. Among them, the groups of order 144 that are not the direct product of two smaller groups are $G_3 := ((C_2 \times C_2) \rtimes C_9) \rtimes C_4$, $G_4 := ((C_4 \times C_2) \rtimes C_2) \rtimes C_9$.

5.1. The group $G_3 := ((C_2 \times C_2) \rtimes C_9) \rtimes C_4$**.** The presentation of G_3 is the following:

$$\begin{aligned}\langle x, y, z, w, t, u \colon\ & x^2y^{-1}, [y, x], [z, x]w^{-1}z^{-1}, [w, x]w^{-1}, [t, x]u^{-1}t^{-1},\\ & [u, x]u^{-1}t^{-1}, y^2, [z, y], [w, y], [t, y], [u, y], z^3w^{-2}, [w, z],\\ & [t, z]u^{-1}t^{-1}, [u, z]t^{-1}, w^3, [t, w], [u, w], t^2, [u, t], u^2\rangle.\end{aligned}$$

Also G_3 has 18 conjugacy classes as shown in the table below.

R	e	x	y	z	w	t	xy	xt	yz	yw	yt	z^2	wt	xyt	yz^2	ywt	zw^2	yzw^2
S	1	18	1	8	2	3	18	18	8	2	3	8	6	18	8	6	8	8
O	1	4	2	9	3	2	4	4	18	6	2	9	6	4	18	6	9	18

Clearly, the exponent of G_3 is 36. Also $G_3' \cong (C_2 \times C_2) \rtimes C_9$ with $G_3/G_3' \cong C_4$. Next, we give the unit group of $\mathbb{F}_qG_3$ when $p > 3$.

Theorem 5.1. *The unit group U_3 of $\mathbb{F}_qG_3$, for $q = p^k$ and $p > 3$, where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements, is as follows:*
(1) *for* $p^k \in \{1, 17\} \bmod 36$, $U_3 \cong (\mathbb{F}_q^*)^4 \oplus GL_2(\mathbb{F}_q)^8 \oplus GL_3(\mathbb{F}_q)^4 \oplus GL_6(\mathbb{F}_q)^2$;
(2) *for* $p^k \in \{7, 11, 23, 31\} \bmod 36$, $U_3 \cong (\mathbb{F}_q^*)^2 \oplus \mathbb{F}_{q^2}^* \oplus GL_2(\mathbb{F}_q)^2 \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_6(\mathbb{F}_q)^2 \oplus GL_3(\mathbb{F}_{q^2}) \oplus GL_2(\mathbb{F}_{q^3})^2$;
(3) *for* $p^k \in \{5, 13, 25, 29\} \bmod 36$, $U_3 \cong (\mathbb{F}_q^*)^4 \oplus GL_2(\mathbb{F}_q)^2 \oplus GL_3(\mathbb{F}_q)^4 \oplus GL_6(\mathbb{F}_q)^2 \oplus GL_2(\mathbb{F}_{q^3})^2$;
(4) *for* $p^k \in \{19, 35\} \bmod 36$, $U_3 \cong (\mathbb{F}_q^*)^2 \oplus \mathbb{F}_{q^2}^* \oplus GL_2(\mathbb{F}_q)^8 \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_6(\mathbb{F}_q)^2 \oplus GL_3(\mathbb{F}_{q^2})$.

Proof. Since $\mathbb{F}_qG_3$ is semisimple, using Lemma 3.1 we have

$$\mathbb{F}_qG_3 \cong \mathbb{F}_q \bigoplus_{r=1}^{m-1} M_{n_r}(\mathbb{F}_r) \quad \text{for some } m \in \mathbb{Z}. \tag{5.1}$$

First, assume that for each $g \in G_3$, $|S(\gamma_g)| = 1$. Hence, (5.1), Theorems 3.1 and 3.2 provide that $\mathbb{F}_qG_3 \cong \mathbb{F}_q \bigoplus_{r=1}^{17} M_{n_r}(\mathbb{F}_q)$. Utilizing Theorem 3.4 with $G_3/G_3' \cong C_4$ to obtain $\mathbb{F}_qG_3 \cong \mathbb{F}_q^4 \bigoplus_{r=1}^{14} M_{n_r}(\mathbb{F}_q)$, where $n_r \geqslant 2$ with $140 = \sum_{r=1}^{14} n_r^2$. To this end, consider the normal subgroup $H_1 := \langle y\rangle$ of G_3. Note that $G_3/H_1 \cong ((C_2 \times C_2) \rtimes C_9) \rtimes C_2)$ and $\mathbb{F}_q(G_3/H_1) \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q)$ (as in Theorem 4.1). This with Theorem 3.5 implies that $\mathbb{F}_qG_3 \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q) \bigoplus_{r=1}^{7} M_{n_r}(\mathbb{F}_q)$, where $n_r \geqslant 2$ with $70 = \sum_{r=1}^{7} n_r^2$. This equation gives the following 3 possibilities for n_r's, $(2^5, 5^2)$, $(2^4, 3, 3, 6)$ and $(3^5, 4)$. In order to deduce the uniqueness, we utilize the normal subgroup $H_2 := \langle t, u \rangle$ of G_3. Note that $G_3/H_2 \cong C_9 \rtimes C_4$. Since G_3/H_2 is a metabelian group, from [1] we know that $\mathbb{F}_q(G_3/H_2) \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^8$. This with Theorem 3.5 concludes that $(3^5, 4)$ cannot be the possibility for n_r's. Further, with a similar argument as in Theorem 4.1, we know that $(2^5, 5^2)$ cannot be a possible choice as it contains the Wedderburn component $M_5(\mathbb{F}_q)$. Consequently, the only possible choice is $(2^4, 3, 3, 6)$, i.e., the Wedderburn decomposition is

$$\mathbb{F}_q G_3 \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^8 \oplus M_3(\mathbb{F}_q)^4 \oplus M_6(\mathbb{F}_q)^2. \tag{5.2}$$

Now, we look for the possible values of p^k. For any k, we have that $p^k \in \{1, 3\} \bmod 4$ and $p^k \in \{1, 2, 4, 5, 7, 8\} \bmod 9$. Due to this, we have 12 possible values of p^k modulo 36. Note that for $p^k \in \{1, 17\} \bmod 36$, $|S(\gamma_g)| = 1$ for all $g \in G_3$ as $I_{\mathbb{F}} = \{1\}$. Hence, the Wedderburn decomposition is given by (5.2). Further, for $p^k \in \{7, 11, 23, 31\} \bmod 36$, we have that $S(\gamma_x) = \{\gamma_x, \gamma_{xy}\}$, $S(\gamma_z) = \{\gamma_z, \gamma_{z^2}, \gamma_{zw^2}\}$, $S(\gamma_{xt}) = \{\gamma_{xt}, \gamma_{xyt}\}$, $S(\gamma_{yz}) = \{\gamma_{yz}, \gamma_{yz^2}, \gamma_{yzw^2}\}$, and for the other representatives g, $S(\gamma_g) = \{\gamma_g\}$. Therefore, (5.1) and Theorems 3.1, 3.2 imply that $\mathbb{F}_q G_1 \cong \mathbb{F}_q \bigoplus_{r=1}^{7} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=8}^{9} M_{n_r}(\mathbb{F}_{q^2}) \bigoplus_{r=10}^{11} M_{n_r}(\mathbb{F}_{q^3})$. Since $G_3/G_3' \cong C_4$ and $\mathbb{F}_q(G_3/G_3') \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}$, Theorem 3.4 yields $\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \bigoplus_{r=1}^{6} M_{n_r}(\mathbb{F}_q) \oplus M_7(\mathbb{F}_{q^2}) \bigoplus_{r=8}^{9} M_{n_r}(\mathbb{F}_{q^3})$ with $n_r \geqslant 2$. At this point, again consider the normal subgroup H_1 of G_3 and $\mathbb{F}_q(G_3/H_1) \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^3})$ (as in Theorem 4.1). This with Theorem 3.5 implies that $\mathbb{F}_q G_3 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^3}) \bigoplus_{r=1}^{2} M_{n_r}(\mathbb{F}_q) \oplus M_{n_3}(\mathbb{F}_{q^2}) \oplus M_{n_4}(\mathbb{F}_{q^3})$, where $n_r \geqslant 2$ with $70 = \sum_{r=1}^{2} n_r^2 + 2n_3^2 + 3n_4^2$. This equation has four solutions given by $(2, 2, 5, 2)$, $(2, 6, 3, 2)$, $(3, 4, 3, 3)$ and $(5, 5, 2, 2)$. For uniqueness, we utilize the normal subgroup H_2 of G_3. Further, we know that $\mathbb{F}_q(G_3/H_2) \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q)^2 \oplus M_2(\mathbb{F}_{q^3})^2$ (cf. [1]). This with Theorem 3.5 ensures that $(3, 4, 3, 3)$ cannot be the possibility for n_r's. Moreover, $(2, 2, 5, 2)$, and $(5, 5, 2, 2)$ cannot be the possible choices due to the similar reason as stated in Theorem 4.1. Consequently, the Wedderburn decomposition is given by

$$\mathbb{F}_q G_3 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_{q^2}) \oplus M_2(\mathbb{F}_{q^3})^2.$$

Next, for $p^k \in \{5, 13, 25, 29\} \bmod 36$, we have $S(\gamma_z) = \{\gamma_z, \gamma_{z^2}, \gamma_{zw^2}\}$, $S(\gamma_{yz}) = \{\gamma_{yz}, \gamma_{yz^2}, \gamma_{yzw^2}\}$, and for the other representatives g, $S(\gamma_g) = \{\gamma_g\}$. Therefore, equation (5.1) and Theorems 3.1, 3.2 imply that $\mathbb{F}_q G_1 \cong \mathbb{F}_q \bigoplus_{r=1}^{11} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=12}^{13} M_{n_r}(\mathbb{F}_{q^3})$. Since $\mathbb{F}_q(G_3/G_3') \cong \mathbb{F}_q^4$, Theorem 3.4 yields $\mathbb{F}_q G_3 \cong \mathbb{F}_q^4 \bigoplus_{r=1}^{8} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=9}^{10} M_{n_r}(\mathbb{F}_{q^3})$

with $n_r \geqslant 2$. We again consider the normal subgroup H_1 of G_3 and $\mathbb{F}_q(G_3/H_1) \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^3})$. This with Theorem 3.5 implies that $\mathbb{F}_qG_3 \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^3}) \bigoplus_{r=1}^{4} M_{n_r}(\mathbb{F}_q) \oplus M_{n_3}(\mathbb{F}_{q^3})$, where $n_r \geqslant 2$ with $70 = \sum_{r=1}^{4} n_r^2 + 3n_5^2$. This equation has three solutions given by $(2,2,5,5,2)$, $(2,3,3,6,2)$, and $(3,3,3,4,3)$. In order to obtain the uniqueness, we consider the normal subgroup H_2 of G_3 and recall that $\mathbb{F}_q(G_3/H_2) \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_2(\mathbb{F}_{q^3})^2$. This and Theorem 3.5 ensure that $(3,3,3,4,3)$ cannot be the possibility for n_r's. Moreover, as $(2,2,5,5,2)$ can not be the possible choice, the required Wedderburn decomposition is given by

$$\mathbb{F}_qG_3 \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_6(\mathbb{F}_q)^2 \oplus M_2(\mathbb{F}_{q^3})^2.$$

Finally, we are left with the choices $p^k \in \{19, 35\} \bmod 36$. We observe that $S(\gamma_x) = \{\gamma_x, \gamma_{xy}\}$, $S(\gamma_{xt}) = \{\gamma_{xt}, \gamma_{xyt}\}$, and for the other representatives g, $S(\gamma_g) = \{\gamma_g\}$. Therefore, (5.1) and Theorems 3.1, 3.2 imply that $\mathbb{F}_qG_3 \cong \mathbb{F}_q \bigoplus_{r=1}^{13} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=14}^{15} M_{n_r}(\mathbb{F}_{q^2})$. Since $\mathbb{F}_q(G_3/G_3') \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}$, Theorem 3.4 yields $\mathbb{F}_qG_3 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \bigoplus_{r=1}^{12} M_{n_r}(\mathbb{F}_q) \oplus M_{13}(\mathbb{F}_{q^2})$ with $n_r \geqslant 2$. At this point, again consider the normal subgroup H_1 of G_3 and $\mathbb{F}_q(G_3/H_1) \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q)$. This together with Theorem 3.5 deduces that $\mathbb{F}_qG_3 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q) \bigoplus_{r=1}^{5} M_{n_r}(\mathbb{F}_q) \oplus M_{n_6}(\mathbb{F}_{q^2})$. We now engage the normal subgroup H_2 and recall that $\mathbb{F}_q(G_3/H_2) \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q)^8$. Again apply Theorem 3.5 with H_2 to see that $\mathbb{F}_qG_3 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q)^8 \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q) \oplus M_{n_1}(\mathbb{F}_q) \oplus M_{n_6}(\mathbb{F}_{q^2})$, where $n_r \geqslant 2$ with $54 = n_1^2 + 2n_2^2$. This equation has two solutions given by $(2,5)$, and $(6,3)$. However, as $(2,5)$ can not be the possible choice, the required Wedderburn decomposition is given by

$$\mathbb{F}_qG_3 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q)^8 \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_{q^2}).$$

This completes the proof. □

5.2. The group $G_4 = ((C_4 \times C_2) \rtimes C_2) \rtimes C_9$**.** The presentation of G_4 is the following:

$$\begin{aligned}\langle x, y, z, w, t, u\colon\ & x^2u^{-1}, [y,x], [z,x], [w,x], [t,x], [u,x], y^2z^{-1}, [z,y],\\ & [w,y]u^{-1}t^{-1}w^{-1}, [t,y]u^{-1}w^{-1}, [u,y], z^3, [w,z], [t,z],\\ & [u,z], w^2u^{-1}, [t,w]u^{-1}, [u,w], t^2u^{-1}, [u,t], u^2\rangle.\end{aligned}$$

Moreover, we see that the group G_4 has 42 conjugacy classes. Further, the exponent of G_4 is 36 and $G_4' \cong Q_8$ with $G_4/G_4' \cong C_{18}$. For this group, we are not giving the full details in the following theorem since the procedure to obtain the unit group is similar to the one in preceding Theorems 4.1, 4.2 and 5.1.

Theorem 5.2. *The unit group U_4 of $\mathbb{F}_q G_4$, for $q = p^k$ and $p > 3$, where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements, is as follows:*

(1) *for $p^k \equiv 1 \bmod 36$, $U_4 \cong (\mathbb{F}_q^*)^{18} \oplus GL_2(\mathbb{F}_q)^{18} \oplus GL_3(\mathbb{F}_q)^6$, for $p^k \equiv 17 \bmod 36$, $U_4 \cong (\mathbb{F}_q^*)^2 \oplus (\mathbb{F}_{q^2}^*)^8 \oplus GL_2(\mathbb{F}_q)^2 \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_2(\mathbb{F}_{q^2})^8 \oplus GL_3(\mathbb{F}_{q^2})^2$, for $p^k \equiv 19 \bmod 36$, $U_4 \cong (\mathbb{F}_q^*)^{18} \oplus GL_2(\mathbb{F}_{q^2})^9 \oplus GL_3(\mathbb{F}_q)^6$ and for $p^k \equiv 35 \bmod 36$, $U_4 \cong (\mathbb{F}_q^*)^2 \oplus (\mathbb{F}_{q^2}^*)^8 \oplus GL_2(\mathbb{F}_{q^2})^9 \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_3(\mathbb{F}_{q^2})^2$;*

(2) *for $p^k \in \{5, 29\} \bmod 36$, $U_4 \cong (\mathbb{F}_q^*)^2 \oplus (\mathbb{F}_{q^2}^*)^2 \oplus (\mathbb{F}_{q^6}^*)^2 \oplus GL_2(\mathbb{F}_q)^2 \oplus GL_2(\mathbb{F}_{q^2})^2 \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_3(\mathbb{F}_{q^2})^2 \oplus GL_2(\mathbb{F}_{q^6})^2$;*

(3) *for $p^k \in \{7, 31\} \bmod 36$, $U_4 \cong (\mathbb{F}_q^*)^6 \oplus (\mathbb{F}_{q^3}^*)^4 \oplus GL_3(\mathbb{F}_q)^6 \oplus GL_2(\mathbb{F}_{q^2})^3 \oplus GL_2(\mathbb{F}_{q^6})^2$;*

(4) *for $p^k \in \{11, 23\} \bmod 36$, $U_4 \cong (\mathbb{F}_q^*)^2 \oplus (\mathbb{F}_{q^2}^*)^2 \oplus (\mathbb{F}_{q^6}^*)^2 \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_2(\mathbb{F}_{q^2})^3 \oplus GL_3(\mathbb{F}_{q^2})^2 \oplus GL_2(\mathbb{F}_{q^6})^2$;*

(5) *for $p^k \in \{13, 25\} \bmod 36$, $U_4 \cong (\mathbb{F}_q^*)^6 \oplus (\mathbb{F}_{q^3}^*)^4 \oplus GL_2(\mathbb{F}_q)^6 \oplus GL_2(\mathbb{F}_{q^3})^4 \oplus GL_3(\mathbb{F}_q)^6$.*

Proof. Since $\mathbb{F}_q G_4$ is semisimple, we have

$$\mathbb{F}_q G_4 \cong \mathbb{F}_q \bigoplus_{r=1}^{m-1} M_{n_r}(\mathbb{F}_r) \quad \text{for some } m \in \mathbb{Z}. \tag{5.3}$$

Let $p^k \equiv 1 \bmod 36$. This means for each $g \in G_4$, $|S(\gamma_g)| = 1$. Hence, (5.3) together with Theorems 3.1 and 3.2 gives $\mathbb{F}_q G_4 \cong \mathbb{F}_q \bigoplus_{r=1}^{41} M_{n_r}(\mathbb{F}_q)$. We utilize Theorem 3.4 with $G_4/G_4' \cong C_{18}$ to obtain $\mathbb{F}_q G_4 \cong \mathbb{F}_q^{18} \bigoplus_{r=1}^{24} M_{n_r}(\mathbb{F}_q)$, where $n_r \geqslant 2$. To this end, consider the normal subgroup $H_1 := \langle u \rangle$ of G_4 and note that $G_4/H_1 \cong C_2 \times ((C_2 \times C_2) \rtimes C_9)$. This quotient group G_4/H_1 is a metabelian group. Therefore, using [1] we know that $\mathbb{F}_q(G_4/H_1) \cong \mathbb{F}_q^{18} \oplus M_3(\mathbb{F}_q)^6$. This with Theorem 3.5 implies that $\mathbb{F}_q G_4 \cong \mathbb{F}_q^{18} \oplus M_3(\mathbb{F}_q)^6 \bigoplus_{r=1}^{18} M_{n_r}(\mathbb{F}_q)$, where $n_r \geqslant 2$. Next, we consider the normal subgroup $H_2 := \langle z \rangle$ of G_4 and note that $G_4/H_2 \cong ((C_4 \times C_2) \rtimes C_2) \rtimes C_3$. The quotient group G_4/H_2 is a non-metabelian group. Therefore, using [14], we have that $\mathbb{F}_q(G_4/H_2) \cong \mathbb{F}_q^6 \oplus M_2(\mathbb{F}_q)^6 \oplus M_3(\mathbb{F}_q)^2$. This with Theorem 3.5 further implies that $\mathbb{F}_q G_4 \cong \mathbb{F}_q^{18} \oplus M_2(\mathbb{F}_q)^6 \oplus M_3(\mathbb{F}_q)^6 \bigoplus_{r=1}^{12} M_{n_r}(\mathbb{F}_q)$, where $n_r \geqslant 2$ with $\sum_{i=1}^{12} n_r^2 = 48$. Consequently, we are left with the only possible choice of $n_r's$ given by (2^{12}). Thus, the Wedderburn decomposition is

$$\mathbb{F}_q G_4 \cong \mathbb{F}_q^{18} \oplus M_2(\mathbb{F}_q)^{18} \oplus M_3(\mathbb{F}_q)^6.$$

Now, we consider $p^k \in \{5, 29\} \bmod 36$. Here, we see that $|S(\gamma_g)| = 2$ for 12 representatives of conjugacy classes of G_4, $|S(\gamma_g)| = 6$ for 24 representatives of conjugacy classes of G_4 and $|S(\gamma_g)| = 1$ for the remaining 6 representatives of conjugacy classes of G_4. Therefore, (5.3) and Theorems 3.1, 3.2 imply that $\mathbb{F}_qG_4 \cong \mathbb{F}_q \bigoplus_{r=1}^{5} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=6}^{11} M_{n_r}(\mathbb{F}_{q^2}) \bigoplus_{r=12}^{15} M_{n_r}(\mathbb{F}_{q^6})$. Since $G_4/G_4' \cong C_{18}$ and $\mathbb{F}_q(G_4/G_4') \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \oplus \mathbb{F}_{q^6}^2$, Theorem 3.4 yields $\mathbb{F}_qG_4 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \oplus \mathbb{F}_{q^6}^2 \bigoplus_{r=1}^{4} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=5}^{8} M_{n_r}(\mathbb{F}_{q^2}) \bigoplus_{r=9}^{10} M_{n_r}(\mathbb{F}_{q^6})$ with $n_r \geqslant 2$. At this point, we again consider the normal subgroup H_1 of G_4 and utilize $\mathbb{F}_q(G_4/H_1) \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \oplus \mathbb{F}_{q^6}^2 \oplus M_3(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_{q^2})^2$. This with Theorem 3.5 implies that $\mathbb{F}_qG_4 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \oplus \mathbb{F}_{q^6}^2 \oplus M_3(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_{q^2})^2 \bigoplus_{r=1}^{2} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=3}^{4} M_{n_r}(\mathbb{F}_{q^2}) \bigoplus_{r=5}^{6} M_{n_r}(\mathbb{F}_{q^6})$. Further, as in the previous case, we consider the normal subgroup H_2 of G_4 and utilize $\mathbb{F}_q(G_4/H_2) \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \oplus M_2(\mathbb{F}_q)^2 \oplus M_2(\mathbb{F}_{q^2})^2 \oplus M_3(\mathbb{F}_q)^2$. This with Theorem 3.5 further shows that $\mathbb{F}_qG_4 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \oplus \mathbb{F}_{q^6}^2 \oplus M_2(\mathbb{F}_q)^2 \oplus M_2(\mathbb{F}_{q^2})^2 \oplus M_3(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_{q^2})^2 \bigoplus_{r=1}^{2} M_{n_r}(\mathbb{F}_{q^6})$, where $n_r \geqslant 2$ with $6(n_1^2 + n_2^2) = 48$. This leaves us with the only possible choice of $n_r's$ given by $(2, 2)$ and therefore the Wedderburn decomposition is

$$\mathbb{F}_qG_4 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \oplus \mathbb{F}_{q^6}^2 \oplus M_2(\mathbb{F}_q)^2 \oplus M_2(\mathbb{F}_{q^2})^2 \oplus M_3(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_{q^2})^2 \oplus M_2(\mathbb{F}_{q^6})^2.$$

Next, we consider $p^k \in \{7, 31\} \bmod 36$. In this case, we have that $|S(\gamma_g)| = 3$ for 12 representatives of conjugacy classes of G_4, $|S(\gamma_g)| = 2$ for 6 representatives of conjugacy classes of G_4, $|S(\gamma_g)| = 6$ for 12 representatives of conjugacy classes of G_4 and $|S(\gamma_g)| = 1$ for the remaining 12 representatives of conjugacy classes of G_4. Therefore, (5.3) and Theorems 3.1, 3.2 imply that $\mathbb{F}_qG_4 \cong \mathbb{F}_q \bigoplus_{r=1}^{11} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=12}^{14} M_{n_r}(\mathbb{F}_{q^2}) \bigoplus_{r=15}^{18} M_{n_r}(\mathbb{F}_{q^3}) \bigoplus_{r=19}^{20} M_{n_r}(\mathbb{F}_{q^6})$. Since $\mathbb{F}_q(G_4/G_4') \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^6 \oplus \mathbb{F}_{q^3}^4$, Theorem 3.4 yields $\mathbb{F}_qG_4 \cong \mathbb{F}_q^6 \oplus \mathbb{F}_{q^3}^4 \bigoplus_{r=1}^{6} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=7}^{9} M_{n_r}(\mathbb{F}_{q^2}) \bigoplus_{r=10}^{11} M_{n_r}(\mathbb{F}_{q^6})$ with $n_r \geqslant 2$. Next, we consider the normal subgroup H_1 of G_4 and utilize $\mathbb{F}_q(G_4/H_1) \cong \mathbb{F}_q^6 \oplus \mathbb{F}_{q^3}^4 \oplus M_3(\mathbb{F}_q)^6$. This with Theorem 3.5 implies that $\mathbb{F}_qG_4 \cong \mathbb{F}_q^6 \oplus \mathbb{F}_{q^3}^4 \oplus M_3(\mathbb{F}_q)^6 \bigoplus_{r=1}^{3} M_{n_r}(\mathbb{F}_{q^2}) \bigoplus_{r=4}^{5} M_{n_r}(\mathbb{F}_{q^6})$. Again, we consider the normal subgroup H_2 of G_4 and utilize $\mathbb{F}_q(G_4/H_2) \cong \mathbb{F}_q^6 \oplus M_2(\mathbb{F}_{q^2})^3 \oplus M_3(\mathbb{F}_q)^2$. This with Theorem 3.5 further deduces that $\mathbb{F}_qG_4 \cong \mathbb{F}_q^6 \oplus \mathbb{F}_{q^3}^4 \oplus M_3(\mathbb{F}_q)^6 \oplus M_2(\mathbb{F}_{q^2})^3 \bigoplus_{r=1}^{2} M_{n_r}(\mathbb{F}_{q^6})$, where $n_r \geqslant 2$ with $6(n_1^2 + n_2^2) = 48$. This leaves us with the only possible choice of $n_r's$ given by $(2, 2)$ and therefore the Wedderburn decomposition is

$$\mathbb{F}_qG_4 \cong \mathbb{F}_q^6 \oplus \mathbb{F}_{q^3}^4 \oplus M_3(\mathbb{F}_q)^6 \oplus M_2(\mathbb{F}_{q^2})^3 \oplus M_2(\mathbb{F}_{q^6})^2.$$

Further, we consider $p^k \equiv 19 \bmod 36$. In this case, we have that $|S(\gamma_g)| = 1$ for 24 representatives of conjugacy classes of G_4 and $|S(\gamma_g)| = 2$ for the remaining 24 representatives of conjugacy classes of G_4. Therefore, (5.3) and Theorems 3.1, 3.2 imply that $\mathbb{F}_q G_4 \cong \mathbb{F}_q \bigoplus_{r=1}^{23} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=24}^{32} M_{n_r}(\mathbb{F}_{q^2})$. Since $\mathbb{F}_q(G_4/G_4') \cong \mathbb{F}_q^{18}$, Theorem 3.4 yields $\mathbb{F}_q G_4 \cong \mathbb{F}_q^{18} \bigoplus_{r=1}^{6} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=7}^{15} M_{n_r}(\mathbb{F}_{q^2})$ with $n_r \geqslant 2$. Next, we consider the normal subgroup H_1 of G_4 and utilize $\mathbb{F}_q(G_4/H_1) \cong \mathbb{F}_q^{18} \oplus M_3(\mathbb{F}_q)^6$. This with Theorem 3.5 implies that $\mathbb{F}_q G_4 \cong \mathbb{F}_q^{18} \oplus M_3(\mathbb{F}_q)^6 \bigoplus_{r=1}^{9} M_{n_r}(\mathbb{F}_{q^2})$, where $n_r \geqslant 2$ with $2\sum_{i=1}^{9} n_r^2 = 72$. This provides us the only possible choice of $n_r's$ given by (2^9). Consequently, the Wedderburn decomposition is $\mathbb{F}_q G_4 \cong \mathbb{F}_q^{18} \oplus M_3(\mathbb{F}_q)^6 \oplus M_2(\mathbb{F}_{q^2})^9$. Finally, we remark that, similar to the procedure already followed in this theorem, the remaining Wedderburn decompositions can be deduced. This completes the proof. □

6. Conclusion

We have completely characterized the unit groups $U(\mathbb{F}_q G)$ of the semisimple group algebra of 17 non-metabelian groups of order 144. The 17 groups considered in this paper either have exponent 72 or 36 or they are direct product of nontrivial subgroups. This paper completes the study of characterization of unit groups of $U(\mathbb{F}_q G)$ for all groups G up to order 144 except that of groups of order 96, 128 and 144 (having exponent 12 and 24). Finally, this paper motivates the study of unit groups of the group algebras of non-metabelian groups of higher order.

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References

[1] *G. K. Bakshi, S. Gupta, I. B. S. Passi*: The algebraic structure of finite metabelian group algebras. Commun. Algebra *43* (2015), 2240–2257. zbl MR doi
[2] *A. A. Bovdi, J. Kurdics*: Lie properties of the group algebra and the nilpotency class of the group of units. J. Algebra *212* (1999), 28–64. zbl MR doi
[3] *C. Dietzel, G. Mittal*: Summands of finite group algebras. Czech. Math. J. *71* (2021), 1011–1014. zbl MR doi
[4] *R. A. Ferraz*: Simple components of the center of $FG/J(FG)$. Commun. Algebra *36* (2008), 3191–3199. zbl MR doi
[5] *S. Gupta, S. Maheshwary*: Finite semisimple group algebra of a normally monomial group. Int. J. Algebra Comput. *29* (2019), 159–177. zbl MR doi

[6] *P. Hurley, T. Hurley*: Codes from zero-divisors and units in group rings. Int. J. Inf. Coding Theory *1* (2009), 57–87. zbl MR doi

[7] *B. Hurley, T. Hurley*: Group ring cryptography. Int. J. Pure Appl. Math. *69* (2011), 67–86. zbl MR

[8] *G. D. James*: The Representation Theory of the Symmetric Groups. Lecture Notes in Mathematics 682. Springer, Berlin, 1978. zbl MR doi

[9] *M. Khan, R. K. Sharma, J. B. Srivastava*: The unit group of FS_4. Acta Math. Hung. *118* (2008), 105–113. zbl MR doi

[10] *R. Lidl, H. Niederreiter*: Introduction to Finite Fields and Their Applications. Cambridge University Press, Cambridge, 1994. zbl MR doi

[11] *N. Makhijani, R. K. Sharma, J. B. Srivastava*: The unit group of $\mathbb{F}_q[D_{30}]$. Serdica Math. J. *41* (2015), 185–198. zbl MR

[12] *N. Makhijani, R. K. Sharma, J. B. Srivastava*: A note on the structure of $\mathbb{F}_{p^k}A_5/J(\mathbb{F}_{p^k}A_5)$. Acta Sci. Math. *82* (2016), 29–43. zbl MR doi

[13] *G. Mittal, S. Kumar, S. Kumar*: A quantum secure ID-based cryptographic encryption based on group rings. Sādhanā *47* (2022), Article ID 35, 16 pages. MR doi

[14] *G. Mittal, R. K. Sharma*: On unit group of finite group algebras of non-metabelian groups up to order 72. Math. Bohem. *146* (2021), 429–455. zbl MR doi

[15] *G. Mittal, R. K. Sharma*: On unit group of finite semisimple group algebras of non-metabelian groups of order 108. J. Algebra Comb. Discrete Struct. Appl. *8* (2021), 59–71. zbl MR doi

[16] *G. Mittal, R. K. Sharma*: Computation of Wedderburn decomposition of groups algebras from their subalgebra. Bull. Korean Math. Soc. *59* (2022), 781–787. zbl MR doi

[17] *G. Mittal, R. K. Sharma*: Unit group of semisimple group algebras of some non-metabelian groups of order 120. Asian-Eur. J. Math. *15* (2022), Article ID 2250059, 11 pages. zbl MR doi

[18] *G. Mittal, R. K. Sharma*: Wedderburn decomposition of a semisimple group algebra $\mathbb{F}_qG$ from a subalgebra of factor group of G. Int. Electron. J. Algebra *32* (2022), 91–100. zbl MR doi

[19] *G. Pazderski*: The orders to which only belong metabelian groups. Math. Nachr. *95* (1980), 7–16. zbl MR doi

[20] *S. Perlis, G. L. Walker*: Abelian group algebras of finite order. Trans. Am. Math. Soc. *68* (1950), 420–426. zbl MR doi

[21] *C. Polcino Milies, S. K. Sehgal*: An Introduction to Group Rings. Algebras and Applications 1. Kluwer Academic, Dordrecht, 2002. zbl MR

[22] *R. K. Sharma, G. Mittal*: On the unit group of semisimple group algebra $\mathbb{F}_qSL(2,\mathbb{Z}_5)$. Math. Bohem. *147* (2022), 1–10. zbl MR doi

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