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Minimal Sets, Union-Closed Families, and Frankl's Conjecture

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MINIMAL SETS, UNION-CLOSED FAMILIES, AND FRANKL'S CONJECTURE

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by

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Abstract

The most common statement of Frankl's conjecture is that for every finite family of sets closed under the union operation, there is some element which belongs to at least half of the sets in the family. Despite its apparent simplicity, Frankl's conjecture has remained open and highly researched since its first mention in 1979. In this paper, we begin by examining the history and previous attempts at solving the conjecture. Using these previous ideas, we introduce the concepts of minimal sets and minimally-generated families, some ideas related to viewing union-closed families as posets, and some constructions of families involving poset-defined parameters such as height and width.

1 Introduction and History

A *set system* or *set family* on a set X is a pair $(X, \mathcal{F})$ where $\mathcal{F}$ is a collection of sets which are subsets of X. Letting $\mathcal{P}(X)$ denote the power set of X, we can write $\mathcal{F} \subseteq \mathcal{P}(X)$. The set X is called the *ground set* of the family and we usually use $X = [n]$ where n is a positive integer and $[n]$ is the set $\{1, 2, 3, \ldots, n\}$. If we are comparing multiple families and they have different ground sets, we can refer to the ground sets via subscripts (i.e. $\mathcal{F}$ and $\mathcal{F}'$ would have ground sets $X_{\mathcal{F}}$ and $X_{\mathcal{F}'}$, respectively). If the ground set is clear from context, we can refer to $\mathcal{F}$ as a set system. The *elements* of a set system are the elements of its ground set.

We say that a set system $\mathcal{F}$ is *union-closed* if for any two sets $S, T \in \mathcal{F}$, $S \cup T \in \mathcal{F}$. In other words, union-closed families are closed under the union operation. For $x \in X$, the *subfamily* of $\mathcal{F}$ containing x, denoted $\mathcal{F}_x$ is $\mathcal{F}_x = \{S \in \mathcal{F} : x \in S\}$. In other words, $\mathcal{F}_x$ is the collection of sets in $\mathcal{F}$ which contain x. We can also define the *complement* of $\mathcal{F}_x$ to be the sets of $\mathcal{F}$ *not* containing x, $\mathcal{F}_{\overline{x}} = \{S \in \mathcal{F} : x \notin S\}$.

We define the *frequency* of x in $\mathcal{F}$ to be $|\mathcal{F}_x|$, the number of sets in which x appears. We say that x is *abundant* if $|\mathcal{F}_x|/|\mathcal{F}| \geq 1/2$. If x is not abundant, meaning $|\mathcal{F}_x|/|\mathcal{F}| < 1/2$, then we say that x is *rare*. Using these definitions, we can define Frankl's conjecture as follows.

Conjecture 1. *Every union-closed family (besides the family containing only the empty set) contains an abundant element.*

Despite typically being called "Frankl's conjecture" and being attributed to Péter Frankl in 1979, the conjecture has a long and relatively uncertain history. According to Frankl himself in [1], he was studying the traces of finite sets in 1979 when he came up with the conjecture. He later told the conjecture to Ron Graham who attempted to find a counterexample [2] and then spread word of the conjecture. The first mention of the conjecture in print is from Dwight Duffus in the Proceedings of the NATO Advanced Study Institute on Graphs and Order in Banff, Canada, published in 1985. During one of the problem sessions of this conference, Duffus listed three variations of the conjecture and referred to the conjecture as a "problem of P. Frankl" [3]. Duffus' three forms of the conjecture are the union-closed version described above, the intersection-closed version, and a formulation on finite lattices. A family $\mathcal{F}$ is called *intersection-closed* if for all $S_1, S_2 \in \mathcal{F}$, $S_1 \cap S_2 \in \mathcal{F}$. The intersection-closed version of the conjecture states that all intersection-closed families contain an element in at most $|\mathcal{F}|/2$ sets.

The next two times the conjecture appeared in literature were in two issues of the Australian Mathematical Society in 1987 [2, 4]. The first issue refers to the conjecture as a "much-travelled conjecture" and the second issue refers to the conjecture as "one of the most embarrassing gaps in combinatorial knowledge". Despite the conjecture being known for several years by many skilled set theorists and combinatorialists, the first research article on the conjecture wasn't published until 1989 in [5], 10 years after Frankl's introduction of the conjecture.

Some significant progress was made towards the conjecture in November 2022 with the publication of a pre-print by Gilmer [6] proving a weakening of Frankl's conjecture via information theory methods [6]. Gilmer showed that if $\mathcal{F}$ is a union-closed family containing at least one set, there exist an element with frequency at least $0.01|\mathcal{F}|$.

Shortly after the discovery of this initial lower bound, three more pre-prints [6, 7, 8] were published which improved this bound from 0.01 to $\frac{3-\sqrt{5}}{2} \approx 0.3819$. This bound was then improved to approximately 0.3823 in a preprint by Cambie [9] and a paper published by Yu [10] which both used an idea from Sawin [8].

Multiple other equivalent formulations of Frankl's conjecture exist besides the union-closed, intersection-closed, and lattice formulations that we have mentioned. One wording of the graph formulation of the conjecture is that in every bipartite graph, there exists a vertex which appears in at most half of the graphs' maximal independent sets. The equivalence of this form of the conjecture is shown in [11]. Of these forms of the conjecture, we will just examine the union-closed version.

Another way to represent Frankl's conjecture is as an extremal problem. For a union-closed family $\mathcal{F}$, let

$$\phi(\mathcal{F}) = \max_{x \in X} |\mathcal{F}_x|$$

and for positive integer m, let

$$\phi(m) = \min\{\phi(\mathcal{F}) : |\mathcal{F}| = m\}.$$

In other words, $\phi(\mathcal{F})$ is the frequency of the most frequent element in $\mathcal{F}$ and $\phi(m)$ is the smallest number such that all union-closed families of m sets contain an element of frequency at least $\phi(m)$. With this representation, Frankl's conjecture becomes that $\phi(m) \geq m/2$ for all positive integers m. This extremal representation was first introduced by Renaud in [12]. We will later show in Theorem 7 that for $m \geq 2$, $\varphi(m-1) \leq \varphi(m) \leq \varphi(m-1) + 1$. As shown in [12], this bound can be used to show that the initial terms of $\varphi(m)$, starting at $m = 1$ are:

$$1 \quad 2 \quad 2 \quad 3 \quad 4 \quad 4 \quad 4 \quad 5 \quad 6 \quad 7 \quad 7 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 9 \quad 10$$

We further explore bounds for $\varphi(m)$ in Theorem 21.

2 Preliminary Results

Before introducing more complicated ideas related to union-closed sets, we can prove several general results. First, we show that given a union-closed family $\mathcal{F}$, the subfamilies of $\mathcal{F}$ and their complements are union-closed.

Proposition 1. *If $\mathcal{F}$ is a union-closed family, then $\mathcal{F}_x$ is a union-closed family for all $x \in X$.*

Proof. Let $\mathcal{F}$ be a union-closed family, $x \in X$, and $S_1, S_2 \in \mathcal{F}_x$. Since $\mathcal{F}$ is union-closed, then $S_1 \cup S_2 \in \mathcal{F}$. Since x is in both of S_1 and S_2, then $x \in S_1 \cup S_2$, meaning $S_1 \cup S_2 \in \mathcal{F}_x$. The union of any two sets in $\mathcal{F}_x$ is still in $\mathcal{F}_x$, so $\mathcal{F}_x$ is union-closed. □

Proposition 2. *If $\mathcal{F}$ is a union-closed family, then $\mathcal{F}_{\overline{x}}$ is a union-closed family for all $x \in X$.*

Proof. Let $\mathcal{F}$ be a union-closed family, $x \in X$, and $S, T \in \mathcal{F}_{\overline{x}}$. Since $\mathcal{F}$ is union-closed, then

$S \cup T \in \mathcal{F}$. Since x is in neither of S or T, then $x \notin S \cup T$, meaning $S \cup T \in \mathcal{F}_{\overline{x}}$. The union of any two sets in $\mathcal{F}_{\overline{x}}$ is still in $\mathcal{F}_{\overline{x}}$, so $\mathcal{F}_{\overline{x}}$ is union-closed. $\square$

Next, we can make some observations related to the frequency of elements in union-closed families. Let $d(\mathcal{F})$ denote the *average abundance* of $\mathcal{F}$, given by

$$d(\mathcal{F}) = \frac{\sum_{x \in X} |\mathcal{F}_x|}{|\mathcal{F}|}.$$

Additionally, let $R(\mathcal{F})$ be the collection of rare elements in $\mathcal{F}$ and so $|R(\mathcal{F})|$ is the number of rare elements in $\mathcal{F}$.

Lemma 1. *If $\mathcal{F} = \mathcal{P}(X)$, then for all $x \in X$,*

$$|\mathcal{F}_x| = 2^{n-1}.$$

Proof. Let $\mathcal{F} = \mathcal{P}(X)$ and select $x \in X$. Any set in $\mathcal{F}_x$ contains x and k other elements for $k \in \{0, 1, \ldots, n-1\}$. Thus

$$|\mathcal{F}_x| = \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}.$$

$\square$

Corollary 1. *If $\mathcal{F} = \mathcal{P}(X)$, then $R(\mathcal{F}) = \varnothing$ and $d(\mathcal{F}) = 2^{n-1}$.*

Proof. Both of these facts follow immediately from Lemma 1. $\square$

Theorem 1. *If $\mathcal{F}$ is a union-closed family which satisfies Frankl's conjecture, then*

$$d(\mathcal{F}) + \frac{|R(\mathcal{F})|}{|X|} \leq \phi(\mathcal{F}).$$

Proof. Let $\mathcal{F}$ be a union-closed family which satisfies Frankl's conjecture and such that $|X| = m + k$ where m elements of X have frequency $\phi(\mathcal{F})$ and the remaining k elements have frequency less than $\phi(\mathcal{F})$. This means k elements of X have frequencies which are less than or equal to

$\phi(\mathcal{F}) - 1$. This means

$$d(\mathcal{F}) \le \frac{m\phi(\mathcal{F}) + k(\phi(\mathcal{F}) - 1)}{m+k}.$$

Since $\mathcal{F}$ satisfies Frankl's conjecture, then the m elements of frequency $\phi(\mathcal{F})$ cannot be rare and so $|R(\mathcal{F})| \le k$

$$\frac{|R(\mathcal{F})|}{|X|} \le \frac{k}{m+k}.$$

Combining inequalities gives

$$\begin{aligned} d(\mathcal{F}) + \frac{|R(\mathcal{F})|}{|X|} &\le \frac{m\phi(\mathcal{F}) + k(\phi(\mathcal{F}) - 1)}{m+k} + \frac{k}{m+k} \\ &= \frac{(m+k)\phi(\mathcal{F}) - k}{m+k} + \frac{k}{m+k} \\ &= \phi(\mathcal{F}) - \frac{k}{m+k} + \frac{k}{m+k} \\ &= \phi(\mathcal{F}) \end{aligned}$$

as desired. $\square$

Proposition 3. *The difference between $d(\mathcal{F}) + |R(\mathcal{F})|/|X|$ and $\phi(\mathcal{F})$ can be made arbitrarily large.*

Proof. We can do this by taking a power set family and adding a large set to it. Select positive integers m and k and let $\mathcal{F}$ be the family $\mathcal{F} = \mathcal{P}([m]) \cup [m+k]$. Proposition 1 tells us $|\mathcal{P}([m])_x| = 2^{m-1}$ and each $x \in [m]$ is in $[m+k]$, then $|\mathcal{F}_x| = 2^{m-1}+1$, so each $x \in [m]$ is abundant. However each $y \in [m+k] \setminus [m]$ has $|\mathcal{F}_y| = 1$ and is thus not abundant. Therefore

$$d(\mathcal{F}) = \frac{m(2^{m-1}+1) + k(1)}{m+k} \quad \text{and} \quad \frac{|R(\mathcal{F})|}{|X|} = \frac{m}{m+k},$$

so

$$\begin{aligned} d(\mathcal{F}) + \frac{|R(\mathcal{F})|}{|X|} &= \frac{n(2^{m-1}+1) + k}{m+k} + \frac{k}{m+k} \\ &= \frac{2k}{m+k} + \frac{2^{m-1}+1)m}{m+k}. \end{aligned}$$

Taking the limit as $k \to \infty$ (i.e. $[m+k]$ becomes $\mathbb{N}$) results in

$$\lim_{k\to\infty} d(\mathcal{F}) + \frac{|R(\mathcal{F})|}{|X|} = \lim_{k\to\infty} \left(\frac{2k}{m+k} + \frac{(2^{m-1}+1)m}{m+k} \right) = 2 - 0 = 2.$$

However, for any k, the only element frequencies in $\mathcal{F}$ are $2^{m-1}+1$ and 1, so $\phi(\mathcal{F}) = 2^{m-1}+1$. $\square$

Proposition 4. *The inequality in Theorem 1 is an equality when $\mathcal{F}$ is a power set missing exactly zero or one of its singletons.*

Proof. Let $\mathcal{F} = \mathcal{P}([n])$. Proposition 1 gives $\phi(\mathcal{F}) = 2^{n-1}$ and Corollary 1 gives that $d(\mathcal{F}) = 2^{n-1}$ and $|R(\mathcal{F})|/|X| = 0$. Thus $d(\mathcal{F}) + |R(\mathcal{F})|/|X| = \phi(\mathcal{F})$.

Now select integer n, select $i \in [n]$, and let $\mathcal{F} = \mathcal{P}([n]) \setminus \{i\}$, so $\mathcal{F}$ is a power set family missing exactly one singleton. We have $|\mathcal{F}| = 2^n - 1$, so for an element to be abundant, its frequency must be greater than or equal to $\frac{2^n-1}{2} = 2^{n-1} - 1/2$. By Proposition 1, every element $j \neq i$ has $|\mathcal{F}_j|$, so each $j \neq i$ is abundant. However, i has $|\mathcal{F}_i| = 2^{n-1} - 1 < 2^{n-1} - 1/2$. We therefore have

$$\begin{aligned} d(\mathcal{F}) + \frac{|R(\mathcal{F})|}{|X|} &= \frac{(n-1)(2^{n-1}) + (1)(2^{n-1}-1)}{n} + \frac{1}{n} \\ &= \frac{n2^{n-1} - 2^{n-1} + 2^{n-1} - 1 + 1}{n} = \frac{n2^{n-1}}{n} \\ &= 2^{n-1} = \phi(\mathcal{F}) \end{aligned}$$

as desired. $\square$

3 Minimal Sets and Minimally-Generated Families

Now we will examine some ideas used in previous work on Frankl's conjecture as motivation towards our definition of minimally-generated families. The following is a result from [5] which shows it is sufficient to consider Frankl's conjecture for union-closed families with an odd number of member sets. Our proof is based on the one given in [5].

Theorem 2. *For integer m, if Frankl's conjecture is true for families of size $2m+1$, then Frankl's* *conjecture is true for families of size* $2m+2$.

Proof. Let $\mathcal{F}$ be a union-closed family of size $2m+2$ written as $\mathcal{F} = \{S_1, S_2, S_3, \ldots, S_{2m+2}\}$ where $|S_i| \leq |S_{i+1}|$. Now consider $\mathcal{G} = \{S_2, S_3, \ldots, S_{2m+2}\}$. The smallest set in a union-closed family cannot be the union of any other sets, so removing S_1 from $\mathcal{F}$ results in a set which is still union-closed. If $\mathcal{G}$ satisfies Frankl's conjecture, then there exists an $x \in X$ where

$$|\mathcal{G}_x| \geq \left\lceil \frac{2m+1}{2} \right\rceil = \left\lceil \frac{2m+2}{2} \right\rceil.$$

Since $\mathcal{G}$ is a subfamily of $\mathcal{F}$, then $\mathcal{G}_x \subseteq \mathcal{F}_x$, meaning

$$|\mathcal{F}_x| \geq |\mathcal{G}_x| \geq \left\lceil \frac{2m+2}{2} \right\rceil = \frac{|\mathcal{F}|}{2}.$$

Therefore x is abundant in $\mathcal{F}$. $\square$

Even though this theorem may seem like it would make Frankl's conjecture far easier to solve, its effectiveness was still limited by the fact that proving that there isn't a known way to show that $\mathcal{F}$ satisfies Frankl's conjecture for general $|\mathcal{F}| = 2m+1$. Sarvate and Renaud [5] used this theorem to show that Frankl's conjecture is true for $|\mathcal{F}| \leq 10$, but going higher required different methods. In 1990, Sarvate and Renaud [13] improved this result up to $|\mathcal{F}| \leq 18$ using the following approach.

For $\mathcal{F} = \{S_1, S_2, S_3, \ldots, S_m\}$ with $|S_i| \leq |S_{i+1}|$, consider $\mathcal{B} = \{S_1 \cup S_i : i \in \{1, 2, \ldots, m\}\}$ and $b = |\mathcal{B}|$. The value b counts the number of distinct sets which can be generated by taking the union of S_1 with another set in the family. Sarvate and Renaud note that for any n, if $b \geq (m-1)/2$, then S_1 is a subset of b distinct sets in $\mathcal{B}$ as well as of S_1 itself, meaning that the elements of A_1 are abundant. This means that if it can be shown that Frankl's conjecture holds for $b \leq (m-2)/2$, then Frankl's conjecture holds for families up to that value of m. Via the even/odd result proved above, it is sufficient to show that Frankl's conjecture holds for the odd values of b with $b \leq (m-3)/2$. Sarvate and Renaud managed to prove that this bound can be further weakened and then found that

Frankl's conjecture is true for $b \le 5$.

Theorem 3. *Frankl's conjecture holds for $b \ge (m-5)/2$.*

Theorem 4. *Frankl's conjecture holds for $b \le 5$.*

The above discussion and Theorem 3 tell us that showing Frankl's conjecture is true for $b \le 5$ means that Frankl's conjecture is true for $m \le 18$. The idea of using the smallest sets in a union-closed family to make statements about the whole family is further explored by Sarvate and Renaud in [5] to obtain the following theorem. We base our proof on that found in [5].

Theorem 5. *Singletons and pairs in union-closed families always contain abundant elements.*

Proof. First we show the singleton case. Let $\mathcal{F}$ be a union-closed family $\mathcal{F} = \{S_1, S_2, \ldots, S_m\}$, $|S_i| \le |S_{i+1}|$, and $S_1 = \{x\}$ for some $x \in X$. For each $S_i \in \mathcal{F}$ such that $x \in S_i$, the set $S_i \cup \{x\}$ is in $\mathcal{F}$. For such S_i, $S_i \neq S_j$ implies that $S_i \cup \{x\} \neq S_j \cup \{x\}$. Therefore x is present in at least half of the sets of $\mathcal{F}$.

Now let $\mathcal{F}$ be a union-closed family with $\mathcal{F} = \{S_1, S_2, \ldots, S_m\}$, $|S_i| \le |S_{i+1}|$, and $S_1 = \{x, y\}$ for $x, y \in X$ and $x \neq y$. Suppose s_0 sets of $\mathcal{F}$ contain neither x nor y, s_{xy} sets contain both x and y, s_x sets contain x but not y, and s_y sets contain y but not x. Every set is counted exactly once by s_0, s_x, s_y, or s_{xy}, so $s_0 + s_{xy} + s_x + s_y = m$. For every set $S_i \in \mathcal{F}$ containing neither x nor y, the set $S_i \cup S_1$ will be in $\mathcal{F}$, meaning $s_0 \le s_{xy}$. This gives that $2_{xy} + s_x + s_y \ge m$ which means either $s_{xy} + s_x \ge m/2$ or $s_{xy} + s_y \ge m/2$. This means that at least one of x or y is abundant in $\mathcal{F}$. $\square$

After seeing this theorem, it is natural to wonder if elements within a set of size three must be abundant in union-closed families. Unfortunately, the answer turns out to be no.

Theorem 6. *Size three sets in union-closed families don't always contain abundant elements.*

Proof. Consider the union-closed family $\mathcal{F}$ with ground set $[9]$ in Figure 1. The only size three set in this family is $\{1, 2, 3\}$. We can see that $|\mathcal{F}_1| = 13$, $|\mathcal{F}_2| = 13$, $|\mathcal{F}_3| = 13$. However, since

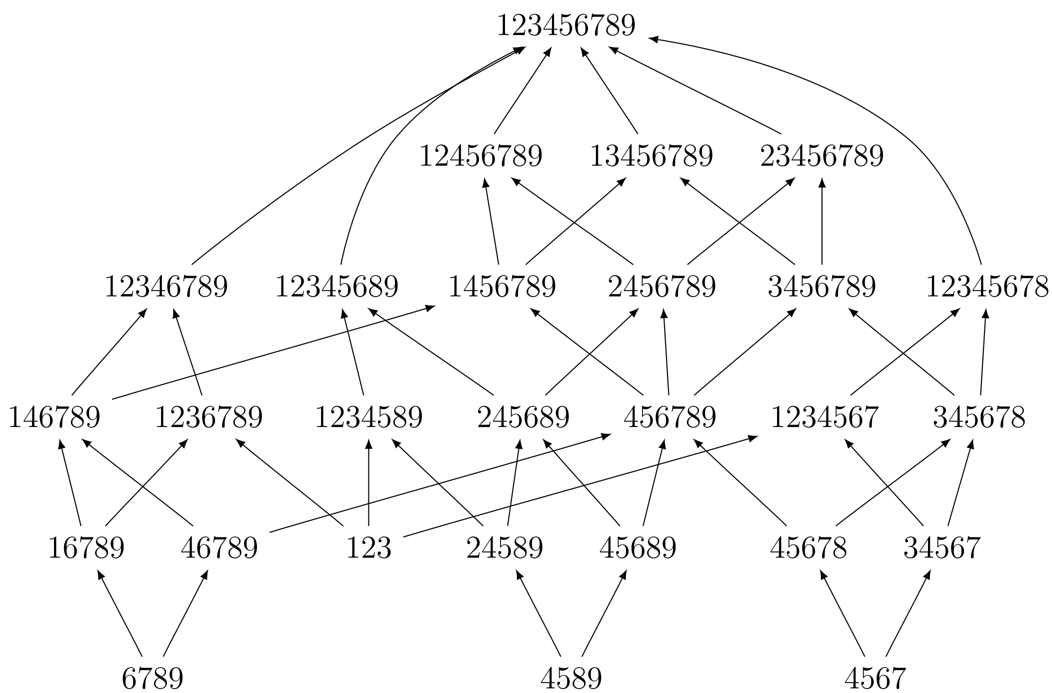

Figure 1: Here, $\mathcal{F}$ is displayed without braces or commas to improve readability.

$|\mathcal{F}| = 27$, an element must have a frequency of at least 14 in order to be abundant, so none of the elements in our size 3 family are abundant. However, $|\mathcal{F}_4| = 23$, so this family still contains an abundant element. □

This counterexample shows that examining only sets of smallest size may not be sufficient. Instead, we can try to generalize the sets we look at with the idea of *minimal* sets. For a union-closed family $\mathcal{F}$, we say that M is a minimal set if M is nonempty and there does not exist a set $M' \in \mathcal{F} \setminus \varnothing$ with $M' \subset M$. We conjecture that for any union-closed family, any element of maximum frequency will appear in at least one of the family's minimal sets.

Conjecture 2. *Every element of maximum frequency in a union-closed family appears in some minimal set.*

Proposition 5. *Conjecture 2 is not true if we only require the elements we consider to be abundant.*

Proof. Consider the union-closed family

$$\mathcal{F} = \{\{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}\}.$$

It is easy to see that 2 is an abundant element, but the only minimal set of this family is $\{1\}$ which does not contain 2. $\square$

One useful property of minimal sets that is that they can always be removed from a family and result in a family which is still union-closed.

Lemma 2. *If $\mathcal{F}$ is a union-closed family, $|\mathcal{F}| \geq 1$, and $M \in \mathcal{F}$ is a minimal set, then $\mathcal{F} \setminus M$ is a union-closed family.*

Proof. Suppose for contradiction that $\mathcal{F}$ is union-closed but $\mathcal{F} \setminus M$ is not. This means that there exist $S, T \in \mathcal{F} \setminus M$ where $S \cup T \notin \mathcal{F} \setminus M$. Since $S, T \in \mathcal{F} \setminus M$, then $S, T \in \mathcal{F}$ and since $\mathcal{F}$ is union-closed, then $S \cup T \in \mathcal{F}$. Since the only set we removed from $\mathcal{F}$ to get $\mathcal{F} \setminus M$ is M, then $S \cup T = M$, contradicting that M is minimal. $\square$

We can use this lemma to prove the following bound on $\varphi(m)$ first described in [12]. Our proof uses a similar method to that used in [12] and we write the result in the form that appears in [1].

Theorem 7. *For $m \geq 2$, $\varphi(m-1) \leq \varphi(m) \leq \varphi(m-1) + 1$.*

Proof. Consider union-closed family $\mathcal{F}$ where $|\mathcal{F}| = m$ and $\varphi(\mathcal{F}) = \varphi(m)$. Select any minimal set M of $\mathcal{F}$. Lemma 2 gives that $\mathcal{F} \setminus M$ is a union-closed family containing $m-1$ sets. Any element $x \in X$ cannot have a higher frequency in $\mathcal{F} \setminus M$ than it did in $\mathcal{F}$, so $\varphi(\mathcal{F} \setminus M) \leq \varphi(\mathcal{F})$. Additionally, since $|\mathcal{F} \setminus M| = m-1$, $\varphi(m-1) \leq \varphi(\mathcal{F} \setminus M)$. We therefore have

$$\varphi(m-1) \leq \varphi(\mathcal{F} \setminus M) \leq \varphi(\mathcal{F}) = \varphi(m)$$

and so $\varphi(m-1) \leq \varphi(m)$.

Next, let $\mathcal{F}$ be a union-closed family with $|\mathcal{F}| = m - 1$ and $\varphi(\mathcal{F}) = \varphi(m-1)$. Let y be an element which does not appear in X and $\mathcal{F}'$ be the union-closed family $\mathcal{F}' = \mathcal{F} \cup \{X \cup \{y\}\}$. Notice that any element $x \in X$ with frequency $\varphi(\mathcal{F})$ in $\mathcal{F}$ will have frequency $\varphi(\mathcal{F}) + 1$ in $\mathcal{F}'$, meaning $\varphi(\mathcal{F}') = \varphi(\mathcal{F}) + 1$. Additionally, since $|\mathcal{F}'| = m$, $\varphi(m) \leq \varphi(\mathcal{F}')$. We therefore have

$$\varphi(m) \leq \varphi(\mathcal{F}') = \varphi(\mathcal{F}) + 1 = \varphi(m-1) + 1$$

and so $\varphi(m) \leq \varphi(m-1) + 1$.

Combining our upper and lower bounds gives

$$\varphi(m-1) \leq \varphi(m) \leq \varphi(m-1) + 1$$

as desired. $\square$

Another use of minimal sets is to define *minimally-generated families*. Given a (not necessarily union-closed) family of sets $\mathcal{G}$, let the family *generated by* $\mathcal{G}$, denoted $\langle \mathcal{G} \rangle$, be the family

$$\langle \mathcal{G} \rangle = \left\{ \bigcup_{S \in \mathcal{S}} S : \varnothing \subset \mathcal{S} \subseteq \mathcal{G} \right\}.$$

In other words, $\langle \mathcal{G} \rangle$ is the union-closed family consisting of all unions of sets from $\mathcal{G}$. If $\mathcal{F} = \langle \mathcal{G} \rangle$, we can say that $\mathcal{G}$ is a *generator* of $\mathcal{F}$. Let $\mathcal{F}$ be a union-closed family and $\mathcal{M}$ be the collection of the minimal sets in $\mathcal{F}$. We say that $\mathcal{F}$ is *minimally-generated* if $\mathcal{F} = \langle \mathcal{M} \rangle$ or $\mathcal{F} = \langle \mathcal{M} \cup \{\varnothing\} \rangle$ if $\varnothing \in \mathcal{F}$. Notice that even if a family is not minimally-generated, it still contains a minimally-generated subfamily.

Proposition 6. *For any union-closed family $\mathcal{F}$ and $\mathcal{G} \subseteq \mathcal{F}$, $\langle \mathcal{G} \rangle \subseteq \mathcal{F}$.*

Proof. Suppose for contradiction that $\langle \mathcal{G} \rangle \not\subseteq \mathcal{F}$. This means that there is some set $T \in \langle \mathcal{G} \rangle$ not in $\mathcal{F}$. However, the definition of generator says that every set in $\langle \mathcal{G} \rangle$ is a union of some subfamily $\mathcal{S} \subseteq \mathcal{G}$. Thus, T is a set which is a union of sets in $\mathcal{G}$, but not in $\mathcal{F}$, contradicting that $\mathcal{F}$ is union-closed. $\square$

We now will discuss two types of union-closed families which are minimally-generated.

Lemma 3. *All power set families are minimally-generated.*

Proof. Let $\mathcal{F} = \mathcal{P}(X)$. It is easy to see that the minimal sets of $\mathcal{F}$ are the singletons: $\{\{x\} : x \in X\}$. Any non-empty set $S = \{s_1, s_2, \ldots, s_k\} \in \mathcal{F}$ can be represented as the union $S = \bigcup_{i \in [k]} \{s_i\}$, so any set in $\mathcal{F}$ is a union of minimal sets. $\square$

For positive integers n and k where $k < n$, the *sliced power set* family $\mathcal{S}(n, k)$ is the family created by starting with $\mathcal{P}([n])$ and then removing the sets from the family of sizes less than or equal to k. We have that $\mathcal{P}([n]) \setminus \varnothing = \mathcal{S}(n, 0)$. We call $\mathcal{S}(n, n-2)$ the *missing-one* family since it contains the ground set $[n]$ and all of the sets which are "missing" exactly one element from $[n]$.

Lemma 4. *All sliced power set families are minimally-generated.*

Proof. Consider sliced power set $\mathcal{S}(n, k)$. The minimal sets of $\mathcal{S}(n, k)$ are the collection of size $k+1$ subsets of X, $X^{(k+1)}$. Select set $S = \{s_1, s_2, \ldots, s_{k+2}\}$ of size $k+2$. We can take $S_1 = \{s_1, s_2, \ldots, s_{k+1}\}$ and $S_2 = \{s_2, s_3, \ldots, s_{k+2}\}$ from $X^{(k+1)}$ to get $S = S_1 \cup S_2$. Thus we can write any set in $X^{(k+2)}$ as a union of sets in $X^{(k+1)}$. By the same logic, we can write any set in $X^{(k+3)}$ as a union of sets in $X^{(k+2)}$ (and thus as a union of sets in $X^{(k+1)}$). Repeating in this way, we can obtain any set in $\mathcal{S}(n, k)$ as a union of sets in $X^{(k+1)}$. $\square$

Theorem 8. *All sliced power set families satisfy Frankl's conjecture.*

Proof. Let $S(n, k)$ be an arbitrary sliced power set family. All elements in sliced power sets have equal frequency, so select any $x \in X$. The number of sets in the family is

$$\sum_{i=k+1}^{n} \binom{n}{i}$$

and so the number of sets which contain x is

$$\sum_{i=k}^{n-1} \binom{n-1}{i}.$$

Therefore, in order for x to be abundant, we need

$$2\sum_{i=k}^{n-1}\binom{n-1}{i} \geq \sum_{i=k+1}^{n}\binom{n}{i}.$$

We start with the fact that

$$\binom{n-1}{k} \geq 0$$

and add

$$\binom{n-1}{k} + 2\sum_{i=k+1}^{n}\binom{n-1}{i}$$

to both sides to get

$$2\sum_{i=k}^{n-1}\binom{n-1}{i} \geq \sum_{i=k+1}^{n}\left[\binom{n-1}{i} + \binom{n-1}{i-1}\right].$$

Using Pascal's formula on the right hand side of this equation gives the desired result. □

One way in which we can strengthen the idea of minimally-generated families is with an idea from Poonen in [14]. For a union-closed family $\mathcal{F}$, we say that a *block* $B \subseteq X$ is an equivalence class of the relation $\sim$ where $x \sim y$ if and only if $\mathcal{F}_x = \mathcal{F}_y$. In other words, a block is a collection of elements of X which always appear together in the sets of $\mathcal{F}$. If all blocks in a family are size one, meaning that $\mathcal{F}_x = \mathcal{F}_y$ implies $x = y$, then we say that the family is *normal*. As shown in [14], if Frankl's conjecture is true for normal union-closed families, then it is true for all union-closed families.

Lemma 5. *If Frankl's conjecture is true for normal union-closed families, then it is true for for all union-closed families.*

Proof. Let $\mathcal{F}$ be a union-closed family. For each block B_i in $\mathcal{F}$ and for each $S \in \mathcal{F}$, if $B_i \subseteq S$, replace the elements of B_i in S with a representative element i. If after doing this process, some element i in the new family is abundant, then each of the elements of the corresponding block B_i in $\mathcal{F}$ was abundant.

By this reasoning, if Frankl's conjecture is true for all normal union-closed families, then for any union-closed $\mathcal{F}$, the family formed by replacing blocks with singletons will contain an abundant element i which corresponds to a block of abundant elements in $\mathcal{F}$. $\square$

We can show that the same idea applies for normal minimally-generated families.

Theorem 9. *If Frankl's conjecture is true for normal minimally-generated families, then it is true for all minimally-generated families.*

Proof. Let $\mathcal{F}$ be a minimally-generated union-closed family, let S be a set in $\mathcal{F}$, and $\mathcal{M}$ be the collection of minimal sets in $\mathcal{F}$. Since $\mathcal{F}$ is minimally-generated, $S = M_1 \cup M_2 \cup \ldots M_k$ for some $M_1, M_2, \ldots, M_k \in \mathcal{M}$. We can write S in terms of blocks as $S = B_{s_1} \cup B_{s_2} \cup \ldots \cup B_{s_\ell}$. Since blocks are disjoint and $S = M_1 \cup M_2 \cup \ldots \cup M_k$, then each of the M_i's is the union of a subset of $\{B_{s_1}, B_{s_2}, \ldots, B_{s_\ell}\}$ with each of the blocks being contained in at least one of the M_i's.

Replace the blocks in $\mathcal{F}$ with singletons in the same way as in Lemma 5 and call this new family $\mathcal{F}'$. We already know from Lemma 5 that if $\mathcal{F}'$ has an abundant element, then $\mathcal{F}$ has one. Thus we just need to show that $\mathcal{F}'$ is minimally-generated. In $\mathcal{F}'$, we have $S = \{s_1, s_2, \ldots, s_\ell\}$ and each of $M_1, M_2, \ldots, M_k$ is a subset of $\{s_1, s_2, \ldots, s_\ell\}$ where each s_j appears in some M_i. Thus, in $\mathcal{F}'$, S is still a union of minimal sets, meaning $\mathcal{F}'$ is minimally-generated. $\square$

We can see that both the power set family and sliced power set family are normal minimally-generated families.

Proposition 7. *Power set and sliced power set families are normal.*

Proof. It is easy to see that $\mathcal{P}(X)$ is normal since for any $i, j \in X$ with $i \neq j$, the sets $\{i\}$ and $\{j\}$ are in $\mathcal{P}(X)$, so $\mathcal{F}_i \neq \mathcal{F}_j$ and so i and j are not in the same block.

Now consider $\mathcal{S}(n, k)$. Select $i, j \in X$ with $i \neq j$. Since $X^{(k+1)} \subseteq \mathcal{S}(n, k)$, we can show $\mathcal{S}(n, k)_i \neq \mathcal{S}(n, k)_j$ by finding a set in $X^{(k+1)}$ containing i but not j. We can do this by taking i and selecting k other elements from $X \setminus \{j\}$ to get a set in $X^{(k+1)}$. Thus $\mathcal{S}(n, k)_i \neq \mathcal{S}(n, k)_j$ and so i and j are not in the same block. $\square$

We conclude this section with two questions that are natural to ask about minimally-generated families. First, notice that if Frankl's conjecture is true in general, then it will be true for minimally-generated families. However, it is unclear if proving Frankl's conjecture for minimally-generated union-closed families would be much easier than proving it for general union-closed families.

Question 1. *Does a simple proof of Frankl's conjecture for minimally-generated families exist?*

Secondly, since we have only described a few constructions for minimally-generated families, it is natural to ask what other constructions for minimally-generated families exist. While Proposition 6 tells us how to find minimally-generated families within larger families, it doesn't tell us how to construct minimally-generated with "interesting" properties.

Question 2. *What other constructions for minimally-generated or normal minimally-generated families exist?*

4 Poset-Defined Invariants and Results

One valuable source of new invariants for union-closed families is to view them as posets ordered by set inclusion. Given a set A, a *partial order* on A, usually denoted $\leq$ or $\subseteq$ is a relation which for any $a, b, c \in A$ satisfies the following.

- The relation is reflexive, so $a \leq a$.
- The relation is antisymmetric, so $a \leq b$ and $b \leq a$ imply that $a = b$.
- The relation is transitive, so $a \leq b$ and $b \leq c$ imply that $a \leq c$.

We define a partially ordered set, or *poset*, as a pair $(A, \leq)$ consisting of a set and a partial order on it. Thus, if $\mathcal{F}$ is a union-closed family, its corresponding poset is $(\mathcal{F}, \subseteq)$ where $\subseteq$ is the usual subset relation for sets.

Two concepts related to posets we will use are chains and antichains. If $(\mathcal{F}, \subseteq)$ is a poset with the set inclusion relation, a *chain* is a subset $\mathcal{C} \subseteq \mathcal{F}$ where for any two sets $C_1, C_2 \in \mathcal{C}$, either $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$. In other words, any two elements in a chain are comparable. The dual concept to a chain is an antichain. An *antichain* is a subset $\mathcal{A} \subseteq \mathcal{F}$ where for any two sets $A_1, A_2 \in \mathcal{A}$, $A_1 \not\subseteq A_2$. In other words, any two elements in an antichain *are not* comparable. Two helpful theorems when searching for antichains and chains are Sperner's theorem [15] and Dilworth's theorem [16].

Theorem 10 (Sperner's Theorem)**.** *Any antichain $\mathcal{A}$ in a set family on X with $|X| = n$ has size at most $\binom{n}{\lfloor n/2 \rfloor}$ where this bound is achieved only if $\mathcal{A}$ is either the collection of all size $\lfloor n/2 \rfloor$ subsets of $[n]$ or all size $\lceil n/2 \rceil$ subsets of $[n]$.*

Theorem 11 (Dilworth's Theorem)**.** *Let $\mathcal{F}$ be a set family and the size of its largest antichain be w. Then $\mathcal{F}$ can be partitioned into w chains and no fewer.*

Another general property of chains and antichains we will use is the the following.

Lemma 6. *The intersection of a chain with an antichain is size at most 1.*

Proof. Let $\mathcal{A}$ be an antichain and $\mathcal{C}$ be a chain in some set family. Suppose for contradiction that $|\mathcal{A} \cap \mathcal{C}| \geq 2$. Let S and T be distinct sets in $\mathcal{A} \cap \mathcal{C}$. Since S and T are in a chain together, we know that either $S \subseteq T$ or $T \subseteq S$. However, since S and T are in an antichain together, then $S \not\subseteq T$ and $T \subseteq S$, a contradiction. □

We will now demonstrate some simple properties of chains and antichains in union-closed families.

Proposition 8. *The collection of minimal sets of a union-closed family forms an antichain.*

Proof. Let $\mathcal{F}$ be a union-closed family and $\mathcal{M}$ is the collection of minimal sets in $\mathcal{F}$. Suppose for contradiction that $\mathcal{M}$ is not an antichain. This means that there exist $M_1, M_2 \in \mathcal{M}$ with $M_1 \subseteq M_2$. However, this would mean M_2 is not a minimal set since M_1 is a set in $\mathcal{F}$ which is a subset of it, a contradiction. □

Proposition 9. *If an antichain contains the ground set of a union-closed family, then the ground set is the only element of the antichain.*

Proof. Let $\mathcal{F}$ be a union-closed family. Suppose for contradiction that $\mathcal{F}$ contains an antichain $\mathcal{A}$ which contains X and another set S. Since all sets in the family are subsets of the ground set, then $S \subseteq X$, contradicting that $\mathcal{A}$ is an antichain. □

Lemma 7. *Any maximal chain in a union-closed family contains the ground set and a minimal set.*

Proof. Consider union-closed family $\mathcal{F}$ and a maximal chain $\mathcal{C} = \{C_1, C_2, C_3, \ldots, C_\ell\}$ with $C_1 \subseteq C_2 \subseteq \ldots \subseteq C_\ell$. Suppose $S_\ell \neq X$. If $S_\ell \subseteq X$, then the chain is not maximal. However, if $C_\ell \nsubseteq X$, then C_ℓ contains an element not in X which is impossible. Similarly, if C_1 is not minimal, then then there must be a set C_0 in $\mathcal{F}$ which is a subset of C_1. However, this again contradicts maximality of $\mathcal{C}$. □

Proposition 10. *If Conjecture 2 is true, then in any union-closed family, every element of maximum frequency appears in every non-empty set within some maximal chain.*

Proof. Let $\mathcal{F}$ be a union-closed family and $x \in X$ be an element of maximum frequency. Suppose that Conjecture 2 is true, meaning there exists a minimal set $M \in \mathcal{F}$ with $x \in M$. Let $\mathcal{C}$ be a maximal chain in $\mathcal{F}$ containing M. From the definition of minimal set, the only set in $\mathcal{C}$ which could be a subset of M is the empty set. Thus for any $C \in \mathcal{C} \setminus \{\varnothing\}$, $M \subseteq C$ and so $x \in C$. □

The *width* of a poset is the size of its largest antichain. We thus define the width of a union-closed family $\mathcal{F}$, denoted $\mathrm{w}(\mathcal{F})$ as the size of the largest antichain in its corresponding poset. The *height* of a poset is the size of its largest chain. We thus define the height of a union-closed family $\mathcal{F}$, denoted $\mathrm{h}(\mathcal{F})$ as the size of the largest chain in its corresponding poset.

Another idea related to the width of a union-closed family is if a family is *width-critical*. The idea of creating new properties from old ones via "criticality" is common in combinatorics such as by Lovász in [17]. We say that a union-closed family $\mathcal{F}$ is width-critical if for every set S in $\mathcal{F}$, either $\mathcal{F} \setminus S$ is not union-closed or $\mathrm{w}(\mathcal{F}) > \mathrm{w}(\mathcal{F} \setminus S)$.

Proposition 11. *The sliced power set $\mathcal{S}(n,k)$ is width-critical if and only if $k > \lceil n/2 \rceil - 2$.*

Proof. Consider arbitrary sliced power set $\mathcal{S}(n,k)$ in which $k > \lceil n/2 \rceil - 2$. Any set A in this family with $|A| > k+1$ may be represented as a union of two sets of cardinality $|A|-1$, so trying to remove any set of cardinality larger than $k+1$ will make the resulting family fail to be union-closed. The width of $\mathcal{S}(n,k)$ is $\binom{n}{k}$. Since $k > \lceil n/2 \rceil - 2$, the largest antichain in the family is the collection of sets of size $k+1$. This means that removing any set of size $k+1$ from the family would reduce the width of the family. Since removing any set from the family either reduces the width or makes the family no longer union-closed, then $\mathcal{S}(n,k)$ is width-critical.

Now consider arbitrary sliced power set $\mathcal{S}(n,k)$ in which $k \leq \lceil n/2 \rceil - 2$. In this case, the antichain of sets of cardinality $k+1$ will either not be the largest antichain or will be of equal size to the antichain of sets of cardinality $k+2$. In both of these cases, removing a set of cardinality $k+1$ will fail to reduce the width of the family, meaning that the family cannot be width-critical. $\square$

With these definitions in place, we can prove that families with small height and width values satisfy Frankl's conjecture.

Lemma 8. *The only union-closed families with height one are the families consisting of only the ground set.*

Proof. Suppose for contradiction that $\mathcal{F}$ is a union-closed family of height one, but it two distinct sets A and B. Since $\mathcal{F}$ has height one, $A \not\subseteq B$ and $B \not\subseteq A$. However, since $\mathcal{F}$ is union-closed, $A \cup B \in \mathcal{F}$ and since A and B are distinct, $A \subset A \cup B$, meaning $\mathcal{F}$ contains a chain of size two, contradicting that $\mathcal{F}$ has height one. $\square$

Lemma 9. *The frequency of any element in a union-closed family $\mathcal{F}$ with height two is at least $|\mathcal{F}| - 1$.*

Proof. Suppose for contradiction that there exists $x \in X$ and distinct sets $S, T \in \mathcal{F}$ with x in neither of S or T. Since $\mathcal{F}$ is union-closed, $S \cup T \in \mathcal{F}$. However, $x \notin S \cup T$, meaning $S \cup T \subset X$, meaning we have the chain $\{S, S \cup T, T\}$ in $\mathcal{F}$. This chain is size three, contradicting that $\mathcal{F}$ has height two. Thus the frequency of any element in $\mathcal{F}$ is at least $|\mathcal{F}| - 1$. $\square$

Corollary 2. *All union-closed families of height at most two satisfy Frankl's conjecture.*

Proof. If $\mathrm{h}(\mathcal{F}) = 1$, then by Lemma 8, $\mathcal{F}$ consists of just the ground set and so every element of the ground set appears in every set of $\mathcal{F}$, meaning $\mathcal{F}$ satisfies Frankl's conjecture.

If $\mathrm{h}(\mathcal{F}) = 2$, then select an element x of $\mathcal{F}$. By Lemma 9, x satisfies $|\mathcal{F}_x| \geq |\mathcal{F}| - 1$. Since $|\mathcal{F}| \geq 2$ in order for $\mathrm{h}(\mathcal{F}) = 2$, then $|\mathcal{F}| - 1 \geq |\mathcal{F}|/2$. Thus $|\mathcal{F}_x| \geq |\mathcal{F}|/2$ and so x is abundant, meaning $\mathcal{F}$ satisfies Frankl's conjecture. $\square$

Theorem 12. *Union-closed families of width three satisfy Frankl's conjecture.*

Proof. Let $\mathcal{F}$ be a union-closed family with width three. By Theorem 11, we can decompose $\mathcal{F}$ into three chains: $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$. Let S_1, S_2, and S_3 be the minimal elements in the chains. There are two cases to consider: where S_1, S_2, and S_3 are mutually disjoint and where they are not.

Case 1: Suppose without loss of generality that there exists $x \in S_1 \cap S_2$. Since x is in the minimal sets of $\mathcal{C}_1$ and $\mathcal{C}_2$, then it must be in every set of $\mathcal{C}_1$ and $\mathcal{C}_2$. If $|\mathcal{C}_1 \cup \mathcal{C}_2| > |\mathcal{F}|/2$, then this means x is abundant. If $|\mathcal{C}_1 \cup \mathcal{C}_2| < |\mathcal{F}|/2$, then any element in S_3 is abundant. If $|\mathcal{C}_1 \cup \mathcal{C}_2| = |\mathcal{C}_3|$, then there are two possibilities to consider. If x is in any set of $\mathcal{C}_3$, then x is abundant. Likewise, if any element y is in S_3 and any set of $\mathcal{C}_1 \cup \mathcal{C}_2$, then y is abundant. If $S_1 \cup S_3$ is either in $\mathcal{C}_3$, then x must be in $\mathcal{C}_3$, making x abundant. If $S_1 \cup S_3$ is in $\mathcal{C}_1 \cup \mathcal{C}_2$, then y must be in $\mathcal{C}_1 \cup \mathcal{C}_2$, making y abundant. Therefore, if S_1, S_2, and S_3 aren't all mutually disjoint, then $\mathcal{F}$ satisfies Frankl's conjecture.

Case 2: Suppose that S_1, S_2, and S_3 are mutually disjoint. Without loss of generality, assume $|\mathcal{C}_1| \geq |\mathcal{C}_2| \geq |\mathcal{C}_3|$. Note that $|\mathcal{C}_1 \cup \mathcal{C}_2| \geq 2|\mathcal{F}|/3$. Since $S_1 \cup S_2$ is disjoint from S_3, then $S_1 \cup S_2$ cannot be in $\mathcal{C}_3$ and must therefore be in either $\mathcal{C}_1$ or $\mathcal{C}_2$. Suppose that $S_1 \cup S_2$ is in $\mathcal{C}_1$. Any set A between S_1 and $S_1 \cup S_2$ in $\mathcal{C}_1$ (i.e. $S_1 \subseteq \ldots \subseteq A \subseteq \ldots \subseteq S_1 \cup S_2$) is of the form $A = S_1 \cup B$ where $B \subseteq S_2$. Let S be the smallest set in $\mathcal{C}_1$ larger than S_1. This set must contain some $x \in S_2$.

Therefore, x is in every set of $\mathcal{C}_1$ except for S_1. We also know x is in every set of $\mathcal{C}_2$ since it is in S_2. If $\mathcal{F} > 6$, then this means $|\mathcal{F}_x| \geq 2|\mathcal{F}|/3 - 1 > |\mathcal{F}|/2$.

Case 2 Subcases: If $|\mathcal{F}| \leq 6$ and $\mathrm{h}(\mathcal{F}) = 2$, then $\mathcal{F}$ satisfies Frankl's conjecture by Lemma 9. If $|\mathcal{F}| < 6$ and $\mathrm{h}(\mathcal{F}) \geq 3$, then any element in the minimal set of the 3-chain will be abundant, meaning $\mathcal{F}$ satisfies Frankl's conjecture. If $|\mathcal{F}| = 6$ and $\mathrm{h}(\mathcal{F}) > 3$, then the abundant element is any element in the minimal set of the longest chain. If $|\mathcal{F}| = 6$ and $\mathrm{h}(\mathcal{F}) = 3$, then we can decompose $\mathcal{F}$ into chains $|\mathcal{C}_1| = 3$, $|\mathcal{C}_2| = 2$, and $|\mathcal{C}_3| = 1$. From the reasoning described above, there must be an element in either all sets of $\mathcal{C}_1$ and 1 set of $\mathcal{C}_2$ or an element in all sets of $\mathcal{C}_2$ and all but one set of $\mathcal{C}_1$. In both cases, we have an element in $4 > |\mathcal{F}|/2 = 3$ sets, meaning $\mathcal{F}$ satisfies Frankl's conjecture. $\square$

Since we were able to prove Frankl's conjecture for union-closed families with height up to two and width up to three, it is natural to ask if we can prove Frankl's conjecture for larger heights and widths. If Frankl's conjecture is true, then it would be true for families of any height or width, so it seems that proofs for these height and width values must be possible. However, it seems that these proofs would require more advanced methods than the ones we used here.

Question 3. *Does a simple proof of Frankl's conjecture for width four or height three families exist? If so, can those same proof stategies show that Frankl's conjecture is true for larger width or height values?*

We next examine two conjectures which involve the maximum frequency of an element in a union-closed family.

Conjecture 3. *If $\mathcal{F}$ is a union-closed family, then $\mathrm{w}(\mathcal{F}) \leq \phi(\mathcal{F})$.*

Conjecture 4. *If $\mathcal{F}$ is a union-closed family and $\mathcal{M}$ is the collection of its minimal sets. Then $|\mathcal{M}| \leq \phi(\mathcal{F})$.*

It turns out that these two conjectures are equivalent to each other which may seem surprising considering that $\mathrm{w}(\mathcal{F})$ may be much larger than $\mathcal{M}$. For example, if $\mathcal{F}$ is the power set family $\mathcal{P}(X)$ with $|X| = n$, Theorem 10 gives that $\mathrm{w}(\mathcal{F}) = \binom{n}{\lfloor n/2 \rfloor}$, but $\mathcal{M}$ for this $\mathcal{F}$ has size n.

Theorem 13. *Conjecture 3 is equivalent to Conjecture 4.*

Proof. First we show that Conjecture 3 implies Conjecture 4. It follows from Proposition 8 that since $\mathcal{M}$ is an antichain, $|\mathcal{M}| \leq \mathrm{w}(\mathcal{F})$. This fact combined with Conjecture 3 telling us that $\mathrm{w}(\mathcal{F}) \leq \phi(\mathcal{F})$ gives

$$|\mathcal{M}| \leq \mathrm{w}(\mathcal{F}) \leq \phi(\mathcal{F})$$

as desired.

Now we show that Conjecture 4 implies Conjecture 3. Let $\mathcal{F}$ be a union-closed family, $\mathcal{A}$ be a maximum antichain in $\mathcal{F}$, and $\mathcal{F}' = \langle \mathcal{A} \rangle$. Notice that $\mathcal{A}$ forms the set of minimal elements in $\mathcal{F}'$ and thus if Conjecture 4 holds, then $|\mathcal{A}| \leq \phi(\mathcal{F}')$. Since $\mathcal{F}' \subseteq \mathcal{F}$, then $\phi(\mathcal{F}') \leq \phi(\mathcal{F})$. Since $\mathcal{A}$ is a maximum antichain in $\mathcal{F}$, then $|\mathcal{A}| = \mathrm{w}(\mathcal{F})$ and so combining this fact with our inequalities gives

$$\mathrm{w}(\mathcal{F}) = |\mathcal{A}| \leq \phi(\mathcal{F}') \leq \phi(\mathcal{F})$$

as desired. □

We conclude this section by stating a conjecture similar in idea to Conjecture 2. In Proposition 10, we showed that Conjecture 2 can be viewed as a statement about how maximum frequency elements interact with maximal chains. It is thus natural to ask if anything similar can be said about maximum frequency elements and antichains.

Conjecture 5. *In a union-closed family $\mathcal{F}$ with element x of maximum frequency, there is a maximum antichain $\mathcal{A}$ in $\mathcal{F}$ where at least half of the sets of $\mathcal{A}$ contain x.*

Proposition 12. *Conjecture 5 is true for power set families.*

Proof. Consider power set family $\mathcal{P}(X)$ with $|X| = n$. Since all elements in power set families have equal frequency, we can select any $x \in X$ as our maximum frequency element. Theorem 10 tells us that the maximum antichains in $\mathcal{P}(X)$ are $X^{(\lceil n/2 \rceil)}$ and $X^{(\lfloor n/2 \rfloor)}$ (where these two antichains are the same if n is even). Let $\mathcal{A} = X^{(\lceil n/2 \rceil)}$. The collection of sets in $\mathcal{A}$ which contain x is the collection of sets formed by starting with x and selecting $\lceil n/2 \rceil - 1$ elements from $X \setminus \{x\}$. Thus we have

$$|\mathcal{A}_x| = \binom{n-1}{\lceil n/2 \rceil - 1} \quad \text{and} \quad |\mathcal{A}| = \binom{n}{\lceil n/2 \rceil}.$$

Rewriting these binomial coefficients gives us

$$\frac{|\mathcal{A}_x|}{|\mathcal{A}|} = \frac{\binom{n-1}{\lceil n/2 \rceil - 1}}{\binom{n}{\lceil n/2 \rceil}} = \frac{\lceil n/2 \rceil}{n} \geq 1/2$$

as desired. $\square$

5 More on Minimal Sets

Next, we will define two ways of "slicing" union-closed families into levels where the number of levels is dependant on the height of the family. Let $\mathcal{F}$ be a union-closed family on ground set with height h and let $\mathcal{M}_0$ be the collection of minimal sets of $\mathcal{F}$. Now let $\mathcal{F}_1 = \mathcal{F} \setminus \mathcal{M}_0$ and $\mathcal{M}_1$ be the collection of minimal sets of $\mathcal{F}_1$. Repeating this process gives for $i \in \{1, 2, \ldots, h-2\}$, $\mathcal{F}_{i+1} = \mathcal{F}_i \setminus \mathcal{M}_i$ where $\mathcal{M}_i$ is the collection of minimal sets of $\mathcal{F}_i$. We say that for each $i \in \{0, 1, \ldots, h-1\}$, $\mathcal{M}_i$ is *minimal level* i of $\mathcal{F}$. Visually, we can think of minimal levels as a way to "slice" union-closed families from the bottom up.

Proposition 13. *If $\mathcal{F}$ is a minimally-generated union-closed family with collection of minimal sets $\mathcal{M}$, then for $M_1, M_2 \in \mathcal{M}$, $M_1 \cup M_2$ is not necessarily minimal in $\mathcal{F} \setminus \mathcal{M}$.*

Proof. Consider the family $\mathcal{F}$ shown in Figure 4. Here, we have that $\mathcal{F}$ is minimally generated with $\mathcal{M} = \{\{1,2\}, \{2,3\}, \{2,4\}, \{3,4\}\}$. We can see that for $M_1 = \{1,2\}$ and $M_2 = \{3,4\}$, that $M_1 \cup M_2 = \{1,2,3,4\}$ is not minimal in $\mathcal{F} \setminus \mathcal{M}$. $\square$

Theorem 14. *The height of a union-closed family is the same as its number of minimal levels.*

Proof. We can show this via induction on height. The claim is trivial for union-closed families with height one since a family with height one is just the ground set by Lemma 8. Let $\mathcal{F}$ be a union-closed family with $\mathrm{h}(\mathcal{F}) > 1$. Our induction hypothesis is that the claim is true for families with height less than $\mathrm{h}(\mathcal{F})$.

Let $\mathcal{M}$ be the collection of minimal sets in $\mathcal{F}$, and let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \ldots, \mathcal{S}_k$ be all of the chains in $\mathcal{F}$ of size $\mathrm{h}(\mathcal{F})$, i.e. all of the maximum size chains in $\mathcal{F}$. By Lemma 7, these chains all contain minimal elements of $\mathcal{F}$. This all of the chains which are size $\mathrm{h}(\mathcal{F})$ in $\mathcal{F}$ are size $\mathrm{h}(\mathcal{F}) - 1$ in $\mathcal{F} \setminus \mathcal{M}$. Thus, $\mathrm{h}(\mathcal{F} \setminus \mathcal{M}) = \mathrm{h}(\mathcal{F}) - 1$ and so our induction hypothesis tells us $\mathcal{F} \setminus \mathcal{M}$ has $\mathrm{h}(\mathcal{F}) - 1$ minimal levels. Since every minimal level in $\mathcal{F} \setminus \mathcal{M}$ is a minimal level in $\mathcal{F}$ and $\mathcal{M}$ is a minimal level in $\mathcal{F}$ not in $\mathcal{F} \setminus \mathcal{M}$, then $\mathcal{F}$ has $\mathrm{h}(\mathcal{F})$ minimal levels. $\square$

Instead of "slicing" from the bottom-up, we can also "slice" from the top-down. We will do this using the *covering relation*. For elements in a general poset, $(X, \leq)$, we say that y covers x if $x \leq y$ and there is no element z (distinct from x and y) in X with $x \leq z \leq y$. In the context of set families, Y covers X if $X \subseteq Y$ and there exists no set Z (distinct from X and Y) in the family with $X \subseteq Z \subseteq Y$.

With this idea in mind, we define *covering levels* as follows. Let $\mathcal{F}$ be a union-closed family with height h and ground set X. We say that X is covering level 0 of the family, denoted $\mathcal{C}_0$. The sets covered by X form covering level 1, $\mathcal{C}_1$. Continuing in this way, we say that the the covering level $i + 1$, $\mathcal{C}_{i+1}$ is the collection of sets covered by covering level i, $\mathcal{C}_i$.

Theorem 15. *The covering level $\mathcal{C}_1$ of any union-closed family with ground set of size n has size at most n.*

Proof. Let $\mathcal{F}$ be a union-closed family on X with $|X| = n$ and write covering level $\mathcal{C}_1$ of $\mathcal{F}$ as $\mathcal{C}_1 = \{C_1, C_2, C_3, \ldots, C_\ell\}$. Let $E_1 = X \setminus C_1$ be the set of elements "excluded from" C_1. From the definition of covering levels, $C_1 \neq X$, so E_1 is non-empty.

In order for C_1 to be in $\mathcal{C}$, we need that every other C_i in the covering levels contains E_1 since otherwise $C_1 \cup C_i$ would cover C_1 and C_i, but $C_1 \cup C_i \neq X$. Let $E_2 = X \setminus C_2$. By definition of covering level and since $E_1 \subseteq C_i$ for all $i \neq 1$, then E_2 is non-empty and disjoint from E_1. In order for $C_2 \cup C_i = X$ and $C_1 \cup C_i = X$, we therefore need $E_1, E_2 \subseteq C_i$ for each $i \notin \{1, 2\}$. Let $E_3 = X \setminus C_3$. By the same reasoning as before, $E_3 \subseteq C_i$ for all $i \neq 3$ and E_3 is non-empty and disjoint from E_1 and E_2. Continuing in this fashion gives a list of excluded sets $E_1, E_2, E_3, \ldots, E_\ell$ that are all pairwise disjoint and non-empty. There are at most n pairwise disjoint non-empty subsets of X and so $|\mathcal{C}_1| \leq n$. $\square$

In general, a union-closed set family may have different numbers of covering levels and minimal levels. The covering and minimal levels for the family displayed in Figure 2 are as follows. For readability, we write the sets within the levels without commas and braces.

$$\mathcal{C}_0 = [6] \qquad \mathcal{M}_0 = \{23, 46, 356, 456\}$$

$$\mathcal{C}_1 = \{12346,\ 12356,\ 23456\} \qquad \mathcal{M}_1 = \{123,\ 2346,\ 2356,\ 456\}$$

$$\mathcal{C}_2 = \{123,\ 2346,\ 2356,\ 3456\} \qquad \mathcal{M}_2 = \{12346,\ 12356,\ 3456\}$$

$$\mathcal{C}_3 = \{23,\ 46,\ 356,\ 456\} \qquad \mathcal{M}_3 = \{23456\}$$

$$\mathcal{M}_4 = [6]$$

Additionally, even if a union-closed set family has equal numbers of covering levels and minimal levels, two sets being together in a covering level might not mean that they are together in a minimal level and vice versa. The covering and minimal levels for the family displayed in figure 3 are as follows. Again we write the sets within the levels without commas and braces to improve readability.

$$\mathcal{C}_0 = [5] \qquad \mathcal{M}_0 = \{125,\ 234,\ 235,\ 245,\ 1345\}$$

$$\mathcal{C}_1 = \{1235,\ 1245,\ 1345,\ 2345\} \qquad \mathcal{M}_1 = \{1235,\ 1245,\ 2345\}$$

$$\mathcal{C}_2 = \{125,\ 234,\ 235,\ 245\} \qquad \mathcal{M}_2 = [5]$$

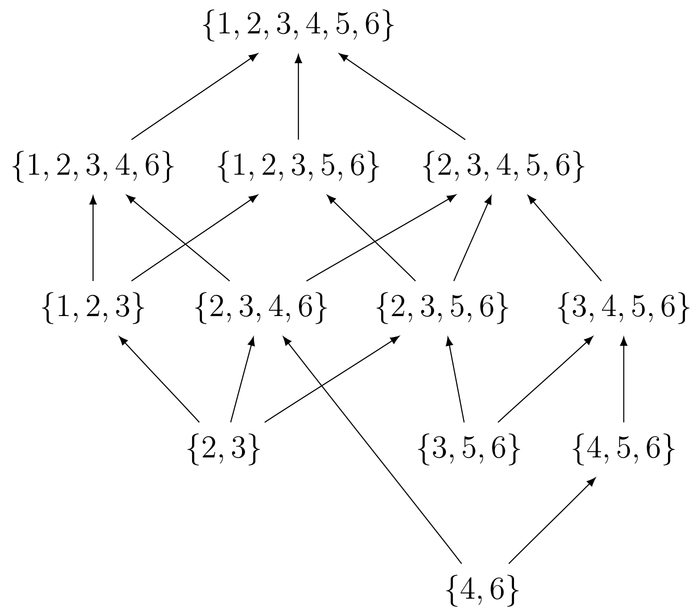

Figure 2: Here, we have four covering levels, but five minimal levels.

Notice that $\{1, 3, 4, 5\}$ is grouped with the three element sets in the minimal levels, but it is grouped with the four element sets in the covering levels.

These two situations raise the question of what condition would make the minimal and covering levels of a family "the same". We say that a union-closed set family $\mathcal{F}$ satisfies the *MC condition* or *is MC* if $\mathcal{F}$ has the same number of minimal and covering levels, say k, and $\bigcup_{i=0}^{k-1} \mathcal{M}_i = \bigcup_{i=0}^{k-1} \mathcal{C}_i$. In other words, $\mathcal{F}$ is MC if it has the same number of minimal and covering levels and

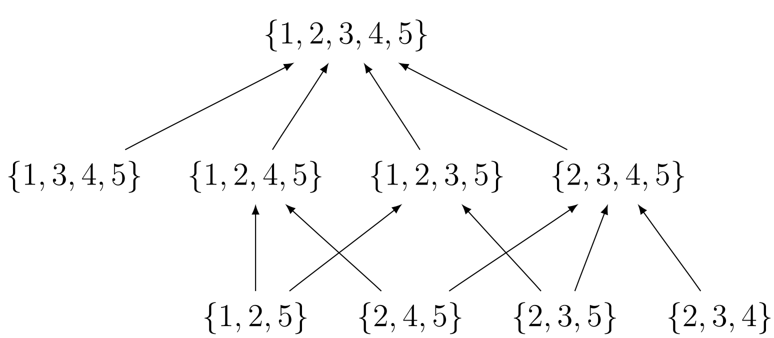

Figure 3: Here, we have three covering levels and three minimal levels, but their contents differ.

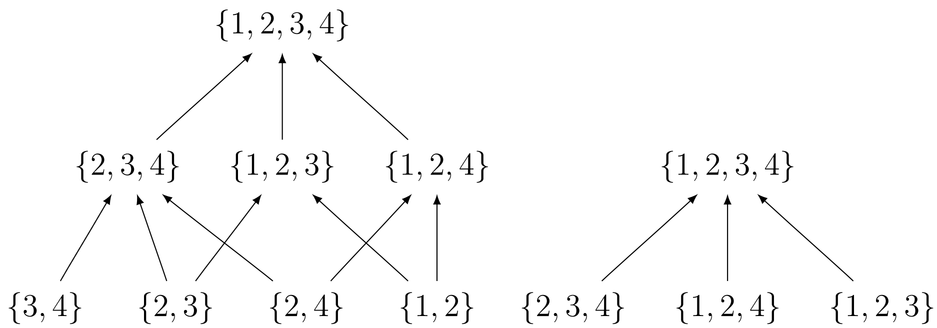

Figure 4: Minimally-generated family $\mathcal{F}$ (left) and the family formed by removing the minimal sets from $\mathcal{F}$ (right).

two sets appear together in a covering level if and only if they appear together in a minimal level.

Proposition 14. *If a union-closed family satisfies the MC property, then for every pair M and M' of minimal sets, the shortest M, X and M', X chains are of the same size.*

Proof. We prove this by showing that if a union-closed family has M, X and M', X chains of different sizes, then the family cannot satisfy the MC property. Suppose that $\mathcal{F}$ is a union-closed family and M and M' are two minimal sets. Let $\mathcal{H}_M$ and $\mathcal{H}_{M'}$ be minimum size M, X and M', X chains of sizes ℓ_M and $\ell_{M'}$ where $\ell_M \neq \ell_{M'}$. Since M and M' are minimal sets, they are both in $\mathcal{M}_0$. However, by minimality of $\mathcal{H}_M$ and $\mathcal{H}_{M'}$, $M \in \mathcal{C}_{\ell_M}$ and $M' \in \mathcal{C}_{\ell_{M'}}$, so M and M' are in different covering levels. Thus $\mathcal{F}$ does not satisfy the MC property. $\square$

Another way in which we can use minimal levels is to expand on our earlier idea of minimally-generated families. Let $\mathcal{F}$ be a union-closed family which is not minimally-generated. Since $\mathcal{F}$ is not minimally-generated, there exists a non-empty collection $\mathcal{D} \subset \mathcal{F}$ which cannot be generated using the minimal sets. We say that $\mathcal{F}$ is *defective* and that the sets in $\mathcal{D}$ are the *defects* of $\mathcal{F}$. If a family is minimally-generated, it is not defective. Using minimal levels, we can strengthen this condition. Let $\mathcal{F}$ be a minimally-generated union-closed family with height h. Let $\mathcal{F}_1 = \mathcal{F} \setminus \mathcal{M}_0$ and for $i \in \{1, 2, \ldots, h-2\}$, let $\mathcal{F}_{i+1} = \mathcal{F}_i \setminus \mathcal{M}_i$. We say that $\mathcal{F}$ is *k-defect-free* if none of $\mathcal{F}, \mathcal{F}_1, \ldots, \mathcal{F}_{k-1}$ is defective. Since a union-closed family has h minimal levels by Theorem 14, then the highest possible value of k for which $\mathcal{F}$ can be k-defect-free is h. The *defectiveness* of a family $\mathcal{F}$ is the highest value of k for which $\mathcal{F}$ is k-defect-free. We write defectiveness of $\mathcal{F}$ as $D(\mathcal{F})$.

Theorem 16. *If $\mathcal{F}$ is a union-closed family with $D(\mathcal{F}) = h$ and for any $M_1, M_2 \in \mathcal{M}_i$, $M_1 \cup M_2 \in \mathcal{M}_{i+1}$ then every element of the ground set is abundant.*

Proof. Let $\mathcal{F}$ be as in the theorem statement. Consider some $x \in X$. For each $i \in \{1, \ldots, h-1\}$, let $\mathcal{M}_{i_x}$ be the collection of sets in minimal level i containing x and $\mathcal{M}_{i_{\overline{x}}}$ be the collection of sets in minimal level i *not* containing x. Let S be a set in $\mathcal{M}_{i_x}$. Since $\mathcal{F}$ is h-defect-free, $\mathcal{F}_i$ is minimally-generated and so S is generated by sets in $\mathcal{M}_{i-1}$. Thus, S has subsets in $\mathcal{M}_{i-1}$. Additionally, the union of any pair of subsets of S in $\mathcal{M}_{i-1}$ must be S. If such a union were a subset or superset of S, then the union would be on a minimal level other than $\mathcal{M}_{i-1}$, which is impossible by assumption. We also notice that the union of any set in $\mathcal{M}_{(i-1)_x}$ and any set in $\mathcal{M}_{(i-1)_{\overline{x}}}$ contains x and will be in $\mathcal{M}_{i_x}$.

Since every set in $\mathcal{M}_{i-1}$ not containing x is a subset of a set in $\mathcal{M}_i$ containing x and each set in $\mathcal{M}_{i_x}$ has at most one subset in M_{i-1} not containing x, then $|\mathcal{M}_{i_x}| \geq |M_{(i-1)_{\overline{x}}}|$.

Now let $\mathcal{F}_x$ be the sets in $\mathcal{F}$ containing x. We want to show that $|\mathcal{F}_x| \geq |\mathcal{F}|/2$. We have

$$|\mathcal{F}_x| = |\mathcal{M}_{0_x}| + |\mathcal{M}_{1_x}| + \ldots + |\mathcal{M}_{h-2_x}| + |\mathcal{M}_{(h-1)_x}|.$$

Since $\mathcal{M}_{h-1}$ is just the ground set, then $|\mathcal{M}_{(h-1)_x}| = 1$. Using this fact and the above argument that $|\mathcal{M}_{i_x}| \geq |M_{(i-1)_{\overline{x}}}|$ gives us

$$|\mathcal{F}_x| \geq |\mathcal{M}_{0_x}| + |\mathcal{M}_{0_{\overline{x}}}| + |\mathcal{M}_{1_{\overline{x}}}| + \ldots + |\mathcal{M}_{(h-3)_{\overline{x}}}| + |\mathcal{M}_{(h-2)_{\overline{x}}}|.$$

We therefore have

$$|\mathcal{F}| = |\mathcal{F}_x| + |\mathcal{M}_{0_{\overline{x}}}| + |\mathcal{M}_{1_{\overline{x}}}| + \ldots + |\mathcal{M}_{(h-2)_{\overline{x}}}| \leq 2|\mathcal{F}_x|,$$

meaning $|\mathcal{F}_x| \geq |F|/2$. $\square$

6 Transformations of Set Families

Next, we will examine multiple ways of transforming union-closed families. The first is via *structure isomorphism*. We say that two union-closed families $\mathcal{F}$ and $\mathcal{F}'$ are *structure isomorphic* if there exists bijective function $\varphi : \mathcal{F} \to \mathcal{F}'$ such that for all $S, T \in \mathcal{F}$, $\varphi(S \cup T) = \varphi(S) \cup \varphi(T)$ and $S \subseteq T$ in $\mathcal{F}$ if and only if $\varphi(S) \subseteq \varphi(T)$. We call such a φ a structure isomorphism.

Proposition 15. *The property of families being structure isomorphic forms an equivalence relation.*

Proof. First, we show this property is reflexive. For union-closed family $\mathcal{F}$, consider the identity function $\mathrm{id} : \mathcal{F} \to \mathcal{F}$. We have $\mathrm{id}(S \cup T) = S \cup T = \mathrm{id}(S) \cup \mathrm{id}(T)$. We also have that if $S \subseteq T$ then $\mathrm{id}(S) = S \subseteq T = \mathrm{id}(T)$ and if $\mathrm{id}(S) \subseteq \mathrm{id}(T)$, then $S = \mathrm{id}(S) \subseteq \mathrm{id}(T) = T$.

Next we show this property is symmetric. Let $\mathcal{F}$ and $\mathcal{F}'$ be structure isomorphic and $\varphi : \mathcal{F} \to \mathcal{F}'$ be a structure isomorphism. We need to show φ^{-1} is a structure isomorphism. Properties of bijective functions tell us

$$\varphi\left(\varphi^{-1}(S \cup T)\right) = S \cup T$$

and since $\varphi^{-1}(S), \varphi^{-1}(T) \subseteq \mathcal{F}$, then

$$\varphi\left(\varphi^{-1}(S) \cup \varphi^{-1}(T)\right) = \varphi\left(\varphi^{-1}(S)\right) \cup \varphi\left(\varphi^{-1}(T)\right) = S \cup T.$$

Since φ is injective, $\varphi\left(\varphi^{-1}(S \cup T)\right) = \varphi\left(\varphi^{-1}(S)\right) \cup \varphi(\varphi^{-1}(T))$ implies that

$$\varphi^{-1}\left(S \cup T\right) = \varphi^{-1}(S) \cup \varphi^{-1}(T).$$

Next, we can write $S \subseteq T$ as $\varphi\left(\varphi^{-1}(S)\right) \subseteq \varphi\left(\varphi^{-1}(T)\right)$ which by properties of structure isomorphisms tells us that $\varphi^{-1}(S) \subseteq \varphi^{-1}(T)$. Thus, φ^{-1} is a structure isomorphism.

Finally we need to show that this property is transitive. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are structure isomorphic, $\mathcal{G}$ and $\mathcal{H}$ are structure isomorphic, and $\varphi_1 : \mathcal{F} \to \mathcal{G}$ and $\varphi_2 : \mathcal{G} \to \mathcal{H}$ are structure isomorphisms. We can show that $\varphi^* : \mathcal{F} \to \mathcal{H}$ given by $\varphi^* = \varphi_2 \circ \varphi_1$ is a structure isomorphism.

Compositions of bijections are bijections, so φ^* is bijective. We also have

$$\begin{aligned}\varphi^*(S \cup T) &= \varphi_2\left(\varphi_1(S \cup T)\right) \\ &= \varphi_2\left(\varphi_1(S) \cup \varphi_1(T)\right) \\ &= \varphi_2\left(\varphi_1(S)\right) \cup \varphi_2\left(\varphi_1(T)\right) \\ &= \varphi^*(S) \cup \varphi^*(T).\end{aligned}$$

Finally, if $S \subseteq T$, then we have

$$\varphi_1(S) \subseteq \varphi_1(T)$$

and thus

$$\varphi_2\left(\varphi_1(S)\right) \subseteq \varphi_2\left(\varphi_1(T)\right)$$

which is $\varphi^*(S) \subseteq \varphi^*(T)$. □

Proposition 16. *If $\mathcal{F}$ and $\mathcal{F}'$ are structure isomorphic and $\varphi : \mathcal{F} \to \mathcal{F}'$ is any structure isomorphism, then $S = T_1 \cup \ldots \cup T_k$ in $\mathcal{F}$ if and only if $\varphi(S) = \varphi(T_1) \cup \ldots \cup \varphi(T_k)$ in $\mathcal{F}'$.*

Proof. Let $\mathcal{F}$ and $\mathcal{F}'$ be structure isomorphic and $\varphi : \mathcal{F} \to \mathcal{F}'$ be a structure isomorphism. Suppose that $S = T_1 \cup \ldots \cup T_k$. This means $S \subseteq T_1 \cup \ldots \cup T_k$ and so by definition of structure isomorphism

$$\varphi(S) \subseteq \varphi(T_1 \cup \ldots \cup T_k) = \varphi(T_1) \cup \ldots \cup \varphi(T_k).$$

We also know that $T_1 \cup \ldots \cup T_k \subseteq S$ and again by definition of structure isomorphism

$$\varphi(T_1) \cup \ldots \cup \varphi(T_k) = \varphi(T_1 \cup \ldots \cup T_k) \subseteq \varphi(S).$$

Thus by double inclusion, $\varphi(S) = \varphi(T_1) \cup \ldots \cup \varphi(T_k)$. □

Another idea related to structure isomorphism is element isomorphism. Let $\mathcal{F}$ and $\mathcal{F}'$ be union-closed families on ground sets $X_{\mathcal{F}}$ and $X_{\mathcal{F}'}$ respectively which satisfy $|X_{\mathcal{F}}| = |X_{\mathcal{F}'}|$. Further consider bijective map $\varphi : X_{\mathcal{F}} \to X_{\mathcal{F}'}$. For $S \in \mathcal{F}$, let $\varphi(S) := \{\varphi(s) : s \in S\}$ and let $\varphi(\mathcal{F}) := \{\varphi(S) : S \in \mathcal{F}\}$. With this notation, we say that φ is an *element isomorphism* if $\mathcal{F}' = \varphi(\mathcal{F})$. If such a function exists, then we say $\mathcal{F}$ and $\mathcal{F}'$ are *element isomorphic*. Intuitively, we can say that element isomorphisms take a family and relabel it get another family and that two families are element isomorphic if they differ only in the labelling of their elements.

Proposition 17. *The property of being element isomorphic forms an equivalence relation.*

Proof. First, we show this property is reflexive. For union-closed family $\mathcal{F}$, consider the identity function $\mathrm{id} : X_{\mathcal{F}} \to X_{\mathcal{F}}$. Then we have $\mathrm{id}(\mathcal{F}) = \mathcal{F}$, so id is an element isomorphism here.

Next, we show this property is symmetric. Suppose that $\mathcal{F}$ is element isomorphic to $\mathcal{F}'$ and let $\varphi : X_{\mathcal{F}} \to X_{\mathcal{F}'}$ be an element isomorphism. We need to show φ^{-1} is an element isomorphism. Since $\varphi(\mathcal{F}) = \mathcal{F}'$, we can just take the inverse of both sides to get

$$\mathcal{F} = \varphi^{-1}\left(\varphi(\mathcal{F})\right) = \varphi^{-1}(\mathcal{F}').$$

Since $\varphi^{-1}(\mathcal{F}') = \mathcal{F}$, then φ^{-1} is an element isomorphism.

Finally, we show this property is transitive. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are element isomorphic, $\mathcal{G}$ and $\mathcal{H}$ are element isomorphic, and $\varphi_1 : X_{\mathcal{F}} \to X_{\mathcal{G}}$ and $\varphi_2 : X_{\mathcal{G}} \to X_{\mathcal{H}}$ are element isomorphisms. We can show that $\varphi^* : X_{\mathcal{F}} \to X_{\mathcal{H}}$ given by $\varphi^* = \varphi_2 \circ \varphi_1$ is an element isomorphism. Since $\varphi_1(\mathcal{F}) = \mathcal{G}$ and $\varphi_2(\mathcal{G}) = \mathcal{H}$, then

$$\varphi^*(\mathcal{F}) = \varphi_2(\varphi_1(\mathcal{F})) = \varphi_2(\mathcal{G}) = \mathcal{H}$$

as desired. $\square$

One powerful property of element isomorphisms which is not shared by structure isomorphisms is that element isomorphisms preserve frequencies of elements. This property is not shared by structure isomorphisms in general since two families may be structure isomorphic but not have equal sized ground sets.

Lemma 10. *Element isomorphisms preserve frequency of elements.*

Proof. Let $\mathcal{F}$ and $\mathcal{F}'$ be element isomorphic families and let $\varphi : X_{\mathcal{F}} \to X_{\mathcal{F}'}$ be an element isomorphism. For any $a \in X_{\mathcal{F}}$, any set in $\mathcal{F}'$ containing $\varphi(a)$ has a corresponding set in $\mathcal{F}$ containing a. Thus $|\mathcal{F}_a| \leq |\mathcal{F}'_a|$. The proof of Proposition 17 gives that φ^{-1} is also an element isomorphism and so the same argument gives $|\mathcal{F}'_{\varphi(a)}| \leq |\mathcal{F}_a|$. Thus $|\mathcal{F}_a| = |\mathcal{F}'_{\varphi(a)}|$ for any $a \in X_{\mathcal{F}}$. □

Lemma 11. *In order for two union-closed families $\mathcal{F}$ and $\mathcal{G}$ to be element isomorphic, the multisets of frequencies of their elements must be equal. In other words,* $\{|\mathcal{F}_i| : i \in X_{\mathcal{F}}\} = \{|\mathcal{G}_i| : i \in X_{\mathcal{G}}\}$.

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be element isomorphic families, $\varphi : X_{\mathcal{F}} \to X_{\mathcal{G}}$ be an element isomorphism, and $P_{\mathcal{F}}$ and $P_{\mathcal{G}}$ be the multisets defined by

$$P_{\mathcal{F}} = \{|\mathcal{F}_i| : i \in X_{\mathcal{F}}\}$$

and

$$P_{\mathcal{G}} = \{|\mathcal{G}_i| : i \in X_{\mathcal{G}}\}.$$

For any $x \in X_{\mathcal{F}}$, Lemma 10 gives that $|\mathcal{F}_x| = |\mathcal{G}_{\varphi(x)}|$. Since φ is bijective, this implies that $P_{\mathcal{F}} = P_{\mathcal{G}}$. □

Proposition 18. *The property of satisfying Frankl's conjecture is preserved under element isomorphism.*

Proof. The definition of element isomorphism gives that element isomorphic families contain the same number of elements. This fact combined with element isomorphism preserving frequency of elements means that if $\mathcal{F}$ and $\mathcal{F}'$ are element isomorphic and $\varphi : X_{\mathcal{F}} \to X_{\mathcal{F}'}$ is an isomorphism, then $a \in X_{\mathcal{F}}$ is abundant in $\mathcal{F}$ if and only if $\varphi(a)$ is abundant in $X_{\mathcal{F}'}$. □

Both element and structure isomorphism are explored in [18]. The authors describe an algorithm which for fixed n, enumerates all element isomorphic and structure isomorphic union-closed set families on ground set $[n]$. They found that for $n = 7$, there are 2,796,163,091,470,050 structure isomorphic union-closed families and 14,087,647,703,920,103,947 element isomorphic union-closed families. Even for this relatively small n, this is approximately 5,038 times as many element isomorphic union-closed sets. On the OEIS [19], these results for element and structure isomorphic families are sequences (A102894) and (A108798).

The fact that there are far more structure isomorphic families than element isomorphic ones suggests that being element isomorphic is a strictly stronger condition than being structure isomorphic. We show that being element isomorphic implies structure isomorphic, but the opposite direction does not hold in general.

Proposition 19. *If two union-closed families are element isomorphic, then they are structure isomorphic.*

Proof. Let $\mathcal{F}$ and $\mathcal{F}'$ be two element isomorphic families and let $\varphi : X_{\mathcal{F}} \to X_{\mathcal{F}'}$ be an element isomorphism. Now consider function $\varphi^* : \mathcal{F} \to \mathcal{F}'$ given by $\varphi^*(\{s_1, s_2, \ldots, s_k\}) = \{\varphi(s_1), \varphi(s_2), \ldots, \varphi(s_k)\}$ for any $S = \{s_1, s_2, \ldots, s_k\} \in \mathcal{F}$. Since φ is bijective, then φ^* is bijective. For $X, Y \in \mathcal{F}$, we have

$$\begin{aligned}
\varphi^*(X) \cup \varphi^*(Y) &= \{\varphi(x) : x \in X\} \cup \{\varphi(y) : y \in Y\} \\
&= \{\varphi(z) : z \in X \cup Y\} \\
&= \varphi^*(X \cup Y).
\end{aligned}$$

Now consider $S, T \in \mathcal{F}$ with $S \subseteq T$ and write $S = \{s_1, \ldots, s_k\}$ and $T = \{s_1, \ldots, s_k, t_1, \ldots, t_\ell\}$. Then we have $\varphi^*(S) = \{\varphi(s_1), \ldots, \varphi(s_k)\}$ and $\varphi^*(T) = \{\varphi(s_1), \ldots, \varphi(s_k), \varphi(t_1), \ldots, \varphi(t_\ell)\}$, so $\varphi^*(S) \subseteq \varphi^*(T)$. Thus, φ^* is a structure isomorphism. $\square$

Proposition 20. *Two union-closed families being structure isomorphic does not guarantee that they are element isomorphic.*

Proof. Consider the pair of families in Figure 5. The families $\mathcal{F}$ and $\mathcal{G}$ are not element isomorphic since they have ground sets of different sizes. However, the families are structure isomorphic

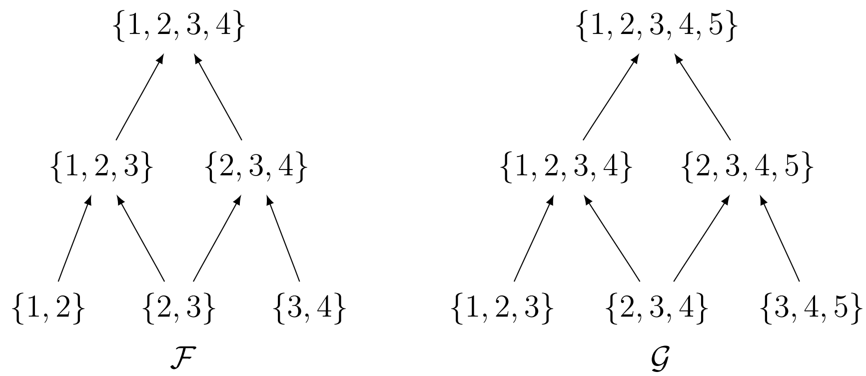

Figure 5: $\mathcal{F}$ and $\mathcal{G}$ are structure isomorphic, but not element isomorphic.

via structure isomorphism $\varphi : \mathcal{F} \to \mathcal{G}$ given by $\varphi(\{1,2\}) = \{2,3\}$, $\varphi(\{2,3\}) = \{2,3,4\}$, $\varphi(\{3,4\}) = \{3,4,5\}$, $\varphi(\{1,2,3\}) = \{1,2,3,4\}$, $\varphi(\{2,3,4\}) = \{2,3,4,5\}$, and $\varphi(\{1,2,3,4\}) = \varphi(\{1,2,3,4,5\})$. □

Notice that one "trivial" way to create counterexamples to the converse of Proposition 19 is to start with two structure isomorphic families both on $[n]$ and then add a new element $x \notin [n]$ to every set of one of the families. The resulting families are still structure isomorphic but no longer element isomorphic since their ground sets differ in size. It is thus natural to ask what conditions are needed for two structure isomorphic families to be element isomorphic. It turns out that simply having equal sized ground sets is not sufficient.

Proposition 21. *Two union-closed families being structure isomorphic and having the same ground set doesn't imply that they are element isomorphic.*

Proof. Consider the pair of families in Figure 6. Lemma 10 tells us that element isomorphisms preserve frequency, so we need that the multisets of frequencies of elements in these families are equal. We have

$$P_{\mathcal{F}} = \{|\mathcal{F}_i| : i \in [6]\} = \{3,3,5,5,6,6\}$$

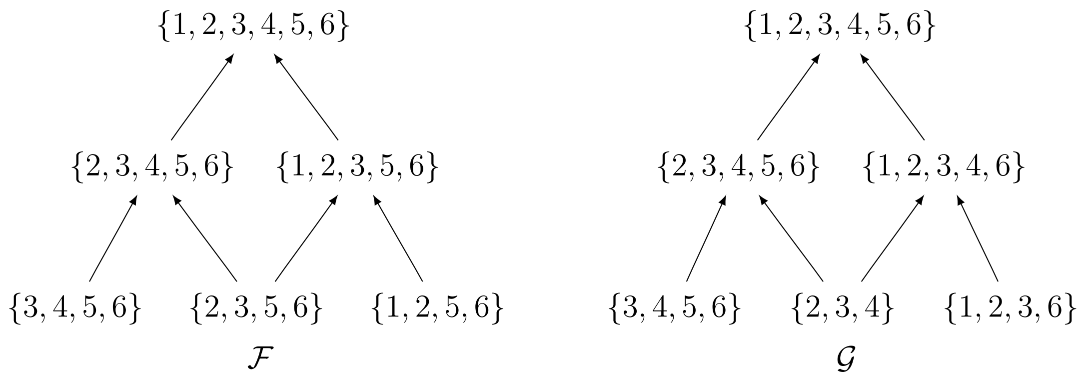

Figure 6: $\mathcal{F}$ and $\mathcal{G}$ are structure isomorphic, but not element isomorphic.

and

$$P_{\mathcal{G}} = \{|\mathcal{G}_i| : i \in [6]\} = \{3, 3, 5, 5, 5, 6\}.$$

Since $P_{\mathcal{F}} \neq P_{\mathcal{G}}$, Lemma 11 gives that $\mathcal{F}$ is not element isomorphic to $\mathcal{G}$. We can see that $\mathcal{F}$ and $\mathcal{G}$ are structure isomorphic since $\varphi : \mathcal{F} \to \mathcal{G}$ given by $\varphi(\{3, 4, 5, 6\}) = \{3, 4, 5, 6\}$, $\varphi(\{2, 3, 5, 6\}) = \{2, 3, 4\}$, $\varphi(\{1, 2, 5, 6\}) = \{1, 2, 3, 6\}$, $\varphi(\{2, 3, 4, 5, 6\}) = \{2, 3, 4, 5, 6\}$, $\varphi(\{1, 2, 3, 5, 6\}) = \{1, 2, 3, 4, 6\}$, $\varphi(\{1, 2, 3, 4, 5, 6\}) = \{1, 2, 3, 4, 5, 6\}$ is a structure isomorphism. Thus, $\mathcal{F}$ and $\mathcal{G}$ are structure isomorphic but not element isomorphic. $\square$

Question 4. *What sufficient conditions exist for two structure isomorphic families on the same ground set to be element isomorphic?*

By Proposition 19, we know that being structure isomorphic is a weaker condition than element isomorphism, so if a property is preserved by structure isomorphisms, it is also preserved by element isomorphism. Thus, the following results that we show for structure isomorphic families are also true for element isomorphic families.

Lemma 12. *If $\mathcal{F}$ and $\mathcal{F}'$ are structure isomorphic, then they have the same height.*

Proof. Let $\mathcal{F}$ and $\mathcal{F}'$ be structure isomorphic union-closed families. Let $S_1 \subseteq S_2 \subseteq \ldots \subseteq S_k$ be a maximum chain in $\mathcal{F}$. Let φ be a structure isomorphism between $\mathcal{F}$ and $\mathcal{F}'$. Then $\varphi(S_1) \subseteq$

$\varphi(S_2) \subseteq \ldots \subseteq \varphi(S_k)$ is a chain in $\mathcal{F}'$. There cannot be a larger chain in $\mathcal{F}'$ since if we had $T_1 \subseteq T_2 \subseteq \ldots \subseteq T_\ell$ where $\ell > k$, then by definition of structure isomorphism, $\varphi^{-1}(T_1) \subseteq \varphi^{-1}(T_2) \subseteq \ldots \subseteq \varphi^{-1}(T_\ell)$ would be a chain in $\mathcal{F}$, a contradiction. □

Lemma 13. *If $\mathcal{F}$ and $\mathcal{F}'$ are structure isomorphic, then they have the same width.*

Proof. Let $\mathcal{F}$ and $\mathcal{F}'$ be structure isomorphic union-closed families. Let $S_1, S_2, \ldots, S_k$ be a maximum antichain in $\mathcal{F}$. Let φ be a structure isomorphism between $\mathcal{F}$ and $\mathcal{F}'$. Then $\varphi(S_i) \not\subseteq \varphi(S_j)$ for each pair $i \neq j$ where $i, j \leq k$. There cannot be a larger antichain in $\mathcal{F}'$ since if we had antichain $T_1, T_2, \ldots, T_\ell$ where $\ell > k$, then by definition of structure isomorphism, $\varphi^{-1}(T_i) \not\subseteq \varphi^{-1}(T_j)$ for each pair $i \neq j$ where $i, j \leq \ell$ and so $T_1, T_2, \ldots, T_\ell$ would be an antichain in $\mathcal{F}$, a contradiction. □

Lemma 14. *If $\mathcal{F}$ and $\mathcal{F}'$ are structure isomorphic, then any structure isomorphism between them maps minimal sets to minimal sets.*

Proof. Let $\mathcal{F}$ and $\mathcal{F}'$ be structure isomorphic and $\varphi : \mathcal{F} \to \mathcal{F}'$ be a structure isomorphism. Suppose for contradiction that M is a minimal set in $\mathcal{F}$ and $\varphi(M)$ is not a minimal set in $\mathcal{F}'$. This means that there exists $S \in \mathcal{F}'$ with $S \subset \varphi(M)$ and so $\varphi^{-1}(S) \subset M$, contradicting that M was minimal in $\mathcal{F}$. □

Theorem 17. *If Conjecture 4 is true, $\mathcal{F}$ and $\mathcal{F}'$ are structure isomorphic, and $\mathcal{M}$ is the collection of minimal sets of $\mathcal{F}$, then $|\mathcal{M}| \leq \min\{\phi(\mathcal{F}), \phi(\mathcal{F}')\}$ and $\mathrm{w}(\mathcal{F}) \leq \min\{\phi(\mathcal{F}), \phi(\mathcal{F}')\}$.*

Proof. Let $\mathcal{F}$ and $\mathcal{F}'$ be structure isomorphic families and let $\mathcal{M}$ and $\mathcal{M}'$ be their respective collections of minimal sets. Lemmas 13 and 14 tell us that $\mathrm{w}(\mathcal{F}) = \mathrm{w}(\mathcal{F}')$ and $|\mathcal{M}| = |\mathcal{M}'|$. We also know from Proposition 8 that $|\mathcal{M}| \leq \mathrm{w}(\mathcal{F})$. Thus we just need to show $\mathrm{w}(\mathcal{F}) \leq \min\{\phi(\mathcal{F}), \phi(\mathcal{F}')\}$ to complete the proof. If 4 is true, then $\mathrm{w}(\mathcal{F}) \leq \phi(\mathcal{F})$ and $\mathrm{w}(\mathcal{F}') \leq \phi(\mathcal{F}')$. Since $\mathrm{w}(\mathcal{F}) = \mathrm{w}(\mathcal{F}')$, then $\mathrm{w}(\mathcal{F}) \leq \phi(\mathcal{F}')$. Therefore we can combine inequality to obtain

$$|\mathcal{M}| \leq \mathrm{w}(\mathcal{F}) \leq \min\{\phi(\mathcal{F}), \phi(\mathcal{F}')\}$$

as desired. $\square$

Proposition 22. *If $\mathcal{F}$ and $\mathcal{F}'$ are structure isomorphic, then $\mathcal{F}$ is minimally-generated if and only if $\mathcal{F}'$ is minimally-generated.*

Proof. Let $\mathcal{F}$ and $\mathcal{F}'$ be structure isomorphic and $\varphi : \mathcal{F} \to \mathcal{F}'$ be a structure isomorphism. Suppose for contradiction that $\mathcal{F}$ is minimally-generated and $\mathcal{F}'$ is not. Let $\mathcal{M}$ be the collection of minimal sets in $\mathcal{F}$ and let S be a defect in $\mathcal{F}'$. Since $\mathcal{F}$ is minimally-generated, we can write $\varphi^{-1}(S) = M_1 \cup M_2 \cup \ldots \cup M_k$ for $M_1, \ldots, M_k \in \mathcal{M}$. By Proposition 16, this means that $S = \varphi(M_1) \cup \ldots \cup \varphi(M_k)$. By Proposition 14, each $\varphi(M_i)$ is minimal in $\mathcal{F}'$, so we have written S as a union of minimal sets in $\mathcal{F}'$, contradicting that S was a defect. $\square$

Lemma 15. *If $\mathcal{F}$ and $\mathcal{F}'$ are structure isomorphic, $\varphi : \mathcal{F} \to \mathcal{F}'$ is a structure isomorphism, and $A \in \mathcal{F}$ such that $\mathcal{F} \setminus A$ is union-closed, then $\mathcal{F} \setminus A$ and $\mathcal{F}' \setminus \varphi(A)$ are structure isomorphic.*

Proof. Let $\mathcal{F}$ and $\mathcal{F}'$ be structure isomorphic, $\varphi : \mathcal{F} \to \mathcal{F}'$ be a structure isomorphism, and $A \in \mathcal{F}$ such that $\mathcal{F} \setminus A$ is union-closed. Let φ^* be the function formed by restricting φ's domain to $\mathcal{F} \setminus A$. Clearly, φ^* is a bijection since φ is. We have $\varphi^*(S \cup T) = \varphi(S \cup T) = \varphi(S) \cup \varphi(T) = \varphi^*(S) \cup \varphi^*(T)$, so $\varphi^*(S \cup T) = \varphi^*(S) \cup \varphi^*(T)$. We also have that if $S \subseteq T$ in $\mathcal{F} \setminus A$, then $\varphi^*(S) = \varphi(S) \subseteq \varphi(T) = \varphi^*(T)$, so $\varphi^*(S) \subseteq \varphi^*(T)$. Thus φ^* is a structure isomorphism between $\mathcal{F} \setminus A$ and $\mathcal{F}' \setminus \varphi(A)$. $\square$

Theorem 18. *If $\mathcal{F}$ and $\mathcal{F}'$ are structure isomorphic and $D(\mathcal{F}) = k$, then $D(\mathcal{F}') = k$.*

Proof. Suppose that $\mathcal{F}$ and $\mathcal{F}'$ are structure isomorphic and let $\varphi : \mathcal{F} \to \mathcal{F}'$ be a structure isomorphism. We prove the statement via induction on $D(\mathcal{F})$. Our base case is when $k = 1$ and this was shown in Proposition 22. Our induction hypothesis is that the claim is true for defectiveness values less than $D(\mathcal{F})$.

Suppose for contradiction that $D(\mathcal{F}) = k$, but $D(\mathcal{F}') \neq k$. Without loss of generality, suppose $D(\mathcal{F}') = \ell < k$. Let $\mathcal{M}$ and $\mathcal{M}'$ be the collection of minimal sets of $\mathcal{F}$ and $\mathcal{F}'$ respectively. We know from the definition of defect-free that $D(\mathcal{F} \setminus \mathcal{M}) = k - 1$ and $D(\mathcal{F}' \setminus \mathcal{M}') = \ell - 1$. We know by Lemmas 2 and 15 that for $M_i \in \mathcal{M}$, $\mathcal{F} \setminus M_i$ is structure isomorphic to $\mathcal{F}' \setminus \varphi(M_i)$. Thus, after subtracting each $\mathcal{M}_i \in \mathcal{M}$ and $\varphi(M_i)$ from $\mathcal{F}$ and $\mathcal{F}'$, we are left with two structure isomorphic families. By Proposition 14, these families are $\mathcal{F} \setminus \mathcal{M}$ and $\mathcal{F}' \setminus \mathcal{M}'$. The induction hypothesis tells us that since $D(\mathcal{F} \setminus \mathcal{M}) = k - 1$ and $\mathcal{F} \setminus \mathcal{M}$ is structure isomorphic to $\mathcal{F}' \setminus \mathcal{M}'$, then $D(\mathcal{F}' \setminus \mathcal{M}') = k - 1$ as well. However, we said that $D(\mathcal{F}' \setminus \mathcal{M}') = \ell - 1 < k - 1$, a contradiction. □

We'll conclude our discussion of structure isomorphism by stating a conjecture equivalent to Frankl's conjecture which uses structure isomorphisms.

Conjecture 6. *If $\mathcal{F}$ and $\mathcal{F}'$ are structure isomorphic, then $\mathcal{F}$ satisfies Frankl's conjecture if and only if $\mathcal{F}'$ does.*

To prove that this conjecture is equivalent to Frankl's conjecture, we will use the following lemma.

Lemma 16. *Every union-closed family is structure isomorphic to a family which satisfies Frankl's conjecture.*

Proof. Let $\mathcal{F}$ be a union-closed family. Select an element y not in X. Let $\mathcal{F}'$ be the family with ground set $X \cup \{y\}$ given by $\mathcal{F}' = \{S \cup \{y\} : S \in \mathcal{F}\}$. Consider the function $\varphi : \mathcal{F} \to \mathcal{F}'$ given by $\varphi(S) = S \cup \{y\}$. It is easy to see that φ is a bijection since each S in $\mathcal{F}$ corresponds uniquely to $S \cup \{y\}$ in $\mathcal{F}'$ and vice versa. We also see that for $S, T \in \mathcal{F}$, $\varphi(S \cup T) = S \cup T \cup \{y\} = (S \cup \{y\}) \cup (T \cup \{y\}) = \varphi(S) \cup \varphi(T)$. Additionally, for $S \subseteq T$ in $\mathcal{F}$,

$$\varphi(S) = S \cup \{y\} \subseteq T \cup \{y\} = \varphi(T).$$

Therefore φ is a structure isomorphism, so $\mathcal{F}$ and $\mathcal{F}'$ are structure isomorphic. Finally, since $|\mathcal{F}'_y| = |\mathcal{F}'|$, y is abundant in $\mathcal{F}'$, meaning $\mathcal{F}'$ satisfies Frankl's conjecture. □

Theorem 19. *Conjecture 6 is equivalent to Frankl's conjecture.*

Proof. ($\Rightarrow$) Suppose that Conjecture 6 is true and let $\mathcal{F}$ be any union-closed family. Lemma 16 tells us that $\mathcal{F}$ is structure isomorphic to a family that satisfies Frankl's conjecture. Conjecture 6 thus gives that $\mathcal{F}$ satisfies Frankl's conjecture.

($\Leftarrow$) Suppose that Frankl's conjecture is true. This implies that any structure isomorphism between two union-closed families is a structure isomorphism between two families which satisfy Frankl's conjecture. $\square$

A final way in which we can transform set families is to find the *expanded family*. The goal of expanding a family is to create a family which has disjoint minimal sets. Given a (not necessarily union-closed) family $\mathcal{F}$, the expanded family $\mathcal{E}(\mathcal{F})$, is a (not necessarily union-closed) family constructed as follows by "expanding" the sets of $\mathcal{F}$. For each $x \in X$, let $\mathcal{M}_x$ be the collection of minimal sets in $\mathcal{F}_x$ and define a the subfamily $\mathcal{E}_x$ using ordered pairs as follows $\mathcal{E}_x = \{(x, M) : M \in \mathcal{M}_x\}$. For each $S \in \mathcal{F}$ and $x \in S$, the expansion of S with respect to x is given by $\mathcal{E}_x(S) = \{(x, M) : M \in \mathcal{M}_x \text{ and } M \subseteq S\}$. Finally, for each set $S \in \mathcal{F}$, we define the set of all variables related to the elements of S with $\mathcal{E}(S) = \bigcup_{x \in S} E_x(S)$. Using these expanded sets, we have that $\mathcal{E}(\mathcal{F}) = \{\mathcal{E}(S) : S \in \mathcal{F}\}$ and we can note that $X_{\mathcal{E}(\mathcal{F})} = \bigcup_{x \in X} \mathcal{E}_x$. Even though expanded families don't necessarily preserve being union-closed, they do preserve other aspects of the family.

Proposition 23. *For any family of sets $\mathcal{F}$, $|\mathcal{F}| = |\mathcal{E}(\mathcal{F})|$.*

Proof. Let $\mathcal{F}$ be a set family and $\mathcal{E}(\mathcal{F})$ be its expansion. Since every set in $\mathcal{E}(\mathcal{F})$ has the form $\mathcal{E}(S)$ for some set in $\mathcal{F}$, we just need to show that for different sets $S, T \in \mathcal{F}$, their images $\mathcal{E}(S), \mathcal{E}(T)$ are different. We can see this is true since due to S and T being different, we can consider without loss of generality some $x \in S \setminus T$. Then $(x, M) \in \mathcal{E}(S)$ for some minimal set $M \in \mathcal{F}_x$ but since $x \notin T$, then no element of T has the form $(x, \cdot)$ (a pair whose first component is x). Therefore $\mathcal{E}(S) \neq \mathcal{E}(T)$. $\square$

Proposition 24. *The expansion map $\mathcal{E}$ is monotone with respect to set-inclusion. That is, if $\mathcal{F}$ is a set family, then $S, T \in \mathcal{F}$ and $S \subseteq T$ implies $\mathcal{E}(S) \subseteq \mathcal{E}(T)$.*

Proof. Let $\mathcal{F}$ be a set family, $S, T \in \mathcal{F}$ be such that $S \subseteq T$. Consider $(x, M) \in \mathcal{E}(S)$ where $x \in X$ and M is a minimal set in $\mathcal{F}_x$. By our construction, $x \in M$ and $(x, M) \in \mathcal{E}(S)$ implies that $M \subseteq S$. Therefore, $M \subseteq T$, $x \in T$, and $(x, M) \in \mathcal{E}(T)$. □

Corollary 3. *For any family of sets $\mathcal{F}$ and expansion $\mathcal{E}(\mathcal{F})$ of $\mathcal{F}$, if M is minimal in $\mathcal{F}$ then $\mathcal{E}(M)$ is minimal in $\mathcal{F}'$.*

Proof. Let $\mathcal{F}$ be a set family with expansion $\mathcal{E}(\mathcal{F})$, let $\mathcal{M}$ be the collection of minimal sets in $\mathcal{F}$, and let $M \in \mathcal{M} \setminus \{\varnothing\}$. $\mathcal{E}(M)$ is the union of sets $\{(x, M') : x \in M, M' \in \mathcal{M}_x, M' \subseteq M\}$. But, for any set M' with $M' \subseteq M$, the minimality of M implies that $M = M'$. Thus $\mathcal{E}(M) = \{(x, M) : x \in M\}$. If there were a set $\mathcal{E}(S) \subseteq \mathcal{E}(M)$ it would have to have the form $\{(x, M) : x \in S, S \subseteq M\}$. Since $S \subseteq M$ implies $S = M$, the only set of the required form is $\mathcal{E}(M)$ itself. □

Proposition 25. *The minimal sets in the expansion $\mathcal{E}(\mathcal{F})$ of a family of sets $\mathcal{F}$ are disjoint.*

Proof. Let $\mathcal{F}$ be a set family with expansion $\mathcal{E}(\mathcal{F})$. Let $\mathcal{E}(S)$ and $\mathcal{E}(T)$ be minimal sets in $\mathcal{E}(F)$ (with $S, T \in \mathcal{F}$ by construction). Suppose S and T have a common element (x, M). $(x, M) \in \mathcal{E}(S)$ means that $x \in S$ and that M is minimal in $\mathcal{F}_x$ and that $M \subseteq S$ (and also that $x \in M$). Similarly $M \subseteq T$.

Since $\mathcal{E}$ is monotone, we have both that $\mathcal{E}(M) \subseteq \mathcal{E}(S)$ and $\mathcal{E}(M) \subseteq \mathcal{E}(T)$. Since $\mathcal{E}(S)$ and $\mathcal{E}(T)$ are minimal it follows that $\mathcal{E}(M) = \mathcal{E}(S)$ and $\mathcal{E}(M) = \mathcal{E}(T)$. That is, $\mathcal{E}(S) = \mathcal{E}(T)$; in other words, two minimal sets in $\mathcal{E}(\mathcal{F})$ which have a common element are in fact identical—and any two different minimal sets must have empty intersection. Thus minimal sets in $\mathcal{E}(\mathcal{F})$ are disjoint. □

Corollary 4. *If $\mathcal{F}$ is a set family whose minimal sets are disjoint, M is a minimal element of $\mathcal{F}$ containing x, then M is a subset of every set containing x.*

Proof. Let $\mathcal{F}$ be a family of sets whose minimal elements are disjoint, and M be a minimal element of $\mathcal{F}$ containing x. Let S be a set in $\mathcal{F}$ containing x. If S is minimal it must be M. If it is not minimal then it has a proper subset S_1 containing x. This process can be continued a finite number of times yielding sets $S, S_1 \ldots, S_k$ all containing x. If this chain is maximum then S_k must be a minimal set containing x and since the minimal sets are disjoint S_k must be M. Then M is a subset of each of $S_k, S_{k-1}, \ldots, S_1, S$. □

7 Computational Results

One valuable tool when studying union-closed families is finding ways to generate interesting families in order to either develop counterexamples or discover new properties of families. Here we will highlight some preexisting constructions of union-closed families and present some new ones.

One type of family, described in [1] and based on a construction from [20] is called the Hungarian family. Consider the collection $\mathbb{N}^{(<\omega)}$ of finite subsets of the positive integers. Now order this collection using the order $<_H$ where for $A, B \subseteq \mathbb{N}$, $A <_H B$ if

- $\max A < \max B$ or
- $\max A = \max B$ and $\max (A \Delta B) \in A$

where $A \Delta B$ is the symmetric difference of A and B. In [1], $<_H$ is equivalently described as the order defined by first sorting by increasing largest element and then by reverse colex order. The Hungarian family $\mathcal{H}^{(n)}$ is then defined as the initial segment of length n of $\mathbb{N}^{(<\omega)}$ under this order $<_H$. As noted in [1], the family $\mathcal{H}^{(n)}$ has ground set $\left[\lceil \log_2(n) \rceil\right]$. We can thus construct $\mathcal{H}^{(n)}$ with a simple algorithm by starting with the power set on $\left[\lceil \log_2(n) \rceil\right]$, sorting it using the order $<_H$, and taking the initial segment of n sets of the result.

Hungarian families are useful when trying to study the average frequency of elements in union-closed families. Notice that a union-closed family $\mathcal{F}$ must have an element of frequency

$1/2|\mathcal{F}|$ if the average frequency of elements in $\mathcal{F}$ is at least $1/2|\mathcal{F}|$. Symbolically, this is

$$\frac{1}{n}\sum_{x\in X}|\mathcal{F}_x| \geq \frac{1}{2}|\mathcal{F}|.$$

However, this is only a sufficient condition for satisfying Frankl's conjecture, not a necessary one, and the following result from [1] demonstrates this.

Theorem 20. *For integer $m > 1$ and $n = \lfloor \frac{2}{3}2^m \rfloor$, the Hungarian family $\mathcal{H}^{(n)}$ satisfies*

$$\frac{1}{m}\sum_{i\in[m]}\left|\mathcal{H}_i^{(n)}\right| < \frac{|\mathcal{H}^{(n)}|}{2}.$$

Another family, introduced in [21] and further described in [1] is called the Renaud-Fitina family. Like the Hungarian family, we define this family via an order on the collection $\mathbb{N}^{(<\omega)}$ of finite subsets of the positive integers. We order this collection using the order $<_R$ where for $A, B \subseteq \mathbb{N}$, $A <_R B$ if

- $\max A < \max B$ or
- $\max A = \max B$ and $|A| > |B|$ or
- $\max A = \max B$ and $|A| = |B|$ and $\max(A\Delta B) \in B$

where $A\Delta B$ again represents the symmetric difference of A and B. In [1], $<_R$ is equivalently described as the order defined by first sorting by increasing largest element, then by decreasing cardinality, and then by colex order. The Renaud-Fitina family $\mathcal{R}^{(n)}$ is then defined as the initial segment of length n of $\mathbb{N}^{(<\omega)}$ under this order $<_R$. We can construct $\mathcal{R}^{(n)}$ using a similar method to the construction of $\mathcal{H}^{(n)}$. The authors of [21] note that the ground set of $\mathcal{R}^{(n)}$ is $[k]$ where k is such that $2^{k-1} \leq n < 2^k$. Thus, we can construct $\mathcal{R}^{(n)}$ by finding k, constructing the power set on $[k]$, sorting it using the order $<_R$, and taking the first n sets of the result. A comparison of the orders $<_H$ and $<_R$ is shown in Figure 7 and a comparison of the families $\mathcal{H}^{(20)}$ and $\mathcal{R}^{(20)}$ is shown in Figure 8.

$<_H$:	$\varnothing$,	1,	12,	2,	123,	**23**,	13,	3,	1234,	234 ...
	134,	34,	124,	24,	14	4,	12345,	2345,	1345,	345 ...
	1245,	245,	145,	45,	1235,	235,	135,	35,	125,	25 ...
	15,	5,	123456,	23456,	13456,	3456,	12456,	2456,	1456,	456
$<_R$:	$\varnothing$,	1,	12,	2,	123,	**13**,	23,	3,	1234,	124 ...
	134,	234,	14,	24,	34,	4,	12345,	1235,	1245,	1345 ...
	2345,	125,	135,	235,	145,	245,	345,	15,	25,	35 ...
	45,	5,	123456,	12346,	12356,	12456,	13456,	23456,	1236,	1246

Figure 7: The first 40 sets of $\mathbb{N}^{(<\omega)}$ arranged via the orders $<_H$ and $<_R$ with 10 sets in each row. The places where these orders first differ are indicated in bold.

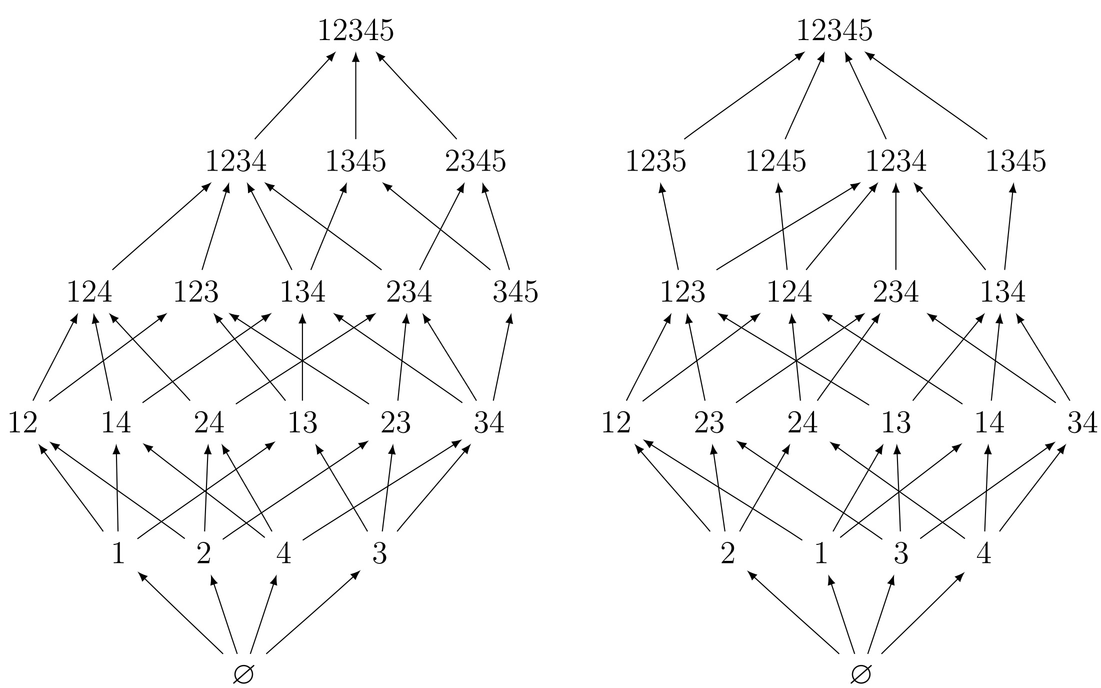

Figure 8: The families $\mathcal{H}^{(20)}$ (left) and $\mathcal{R}^{(20)}$ (right).

The usefulness of this family appears when viewing Frankl's conjecture as an extremal problem since the Renaud-Fitina families allow us to find an upper bound of $\phi(n)$. Let $a(n)$ be the sequence defined by $a(1) = a(2) = 1$ and by

$$a(n) = a(a(n-1)) + a(n - a(n-1))$$

for $n \geq 2$. This sequence is sometimes called Conway's challenge sequence (sequence A004001 in the OEIS) and was shown in [22] to satisfy $a(n) \geq n/2$. This means if we can show that $\phi(n) \geq a(n)$, then we prove Frankl's conjecture. However, the Renaud-Fitina families are a counterexample to this.

Theorem 21 (Renaud and Fitina)**.** *For every $n \geq 2$, the most frequent element of $\mathcal{R}(n)$ has frequency $a(n)$, meaning*

$$\phi(n) \leq a(n).$$

Furthermore, Renaud showed in [23] that there exists families with $\phi(n) < a(n)$. One simple family they provided is

$$\mathcal{B}(23) = \mathcal{P}(4) \cup \{12345,\ 1235,\ 1245,\ 1345,\ 2345,\ 125,\ 345\}.$$

This family $\mathcal{B}(23)$, shown in Figure 9 contains 23 sets and has $\phi(\mathcal{B}(23)) = 13$, but Conway's sequence has $a(23) = 14$, so $\phi(23) < a(23)$. This family is part of a class of families $\mathcal{B}(n)$ defined in [23]. The following is an explanation of how to construct these families. First, write n as

$$n = 2^k - \sum_{i=0}^{r-1} \binom{k-1}{i} - v$$

where $1 \leq r < k-1$ and $0 \leq v < \binom{k-1}{r}$. The values of k, r, and v can be found via a greedy algorithm. We will construct $\mathcal{B}(n)$ by beginning with the family $\mathcal{F}(k) = \{A \cup \{k\} \mid A \in \mathcal{P}(k-1)\}$ and removing $\sum_{i=0}^{r-1} \binom{k-1}{i} + v$ sets.

Remove the sets from $\mathcal{F}(k)$ with cardinality less than or equal to r, of which there are

$\sum_{i=0}^{r-1} \binom{k-1}{i}$. If $v = 0$, we are done. Otherwise, we need to remove v sets from the collection of size $r+1$ sets in $\mathcal{F}(k)$.

Let $\mathcal{C} = \{A \setminus \{k\} \mid A \in \mathcal{F}(k)^{(r+1)}\}$. We will remove sets from $\mathcal{C}$ and $\mathcal{F}(k)$ by splitting $\mathcal{C}$ into equivalence classes under the following equivalence relation, $\sim$. For $A, B \in \mathcal{C}$ with $A = \{a_1, \ldots, a_r\}$ and $B = \{b_1, \ldots, b_r\}$, $A \sim B$ if $\{a_1 + j, \ldots, a_r + j\} = \{b_1, \ldots, b_r\}$ for some $j \in [s]$ where addition is modulo s. Let $\mathcal{D}_1$ be the equivalence class of $\sim$ which contains the set $\{1, 2, \ldots, r\}$ and let $\mathcal{D}_2, \ldots, \mathcal{D}_m$ be the other equivalence classes of $\sim$.

Let $\mathcal{D} = \{\mathcal{D}_2, \ldots, \mathcal{D}_m\}$. While there exists a $\mathcal{D}_i$ in $\mathcal{D}$ with $|\mathcal{D}_i| \leq |\mathcal{C}|$, remove $\mathcal{D}_i$ from $\mathcal{D}$, remove the sets of $\mathcal{D}_i$ from $\mathcal{C}$, and remove the sets of $\mathcal{F}(k)$ corresponding to the sets of $\mathcal{D}_i$. After this process finishes, we will have removed $v - t$ sets from $\mathcal{F}(k)$ and $\mathcal{C}$ for some $t \leq s$. We finish by removing t sets from $\mathcal{C}$ and $\mathcal{F}(k)$ corresponding to t sets from $\mathcal{D}_1$.

We will decide which sets to remove by taking a segment of size t from an ordering of $\mathcal{D}_1$. Note that $\mathcal{D}_1$ has size s. Lemma 2 of [23] states that there exists an ordering of $\mathcal{D}_1$ such that if the ordering of $\mathcal{D}_1$ is subdivided as ordered into $\gcd(r, s)$ different subclasses, each containing $s/\gcd(r, s)$ sets, then each element has frequency $r/\gcd(r, s)$ in each subclass and if element frequencies across the sets are counted from left to right, the difference between the frequencies of any two elements is at most 1. After finding such an ordering, remove the initial segment of length t from $\mathcal{C}$ and remove the corresponding sets from $\mathcal{F}(k)$.

Call the resulting family $\mathcal{B}(n)$. This family has

$$\phi(\mathcal{B}(n)) = \beta(n) = 2^{k-1} - \sum_{i=0}^{r-2} \binom{k-2}{i} - \left\lfloor \frac{rv}{k-1} \right\rfloor.$$

While this function $\beta(n)$ may look strange, Theorem 4 of [23] states that $\beta(n) \leq a(n)$ for $n \geq 1$ and we already saw with $\beta(23) < a(23)$. Thus, the $\mathcal{B}(n)$ families provide us with a sharper bound on the value of $\phi(n)$. Unfortunately, even this bound is not perfect since as Renaud points out,

$$\mathcal{F} = \mathcal{P}([6]) \setminus \{\{5\}, \{6\}, \{1,6\}, \{2,5\}, \{3,6\}, \{4,5\}, \{1,3,6\}, \{2,4,5\}\}$$

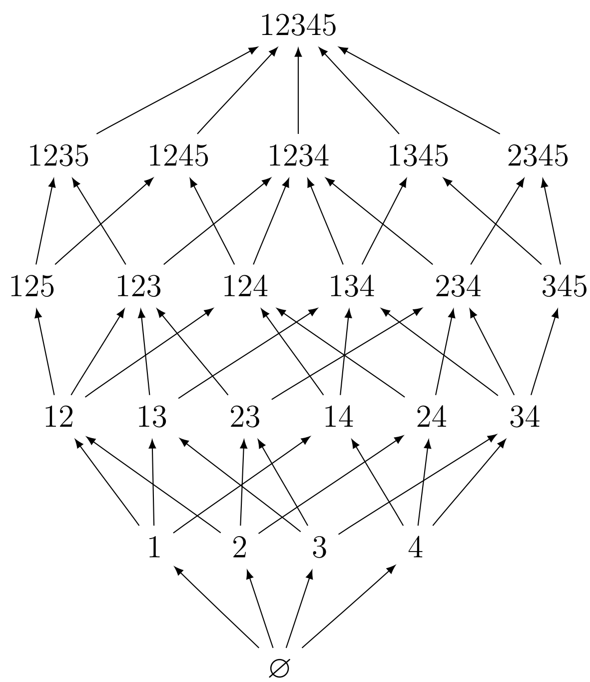

Figure 9: The family $\mathcal{B}(23)$.

is a family with 56 sets and $\phi(\mathcal{F}) = 30$, but $\beta(56) = 31$, so we have $\phi(56) < \beta(56)$.

Next, we describe methods to create union-closed families with any height, any width, and of a range of cardinalities based on the height and width. For these constructions, we will use the sliced power set family as well as two new families: the chain family and the antichain segment. For integers $n \le m$, we define the *chain family* $\mathcal{C}(n,m)$ as the family

$$\mathcal{C}(n,m) = \{[n],[n+1],\ldots,[n+m]\}.$$

It is easy to see that the chain family is union-closed, since if $S,T \in \mathcal{C}(n,m)$, then $S = [n+i]$ and $T = [n+j]$ for some $i,j \in \{0,1,\ldots,m\}$ which means either $S \cup T = S$ or $S \cup T = T$. Notice that $\mathcal{C}(n,m)$ consists of a single chain of size $m+1$.

We also define the *antichain segment* $\mathcal{A}(n,m)$ as follows. Start with the missing-one family $\mathcal{S}(n,n-2)$ and write $\mathcal{S}(n,n-2) = \{[n]\} \cup \mathcal{F}$ (so $\mathcal{F}$ is the collection of size $n-1$ sets in $\mathcal{S}(n,n-2)$). Order $\mathcal{F}$ using the ordering $<_T$ where $A <_T B$ if $\max(A \setminus B) < \max(B \setminus A)$

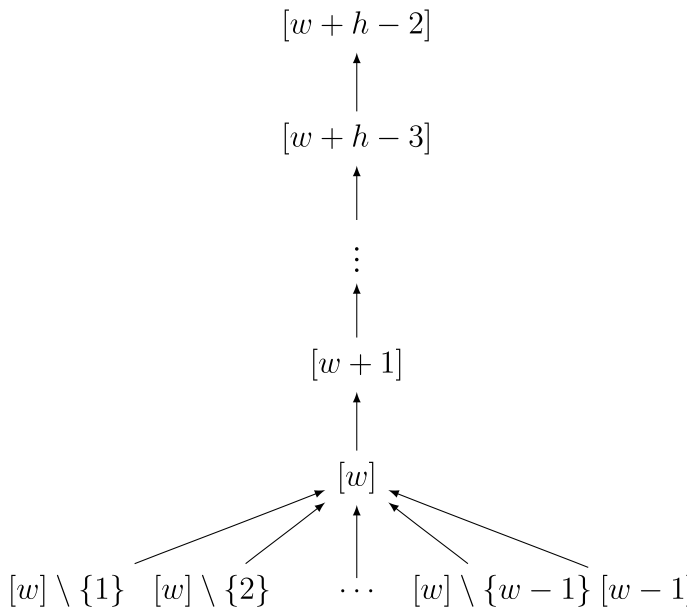

Figure 10: The family $\mathcal{C}(w-1,h-1)\cup\mathcal{A}(w,w)$

and take the initial segment of length m of the result. The resulting (not union-closed) family is $\mathcal{A}(n,m)$. Notice that $\mathcal{A}(n,m)$ consists of a single antichain of size m.

Theorem 22. *If $\mathcal{F}$ is a union-closed family with $\mathrm{h}(\mathcal{F}) = h > 1$ and $\mathrm{w}(\mathcal{F}) = w$, then $|\mathcal{F}| \geq h + w - 1$ and this bound is sharp.*

Proof. In order for a union-closed set family to have height h and width w, it must contain a chain of size h and an antichain of size w. Thus, the smallest possible family with this height and width would be one consisting of just a chain and antichain with the chain and antichain overlapping as much as possible. Recall from Lemma 6 that chains and antichains intersect in at most one set. Thus union-closed family $\mathcal{F}$ with $\mathrm{h}(\mathcal{F}) = h$ and $\mathrm{w}(\mathcal{F})$ satisfies $|\mathcal{F}| \geq h + w - 1$. We show this bound is sharp by constructing the a family of the desired height, width, and size.

Consider the family $\mathcal{F} = \mathcal{C}(w-1,h-1)\cup\mathcal{A}(w,w)$, shown in Figure 10. As noted before, $\mathcal{C}(w-1,h-1)$ is a chain of size h and $\mathcal{A}(w,w)$ is an antichain of size w. We can also see that $\mathcal{C}(w-1,h-1)$ intersect in $[w-1]$. Finally, we show that $\mathcal{F}$ is union-closed. There are three ways to take unions of two sets in $\mathcal{F}$: a union of two sets from $\mathcal{C}(w-1,h-1)$, two sets from $\mathcal{A}(w,w)$, or a set from each. We already explained that $\mathcal{C}(w-1,h-1)$ is union-closed, so we don't need to check that case. If we take the union of any two sets from $\mathcal{A}(w,w)$, we get $[w]$ which is part of $\mathcal{C}(w-1,h-1)$. Finally, if we take a set from $\mathcal{C}(w-1,h-1)$ and a set from $\mathcal{A}(w,w)$, either both sets are $[w-1]$, or one set is a superset of the other. Thus the unions in each case are still in $\mathcal{F}$, so $\mathcal{F}$ is a union-closed family with the desired parameters. □

Now we will describe a construction where that for positive integers h, w, and ℓ with $h > 1$ and $h+w-1 \le \ell \le w(h-1)+1$, creates a union-closed family $\mathcal{F}$ with $\mathrm{h}(\mathcal{F}) = h$, $\mathrm{w}(\mathcal{F}) = w$, and $|\mathcal{F}| = \ell$.

Theorem 23. *For positive integers h, w, ℓ with $h > 1$ and $h+w-1 \le \ell \le w(h-1)+1$, there exists a union-closed family $\mathcal{F}$ with $\mathrm{h}(\mathcal{F}) = h$, $\mathrm{w}(\mathcal{F}) = w$, and $|\mathcal{F}| = \ell$.*

Proof. We will construct the desired family by starting with the family $\mathcal{C}(w-1,h-1) \cup \mathcal{A}(w,w)$ that we used in the proof of Theorem 22 and add sets to it while keeping it union-closed and not increasing its height or width. Start by writing ℓ in the form $\ell = (h+w-1) + k(h-2) + r$ where k is as large as possible. If both k and r are 0, then the family $\mathcal{C}(w-1,h-1) \cup \mathcal{A}(w,w)$ has the desired parameters.

Now consider the case where $k = 0$ and $r > 0$. Here, we will add r sets of size w to the family. We will add the sets $[w+1] \setminus \{w-1\}, [w+1] \setminus \{w-2\}, \ldots [w+1] \setminus \{w-r\}$. This collection is $\mathcal{A}(w+1,r+1) \setminus [w]$. Since $\mathcal{C}(w-1,h-1) \cap \mathcal{A}(w+1,r+1) = [w]$, we can write the resulting family as $\mathcal{C}(w-1,h-1) \cup \mathcal{A}(w,w) \cup \mathcal{A}(w+1,r+1)$. It is easy to check that the resulting family still has height h and width w, so we just need to check that the resulting family is union-closed.

The union of any two sets in $\mathcal{A}(w+1,r+1)$ will be $[w+1]$. The union of a set A from $\mathcal{A}(w+1,r+1)$ with a set C from $\mathcal{C}(w-1,h-1)$ will be whichever or A or C is larger. The union of a set A_1 from $\mathcal{A}(w+1,r+1)$ with a set A_2 from $\mathcal{A}(w,w)$ will either be $[w+1]$ or A_1. Thus the resulting family is union-closed as desired.

Finally, consider the case where $k > 0$ and r may be positive or 0. Here we will add k sets of each of the sizes $w, w+1, \ldots, w+k-1$ to the family and then r sets of size $w+k$. For each $i \in \{0, \ldots, k-1\}$, add $\mathcal{A}(w+i+1, k)$ to the family. The same logic as in the $k = 0$ case tells us that the resulting family is union-closed. Finally, add $\mathcal{A}(w+k+1, r)$ to the family. Again, the logic from the $k = 0$ case tells us that the resulting family is union-closed. $\square$

Conclusion and Future Work

We will now present a summary of the conjectures that we discussed with the purpose of ideally inspiring future research into solving them. Our first original conjecture, Conjecture 2, served as a starting point for our discussion of minimal sets by generalizing the prior ideas of Sarvate and Renaud. In Proposition 10, we further showed that this conjecture being true would allow us to make statements about the location of maximum frequency elements within maximal chains of families. Additionally, if this conjecture is true and there exists a union-closed family such that no minimal set containins an abundant element, then we would have a family where the most frequent element is not abundant, disproving Frankl's conjecture. Conjecture 5 had a similar goal to the chain version of Conjecture 2 of trying to locate maximum frequency elements using antichains. Thus, if both Conjectures 2 and 5 are true, we would have a much better understanding of how maximum frequency elements interact with chains and antichains.

The other three conjectures we discussed can be viewed in terms of structure isomorphisms. For union-closed family $\mathcal{F}$ with collection of minimal sets $|\mathcal{M}|$, Conjectures 3 and 4 both give lower bounds on $\phi(\mathcal{F})$ in terms of $|\mathcal{M}|$ and $\mathrm{w}(\mathcal{F})$ respectively. As shown in Theorem 13, in order to prove Conjecture 4, one would simply need to prove Conjecture 3. Later, we showed in Theorem 17 that if Conjecture 4 is true, we can improve our bound from $\mathrm{w}(\mathcal{F}) \leq \phi(\mathcal{F})$ to $\mathrm{w}(\mathcal{F}) \leq \min\{\phi(\mathcal{F}), \phi(\mathcal{F}')\}$ where $\mathcal{F}'$ is any family that is structure isomorphic to $\mathcal{F}$. This improved bound suggests that it may be worth examining *equivalence classes* of union-closed families, i.e. the collection of all union-closed families which are structure isomorphic to a given union-closed family. This idea of equivalence classes is relevant to our final conjecture, Conjecture 6, which implies that if any family within an equivalence class satisfies Frankl's conjecture, then every family in the equivalence class does. As shown in Theorem 19, this conjecture turns out to be equivalent to Frankl's conjecture, suggesting that equivalence classes of structure isomorphisms may be a very valuable tool for future work on Frankl's conjecture.

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