The Splitting Field of $Y^n - 2$, Two-Variable NTT and Lattice-Based Cryptography

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Abstract

The splitting field F of the polynomial $Y^n - 2$ is an extension over $\mathbb{Q}$ generated by $\zeta_n = \exp(2\pi\sqrt{-1}/n)$ and $\sqrt[n]{2}$. When n (≥ 8) is a power-of-two integer, the degree of F over $\mathbb{Q}$ is $n^2/4$. In this paper, we lay the foundation for applying the Order-LWE in $\mathcal{R} = \mathbb{Z}[\zeta_n, \sqrt[n]{2}]$ to cryptographic uses. More specifically, We will compute the Galois group $\mathrm{Gal}\,(F/\mathbb{Q})$ and the canonical embedding of F into $\mathbb{C}^{n^2/4}$. Then we study the trace pairings of the integral basis $\zeta_n^{k_0} \sqrt[n]{2}^{k_1}$ and obtain its dual explicitly, which will be crucial when we study the error distributions on the ideal lattices associated with $\mathcal{R}$.

Moreover, we design a Two-Variable Number Theoretic Transform (2NTT) algorithm for the quotient $\mathcal{R}_p = \mathcal{R}/p\mathcal{R}$, where p is a prime number such that $Y^n \equiv 2 \bmod p$ has n distinct solutions. Compared to the one-variable NTT, a crucial advantage of 2NTT is that it enjoys a quadratic saving of twiddle factors. Hence, it is very interesting to see how to leverage this quadratic saving to boost the performance of 2NTT in practical implementations.

Keywords: Splitting Field, Galois Group, Trace Pairing, Order-LWE, 2NTT.

1   Introduction

In the last three decades, lattice-based cryptography has played a crucial role in the development of cryptography. Initially, it gained interests in the communities of cryptographers because the constructions in lattice-based cryptography were often accompanied by strong security proofs based on worst-case instances in the problems of lattice theory [1]. Later, this type of scheme was significantly improved by Regev when he introduced a new intermediate problem: Learning With Errors (LWE) [28, 30], which is flexible to use in practice, while also asymptotically being at least as hard as certain worst-case lattice problems [7, 26]. In early 2010s, a more efficient variant of LWE, the Ring Learning With Errors (RLWE) problem, was introduced by researchers [19], the public keys of which have a much smaller size, thus making RLWE more practical. Perhaps the most important class of ring in RLWE is the ring of algebraic integers of a cyclotomic field [19]. The RLWE problem in these rings enjoy strong security proofs based on the worst-case instances in the computational problems of ideal lattices of the cyclotomic fields [19].

A highly crucial property of the lattice-based cryptography is that it is quantum resistant, thus it has attracted rapidly increasing interest because of the recent important advances in quantum computing [2, 13]. In 2016, the National Institute of Standards and Technology (NIST) announced to update

their standards to include post-quantum digital-signature, encryption and key-establishment protocols [8]. In July 2022, NIST announced the candidate algorithms for standardization [25]. In particular, it recommended two primary algorithms to be implemented for most use cases: CRYSTALS-Kyber [5] for key-establishment and CRYSTALS-Dilithium for digital signature [14]. The constructions of both schemes are based on the RLWE (and Module-LWE) problem in the ring of algebraic integers of the 512-th cyclotomic field: $\mathbb{Z}[X]/\langle X^{256}+1\rangle$.

Lattice-based cryptography has also been highly crucial in the development of Homomorphic Encryption (HE), another important theme in cryptography in recent years. An HE scheme enables homomorphic operations to be performed on encrypted data without decryption [16]. As a result, data is encrypted throughout its entire lifecycle, which guarantees the security of sensitive data in scenarios such as cloud computing. The first Fully Homomorphic Encryption (FHE) scheme was constructed by Gentry in his thesis [16]. Following Gentry's original blueprint, researchers have come up with many new FHE schemes. Many of the currently popular schemes, e.g., BGV [6], BFV [15], CKKS [10] and TFHE [11], are based on the RLWE problem in the ring $\mathbb{Z}[X]/\langle X^{M/2}+1\rangle$, where M is a power-of-two integer. The value of M depends on many factors, e.g., the level of security needed in practice. For example, in implementations of CKKS, M could be as large as 2^{16}, which makes it very challenging to compute polynomial multiplications even using Number Theoretic Transform (NTT) on special hardware [18, 31, 32, 33]. This is a major reason why these FHE schemes are so slow.

The main motivation of this paper is to propose a new ring, a subring of the ring of algebraic integers of the splitting field of $Y^n - 2$, for cryptographic uses. This new ring admits a crucial Two-Variable Number Theoretic Transform (2NTT) that has a different algebraic structure. In particular, it enjoys a quadratic saving of twiddle factors. As a result, the multiplications of polynomials in it can be further boosted through leveraging the new properties of 2NTT. Moreover, the Order-LWE problem in the new ring has similar hardness results as that of $\mathbb{Z}[X]/\langle X^{M/2}+1\rangle$. Hence, it offers a new promising way to improve the performance of cryptographic schemes by using this new ring instead, especially when the polynomials used have very high degrees.

1.1 The Cyclotomic Field

Let us briefly review the appealing properties of the cyclotomic field that makes it highly important in the lattice-based cryptography. The M-th cyclotomic field is a number field that is constructed by adjoining a complex root of unity to the field of rational numbers $\mathbb{Q}$ [19, 20, 22]. More precisely, for a positive integer M, $\zeta_M = \exp\left(2\pi\sqrt{-1}/M\right)$ is a primitive M-th root of unity. The M-th cyclotomic field is the extension $\mathbb{Q}(\zeta_M)$ generated by ζ_M. The ring of algebraic integers of $\mathbb{Q}(\zeta_M)$ is the integral ring $\mathbb{Z}[\zeta_M]$ [20, 24].

In practice, the most important case is where M is a power-of-two integer. Then via sending ζ_M to X, $\mathbb{Z}[\zeta_M]$ is isomorphic to $R = \mathbb{Z}[X]/\langle X^{M/2}+1\rangle$. The algebraic properties of $\mathbb{Z}[\zeta_M]$ and $\mathbb{Q}[\zeta_M]$ are very well studied by mathematicians [17]. In particular, its Galois group, canonical embedding and trace pairings are well understood [19, 22]. Based on these results, the hardness of the RLWE problem in R was carefully studied in [19]. Another crucial property of R is that if we choose a prime number p such that $X^{M/2} + 1 = 0 \bmod p$ has

a solution in the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, then polynomial multiplication in $R_p = R/pR$ can be computed by using NTT, whose complexity is $O(M\log_2 M)$ [18, 31]. This property is widely used in practice to boost the performance of the cryptographic schemes based on the RLWE problem in R. Besides, the Galois group $\mathrm{Gal}\left(\mathbb{Q}\left(\zeta_M\right)/\mathbb{Q}\right)$ is naturally isomorphic to the multiplicative group of units modulo M, i.e., $(\mathbb{Z}/M\mathbb{Z})^\times$, which is very important in FHE schemes [6, 10]. For example, the construction of the bootstrapping of CKKS crucially depends on the action of $\mathrm{Gal}\left(\mathbb{Q}\left(\zeta_M\right)/\mathbb{Q}\right)$ on the ciphertexts [9].

1.2 Our Contributions

Throughout this paper, $n\ (\geq 8)$ is a power-of-two integer. Recall that ζ_n is the n-th root of unity $\exp\left(2\pi\sqrt{-1}/n\right)$ and $\sqrt[n]{2}\in\mathbb{R}$ is the real positive n-th root of 2. The splitting field F of $Y^n - 2$ is constructed by adjoining ζ_n and $\sqrt[n]{2}$ to $\mathbb{Q}$, namely $F = \mathbb{Q}\left(\zeta_n, \sqrt[n]{2}\right)$. The integral ring $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$ is only known to be an order of F [4, 22]. Via sending ζ_n to X and $\sqrt[n]{2}$ to Y, $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$ is naturally isomorphic to the quotient

$$\mathcal{R} = \mathbb{Z}[X,Y]/\langle X^{n/2} + 1, Y^{n/2} - (X^{n/8} - X^{3n/8})\rangle.$$

Our major contributions in this paper include the following important results.

1.2.1 Number Theoretic Results

We first compute the Galois group $\mathrm{Gal}\left(F/\mathbb{Q}\right)$ of F over $\mathbb{Q}$. Through studying the immediate fields $\mathbb{Q}\left(\sqrt{-1}\right)$ and $\mathbb{Q}\left(\zeta_n\right)$ of F using the fundamental theorem of Galois theory, we obtain an explicit algebraic structure of $\mathrm{Gal}\left(F/\mathbb{Q}\right)$. Based on these results, we construct the canonical embedding of F into $\mathbb{C}^{n^2/4}$ and give a proof that it sends the natural integral basis $\zeta_n^{k_0}\sqrt[n]{2}^{k_1}$, $0 \leq k_0, k_1 < n/2$, of F to an orthogonal basis of $\mathbb{C}^{n^2/4}$.

Moreover, we compute the trace pairings of the integral basis $\zeta_n^{k_0}\sqrt[n]{2}^{k_1}$, from which we construct its dual basis that forms an integral basis for $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]^\vee$ [22, 34]. We also compute the absolute discriminant of the integral ring $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$, which is a power-of-two integer.

Then, we study the behaviors of the elliptic Gaussian distributions on the ideal lattices associated with $\mathcal{R}$. In particular, we analyze how $\mathrm{Gal}(F/\mathbb{Q})$ acts on these probability distributions. We also compute the first minimums of $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$ and its dual $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]^\vee$, which are very crucial when analyzing the error distributions on ideal lattices.

1.2.2 Order-LWE

We formulate the Search and Average-case Decision Order-LWE problems in $\mathcal{R}$. We study the worst-case hardness of the Search Order-LWE using the algebraic properties of F and the results in [19]. Then we adapt the proof in [19] to give reductions from the Search Order-LWE to Average-Case Decision Order-LWE, thereby showing that the Order-LWE distribution over $\mathcal{R}$ is pseudorandom. As in [4, 19], we give two variants of the reductions. The first reduction is to the Average-Case Decision Order-LWE problem with a nonsperical error distribution defined with respect to the canonical embedding. The second reduction is

to the Average-Case Decision Order-LWE problem with a spherical error distribution with respect to the canonical embedding, but with only a bounded number of samples [19, 30]. The proof of certain technical results are given in the appendix.

1.2.3   2NTT

Perhaps the most important new feature of the ring $\mathcal{R}$ is the existence of a 2NTT. We choose a prime number p such that $X^n \equiv 1 \bmod p$ has a primitive solution $\alpha \in \mathbb{F}_p$ and $Y^n \equiv 2 \bmod p$ has a solution $\beta \in \mathbb{F}_p$. Then the equations

$$X^{n/2} + 1 \equiv 0 \bmod p \text{ and } Y^{n/2} - \left(X^{n/8} - X^{3n/8} \right) \equiv 0 \bmod p \qquad (1.1)$$

have $n^2/4$ solutions in total. The first equation of Eq. (1.1) has $n/2$ solutions: $\{\alpha^{2i+1} : 0 \le i < n/2\}$. For every solution $X = \alpha^{2i+1}$, the second equation of Eq. (1.1) also has $n/2$ solutions: $\{\alpha^{2j}\beta : 0 \le j < n/2\}$ if $i \equiv 0, 3 \bmod 4$; and $\{\alpha^{2j+1}\beta : 0 \le j < n/2\}$ if $i \equiv 1, 2 \bmod 4$. Given a two-variable polynomial $\mathbf{F}(X, Y) = \sum_{k,l=0}^{n/2-1} f_{k,l} X^k Y^l$ in $\mathcal{R}_p = \mathcal{R}/p\mathcal{R}$. The idea of 2NTT is to evaluate $\mathbf{F}$ at the $n^2/4$ solutions of Eq. (1.1). The 2NTT of $\mathbf{F}$ consists of transverse and longitudinal vector butterflies:

1. Transverse vector butterfly. This phase uses a vector butterfly to evaluate $\mathbf{F}$ at the $n/2$ roots $X = \alpha^{2i+1}$, $0 \le i < n/2$, the output of which are $n/2$ vectors $\mathbf{F}\left(\alpha^{2i+1}, Y\right)$. There are $\log_2(n/2)$ stages in this phase, and each stage consumes $O(n)$ vector operations (vector summation and scalar multiplication).

2. Transpose and longitudinal vector butterfly. The output of the transverse vector butterfly naturally falls into two groups: Group 1 with $i \equiv 0, 3 \bmod 4$ and Group 2 with $i \equiv 1, 2 \bmod 4$. Each group forms an $n/4 \times n/2$ matrix. Now we evaluate the vector-valued polynomial constructed from the $n/2$ column vectors of Group 1 at the roots $\{\alpha^{2j}\beta : 0 \le j < n/2\}$ using a vector butterfly with a new set of twiddle factors. We also evaluate the vector-valued polynomial constructed from the $n/2$ column vectors of Group 2 at the roots $\{\alpha^{2j+1}\beta : 0 \le j < n/2\}$ using a vector butterfly with another set of twiddle factors. Each evaluation consists of $\log_2(n/2)$ stages and each stage consumes $O(n)$ vector operations.

A natural comparison is between the 2NTT of $\mathcal{R}$ and the negacyclic NTT of the ring $\mathbb{Z}[X]/\langle X^{n^2/4}+1\rangle$, since polynomials in both rings have $n^2/4$ coefficients. The computational complexities of 2NTT and NTT is the same. Compared to NTT, 2NTT has two important advantages:

1. For NTT, we need the prime numbers such that $X^{n^2/4} \equiv -1 \bmod p'$ has $n^2/4$ solutions. While for 2NTT, we need the prime numbers such that $Y^n \equiv 2 \bmod p$ has n solutions, which are generally different from the NTT case. Hence when design an algorithm, we have more chances to find prime numbers that suit our needs.

2. The space consumed by the twiddle factors of NTT is $O(n^2/4)$, while the space consumed by the twiddle factors of 2NTT is only $O(n)$. Thus, there

is a quadratic saving of twiddle factor space. This saving is especially desirable for large polynomials. As a result, this offers new possibilities for optimizations in practical implementations.

1.2.4 Applications and Generalizations

As an application of the new Order-LWE, we discuss the properties of a new variant of CKKS based on $\mathcal{R} = \mathbb{Z}[\zeta_n, \sqrt[n]{2}]$. Our new contributions include the construction of a careful choice of encoding and decoding maps for $\mathcal{R}$ using the algebraic structure of $\mathrm{Gal}\,(F/\mathbb{Q})$, and the actions of $\mathrm{Gal}\,(F/\mathbb{Q})$ on the plaintexts and ciphertexts in this new variant. We also show that this new variant of CKKS is bootstrappable.

In the appendix, we show that all the results obtained in this paper can be generalized to the ring $\mathbb{Z}[\zeta_n, \sqrt[n]{-2}]$, which is an order of the splitting field $\widehat{F}$ of $Y^n + 2$. We also briefly touch on the splitting field $\mathbb{Q}(\zeta_n, \sqrt[n]{r})$ of $Y^n - r$, where $|r|\ (\geq 3)$ is a prime number.

1.3 Outline

In Section 2, we compute the Galois group of F over $\mathbb{Q}$ and study its algebraic structures using the fundamental theorem of Galois theory. In Section 3, we construct the canonical embedding of F into $\mathbb{C}^{n^2/4}$ and prove an orthogonality property for it. In Section 4, we compute the trace pairings of the natural integral basis of $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$ and compute its absolute discriminant. We also find the first minimums of $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$ and $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]^\vee$. In Section 5, we discuss the error distributions on the ideal lattices associated with $\mathcal{R}$. We formulate the Search and Average-case Decision Order-LWE in $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$ and obtain the hardness of Search Order-LWE. In Section 6, we introduce the 2NTT for $\mathcal{R}$ and give a vector butterfly algorithm. In Section 7, we study a new variant of CKKS based on $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$. The appendix is devoted to discussing some technical details needed in this paper. In particular, we show that all the results obtained for $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$ can be generalized to $\mathbb{Z}[\zeta_n, \sqrt[n]{-2}]$.

2 The Splitting Field and Its Galois Group

In this section, we give a detailed description of the splitting field F of $Y^n - 2$, where $n\ (\geq 8)$ is a power-of-two integer. In particular, we explicitly construct the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ of F over $\mathbb{Q}$ [12] and study its structures using the fundamental theorem of Galois theory [24].

2.1 The Construction of the Splitting Field

The splitting field F of $Y^n - 2$ is a finite extension over $\mathbb{Q}$ generated by two numbers $\zeta_n = \exp(2\pi\sqrt{-1}/n)$ and $\sqrt[n]{2}$, i.e., $F = \mathbb{Q}\left(\zeta_n, \sqrt[n]{2}\right)$ [20, 24]. The two numbers ζ_n and $\sqrt[n]{2}$ satisfy an algebraic relation

$$\sqrt[n]{2}^{n/2} = \zeta_n^{n/8} - \zeta_n^{3n/8}, \tag{2.1}$$

which is just the equation $\sqrt{2} = \exp\left(2\pi\sqrt{-1}/8\right) - \exp\left(6\pi\sqrt{-1}/8\right)$.

Lemma 2.1. *For a power-of-two integer n (≥ 8), the degree of the field extension of F over $\mathbb{Q}$, denoted by $[F : \mathbb{Q}]$, is $n^2/4$.*

The readers are referred to the webpage [35] for a detailed proof of this lemma. Thus viewed as a vector space over $\mathbb{Q}$, the dimension of F is $n^2/4$, with a natural basis given by

$$\left\{ \zeta_n^{k_0} \sqrt[n]{2}^{k_1} : 0 \leq k_0, k_1 < n/2 \right\}. \tag{2.2}$$

2.2 The Galois Group of the Splitting Field

The construction of the Galois group of F over $\mathbb{Q}$, denoted by $\mathrm{Gal}(F/\mathbb{Q})$, uses a method from Conrad's note [12]. The order of $\mathrm{Gal}(F/\mathbb{Q})$ is equal to the dimension of F over $\mathbb{Q}$, hence $\mathrm{Gal}(F/\mathbb{Q})$ has $n^2/4$ elements. Since F is generated by ζ_n and $\sqrt[n]{2}$ over $\mathbb{Q}$, any element of $\mathrm{Gal}(F/\mathbb{Q})$ is determined by its actions on them, which must be of the form

$$\sigma_{a,b}(\zeta_n) = \zeta_n^a \text{ and } \sigma_{a,b}(\sqrt[n]{2}) = \zeta_n^b \sqrt[n]{2} \text{ with } a, b \in \mathbb{Z}/n\mathbb{Z}. \tag{2.3}$$

Notice that here we have used the property that $\sigma_{a,b}$ must send a root of $X^n - 1$ (resp. $Y^n - 2$) to another root of the same equation [24]. In this paper, the notation $\mathbb{Z}/n\mathbb{Z}$ has an additional meaning that its representative is chosen to be

$$\mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}.$$

The action of $\sigma_{a,b}$ on a general basis element $\zeta_n^{k_0} \sqrt[n]{2}^{k_1}$ is $\zeta_n^{ak_0 + bk_1} \sqrt[n]{2}^{k_1}$.

Since $\sigma_{a,b}(\zeta_n)$ is another primitive root of $X^n - 1$, a must be an odd integer in $\mathbb{Z}/n\mathbb{Z}$. Moreover, a and b are not totally independent from each other, as $\sigma_{a,b}$ must preserve the algebraic relation in Eq. (2.1), which can be rewritten into

$$\sqrt[n]{2}^{n/2} = \zeta_n^{n/8} + \zeta_n^{-n/8}.$$

Under the action of $\sigma_{a,b}$, this equation becomes

$$(-1)^b \sqrt[n]{2}^{n/2} = \zeta_n^{na/8} + \zeta_n^{-na/8}.$$

For this equation to be valid, we must require

$$a \equiv \begin{cases} 1, 7 \bmod 8, & \text{if } b \text{ is even}, \\ 3, 5 \bmod 8, & \text{if } b \text{ is odd}. \end{cases} \tag{2.4}$$

For later convenience, we define the finite set G to be

$$G = \{(a, b) \in (\mathbb{Z}/n\mathbb{Z})^2 : (a, b) \text{ satisfies the condition in Eq. (2.4)}\}, \tag{2.5}$$

which has $n^2/4$ elements. Then the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ is

$$\mathrm{Gal}(F/\mathbb{Q}) = \{\sigma_{a,b} : (a, b) \in G\}.$$

Moreover, $\sigma_{1,0}$ is the identity element of $\mathrm{Gal}(F/\mathbb{Q})$ and $\sigma_{n-1,0}$ is the complex conjugation. A quick computation shows that the composition of two elements $\sigma_{a,b}$ and $\sigma_{c,d}$ is $\sigma_{a,b} \circ \sigma_{c,d} = \sigma_{ac,ad+b}$.

2.3 The Algebraic Structure of the Galois Group

We need the fundamental theorem of Galois theory to study the algebraic structure of the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ [24]. In its most basic form, this theorem tells us that there is a one-to-one correspondence between the immediate fields of F and the subgroups of $\mathrm{Gal}(F/\mathbb{Q})$, while the intermediate fields that are Galois extensions of $\mathbb{Q}$ correspond to the normal subgroups of $\mathrm{Gal}(F/\mathbb{Q})$. Here, an immediate field is a subfield of F [24].

2.3.1 The Subfield of Gaussian Rationals

The first important immediate field is $\mathbb{Q}(\zeta_n^{n/4})$, which is just $\mathbb{Q}(\sqrt{-1})$ as $(\zeta_n^{n/4})^2$ is -1. Since $\mathbb{Q}(\sqrt{-1})$ is a quadratic extension of $\mathbb{Q}$, the degree of F over $\mathbb{Q}(\sqrt{-1})$ is $n^2/8$. The Galois group $\mathrm{Gal}\left(F/\mathbb{Q}(\sqrt{-1})\right)$ is a subgroup of $\mathrm{Gal}(F/\mathbb{Q})$ that fixes $\mathbb{Q}(\sqrt{-1})$:

$$\mathrm{Gal}\left(F/\mathbb{Q}(\sqrt{-1})\right) = \{\sigma_{a,b} \in \mathrm{Gal}(F/\mathbb{Q}) : \sigma_{a,b}(x) = x, \ \forall x \in \mathbb{Q}(\sqrt{-1})\}.$$

Let N be the subset of G of the form

$$N = \{(a,b) \in G : a \equiv 1 \text{ or } 5 \bmod 8\}, \tag{2.6}$$

which has $n^2/8$ elements. Then $\mathrm{Gal}\left(F/\mathbb{Q}(\sqrt{-1})\right)$ is the subgroup

$$\mathrm{Gal}\left(F/\mathbb{Q}(\sqrt{-1})\right) = \{\sigma_{a,b} \in \mathrm{Gal}(F/\mathbb{Q}) : (a,b) \in N\}.$$

Because $\mathbb{Q}(\sqrt{-1})$ is a Galois extension of $\mathbb{Q}$, $\mathrm{Gal}\left(F/\mathbb{Q}(\sqrt{-1})\right)$ is a normal subgroup of $\mathrm{Gal}(F/\mathbb{Q})$. The group $\mathrm{Gal}\left(\mathbb{Q}(\sqrt{-1})/\mathbb{Q}\right)$ is naturally identified with the subgroup $\{\sigma_{1,0}, \sigma_{n-1,0}\}$, so $\mathrm{Gal}(F/\mathbb{Q})$ is the disjoint union of cosets

$$\mathrm{Gal}(F/\mathbb{Q}) = \mathrm{Gal}\left(F/\mathbb{Q}(\sqrt{-1})\right) \cup \left(\sigma_{n-1,0} \circ \mathrm{Gal}\left(F/\mathbb{Q}(\sqrt{-1})\right)\right).$$

From the relation $\sigma_{n-1,0} \circ \sigma_{a,b} = \sigma_{(n-1)a,(n-1)b}$, we deduce that $G \setminus N$ can be expressed as

$$G \setminus N = \{((n-1)a, (n-1)b) : (a,b) \in N\}. \tag{2.7}$$

Furthermore, since the restriction of the quotient map on $\{\sigma_{1,0}, \sigma_{n-1,0}\}$ is identity, $\mathrm{Gal}(F/\mathbb{Q})$ can be represented as a semidirect [23]

$$\mathrm{Gal}(F/\mathbb{Q}) = \mathrm{Gal}\left(\mathbb{Q}(\sqrt{-1})/\mathbb{Q}\right) \ltimes \mathrm{Gal}\left(F/\mathbb{Q}(\sqrt{-1})\right). \tag{2.8}$$

2.3.2 The Cyclotomic Subfield

Another important immediate field of F is $\mathbb{Q}(\zeta_n)$, the n-th cyclotomic field [22]. The Galois group of $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$ is

$$\mathrm{Gal}\left(\mathbb{Q}(\zeta_n)/\mathbb{Q}\right) = \{\varsigma_a : a = 1, 3, \ldots, n-1\},$$

where ς_a is the automorphism of $\mathbb{Q}(\zeta_n)$ that sends ζ_n to ζ_n^a. The field $\mathbb{Q}(\sqrt{-1})$ is a subfield of $\mathbb{Q}(\zeta_n)$, and the Galois group of $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}(\sqrt{-1})$ is

$$\mathrm{Gal}\left(\mathbb{Q}(\zeta_n)/\mathbb{Q}(\sqrt{-1})\right) = \langle \varsigma_5 \rangle = \{\varsigma_5^j : 0 \le j < n/4\},$$

which has $n/4$ elements since $5^{n/4} \equiv 1 \bmod n$ [9]. Moreover, ς_{n-1} acts on $\mathbb{Q}(\zeta_n)$ as complex conjugation and the Galois group of $\mathbb{Q}(\sqrt{-1})$ over $\mathbb{Q}$ is naturally $\{\varsigma_1, \varsigma_{n-1}\}$. In conclusion, $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ can be represented as the direct product

$$\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) = \langle \varsigma_5 \rangle \times \langle \varsigma_{n-1} \rangle.$$

Since $\mathbb{Q}(\zeta_n)$ is a Galois extension over $\mathbb{Q}$ with degree $n/2$, $\mathrm{Gal}\left(F/\mathbb{Q}(\zeta_n)\right)$ is a normal subgroup of $\mathrm{Gal}(F/\mathbb{Q})$ that fixes $\mathbb{Q}(\zeta_n)$:

$$\mathrm{Gal}\left(F/\mathbb{Q}(\zeta_n)\right) = \left\{\sigma_{a,b} \in \mathrm{Gal}(F/\mathbb{Q}) : \sigma_{a,b}(x) = x, \ \forall x \in \mathbb{Q}(\zeta_n)\right\}.$$

Thus it is given by

$$\mathrm{Gal}\left(F/\mathbb{Q}(\zeta_n)\right) = \{\sigma_{1,b} \in \mathrm{Gal}(F/\mathbb{Q}) : b = 0, 2, \ldots, n-2\} = \langle \sigma_{1,2} \rangle.$$

From the inclusion $\mathbb{Q}(\sqrt{-1}) \subset \mathbb{Q}(\zeta_n) \subset F$, we obtain an isomorphism [24]

$$\mathrm{Gal}\left(F/\mathbb{Q}(\sqrt{-1})\right)/\mathrm{Gal}\left(F/\mathbb{Q}(\zeta_n)\right) \cong \mathrm{Gal}\left(\mathbb{Q}(\zeta_n)/\mathbb{Q}(\sqrt{-1})\right). \qquad (2.9)$$

Let us look at the isomorphism in Eq. (2.9) more carefully. Since $\sigma_{5,1}^{n/4}$ lies in $\mathrm{Gal}(F/\mathbb{Q}(\zeta_n))$ and $\sigma_{5,1}^i \notin \mathrm{Gal}(F/\mathbb{Q}(\zeta_n))$ for $1 \leq i < n/4$, $\mathrm{Gal}\left(F/\mathbb{Q}(\sqrt{-1})\right)$ is the disjoint union of cosets of the form

$$\mathrm{Gal}\left(F/\mathbb{Q}(\sqrt{-1})\right) = \cup_{i=0}^{n/4-1}\left(\sigma_{5,1}^i \circ \mathrm{Gal}\left(F/\mathbb{Q}(\zeta_n)\right)\right).$$

Thus every element of $\mathrm{Gal}\left(F/\mathbb{Q}(\sqrt{-1})\right)$ can be uniquely written as

$$\sigma_{5,1}^i \circ \sigma_{1,2}^j \text{ with } 0 \leq i < n/4 \text{ and } 0 \leq j < n/2.$$

To multiply two elements of $\mathrm{Gal}\left(F/\mathbb{Q}(\sqrt{-1})\right)$, we need the relation

$$\sigma_{5,1}^{-1} \circ \sigma_{1,2} \circ \sigma_{5,1} = \sigma_{1,2}^{5^{-1}}, \qquad (2.10)$$

where the inverse 5^{-1} is computed modulo $n/2$ as $\sigma_{1,2}^{n/2} = \sigma_{1,0}$. From the semidirect product in Eq. (2.8), we deduce that every element of $\mathrm{Gal}\left(F/\mathbb{Q}\right)$ can be uniquely written as

$$\sigma_{n-1,0}^k \circ \sigma_{5,1}^i \circ \sigma_{1,2}^j, \text{ where } 0 \leq i < n/4, \ 0 \leq j < n/2 \text{ and } k = 0, 1.$$

To multiply two elements of $\mathrm{Gal}\left(F/\mathbb{Q}\right)$, we further need the relations

$$\sigma_{n-1,0} \circ \sigma_{5,1} \circ \sigma_{n-1,0} = \sigma_{5,1} \circ \sigma_{1,2}^{-5^{-1}} \text{ and } \sigma_{n-1,0} \circ \sigma_{1,2} \circ \sigma_{n-1,0} = \sigma_{1,2}^{-1}. \quad (2.11)$$

These results will be crucial when we study the error distributions over the ideal lattices of F and a new variant of CKKS based on the ring $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$ in Section 7.

3 The Canonical Embedding

In this section, we construct the canonical embedding of F into $\mathbb{C}^{n^2/4}$ and study its properties [20, 22]. We prove that the canonical embedding sends the natural integral basis in Eq. (2.2) to an orthogonal basis of $\mathbb{C}^{n^2/4}$, a property that will become important when we study the error distributions over the ideal lattices associated with the integral ring $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$.

3.1 The Construction of the Canonical Embedding

Since the degree of F over $\mathbb{Q}$ is $n^2/4$, there exist $n^2/4$ distinct embeddings of F into $\mathbb{C}$ [20, 22, 24]. The inclusion $F \hookrightarrow \mathbb{C}$ is naturally an embedding. Furthermore, for every $\sigma_{a,b} \in \mathrm{Gal}\,(F/\mathbb{Q})$, there is an embedding of the form

$$\sigma_{a,b} : F \xrightarrow{\sigma_{a,b}} F \hookrightarrow \mathbb{C},$$

which gives us all the embeddings of F into $\mathbb{C}$. Similar to the cyclotomic fields, F does not admit any real embedding [19]. In this paper, a vector of $\mathbb{C}^{n^2/4}$ is denoted by $\mathbf{z} = (z_{a,b})_{(a,b)\in G}$, namely its coordinate is indexed by the elements of G. Given two vectors $\mathbf{z} = (z_{a,b})$ and $\mathbf{z}' = (z'_{a,b})$, their standard inner product is given by $\langle \mathbf{z}, \mathbf{z}' \rangle = \sum_{(a,b)\in G} z_{a,b} \cdot \overline{z'}_{a,b}$, which is positive definite, linear in the first argument and antilinear in the second argument. Hence the canonical embedding of F into $\mathbb{C}^{n^2/4}$ is given by

$$\sigma : x \in F \mapsto (\sigma_{a,b}(x))_{(a,b)\in G} \in \mathbb{C}^{n^2/4}. \tag{3.1}$$

In fact, σ induces an injective real linear map from $F_{\mathbb{R}} = F \otimes_{\mathbb{Q}} \mathbb{R}$ to $\mathbb{C}^{n^2/4}$:

$$\sigma : x \otimes r \mapsto (\sigma_{a,b}(x) \cdot r)_{(a,b)\in G}, \text{ where } x \in F \text{ and } r \in \mathbb{R}. \tag{3.2}$$

The complex conjugation of $\sigma_{a,b}(x)$, denoted by $\overline{\sigma_{a,b}(x)}$, is $\sigma_{(n-1)a,(n-1)b}(x)$ for every $x \in F$ and $(a,b) \in G$. Therefore, the image of $F_{\mathbb{R}}$ under σ is the real subspace

$$H = \left\{ (z_{a,b})_{(a,b)\in G} \in \mathbb{C}^{n^2/4} : z_{(n-1)a,(n-1)b} = \overline{z_{a,b}} \right\},$$

which is isomorphic to $F_{\mathbb{R}}$. In this paper, we often implicitly identify H with $F_{\mathbb{R}}$. The homomorphism in Eq. (3.2) can also be written as

$$\sigma : x \otimes r \mapsto \big(\sigma_{a,b}(x) \cdot r, \sigma_{(n-1)a,(n-1)b}(x) \cdot r \big)_{(a,b)\in N}.$$

An automorphism $\sigma_{c,d} \in \mathrm{Gal}(F/\mathbb{Q})$ of F induces an automorphism of H by making the following diagram commute

$$
\begin{array}{ccc}
x \otimes r & \xrightarrow{\ \sigma\ } & \big(\sigma_{a,b}(x) \cdot r, \sigma_{(n-1)a,(n-1)b}(x) \cdot r \big)_{(a,b)\in N} \\
\downarrow{\scriptstyle \sigma_{c,d}} & & \downarrow{\scriptstyle \sigma_{c,d}} \\
\sigma_{c,d}(x) \otimes r & \xrightarrow{\ \sigma\ } & \big(\sigma_{a,b}(\sigma_{c,d}(x)) \cdot r, \sigma_{(n-1)a,(n-1)b}(\sigma_{c,d}(x)) \cdot r \big)_{(a,b)\in N}
\end{array} \tag{3.3}
$$

Therefore, the effect of the automorphism $\sigma_{c,d}$ is to permute the components of vectors in H. Let $\mathbf{e}_{a,b} \in \mathbb{C}^{n^2/4}$ be the vector with 1 in its (a,b)-th component and 0 elsewhere, which forms a basis for $\mathbb{C}^{n^2/4}$. For every $(a,b) \in N$, we define

$$
\begin{aligned}
\mathbf{h}_{(a,b)} &= \frac{1}{\sqrt{2}} \left(\mathbf{e}_{a,b} + \mathbf{e}_{(n-1)a,(n-1)b} \right), \\
\mathbf{h}_{((n-1)a,(n-1)b)} &= \frac{\sqrt{-1}}{\sqrt{2}} \left(\mathbf{e}_{a,b} - \mathbf{e}_{(n-1)a,(n-1)b} \right),
\end{aligned} \tag{3.4}
$$

which forms a real basis for H. The restriction of the inner product $\langle \cdot, \cdot \rangle$ of $\mathbb{C}^{n^2/4}$ on H is a real positive definite inner product, with respect to which $\{ \mathbf{h}_{a,b} : (a,b) \in G \}$ forms an orthonormal basis.

3.2 The Orthogonality of the Basis Vectors

Now we give a proof to an important orthogonality property of the canonical embedding.

Proposition 3.1. *Under canonical embedding, the natural basis of F in Eq. (2.2) is sent to an orthogonal basis of $\mathbb{C}^{n^2/4}$. More precisely,*

$$\left\langle \sigma\left(\zeta_n^{k_0} \sqrt[n]{2}^{k_1}\right), \sigma\left(\zeta_n^{l_0} \sqrt[n]{2}^{l_1}\right)\right\rangle = \begin{cases} \left(\sqrt[n]{2}\right)^{2k_1} \cdot n^2/4, & \text{if } k_0 = l_0 \text{ and } k_1 = l_1. \\ 0, & \text{otherwise.} \end{cases}$$

Proof. From the construction of σ, we have

$$\left\langle \sigma\left(\zeta_n^{k_0} \sqrt[n]{2}^{k_1}\right), \sigma\left(\zeta_n^{l_0} \sqrt[n]{2}^{l_1}\right)\right\rangle = \left(\sqrt[n]{2}\right)^{k_1+l_1} \cdot \sum_{(a,b)\in G} \zeta_n^{a(k_0-l_0)+b(k_1-l_1)}. \quad (3.5)$$

If $a \equiv 1, 7 \bmod 8$, then b is even, i.e., $b = 2i$ with $0 \le i < n/2$. In this case, keep the value of a fixed and let us look at the partial sum

$$\sum_{i=0}^{n/2-1} \zeta_n^{a(k_0-l_0)+2i(k_1-l_1)} = \zeta_n^{a(k_0-l_0)} \sum_{i=0}^{n/2-1} \zeta_n^{2i(k_1-l_1)}, \quad (3.6)$$

which is 0 unless $\zeta_n^{2(k_1-l_1)} = 1$. Since both $0 \le k_1 < n/2$ and $0 \le l_1 < n/2$, we have $-n/2 < k_1 - l_1 < n/2$, from which we deduce that $\zeta_n^{2(k_1-l_1)} = 1$ if and only if $k_1 = l_1$. Therefore, Eq. (3.6) is equal to 0 unless $k_1 = l_1$. Similarly, if $a \equiv 3, 5 \bmod 8$, then $b = 2i + 1$ with $0 \le i < n/2$. In this case, keep the value of a fixed and let us look at the partial sum

$$\sum_{i=0}^{n/2-1} \zeta_n^{a(k_0-l_0)+(2i+1)(k_1-l_1)} = \zeta_n^{a(k_0-l_0)+(k_1-l_1)} \sum_{i=0}^{n/2-1} \zeta_n^{2i(k_1-l_1)},$$

which again is equal to 0 unless $k_1 = l_1$. Therefore, Eq. (3.5) is always equal to 0 unless $k_1 = l_1$. Now assume $k_1 = l_1$, then Eq. (3.5) becomes

$$\left(\sqrt[n]{2}\right)^{2k_1} \cdot \sum_{(a,b)\in G} \zeta_n^{a(k_0-l_0)} = \frac{n}{2} \cdot \left(\sqrt[n]{2}\right)^{2k_1} \sum_{i=0}^{n/2-1} \zeta_n^{(2i+1)(k_0-l_0)}. \quad (3.7)$$

Similarly, since $-n/2 < k_0 - l_0 < n/2$, Eq. (3.7) is equal to 0 unless $k_0 = l_0$. When both $k_0 = l_0$ and $k_1 = l_1$, Eq. (3.5) is equal to $\left(\sqrt[n]{2}\right)^{2k_1} \cdot n^2/4$. $\qquad \square$

It is very illustrating to compare this orthogonality property of the canonical embedding of F to that of the cyclotomic fields. For the M-th cyclotomic field, the integral basis $\{\zeta_M^{2i+1} : 0 \le i < M/2\}$, where M is a power-of-two integer, is sent to an orthogonal basis of $\mathbb{C}^{M/2}$ under the canonical embedding of $\mathbb{Q}(\zeta_M)$. But the norm of the image of ζ_M^i under canonical embedding is always $\sqrt{M/2}$. Hence up to the overall constant $\sqrt{M/2}$, the canonical embedding of $\mathbb{Q}(\zeta_M)$ is an isometry [19]. But for the splitting field F, the norm of $\sigma(\zeta_n^{k_0} \sqrt[n]{2}^{k_1})$ is $2^{k_1/n} \cdot n/2$, which depends on the value of k_1, so certain basis vectors are elongated. But since $k_1 < n/2$, the ratio between the longest vector and shortest vector is smaller than $\sqrt{2}$.

4   The Trace Pairings and First Minimums

In this section, we compute the trace pairings of the natural integral basis $\zeta_n^{k_0} \sqrt[n]{2}^{k_1}$ of $\mathcal{R} = \mathbb{Z}[\zeta_n, \sqrt[n]{2}]$ and construct a dual basis for

$$\mathcal{R}^\vee = \{x \in F : \mathrm{Tr}(x\mathcal{R}) \subset \mathbb{Z}\}.$$

We also compute the absolute discriminant of $\mathcal{R}$ [22, 34]. We also compute the first minimums of $\mathcal{R} = \mathbb{Z}[\zeta_n, \sqrt[n]{2}]$ and its dual $\mathcal{R}^\vee$.

4.1   The Computations of the Trace Pairings

Recall that given an element $x \in F$, its trace $\mathrm{Tr}(x)$ can be computed by $\sum_{(a,b)\in G} \sigma_{a,b}(x)$ [19, 22]. The trace-pairing matrix is given by $\mathrm{Tr}(\zeta_n^{k_0} \sqrt[n]{2}^{k_1} \cdot \zeta_n^{l_0} \sqrt[n]{2}^{l_1})$ for every pair $(\zeta_n^{k_0} \sqrt[n]{2}^{k_1}, \zeta_n^{l_0} \sqrt[n]{2}^{l_1})$ with $0 \leq k_0, k_1, l_0, l_1 < n/2$. We have the following important theorem, whose proof is divided into three lemmas.

Theorem 4.1. $Tr(\zeta_n^{k_0+l_0} \sqrt[n]{2}^{k_1+l_1})$ is always 0 unless $k_1 = l_1 = 0$ or $k_1 + l_1 = n/2$. When $k_1 = l_1 = 0$, we have

$$Tr(\zeta_n^{k_0+l_0}) = \begin{cases} n^2/4, & \text{if } k_0 = l_0 = 0; \\ -n^2/4, & \text{if } k_0 + l_0 = n/2. \end{cases}$$

when $k_1 + l_1 = n/2$, we have

$$Tr(\zeta_n^{k_0+l_0} \sqrt[n]{2}^{n/2}) = \begin{cases} n^2/4, & \text{if } k_0 + l_0 = n/8 \text{ or } 7n/8; \\ -n^2/4, & \text{if } k_0 + l_0 = 3n/8 \text{ or } 5n/8. \end{cases}$$

Lemma 4.2. If $k_1 + l_1 \neq 0$ or $n/2$, then $Tr(\zeta_n^{k_0+l_0} \sqrt[n]{2}^{k_1+l_1}) = 0$.

Proof. The finite set G can be expressed as a disjoint union

$$G = H_1 \cup H_3 \cup H_5 \cup \cdots \cup H_{n-1},$$

where each subset H_a is

$$H_a = \begin{cases} \{(a,0), (a,2), (a,4), \ldots, (a, n-2)\}, & \text{if } a \equiv 1,7 \bmod 8, \\ \{(a,1), (a,3), (a,5), \ldots, (a, n-1)\}, & \text{if } a \equiv 3,5 \bmod 8. \end{cases}$$

The trace $\mathrm{Tr}(\zeta_n^{k_0+l_0} \sqrt[n]{2}^{k_1+l_1})$ can be computed by

$$\mathrm{Tr}(\zeta_n^{k_0+l_0} \sqrt[n]{2}^{k_1+l_1}) = \sum_{a \text{ odd}} \sum_{(a,b)\in H_a} \sigma_{a,b}(\zeta_n^{k_0+l_0} \sqrt[n]{2}^{k_1+l_1}). \tag{4.1}$$

If $a = 1, 7 \bmod 8$, the inner layer of the sum in Eq. (4.1) becomes

$$\sum_{(a,b)\in H_a} \sigma_{a,b}(\zeta_n^{k_0+l_0} \sqrt[n]{2}^{k_1+l_1}) = \zeta_n^{a(k_0+l_0)} (\sqrt[n]{2})^{k_1+l_1} \sum_{i=0}^{n/2-1} \zeta_n^{2i(k_1+l_1)}.$$

Hence this sum must be 0 unless $\zeta_n^{2(k_1+l_1)} = 1$, i.e., $k_1 + l_1 = 0, n/2$. Similarly, if $a = 3, 5 \bmod 8$, the inner layer of the sum in Eq. (4.1) becomes

$$\sum_{(a,b)\in H_a} \sigma_{a,b}(\zeta_n^{k_0+l_0} \sqrt[n]{2}^{k_1+l_1}) = \zeta_n^{a(k_0+l_0)}(\sqrt[n]{2})^{k_1+l_1} \sum_{i=0}^{n/2-1} \zeta_n^{(2i+1)(k_1+l_1)}.$$

Again it must be 0 unless $\zeta_n^{2(k_1+l_1)} = 1$, i.e., $k_1 + l_1 = 0, \ n/2$. Combine the two cases we proves this lemma. $\qquad\square$

Lemma 4.3. $Tr(\zeta_n^{k_0+l_0}) = 0$ unless $k_0 = l_0 = 0$ or $k_0 + l_0 = n/2$. If so, we trivially have $Tr(1) = n^2/4$ and $Tr(-1) = -n^2/4$.

Proof. From the proof of Lemma 4.2, we obtain

$$\sum_{(a,b)\in H_a} \sigma_{a,b}(\zeta_n^{k_0+l_0}) = \frac{n}{2}\zeta_n^{a(k_0+l_0)}.$$

Therefore, the trace $\text{Tr}(\zeta_n^{k_0+l_0})$ becomes

$$\text{Tr}(\zeta_n^{k_0+l_0}) = \sum_{a \text{ odd}} \frac{n}{2}\zeta_n^{a(k_0+l_0)} = \frac{n}{2} \cdot \sum_{i=0}^{n/2-1} \zeta_n^{(2i+1)(k_0+l_0)},$$

which is 0 unless $k_0 + l_0 = 0$ (i.e., $k_0 = l_0 = 0$) or $k_0 + l_0 = n/2$. If $k_0 = l_0 = 0$, $\text{Tr}(1)$ is just the order of G, i.e., $n^2/4$. If $k_0 + l_0 = n/2$, then $\text{Tr}(\zeta_n^{k_0+l_0})$ is just $\text{Tr}(-1) = -n^2/4$. $\qquad\square$

Lemma 4.4.

$$Tr(\zeta_n^{k_0+l_0} \sqrt[n]{2}^{n/2}) = \begin{cases} n^2/4, & \text{if } k_0 + l_0 = n/8 \text{ or } 7n/8, \\ -n^2/4, & \text{if } k_0 + l_0 = 3n/8 \text{ or } 5n/8, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. For each $b \in \mathbb{Z}_n$, let the subset H_b' of G be defined as

$$H_b' = \begin{cases} \{(a,b) : a \equiv 1,7 \bmod 8\}, & \text{if } b \text{ is even;} \\ \{(a,b) : a \equiv 3,5 \bmod 8\}, & \text{if } b \text{ is odd.} \end{cases}$$

Therefore G is equal to the disjoint union $\cup_{b=0}^{n-1} H_b'$, so the trace can be computed via

$$\text{Tr}(\zeta_n^{k_0+l_0} \sqrt[n]{2}^{n/2}) = \sum_{b=0}^{n-1} \sum_{(a,b)\in H_b'} \sigma_{a,b}(\zeta_n^{k_0+l_0} \sqrt[n]{2}^{n/2}). \tag{4.2}$$

When b is even, we have

$$\sum_{(a,b)\in H_b'} \sigma_{a,b}(\zeta_n^{k_0+l_0} \sqrt[n]{2}^{n/2}) = \sqrt{2} \cdot \left(\zeta_n^{k_0+l_0} + \zeta_n^{7(k_0+l_0)}\right) \sum_{i=0}^{n/8-1} \zeta_n^{8i(k_0+l_0)}.$$

Similarly, when b is odd, we have

$$\sum_{(a,b)\in H_b'} \sigma_{a,b}(\zeta_n^{k_0+l_0} \sqrt[n]{2}^{n/2}) = -\sqrt{2} \cdot \left(\zeta_n^{3(k_0+l_0)} + \zeta_n^{5(k_0+l_0)}\right) \sum_{i=0}^{n/8-1} \zeta_n^{8i(k_0+l_0)}.$$

Both equations are 0 unless $\zeta_n^{8(k_0+l_0)} = 1$, i.e., $k_0 + l_0 = jn/8$ with $j = 0, 1, 2, \ldots, 7$. In these cases, we have

$$\mathrm{Tr}(\zeta_n^{jn/8} \sqrt[n]{2}^{n/2}) = \frac{\sqrt{2}n^2}{16} \left(\zeta_n^{jn/8} - \zeta_n^{3jn/8} - \zeta_n^{5jn/8} + \zeta_n^{7jn/8} \right),$$

whose value can be easily computed for different j:

$$\mathrm{Tr}(\zeta_n^{jn/8} \sqrt[n]{2}^{n/2}) = \begin{cases} n^2/4, & \text{if } j = 1, 7; \\ -n^2/4, & \text{if } j = 3, 5. \end{cases}$$

Thus completes the proof of this lemma.

$\square$

4.2 The Absolute Discriminant and the Dual Basis

From Theorem 4.1, if we choose a special order of the indices k_0, k_1, l_0 and l_1, the $n^2/4 \times n^2/4$ trace-pairing matrix $\mathrm{Tr}(\zeta_n^{k_0} \sqrt[n]{2}^{k_1} \cdot \zeta_n^{l_0} \sqrt[n]{2}^{l_1})$ becomes a block diagonal matrix, which consists of $n/2$ blocks of size 1×1 and $n(n-2)/8$ blocks of size 2×2. The 1×1 block is either $n^2/4$ or $-n^2/4$, and the 2×2 block can be chosen to be

$$\begin{pmatrix} n^2/4 & -n^2/4 \\ -n^2/4 & -n^2/4 \end{pmatrix}.$$

Recall that the absolute discriminant $\mathrm{disc}(\mathcal{L})$ of a lattice $\mathcal{L}$ is the square of the fundamental volume of $\mathcal{L}$ [19, 22, 34]. The absolute discriminant of $\mathcal{R}$, $\mathrm{disc}(\sigma(\mathcal{R})/\mathbb{Z})$, is also written as $\mathrm{disc}(\mathcal{R}/\mathbb{Z})$ [22]. Equivalently, it is the absolute value of the determinant of the trace-pairing matrix of an integral basis [19]:

$$\mathrm{disc}(\mathcal{R}/\mathbb{Z}) = \left| \det \left(\mathrm{Tr} \left(\zeta_n^{k_0} \sqrt[n]{2}^{k_1} \cdot \zeta_n^{l_0} \sqrt[n]{2}^{l_1} \right) \right) \right|.$$

Therefore, it is equal to

$$\mathrm{disc}(\mathcal{R}/\mathbb{Z}) = 2^{n(n-2)/8} \cdot \left(n^2/4 \right)^{n^2/4},$$

which is a power of 2.

By definition the dual of the integral basis of $\mathcal{R}$ is a set of elements of F

$$\{ e_{l_0,l_1} \in F : 0 \le l_0, l_1 < n/2 \}$$

such that $\mathrm{Tr}(e_{l_0,l_1} \cdot \zeta_n^{k_0} \sqrt[n]{2}^{k_1})$ equals 1 if $l_0 = k_0$, $l_1 = k_1$, and 0 otherwise. Hence e_{l_0,l_1} forms an integral basis for $\mathcal{R}^\vee$. The construction of e_{l_0,l_1} follows immediately from Theorem 4.1.

Proposition 4.5. *When $l_1 = 0$, $e_{l_0,0}$ is given by*

$$e_{0,0} = \frac{4}{n^2} \text{ and } e_{l_0,0} = -\frac{4}{n^2} \cdot \zeta_n^{n/2-l_0} \text{ if } l_0 > 0.$$

When $0 < l_1 < n/2$, e_{l_0,l_1} is given by

$$e_{l_0,l_1} = \begin{cases} \frac{2}{n^2} \zeta_n^{n/8-l_0} \sqrt[n]{2}^{n/2-l_1} - \frac{2}{n^2} \zeta_n^{3n/8-l_0} \sqrt[n]{2}^{n/2-l_1}, & \text{if } 0 \le l_0 \le \frac{n}{8}; \\ -\frac{2}{n^2} \zeta_n^{3n/8-l_0} \sqrt[n]{2}^{n/2-l_1} - \frac{2}{n^2} \zeta_n^{5n/8-l_0} \sqrt[n]{2}^{n/2-l_1}, & \text{if } \frac{n}{8} < l_0 \le \frac{3n}{8}; \\ -\frac{2}{n^2} \zeta_n^{5n/8-l_0} \sqrt[n]{2}^{n/2-l_1} + \frac{2}{n^2} \zeta_n^{7n/8-l_0} \sqrt[n]{2}^{n/2-l_1}, & \text{if } \frac{3n}{8} < l_0 < \frac{n}{2}. \end{cases}$$

4.3 The First Mininums of Ideal Lattices

A lattice is a discrete subgroup of a real vector space $\mathbb{R}^m$, $m \in \mathbb{N}_+$ [29]. In this paper, we are only concerned with the full-rank lattices, i.e., the lattices generated by a basis of $\mathbb{R}^m$: $\mathcal{L} = \{\sum_{i=1}^m x_i \mathbf{v}_i : (x_i)_{1 \leq i \leq m} \in \mathbb{Z}^m\}$, where $\{\mathbf{v}_i : 1 \leq i \leq m\}$ forms a basis of $\mathbb{R}^m$. The dual lattice of $\mathcal{L}$ is defined as $\mathcal{L}^* = \{\mathbf{x} \in \mathbb{R}^m : \langle \mathcal{L}, \mathbf{x} \rangle \subset \mathbb{Z}\}$, where $\langle \cdot, \cdot \rangle$ is the natural inner product on $\mathbb{R}^m$. From the definition, we immediately deduce that $(\mathcal{L}^*)^* = \mathcal{L}$. The i-th minimum $\lambda_i(\mathcal{L})$ is defined as the minimum of $\max_{1 \leq j \leq i} \|\mathbf{x}_j\|$ over all choices of i linearly independent vectors $\mathbf{x}_1, \ldots, \mathbf{x}_i \in \mathcal{L}$ [19]. In particular, $\lambda_1(\mathcal{L})$ is the shortest nonzero vector of $\mathcal{L}$.

In this paper, we are mostly interested in the lattices of H that come from the fractional ideals associated with $\mathcal{R}$ [20, 22]. Recall that a fractional ideal $\mathcal{I}$ associated with $\mathcal{R}$ is an $\mathcal{R}$-submodule such that there exists a nonzero element $r \in \mathcal{R}$ such that $r\mathcal{I} \subset \mathcal{R}$ [22]. A priori, an ideal of $\mathcal{R}$ is a fractional ideal. Because the degree of F over $\mathbb{Q}$ is $n^2/4$, a fractional ideal $\mathcal{I}$ always has an integral basis consisting of $n^2/4$ elements: $\{u_i \in \mathcal{I} : 1 \leq i \leq n^2/4\}$ [22]. Then, under the canonical embedding, the image of $\mathcal{I}$, denoted by $\sigma(\mathcal{I})$, is a full-rank lattice in H with a basis $\{\sigma(u_i) : 1 \leq i \leq n^2/4\}$. In particular, $\sigma(\mathcal{R})$ is a full-rank lattice of H with a basis $\{\sigma(\zeta_n^{k_0} \sqrt[n]{2}^{k_1}) : 0 \leq k_0, k_1 < n/2\}$. In this paper, when we say the lattice $\mathcal{I}$ what we really mean is the lattice $\sigma(\mathcal{I})$ in H. Furthermore, the norm of an element $x \in F$ is the norm of the vector $\sigma(x)$ in $\mathbb{C}^{n^2/4}$, i.e., $\|x\| = \|\sigma(x)\|$.

If $\mathcal{I}$ is an ideal in $\mathcal{R}$, then the quotient ring $\mathcal{R}/\mathcal{I}$ only has finitely many elements, whose order by definition is the norm of $\mathcal{I}$, i.e., $\mathrm{Nm}(\mathcal{I}) = |\mathcal{R}/\mathcal{I}|$. To define the norm of a fractional ideal, we first choose an $r \in \mathcal{R}$ such that $r\mathcal{I} \subset \mathcal{R}$, then $\mathrm{Nm}(\mathcal{I})$ is defined to be $\mathrm{Nm}(r\mathcal{I})/\mathrm{Nm}(\langle r \rangle)$, where $\langle r \rangle$ is the principal ideal generated by r. This definition is independent of the choice of r [20, 22]. Given a fractional ideal $\mathcal{I} \subset F$, its inverse $\mathcal{I}^{-1}$ and dual $\mathcal{I}^\vee$ are defined to be

$$\mathcal{I}^{-1} = \{x \in F : x\mathcal{I} \subset \mathcal{R}\}, \ \mathcal{I}^\vee = \{x \in F : \mathrm{Tr}(x\mathcal{I}) \subset \mathbb{Z}\},$$

where the trace $\mathrm{Tr}(x)$ of $x \in F$ can be computed by $\sum_{(a,b) \in G} \sigma_{a,b}(x)$ [19, 22]. Notice that $\mathcal{I}^\vee$ is also a fractional ideal and $(\mathcal{I}^\vee)^\vee = \mathcal{I}$ [22]. However, $\sigma(\mathcal{I}^\vee)$ is the complex conjugate of the dual lattice of $\sigma(\mathcal{I})$ in H [19]. Let $\Delta_\mathcal{R}$ be the absolute discriminant of F [22]. From Lemma 2.9 of [19], we obtain the upper and lower bounds on the first minimums of the ideal lattices.

Lemma 4.6. *The first minimum of any fractional ideal $\mathcal{I}$ satisfies*

$$\frac{n}{2} \cdot \mathrm{Nm}(\mathcal{I})^{4/n^2} \leq \lambda_1(\mathcal{I}) \leq \frac{n}{2} \cdot \mathrm{Nm}(\mathcal{I})^{4/n^2} \cdot \sqrt{\Delta_\mathcal{R}^{4/n^2}}.$$

From this lemma, we immediately have $\lambda_1(\mathcal{R}) \geq n/2$. But the norm of $\zeta_n^{k_0} \in \mathcal{R}$, $0 \leq k_0 < n/2$, is $n/2$, hence we deduce that $\lambda_1(\mathcal{R}) = n/2$. Since the norm of $\zeta_n^{k_0} \sqrt[n]{2}^{k_1}$ is $2^{k_1/n} \cdot n/2$, we learn that $\lambda_{n^2/4}(\mathcal{R}) \leq (\sqrt{2})^{1-2/n} \lambda_1(\mathcal{R})$. From this lemma, we also have $\lambda_1(\mathcal{R}^\vee) \geq \frac{n}{2} \cdot \mathrm{Nm}(\mathcal{R}^\vee)^{4/n^2}$. The norm of $\mathcal{R}^\vee$ can be computed directly from the dual basis e_{l_0,l_1} (Proposition 4.5) [22, 34]. More precisely, since $(n^2/2) \cdot \mathcal{R}^\vee \subset \mathcal{R}$, the norm of $\mathcal{R}^\vee$ can be computed by [19]

$$\mathrm{Nm}(\mathcal{R}^\vee) = \mathrm{Nm}\left(\frac{n^2}{2} \cdot \mathcal{R}^\vee\right) \cdot \mathrm{Nm}^{-1}\left(\frac{n^2}{2}\right).$$

The norm $\mathrm{Nm}(n^2/2)$ of $n^2/2$ is just $(n^2/2)^{n^2/4}$ [22, 34]. The norm of $(n^2/2) \cdot \mathcal{R}^\vee$ is the number of elements of the finite group $\mathcal{R}/\left((n^2/2) \cdot \mathcal{R}^\vee\right)$, which can be directly computed from the integral basis e_{l_0,l_1} of $\mathcal{R}^\vee$ given in Proposition 4.5:

$$\left|\mathcal{R}/\left((n^2/2) \cdot \mathcal{R}^\vee\right)\right| = 2^{n/2} \cdot 2^{n(n-2)/8}.$$

Hence the norm $\mathrm{Nm}(\mathcal{R}^\vee)$ of $\mathcal{R}^\vee$ is $2^{-n(n-2)/8} \cdot (n^2/4)^{-n^2/4}$, i.e., $\Delta_{\mathcal{R}}^{-1}$, from which we immediately deduce that $\lambda_1(\mathcal{R}^\vee) \geq 2^{\frac{1}{2}+\frac{1}{n}}/n$. But the norm of $e_{l_0,n/2-1} \in \mathcal{R}^\vee$ is $2^{\frac{1}{2}+\frac{1}{n}}/n$, hence we must have $\lambda_1(\mathcal{R}^\vee) = 2^{\frac{1}{2}+\frac{1}{n}}/n$. Namely, $\mathcal{R}^\vee$ achieves its first minimum at the element $e_{l_0,n/2-1}$. Moreover, since the norm of $e_{l_0,0}$ is $2/n$, we have $\lambda_{n^2/4}(\mathcal{R}^\vee) \leq 2/n$.

For a general fractional ideal $\mathcal{I}$, suppose $v \in \mathcal{I}$ is a nonzero element at which $\mathcal{I}$ achieves its first minimum, i.e., $\|v\| = \lambda_1(\mathcal{I})$, then the elements $(\zeta_n^{k_0} \sqrt[n]{2}^{k_1}) \cdot v$ with $0 \leq k_0, k_1 < n/2$ will be $n^2/4$ linearly independent elements of $\mathcal{I}$. From Section 3, the norm of the element $(\zeta_n^{k_0} \sqrt[n]{2}^{k_1}) \cdot v$ is $(\sqrt[n]{2})^{k_1} \cdot \|v\|$, from which we deduce

$$\lambda_1(\mathcal{I}) = \cdots = \lambda_{n/2}(\mathcal{I}) \text{ and } \lambda_{n^2/4}(\mathcal{I}) \leq \left(\sqrt{2}\right)^{1-2/n} \lambda_1(\mathcal{I}). \qquad (4.3)$$

5 Order Learning With Errors

In this section, we first study the error distributions on the ideal lattices associated with $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$. Then we formulate the Search and Average-Case Decision Order-LWE problem in $\mathcal{R}$ and study their hardness using available results in literature [4, 19].

5.1 Computational Problems in Lattice Theory

In lattice theory, there are several important computational problems, among which are the shortest vector problem (SVP), the shortest independent vectors problem (SIVP), the bounded distance decoding (BDD) problem and the discrete Gaussian sampling (DGS) problem. Suppose $\gamma(m) \geq 1$ is a function of m. The goal of SVP_γ is to find a nonzero vector with length at most $\gamma(m) \cdot \lambda_1(\mathcal{L})$. The goal of SIVP_γ is to find m linearly independent lattice vectors with length at most $\gamma(m) \cdot \lambda_m(\mathcal{L})$. Given a number $0 < d < \lambda_1(\mathcal{L})/2$ and assume $\mathbf{y} \in \mathbb{R}^m$ is of the form $\mathbf{y} = \mathbf{x} + \mathbf{e}$ for some $\mathbf{x} \in \mathcal{L}$ and $\|\mathbf{e}\| \leq d$, then the goal of BDD_d is to find $\mathbf{x}$ [19, 29].

The spherical Gaussian distribution D_r with width r on $\mathbb{R}^m$ is a distribution whose probability density function is proportional to $\rho_r(\mathbf{x}) = \exp\left(-\pi\|\mathbf{x}\|^2/r^2\right)$. A discrete Gaussian distribution over $\mathcal{L}$ with width r, denoted by $D_{\mathcal{L},r}$, is a distribution in which the probability of the event $\mathbf{x} \in \mathcal{L}$ is proportional to $\rho_r(\mathbf{x})$. Suppose φ is a positive real-valued function defined on the set of all lattices. Given a lattice $\mathcal{L}$ and a real number $r > \varphi(\mathcal{L})$, the goal of DGS_φ is to output a sample from $D_{\mathcal{L},r}$ [19, 29].

The smoothing parameter is an important concept in lattice theory [21]. Intuitively speaking, it is the smallest positive real number r starting from which the distribution $D_{\mathcal{L},r}$ behaves like a continuous Gaussian distribution [19].

Definition 5.1. *Given a lattice $\mathcal{L}$ and a real number $\epsilon > 0$, the smoothing parameter $\eta_\epsilon(\mathcal{L})$ is the smallest number $r > 0$ such that $\rho_{1/r}(\mathcal{L}^* \setminus \{\mathbf{0}\}) \leq \epsilon$.*

From Lemmas 3.2 and 3.3 of [21], we have the following lemma [19].

Lemma 5.2. *For any m-dimensional lattice $\mathcal{L}$, we have*

$$\eta_{2^{-2m}}(\mathcal{L}) \leq \sqrt{m}/\lambda_1(\mathcal{L}^*) \text{ and } \eta_\epsilon(\mathcal{L}) \leq \sqrt{\log(m/\epsilon)}\lambda_n(\mathcal{L}) \text{ for all } 0 \leq \epsilon \leq 1.$$

The computational problems in lattice theory introduced in Section 5.1 are also defined over the special lattices that come from the fractional ideals associated with $\mathcal{R}$. For a real number $\gamma \geq 1$, the $\mathcal{R}$-SVP_γ problem is: given a fractional ideal $\mathcal{I}$ in F, find a nonzero element $x \in \mathcal{I}$ such that $\|x\| \leq \gamma \cdot \lambda_1(\mathcal{I})$. The $\mathcal{R}$-SIVP_γ problem is: given a fractional ideal $\mathcal{I}$, find $n^2/4$ linearly independent elements in $\mathcal{I}$ with norms at most $\gamma \cdot \lambda_{n^2/4}(\mathcal{I})$. The $\mathcal{R}$-BDD_γ problem is: given a fractional ideal $\mathcal{I}$, a positive real number $d < \lambda_1(\mathcal{I})/2$ and an element y of the form $y = x + e$ with $x \in \mathcal{I}$ and $\|e\| \leq d$, find x.

5.2 The Error Distributions over Ideal Lattices

An error distribution ψ over the ideal lattices of $\mathcal{R}$ (or over $F_\mathbb{R}$) is not directly defined with respect to the naive coefficient embedding, i.e., the coefficients of an element of $F_\mathbb{R}$ with respect to the integral basis $\{\zeta_n^{k_0} \sqrt[n]{2}^{k_1} : 0 \leq k_0, k_1 < n/2\}$. Instead, ψ is first defined on H, which then pulls back to a probability distribution on the ideal lattices of F under the canonical embedding σ. The readers are referred to the paper [19] for an explanation about the motivations behind this definition.

One way to define the elliptical Gaussian distributions on H is through the orthonormal basis $\{\mathbf{h}_{a,b}\}_{(a,b) \in G}$ (Eq. (3.4)). Let $\mathbf{r} = (r_{a,b})_{(a,b) \in G} \in (\mathbb{R}^+)^{n^2/4}$ be a vector of positive real numbers such that $r_{a,b} = r_{((n-1)a,(n-1)b)}$ for every pair $(a,b) \in N$. Then a sample from the elliptical Gaussian distribution $D_\mathbf{r}$ is a vector $\sum_{(a,b) \in G} x_{a,b}\mathbf{h}_{a,b}$, where each $x_{a,b}$ is sampled independently from the one-dimensional Gaussian distribution $D_{r_{a,b}}$ [19].

Recall that the Proposition 3.1 tells us that the canonical embedding σ sends the integral basis $\{\zeta_n^{k_0} \sqrt[n]{2}^{k_1} : 0 \leq k_0, k_1 < n/2\}$ to an orthogonal basis of $\mathbb{C}^{n^2/4}$. Hence an elliptical Gaussian distribution on H pulls back to an elliptical Gaussian distribution for the coefficient embedding. However, the norm of the basis element $\sigma\left(\zeta_n^{k_0} \sqrt[n]{2}^{k_1}\right)$ is $2^{k_1/n} \cdot n/2$, which depends on the value of k_1. But since $k_1 < n/2$, the ratio between the longest vector and shortest vector is smaller than $\sqrt{2}$. Therefore, there are small distortions of the elliptical Gaussian distributions under this pull-back. In particular, the pull-back of a spherical Gaussian distribution is only elliptical. This is different from the canonical embedding of the cyclotomic field case, which is an isometry up to an overall constant [19].

Definition 5.3. *Given a positive real number ϱ, let $\Psi_{\leq \varrho}$ be the set of all elliptical Gaussian distribution $D_\mathbf{r}$ over H ($\cong F_\mathbb{R}$) where for every $(a,b) \in N$ we have $r_{a,b} = r_{((n-1)a,(n-1)b)}$ and $r_{a,b} \leq \varrho$.*

Lemma 5.4. *For any positive real number ϱ, the family of distributions $\Psi_{\leq \varrho}$ is closed under the action of any element of $\mathrm{Gal}(F/\mathbb{Q})$.*

Proof. The action of $\sigma_{c,d} \in \mathrm{Gal}(F/\mathbb{Q})$ on $\Psi_{\leq \varrho}$ is induced by the action of $\sigma_{c,d}$ on H. For any distribution $D_\mathbf{r} \in \Psi_{\leq \varrho}$, let $\sigma_{c,d}(D_\mathbf{r})$ be $D_{\mathbf{r}'}$. From the commutative

diagram in Eq. (3.3), $\sigma_{c,d}$ permutes the coordinates of the canonical embedding. Moreover, $\sigma_{c,d}$ preserves the complex conjugation pairs in the canonical embedding. Therefore, $r'_{a,b} = r'_{((n-1)a,(n-1)b)}$ and $(r'_{a,b})_{(a,b)\in N}$ is a permutation of $(r_{a,b})_{(a,b)\in N}$, which implies $r'_{a,b}$ is also at most ϱ. Hence we learn that $D_{\mathbf{r}'} \in \Psi_{\leq\varrho}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The discrete Gaussian sampling problem can also be defined over F [30]. The F-DGS_γ problem is: given a fractional ideal $\mathcal{I}$ and a number $s \geq \gamma = \gamma(\mathcal{I})$, output a sample from the distribution $D_{\mathcal{I},s}$.

Definition 5.5. *Given a positive real number δ, Υ_δ is a distribution over distributions: a sample from Υ_δ is an elliptical Gaussian distribution $D_{\mathbf{r}}$ on H with parameters*

$$r_{a,b}^2 = r_{(n-1)a,(n-1)b}^2 = \delta^2 \left(1 + nx_{a,b}/2\right),\ (a,b) \in N,$$

where $x_{a,b}$ is sampled independently from the distribution $\Gamma(2,1)$.

Recall that the probability density function for the Gamma distribution $\Gamma(2,1)$ is $x\exp(-x)$ for $x \geq 0$. The mean of $\Gamma(2,1)$ is 2, thus the noises sampled from Υ_δ are of size roughly $O(\delta \cdot n^{1/2})$ when n is large [19, 30].

5.3 The Order-LWE Problem

Given a probability distribution ψ, the notation $\mathbf{x} \leftarrow \psi$ means we draw a sample $\mathbf{x}$ according to ψ. For a finite set $\mathcal{S}$, $\mathbf{x} \leftarrow \mathcal{U}(\mathcal{S})$ means we draw a sample $\mathbf{x}$ uniformly randomly from $\mathcal{S}$. Given a positive integer $q \geq 2$, let $\mathcal{R}_q = \mathcal{R}/q\mathcal{R}$, $\mathcal{R}_q^\vee = \mathcal{R}^\vee/q\mathcal{R}^\vee$, and $\mathbb{T} = F_\mathbb{R}/\mathcal{R}^\vee$.

Definition 5.6 (Order-LWE **Distribution**). *Suppose we are given an element $\mathbf{s} \in \mathcal{R}_q^\vee$ (the secret key) and a noise distribution ψ defined over $F_\mathbb{R}$. The* Order-LWE *distribution $A_{\mathbf{s},\psi}$ is defined over $\mathcal{R}_q \times \mathbb{T}$, a sample of which is generated by choosing $\mathbf{a} \leftarrow \mathcal{R}_q$ uniformly randomly and $\mathbf{e} \leftarrow \psi$ according to ψ, and then outputting $(\mathbf{a}, \mathbf{b} = \mathbf{a} \cdot \mathbf{s}/q + \mathbf{e} \bmod \mathcal{R}^\vee)$.*

Notice that since $\mathbf{a} \cdot \mathbf{s}/q$ is in the abelian group $q^{-1}\mathcal{R}^\vee/\mathcal{R}^\vee$, the reduction mod $\mathcal{R}^\vee$ in the second entry of the sample is well-defined.

Definition 5.7 (Order-LWE, **Search**). *Let Ψ be a family of probability distributions over $F_\mathbb{R}$. Suppose we are given access to arbitrarily many independent samples from $A_{\mathbf{s},\psi}$ for arbitrary $\mathbf{s} \in \mathcal{R}_q^\vee$ and $\psi \in \Psi$, then the goal of the Search Order Learning With Errors problem, denoted by $\mathsf{OLWE}_{q,\Psi}$, is to find $\mathbf{s}$.*

Definition 5.8 (Order-LWE, **Average-Case Decision**). *Suppose Υ is a distribution over a family of error distributions over $F_\mathbb{R}$. The Average-Case Decision Order Learning With Errors problem, denoted by $\mathsf{DOLWE}_{q,\Upsilon}$, is to distinguish with non-negligible advantage between the following two cases*

1. *arbitrarily many independent samples from $A_{\mathbf{s},\psi}$ for a random choice of $(\mathbf{s}, \psi) \leftarrow \mathcal{U}(\mathcal{R}_q^\vee) \times \Upsilon$,*

2. *the same number of uniformly random and independent samples from $\mathcal{R}_q \times \mathbb{T}$.*

As the error distribution is added modulo $\mathcal{R}^\vee$, the error must not exceed the smoothing parameter of $\mathcal{R}$. Otherwise the Order-LWE distribution is statistically indistinguishable from the uniform distribution, thus making the Order-LWE problem impossible to solve. Since $\lambda_{n^2/4}(\mathcal{R}^\vee) \le 2/n$, Lemma 5.2 gives us an upper bound of the smoothing parameter: $O(2\sqrt{\log(n^2/4)}/n)$ [19].

On the other hand, as $\zeta_n^{n/8} + \zeta_n^{-n/8} = \sqrt{2}$ and $\zeta_n^{n/8} - \zeta_n^{-n/8} = \zeta_n^{n/4}\sqrt{2}$, from Proposition 4.5, we immediately deduce $(\sqrt{2}n^2/4) \cdot \mathcal{R}^\vee \subset \mathcal{R}$. In fact, $\sqrt{2}n^2/4$ is the smallest real number of $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$ that satisfies this property. If we instead work with the non-dual form of Order-LWE in practice, this property is helpful in choosing error distributions over $F_\mathbb{R}/\mathcal{R}$ [27].

5.4 The Reductions for Order-LWE Problem

The worst-case hardness of $\mathsf{OLWE}_{q,\Psi}$ comes from proof of Theorem 4.1 of [19] and the paper [4].

Proposition 5.9. *Let $\varrho = \varrho(n^2/4)$ (> 0) and $q = q(n^2/4)$ be such that $\varrho q \ge 2 \cdot \omega\left(\sqrt{\log(n^2/4)}\right)$. For some negligible function $\epsilon = \epsilon\left(n^2/4\right)$, there exists a probabilistic polynomial-time reduction from $\mathcal{R}\text{-}\mathsf{DGS}_\gamma$ to the problem $\mathsf{OLWE}_{q,\Psi_{\le\varrho}}$, where γ is*

$$\gamma = \max\left\{\eta_\epsilon\left(\mathcal{I}\right) \cdot \left(\sqrt{2}/\varrho\right) \cdot \omega\left(\sqrt{\log(n^2/4)}\right), \sqrt{n^2/2}/\lambda_1\left(\mathcal{I}^\vee\right)\right\}.$$

Here $\omega(\sqrt{\log(n^2/4)})$ denotes a fixed but arbitrary function that grows asymptotically faster than $\sqrt{\log(n^2/4)}$. There exist reductions from the standard lattice problems to $\mathcal{R}\text{-}\mathsf{DGS}_\gamma$ using lattice theory [19, 30]. In particular, an oracle for $\mathcal{R}\text{-}\mathsf{DGS}_\gamma$ with $\gamma = \eta_\epsilon(\mathcal{I}) \cdot \widetilde{O}(1/\varrho)$ implies the existence of an oracle for $\widetilde{O}(n/(2\varrho))$-approximate SIVP on the ideal lattices associated with $\mathcal{R}$. Moreover, from Eq. (4.3) this oracle also implies the existence of an oracle for $\widetilde{O}(n/(2\varrho))$-approximate SVP on the ideal lattices of F. The main theorem concerning the hardness of $\mathsf{DOLWE}_{q,\Upsilon}$ is Theorem 5.10, which follows from [4, 19]. A detailed proof of this theorem is given in Appendix A.

Theorem 5.10. *Let $\delta < 2\sqrt{\log(n^2/4)}/n$ and let $q = q(n^2) \ge 3$ be a poly(n)-bounded prime number such that both $X^{n/2} \equiv -1 \bmod q$ and $Y^n \equiv 2 \bmod q$ have solutions. Then there exists a polynomial-time reduction from $\widetilde{O}(n/(2\delta))$-approximate SIVP (or SVP) on the ideal lattices associated with $\mathcal{R}$ to $\mathsf{DOLWE}_{q,\Upsilon_\delta}$. Moreover, for any integer $\ell \ge 1$, we can replace the target problem in the reduction by the problem of solving DOLWE_{q,D_ξ} given only ℓ samples, where $\xi = \delta \cdot \left(n^2\ell/(4\log(n^2\ell/4))\right)^{1/4}$.*

6 The Two-Variable Number Theoretic Transform

In this section, we design a Two-Variable Number Theoretic Transform (2NTT) for the ring

$$\mathcal{R}_p = \mathcal{R}/p\mathcal{R} = \mathbb{F}_p[X,Y]/\langle X^{n/2}+1, Y^{n/2} - (X^{n/8} - X^{3n/8})\rangle, \qquad (6.1)$$

where p is a prime number such that $Y^n \equiv 2 \bmod p$ has n distinct solutions. The process of 2NTT consumes $O(n \log_2 n)$ vector operations (vector summation and scalar multiplication), thus it has an intrinsic parallel algebraic structure compared to the one-variable NTT [18, 31]. This property is perhaps the most important new feature of $\mathcal{R}$.

6.1 The Splittings of Polynomials over a Finite Field

We choose a prime number p such that $X^n \equiv 1 \bmod p$ has a primitive solution $\alpha \in \mathbb{F}_p$ and $Y^n \equiv 2 \bmod p$ has a solution $\beta \in \mathbb{F}_p$. Since α is a primitive root, we must have $\alpha^{n/2} \equiv -1 \bmod p$, thus $X^n \equiv 1 \bmod p$ has n distinct solutions in $\mathbb{F}_p$: $\{\alpha^i : 0 \le i < n\}$. The n distinct solutions of $Y^n \equiv 2 \bmod p$ are $\{\alpha^i \beta : 0 \le i < n\}$. Replace β by $\alpha\beta$ if necessary, we can always assume $\beta^{n/2} \equiv \alpha^{n/8} - \alpha^{3n/8} \bmod p$. Therefore, the polynomial equations

$$X^{n/2} \equiv -1 \bmod p \text{ and } Y^{n/2} \equiv X^{n/8} - X^{3n/8} \bmod p \qquad (6.2)$$

have $n^2/4$ solutions in $\mathbb{F}_p$. The first equation has $n/2$ solutions $\{\alpha^{2i+1} : 0 \le i < n/2\}$. For every solution $X = \alpha^{2i+1}$, the second equation also has $n/2$ solutions for Y:

$$Y = \begin{cases} \alpha^{2j}\beta, \ 0 \le j < n/2, \text{ if } i \equiv 0, 3 \bmod 4, \\ \alpha^{2j+1}\beta, \ 0 \le j < n/2, \text{ if } i \equiv 1, 2 \bmod 4. \end{cases}$$

6.2 The One-Variable Negacyclic NTT

Let us now review the negacyclic NTT for the ring $\mathbb{F}_p[X]/\langle X^{n/2}+1 \rangle$, where p is a prime number such that $X^{n/2}+1 \equiv 0 \bmod p$ has $n/2$ solutions: $\{\alpha^{2i+1} : 0 \le i < n/2\}$. Given a polynomial $\mathbf{f}(X) = \sum_{k=0}^{n/2-1} f_k X^k$ with $f_k \in \mathbb{F}_p$, the NTT of $\mathbf{f}$, denoted by $\widetilde{\mathbf{f}}$, is the evaluations of $\mathbf{f}$ at the $n/2$ solutions of $X^{n/2}+1 \equiv 0 \bmod p$:

$$\tilde{\mathbf{f}} = \left(\mathbf{f}(\alpha), \mathbf{f}(\alpha^3), \ldots, \mathbf{f}(\alpha^{n-1}) \right) \in \mathbb{F}_p^{n/2}. \qquad (6.3)$$

Hence we deduce that for any two polynomials $\mathbf{f}$ and $\mathbf{f}'$, $\widetilde{\mathbf{f} \cdot \mathbf{f}'} = \widetilde{\mathbf{f}} \odot \widetilde{\mathbf{f}'}$, where $\odot$ is the Hadamard product (entrywise product) of vectors.

If we naively evaluate $\mathbf{f}$ at the $n/2$ roots, then the complexity is $O(n^2)$. Luckily, there is an efficient algorithm to compute NTT with complexity $O(n \log_2 n)$: the Cooley-Tukey butterfly, which is essentially based on the Chinese Remainder Theorem (CRT) [3]. In $\mathbb{F}_p$, $X^{n/2}+1$ has a factorization $\prod_{i=0}^{n/2-1} (X - \alpha^{2i+1}) \bmod p$, which induces an isomorphism [3]

$$\mathbb{F}_p[X]/\langle X^{n/2}+1 \rangle \cong \prod_{i=0}^{n/2-1} \mathbb{F}_p[X]/\langle X - \alpha^{2i+1} \rangle.$$

Notice that the right hand side is isomorphic to $\mathbb{F}_p^{n/2}$. Under the CRT, a polynomial $\mathbf{f}(X)$ is sent to the vector

$$\mathbf{f}(X) \mapsto \left(\mathbf{f}(X) \bmod \left(X - \alpha^{2i+1} \right) \right)_{0 \le i < n/2} = \left(\mathbf{f}(\alpha^{2i+1}) \right)_{0 \le i < n/2},$$

which is just the NTT of $\mathbf{f}$. The idea of the Cooley-Tukey butterfly is that CRT can be computed using a sequence of polynomial mods.

The butterfly consists of $\log_2(n/2)$ stages, and let us look at each stage in detail. The first stage depends on the factorization

$$X^{n/2} + 1 = \left(X^{n/4} - \alpha^{n/4}\right)\left(X^{n/4} + \alpha^{n/4}\right) \bmod p,$$

from which we obtain an isomorphism

$$\mathbb{F}_p[X]/\langle X^{n/2} + 1 \rangle \cong \mathbb{F}_p[X]/\langle X^{n/4} - \alpha^{n/4} \rangle \times \mathbb{F}_p[X]/\langle X^{n/4} + \alpha^{n/4} \rangle.$$

The image of $\mathbf{f}$ under this isomorphism is

$$\mathbf{f}_0 = \mathbf{f} \bmod \left(X^{n/4} - \alpha^{n/4}\right) = \sum_{k=0}^{n/4-1} \left(f_k + \alpha^{n/4} f_{k+n/4}\right) X^k,$$

$$\mathbf{f}_1 = \mathbf{f} \bmod \left(X^{n/4} + \alpha^{n/4}\right) = \sum_{k=0}^{n/4-1} \left(f_k - \alpha^{n/4} f_{k+n/4}\right) X^k;$$

the computation of which consumes $O(n)$ (integer) operations. In the second stage, we need the factorizations

$$X^{n/4} - \alpha^{n/4} = \left(X^{n/8} - \alpha^{n/8}\right)\left(X^{n/8} - \alpha^{5n/8}\right) \bmod p,$$

$$X^{n/4} + \alpha^{n/4} = \left(X^{n/8} - \alpha^{3n/8}\right)\left(X^{n/8} - \alpha^{7n/8}\right) \bmod p;$$

which gives us isomorphisms

$$\mathbb{F}_p[X]/\langle X^{n/4} - \alpha^{n/4} \rangle \cong \mathbb{F}_p[X]/\langle X^{n/8} - \alpha^{n/8} \rangle \times \mathbb{F}_p[X]/\langle X^{n/8} - \alpha^{5n/8} \rangle,$$

$$\mathbb{F}_p[X]/\langle X^{n/4} + \alpha^{n/4} \rangle \cong \mathbb{F}_p[X]/\langle X^{n/8} - \alpha^{3n/8} \rangle \times \mathbb{F}_p[X]/\langle X^{n/8} - \alpha^{7n/8} \rangle.$$

We can compute the polynomial mods from the output of the first stage:

$$
\begin{aligned}
\mathbf{f}_{00} &= \mathbf{f}_0 \bmod \left(X^{n/8} - \alpha^{n/8}\right) &= \mathbf{f} \bmod \left(X^{n/8} - \alpha^{n/8}\right), \\
\mathbf{f}_{01} &= \mathbf{f}_0 \bmod \left(X^{n/8} - \alpha^{5n/8}\right) &= \mathbf{f} \bmod \left(X^{n/8} - \alpha^{5n/8}\right), \\
\mathbf{f}_{10} &= \mathbf{f}_1 \bmod \left(X^{n/8} - \alpha^{3n/8}\right) &= \mathbf{f} \bmod \left(X^{n/8} - \alpha^{3n/8}\right), \\
\mathbf{f}_{11} &= \mathbf{f}_1 \bmod \left(X^{n/8} - \alpha^{7n/8}\right) &= \mathbf{f} \bmod \left(X^{n/8} - \alpha^{7n/8}\right);
\end{aligned}
$$

which also consumes $O(n)$ operations.

Continue in this way, and suppose the output of the m-th stage is

$$\mathbf{f} \bmod \left(X^{n/2^{m+1}} - \alpha^{(2i+1)\cdot n/2^{m+1}}\right) \text{ with } 0 \le i < 2^m.$$

For every $0 \le i < 2^m$, the polynomial $X^{n/2^{m+1}} - \alpha^{(2i+1)\cdot n/2^{m+1}}$ has a factorization

$$\left(X^{n/2^{m+2}} - \alpha^{(2i+1)\cdot n/2^{m+2}}\right) \cdot \left(X^{n/2^{m+2}} - \alpha^{(2i+1+2^{m+1})\cdot n/2^{m+2}}\right). \tag{6.4}$$

Then in the $(m+1)$-th stage we compute

$$\mathbf{f} \bmod \left(X^{n/2^{m+2}} - \alpha^{(2i+1)\cdot n/2^{m+2}}\right) \text{ with } 0 \le i < 2^{m+1}$$

using the output of the m-th stage, which consumes $O(n)$ operations.

The butterfly comes to an end in the $\log_2(n/2)$-th stage, the output of which is Eq. (6.3). It is important to notice that the order of the components of the output vector depends on the implementation used, and in general it will not be in the order shown in Eq. (6.3). Since the complexity of each stage is $O(n)$, the total complexity of the butterfly is $O(n \log_2 n)$. Moreover, each stage of the butterfly is an invertible linear transformation. Hence if we take the inverse of each stage, we obtain INTT, whose complexity is also $O(n \log_2 n)$.

6.3 A Generalization of the One-Variable NTT

Given a prime number p that satisfies the conditions in Section 6.1, we generalize the butterfly in Section 6.2 to the ring $\mathbb{F}_p[Y]/\langle Y^{n/2} + \beta^{n/2}\rangle$. A polynomial in this ring is of the form $\mathbf{g}(Y) = \sum_{l=0}^{n/2-1} g_l Y^l$, $g_l \in \mathbb{F}_p$. In $\mathbb{F}_p$, $Y^{n/2} + \beta^n$ has a factorization

$$Y^{n/2} + \beta^{n/2} = \prod_{j=0}^{n/2-1} (Y - \alpha^{2j+1}\beta) \bmod p.$$

From the CRT [3], we have an isomorphism

$$\mathbb{F}_p[Y]/\langle Y^{n/2} + \beta^{n/2}\rangle \cong \prod_{j=0}^{n/2-1} \mathbb{F}_p[Y]/\langle Y - \alpha^{2j+1}\beta\rangle,$$

the right hand side of which is isomorphic to $\mathbb{F}_p^{n/2}$. Under the CRT, the image of $\mathbf{g}(Y)$ is

$$\mathbf{g}(Y) \mapsto \big(\mathbf{g}(Y) \bmod (Y - \alpha^{2j+1}\beta)\big)_{0 \le j < n/2} = \big(\mathbf{g}(\alpha^{2j+1}\beta)\big)_{0 \le j < n/2},$$

which is defined to be the NTT of $\mathbf{g}(Y)$. From the definition, we deduce

$$\widetilde{\mathbf{g} \cdot \mathbf{g}'} = \widetilde{\mathbf{g}} \odot \widetilde{\mathbf{g}'}.$$

The butterfly for computing the new NTT also consists of $\log_2(n/2)$ stages of polynomial mods. In the first stage, we need the factorization

$$Y^{n/2} + \beta^{n/2} = \left(Y^{n/4} - \alpha^{n/4}\beta^{n/4}\right)\left(Y^{n/4} - \alpha^{3n/4}\beta^{n/4}\right) \bmod p.$$

The output of the first stage is

$$\begin{aligned}
\mathbf{g}_0 &= \mathbf{g} \bmod \left(Y^{n/4} - \alpha^{n/4}\beta^{n/4}\right) = \sum_{l=0}^{n/4-1} \left(g_l + \alpha^{n/4}\beta^{n/4} \cdot g_{l+n/4}\right) Y^l, \\
\mathbf{g}_1 &= \mathbf{g} \bmod \left(Y^{n/4} - \alpha^{3n/4}\beta^{n/4}\right) = \sum_{l=0}^{n/4-1} \left(g_l + \alpha^{3n/4}\beta^{n/4} \cdot g_{l+n/4}\right) Y^l.
\end{aligned} \tag{6.5}$$

Notice that $O(n)$ operations are needed for the first stage. Suppose the output of the m-th stage is

$$\mathbf{g} \bmod \left(Y^{n/2^{m+1}} - \alpha^{(2j+1)\cdot n/2^{m+1}}\beta^{n/2^{m+1}}\right) \text{ with } 0 \le j < 2^m.$$

In the $(m+1)$-th stage, for every $0 \le j < 2^m$, we have a factorization

$$Y^{n/2^{m+1}} - \alpha^{(2j+1)\cdot n/2^{m+1}}\beta^{n/2^{m+1}} = \left(Y^{n/2^{m+2}} - \alpha^{(2j+1)\cdot n/2^{m+2}}\beta^{n/2^{m+2}}\right)$$
$$\cdot\left(Y^{n/2^{m+2}} - \alpha^{(2j+1+2^{m+1})\cdot n/2^{m+2}}\beta^{n/2^{m+2}}\right).$$

Then we compute

$$\mathbf{g} \bmod \left(Y^{n/2^{m+2}} - \alpha^{(2j+1)\cdot n/2^{m+2}}\beta^{n/2^{m+2}}\right) \text{ with } 0 \le j < 2^{m+1}$$

using the output of the m-th stage, which consumes $O(n)$ operations. The algorithm comes to an end in the $\log_2(n/2)$-th stage. From an algorithmic point of view, the butterfly for this generalization is almost the same as that in Section 6.2, while the only difference is that the twiddle factors in the m-th stage are multiplied by an overall factor $\beta^{n/2^{m+1}}$.

At the same time, the polynomial $Y^{n/2} - \beta^{n/2}$ has a factorization

$$Y^{n/2} - \beta^{n/2} = \prod_{j=0}^{n/2-1} (Y - \alpha^{2j}\beta) \bmod p,$$

which implies

$$\mathbb{F}_p[Y]/\langle Y^{n/2} - \beta^{n/2}\rangle \cong \prod_{j=0}^{n/2-1} \mathbb{F}_p[Y]/\langle Y - \alpha^{2j}\beta\rangle.$$

Under this isomorphism, $\mathbf{h}(Y) = \sum_{l=0}^{n/2-1} h_l Y^l$ in $\mathbb{F}_p[Y]/\langle Y^{n/2} - \beta^{n/2}\rangle$ is sent to

$$\mathbf{h}(Y) \mapsto \left(\mathbf{h}(Y) \bmod (Y - \alpha^{2j}\beta)\right)_{0\le j<n/2} = \left(\mathbf{h}(\alpha^{2j}\beta)\right)_{0\le j<n/2},$$

which is defined to be the NTT of $\mathbf{h}(Y)$. From the definition, we have

$$\widetilde{\mathbf{h}\cdot\mathbf{h}'} = \widetilde{\mathbf{h}} \odot \widetilde{\mathbf{h}'}.$$

Similarly, the butterfly for computing the NTT of $\mathbf{h}(Y)$ also consists of $\log_2(n/2)$ stages. The output of the m-th stage is

$$\mathbf{h} \bmod \left(Y^{n/2^{m+1}} - \alpha^{j\cdot n/2^m}\beta^{n/2^{m+1}}\right) \text{ with } 0 \le j < 2^m.$$

In the $(m+1)$-th stage, for every $0 \le j < 2^m$, we have a factorization

$$Y^{n/2^{m+1}} - \alpha^{j\cdot n/2^m}\beta^{n/2^{m+1}} = \left(Y^{n/2^{m+2}} - \alpha^{j\cdot n/2^{m+1}}\beta^{n/2^{m+2}}\right)$$
$$\cdot\left(Y^{n/2^{m+2}} - \alpha^{(j+2^m)\cdot n/2^{m+1}}\beta^{n/2^{m+2}}\right).$$

Then we compute

$$\mathbf{h} \bmod \left(Y^{n/2^{m+2}} - \alpha^{j\cdot n/2^{m+1}}\beta^{n/2^{m+2}}\right) \text{ with } 0 \le j < 2^{m+1}$$

using the output of the m-th stage. Now compared to the butterfly in Section 6.2, the twiddle factors in the m-th stage is multiplied by an overall factor $(\alpha^{-1}\beta)^{n/2^{m+1}}$.

6.4 The 2NTT

Now, we are ready to introduce the 2NTT for the ring $\mathcal{R}_p$, where the prime number p satisfies the conditions in Section 6.1. We design an efficient algorithm (vector butterfly) to compute the 2NTT (and its inverse). An element of $\mathcal{R}_p$ is of the form

$$\mathbf{F}(X,Y) = \sum_{k=0}^{n/2-1} \sum_{l=0}^{n/2-1} f_{k,l} X^k Y^l \text{ with } f_{k,l} \in \mathbb{F}_p.$$

For later convenience, the $n/2 \times n/2$ coefficient matrix of $\mathbf{F}(X,Y)$ is also denoted by $\mathbf{F}$:

$$\mathbf{F} = (f_{k,l})_{0 \le k,l < n/2}.$$

Definition 6.1. *The 2NTT of $\mathbf{F} \in \mathcal{R}_p$, denoted by $\widetilde{\mathbf{F}}$, is the evaluation of $\mathbf{F}$ at the $n^2/4$ solutions of the two polynomial equations in Eq. (6.2).*

Thus $\widetilde{\mathbf{F}}$ is an $n/2 \times n/2$ matrix with entries in $\mathbb{F}_p$. From the definition, we immediately deduce that $\widetilde{\mathbf{F} \cdot \mathbf{F}'} = \widetilde{\mathbf{F}} \odot \widetilde{\mathbf{F}'}$ for any two polynomials $\mathbf{F}$ and $\mathbf{F}'$ in $\mathcal{R}_p$, where $\odot$ is the Hadamard product (entrywise product) of two matrices. We now give an efficient butterfly algorithm to compute 2NTT, which consists of three phases: the transverse vector butterfly, transpose and the longitudinal vector butterfly.

6.4.1 The Transverse Vector Butterfly

Let $\mathbf{F}_{k,-}$ be the k-th row of the matrix $\mathbf{F}$, i.e., $\mathbf{F}_{k,-} = (f_{k,0}, \ldots, f_{k,n/2-1})$. Let $\mathbf{G}$ be a polynomial with vector-valued coefficients:

$$\mathbf{G}(X) = \sum_{k=0}^{n/2-1} X^k \cdot \mathbf{F}_{k,-}.$$

We generalize the butterfly in Section 6.2 to evaluate $\mathbf{G}$ at $X = \alpha^{2i+1}$ for $0 \le i < n/2$, which is called the transverse vector butterfly. It consists of $\log_2(n/2)$ stages and the first stage is given by

$$\mathbf{G}_0 = \mathbf{G} \bmod \left(X^{n/4} - \alpha^{n/4}\right) = \sum_{k=0}^{n/4-1} X^k \cdot \left(\mathbf{F}_{k,-} + \alpha^{n/4} \cdot \mathbf{F}_{k+n/4,-}\right),$$

$$\mathbf{G}_1 = \mathbf{G} \bmod \left(X^{n/4} + \alpha^{n/4}\right) = \sum_{k=0}^{n/4-1} X^k \cdot \left(\mathbf{F}_{k,-} - \alpha^{n/4} \cdot \mathbf{F}_{k+n/4,-}\right).$$

Notice that the integer summation and multiplication in the negacyclic NTT in Sections 6.2 are replaced by vector summation and scalar multiplication in the transverse vector butterfly. Thus the first stage costs $O(n)$ vector operations. Similarly, the output of the second stage is

$$
\begin{aligned}
\mathbf{G}_{00} &= \mathbf{G}_0 \bmod X^{n/8} - \alpha^{n/8}, & \mathbf{G}_{01} &= \mathbf{G}_0 \bmod X^{n/8} - \alpha^{5n/8}, \\
\mathbf{G}_{10} &= \mathbf{G}_1 \bmod X^{n/8} - \alpha^{3n/8}, & \mathbf{G}_{11} &= \mathbf{G}_1 \bmod X^{n/8} - \alpha^{7n/8},
\end{aligned}
\tag{6.6}
$$

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the computation of which also costs $O(n)$ vector operations. We can continue in the same way as the negacyclic NTT in Section 6.2. Suppose the output of the m-th stage is

$$\mathbf{G}(X) \bmod \left(X^{n/2^{m+1}} - \alpha^{(2i+1)\cdot n/2^{m+1}}\right) \text{ with } 0 \le i < 2^m.$$

Then in the $(m+1)$-th stage we compute

$$\mathbf{G}(X) \bmod \left(X^{n/2^{m+2}} - \alpha^{(2i+1)\cdot n/2^{m+2}}\right) \text{ with } 0 \le i < 2^{m+1}$$

using the factorization in Eq. (6.4) and the output of the m-th stage, which consumes $O(n)$ vector operations. The transverse vector butterfly comes to an end in the $\log_2(n/2)$-th stage, the output of which are $n/2$ vectors of length $n/2$:

$$\mathbf{G}(X) \bmod \left(X - \alpha^{2i+1}\right) = \mathbf{G}(\alpha^{2i+1}) \in \mathbb{F}_p^{n/2} \text{ with } 0 \le i < n/2.$$

The order of the $n/2$ output vectors depends on the implementation of the algorithm, which is similar to the negacyclic NTT in Section 6.2. Each stage costs $O(n)$ vector operations, so in total the transverse vector butterfly costs $O(n \log_2 n)$ vector operations.

6.4.2 The Transpose

The output vectors of the transverse vector butterfly fall into four groups:

1. $\mathbf{B}_{00}$, whose rows are vectors of the form $\mathbf{G}(\alpha^{2i+1})$ with $i = 0 \bmod 4$. It comes from the branch $\mathbf{G}_{00}$ in Eq. (6.6).

2. $\mathbf{B}_{01}$, whose rows are vectors of the form $\mathbf{G}(\alpha^{2i+1})$ with $i = 2 \bmod 4$. It comes from the branch $\mathbf{G}_{01}$ in Eq. (6.6).

3. $\mathbf{B}_{10}$, whose rows are vectors of the form $\mathbf{G}(\alpha^{2i+1})$ with $i = 1 \bmod 4$. It comes from the branch $\mathbf{G}_{10}$ in Eq. (6.6).

4. $\mathbf{B}_{11}$, whose rows are vectors of the form $\mathbf{G}(\alpha^{2i+1})$ with $i = 3 \bmod 4$. It comes from the branch $\mathbf{G}_{11}$ in Eq. (6.6).

Notice that each of $\mathbf{B}_{00}$, $\mathbf{B}_{01}$, $\mathbf{B}_{10}$ and $\mathbf{B}_{11}$ is an $n/8 \times n/2$ matrix. The concatenations of these four matrices give us two $n/4 \times n/2$ matrices:

$$\mathbf{C}' = \begin{pmatrix} \mathbf{B}_{00} \\ \mathbf{B}_{11} \end{pmatrix}, \quad \mathbf{C}'' = \begin{pmatrix} \mathbf{B}_{01} \\ \mathbf{B}_{10} \end{pmatrix}.$$

The l-th column of $\mathbf{C}'$ (resp. $\mathbf{C}''$) is denoted by $\mathbf{C}'_{-,l}$ (resp. $\mathbf{C}''_{-,l}$), which is a column vector of length $n/4$. Define polynomials (with vector-valued coefficients) $\mathbf{H}'$ and $\mathbf{H}''$ to be

$$\mathbf{H}'(Y) = \sum_{l=0}^{n/2-1} \mathbf{C}'_{-,l} Y^l \text{ and } \mathbf{H}''(Y) = \sum_{l=0}^{n/2-1} \mathbf{C}''_{-,l} Y^l.$$

6.4.3 The Longitudinal Vector Butterfly

Now comes the longitudinal vector butterfly. For the polynomial $\mathbf{H}'(Y)$, we need to evaluate it at the roots $Y = \alpha^{2j}\beta$ with $0 \le j < n/2$. The vector butterfly for this evaluation is a vector version of the butterfly in Section 6.3, which costs $O(n\log_2 n)$ vector operations. The output are $n/2$ vectors with length $n/4$, which form an $n/4 \times n/2$ matrix $\widetilde{\mathbf{F}}_{\text{even}}$.

For the polynomial $\mathbf{H}''(Y)$, we need to evaluate it at the roots $Y = \alpha^{2j+1}\beta$ with $0 \le j < n/2$. The vector butterfly for this evaluation is also a vector version of butterfly in Section 6.3, which costs $O(n\log_2 n)$ vector operations. The output are also $n/2$ vectors with length $n/4$, which form another $n/4 \times n/2$ matrix $\widetilde{\mathbf{F}}_{\text{odd}}$.

The output of the 2NTT of $\mathbf{F}$ is the concatenation

$$\widetilde{\mathbf{F}} = \begin{pmatrix} \widetilde{\mathbf{F}}_{\text{even}} \\ \widetilde{\mathbf{F}}_{\text{odd}} \end{pmatrix},$$

which is an $n/2 \times n/2$ matrix with entries in $\mathbb{F}_p$. The vector butterfly algorithm costs $O(n\log_2 n)$ vector operations (plus transposes). Each stage in the transverse and longitudinal vector butterflies is invertible, hence the inverse of 2NTT, denoted by 2INTT, can be constructed by inverting each stage of the 2NTT, whose complexity is the same as 2NTT.

6.5 The Advantages of 2NTT

As we have mentioned before, a natural comparison is between the 2NTT of the new $\mathcal{R}$ and the negacyclic NTT of the ring $\mathbb{Z}[X]/\langle X^{n^2/4} + 1\rangle$, since polynomials in both rings have $n^2/4$ coefficients. Both the 2NTT and NTT consist of $\log_2\left(n^2/4\right)$ stages, and the computational complexity at each stage is the same. Therefore, the computational complexity of 2NTT is the same as negacyclic NTT.

Let us now discuss the advantages of 2NTT. First, for negacyclic NTT, we need prime numbers such that

$$X^{n^2/4} + 1 \equiv 0 \bmod p' \tag{6.7}$$

has $n^2/4$ solutions. While for 2NTT, we need prime numbers such that $Y^n \equiv 2 \bmod p$ has n solutions, which are (generally) different from the negacyclic NTT case. Hence when design an algorithm, we have more chances to find prime numbers that suit our needs. More precisely, we have more chances to find "good" prime numbers at which the modulo operations can be much more efficiently evaluated.

The more important advantage of 2NTT is that it enjoys a quadratic saving of twiddle factors. The space consumed by the twiddle factors of the negacyclic NTT is $O(n^2/4)$, which is essentially the space of the $n^2/4$ solutions of Eq. (6.7). The space consumed by the twiddle factors of 2NTT is only $O(n)$, which is essentially the space of the solutions of $X^{n/2} \equiv -1 \bmod p$ and $Y^n \equiv 2 \bmod p$, $3n/2$ roots in total. Thus, there is a quadratic saving of twiddle factor space. This saving is especially desirable for large polynomials. As a result, this offers new possibilities for optimizations in practical implementations on platforms such as GPU and FPGA.

7 A New Variant of CKKS

As one important application of the theory that we have obtained, we discuss how to construct a new variant of CKKS based on the ring $\mathcal{R}$ [9, 10]. In this section, we show the constructions of a new decoding and encoding map for $\mathcal{R}$, and study the geometric properties of the actions of the Galois group $\mathrm{Gal}\,(F/\mathbb{Q})$. We also show that this new variant is bootstrappable, thus it is in fact a Fully Homomorphic Encryption (FHE) scheme.

7.1 The Decoding and Encoding Map

The plaintext space of the new variant of CKKS is the ring $\mathcal{R}$, thus a plaintext is a two-variable polynomial of the form

$$\mathbf{M}(X, Y) = \sum_{i=0}^{n/2-1} \sum_{i=0}^{n/2-1} m_{i,j} X^i Y^j \text{ with } m_{ij} \in \mathbb{Z}.$$

The coefficients of $\mathbf{M}(X, Y)$ naturally form an $n/2 \times n/2$ matrix that is also denoted by $\mathbf{M} = (m_{i,j})_{0 \le i,j < n/2-1} \in \mathbb{Z}^{n/2 \times n/2}$.

Under the canonical embedding σ in Eq. (3.1), a polynomial $\mathbf{M}$ is sent to its evaluations at all the root-pairs $\left(X = \zeta_n^a,\ Y = \zeta_n^b \sqrt[n]{2}\right)$ with $(a, b) \in G$. The decoding map is essentially the canonical embedding, except that since the coefficients of $\mathbf{M}$ are integers, we only need to consider the evaluation of $\mathbf{M}$ at $\left(X = \zeta_n^a,\ Y = \zeta_n^b \sqrt[n]{2}\right)$ with $(a, b) \in N$. Namely, the evaluation of $\mathbf{M}$ at half of root-pairs that come from N, while the evaluation of $\mathbf{M}$ at the other half is determined by the complex conjugation of the first half, which is similar to CKKS [10].

From Section 2.3, any element of $\{\sigma_{a,b} : (a, b) \in N\}$ can be uniquely written as $\sigma_{5,1}^i \circ \sigma_{1,2}^j$ with $0 \le i < n/4$ and $0 \le j < n/2$. Therefore, we define the decoding of $\mathbf{M}$ to be the matrix $\mathbf{z} = (z_{i,j}) \in \mathbb{C}^{n/4 \times n/2}$, where $z_{i,j}$ is the evaluation $\mathbf{M}$ at

$$X = \sigma_{5,1}^i \circ \sigma_{1,2}^j(\zeta_n) = \zeta_n^{5^i} \text{ and } Y = \sigma_{5,1}^i \circ \sigma_{1,2}^j(\sqrt[n]{2}) = \zeta_n^{2j \cdot 5^i + (5^i - 1)/4} \sqrt[n]{2}.$$

More explicitly, $z_{i,j}$ is given by

$$z_{i,j} = \mathbf{M}(\zeta_n^{5^i}, \zeta_n^{2j \cdot 5^i + (5^i-1)/4} \sqrt[n]{2}), \text{ with } 0 \le i < n/4 \text{ and } 0 \le j < n/2.$$

By abuse of notations, we will also use σ to denote this decoding map:

$$\sigma : \mathbf{M} \in \mathcal{R} \mapsto \mathbf{z} = (z_{i,j}) \in \mathbb{C}^{n/4 \times n/2}. \tag{7.1}$$

Hence, we will follow CKKS and introduce a scale Δ to control the errors [10]. The decoding map is defined to be

$$\mathbf{z} = \mathbf{Dcd}(\mathbf{M}, \Delta) = \Delta^{-1} \cdot \sigma(\mathbf{M}).$$

The encoding map is given by the inverse of σ:

$$\mathbf{M} = \mathbf{Ecd}(\mathbf{z}, \Delta) = \lfloor \sigma^{-1}(\Delta \cdot \mathbf{z}) \rceil,$$

which can be computed using linear algebra. Here $\lfloor \cdot \rceil$ denotes the rounding of a number to the nearest integer, and rounds upward in case of a tie.

7.2 The Actions of the Galois Group

From Section 2.3, there is an action of the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ on $\mathcal{R}$ via

$$\sigma_{a,b}(X) = X^a \text{ and } \sigma_{a,b}(Y) = X^b \cdot Y, \ \forall \sigma_{a,b} \in \mathrm{Gal}(F/\mathbb{Q}).$$

Thus for a polynomial $\mathbf{M} \in \mathcal{R}$, we immediately deduce that

$$\sigma_{a,b} \circ \sigma_{c,d}(\mathbf{M}) = \sigma_{a,b}(\sigma_{c,d}(\mathbf{M})),$$
$$\sigma_{c,d}(\mathbf{M})(\sigma_{a,b}(\zeta_n), \sigma_{a,b}(\sqrt[n]{2})) = \mathbf{M}(\sigma_{a,b} \circ \sigma_{c,d}(\zeta_n), \sigma_{a,b} \circ \sigma_{c,d}(\sqrt[n]{2})).$$

Let us now look at the "geometric" meaning of the actions of $\mathrm{Gal}(F/\mathbb{Q})$. Suppose $\mathbf{M}$ is the encoding of a matrix $\mathbf{z} = (z_{i,j}) \in \mathbb{C}^{n/4 \times n/2}$, then the decoding of $\sigma_{a,b}(\mathbf{M})$ satisfies

$$\sigma_{a,b}(\mathbf{M})\left(\sigma_{5,1}^i \circ \sigma_{1,2}^j(\zeta_n), \sigma_{5,1}^i \circ \sigma_{1,2}^j(\sqrt[n]{2})\right)$$
$$= \mathbf{M}\left(\sigma_{5,1}^i \circ \sigma_{1,2}^j \circ \sigma_{a,b}(\zeta_n), \sigma_{5,1}^i \circ \sigma_{1,2}^j \circ \sigma_{a,b}(\sqrt[n]{2})\right). \tag{7.2}$$

From Section 2.3, $\mathrm{Gal}(F/\mathbb{Q})$ can be generated by three elements: $\sigma_{1,2}$, $\sigma_{5,1}$ and $\sigma_{n-1,0}$, thus we only need to understand the geometric meaning of the actions of these three elements. The action of $\sigma_{1,2}$ is straightforward. From Eq. (7.2), the evaluation of $\sigma_{1,2}(\mathbf{M})$ at $X = \sigma_{5,1}^i \circ \sigma_{1,2}^j(\zeta_n)$ and $Y = \sigma_{5,1}^i \circ \sigma_{1,2}^j(\sqrt[n]{2})$ is

$$\mathbf{M}\left(\sigma_{5,1}^i \circ \sigma_{1,2}^{j+1}(\zeta_n), \sigma_{5,1}^i \circ \sigma_{1,2}^{j+1}(\sqrt[n]{2})\right).$$

So the decoding of $\sigma_{1,2}(\mathbf{M})$ is $\mathbf{z}' = (z'_{i,j}) \in \mathbb{C}^{n/2 \times n/4}$ with $z'_{i,j} = z_{i,j+1}$. Thus $\sigma_{1,2}$ induces a rotation of the columns of the matrix $\mathbf{z}$.

To understand the action of $\sigma_{5,1}$ on $\mathbf{z}$, we will need the relations $\sigma_{5,1}^{n/4} = \sigma_{1,2}^{3n/8}$ and $\sigma_{1,2}^{n/2} = \sigma_{1,0}$. So we immediately learn that the action of $\sigma_{5,1}^{n/4}$ is the same as that of $\sigma_{1,2}^{3n/8}$, i.e., the effect of $\sigma_{5,1}$ repeatedly acting on a plaintext $n/4$ times is the same as $3n/8$ rotations of columns. To proceed, we also need Eq. (2.10), which implies

$$\sigma_{5,1}^i \circ \sigma_{1,2}^j \circ \sigma_{5,1} = \sigma_{5,1}^{i+1} \circ \sigma_{1,2}^{j/5}.$$

Hence together with Eq. (7.2), we learn that the evaluation of $\sigma_{5,1}(\mathbf{M})$ at $X = \sigma_{5,1}^i \circ \sigma_{1,2}^j(\zeta_n)$ and $Y = \sigma_{5,1}^i \circ \sigma_{1,2}^j(\sqrt[n]{2})$ is given by

$$\mathbf{M}\left(\sigma_{5,1}^{i+1} \circ \sigma_{1,2}^{j/5}(\zeta_n), \sigma_{5,1}^{i+1} \circ \sigma_{1,2}^{j/5}(\sqrt[n]{2})\right).$$

Thus $\sigma_{5,1}(\mathbf{M})$ decodes to a matrix $\mathbf{z}'' = (z''_{i,j}) \in \mathbb{C}^{n/4 \times n/2}$ that satisfies

$$z''_{i,j} = \begin{cases} z_{i+1,j/5}, & \text{if } i < n/4 - 1, \\ z_{0,j/5+3n/8}, & \text{if } i = n/4 - 1. \end{cases}$$

Besides row-rotation, $\sigma_{5,1}$ also permutes the elements of each row at the same time.

To understand the effect of $\sigma_{n-1,0}$, we will need Eq. (2.11), from which we deduce

$$\sigma_{5,1}^i \circ \sigma_{1,2}^j \circ \sigma_{n-1,0} = \sigma_{n-1,0} \circ \sigma_{5,1}^i \circ \sigma_{1,2}^{-j-(1-1/5^i)/4}.$$

So from Eq. (7.2), we deduce that the evaluation of $\sigma_{n-1,0}(\mathbf{M})$ at $X = \sigma_{5,1}^i \circ \sigma_{1,2}^j(\zeta_n)$ and $Y = \sigma_{5,1}^i \circ \sigma_{1,2}^j(\sqrt[n]{2})$ is the complex conjugation of

$$\mathbf{M}\left(\sigma_{5,1}^i \circ \sigma_{1,2}^{-j-(1-1/5^i)/4}(\zeta_n), \sigma_{5,1}^i \circ \sigma_{1,2}^{-j-(1-1/5^i)/4}(\sqrt[n]{2})\right),$$

where we have used the property that $\sigma_{n-1,0}$ is the complex conjugation. Thus $\sigma_{n-1,0}(\mathbf{M})$ decodes to $\mathbf{z}^c = \left(z_{i,j}^c\right)$ where $z_{i,j}^c = \overline{z}_{i,-j-(1-1/5^i)/4}$.

7.3 The New Variant is Bootstrappable

We now show this new variant of CKKS is bootstrappable from the bootstrapping of CKKS [9]. The process consists of five steps: 1, Modulus Raising; 2, Putting polynomial coefficients into plaintext slots; 3, Evaluation of the complex exponential function; 4, Extraction of the imaginary part ; 5, Switching back to the coefficient representation. The steps 1, 3 and 4 are exactly the same as in [9], hence will not be discussed here. We only need to show how to construct the new Coefficients-to-Slots map, whose inverse is step 5.

Let us first introduce a linear algebra trick. Suppose m is a positive integer, $\mathbf{A} = (a_{i,j})_{0 \le i,j < m}$ is an $m \times m$ matrix, and $\mathbf{x} = (x_i)_{0 \le i < m}$ is a column vector. Suppose we have m distinct permutations σ_i with $0 \le i < m$ of $\{0, 1, \ldots, m - 1\}$, where σ_0 is the identity. We further assume that for every $0 \le j < m$, $(\sigma_0(j), \sigma_1(j), \ldots, \sigma_{m-1}(j))$ is also a permutation of $\{0, 1, \cdots, m - 1\}$. Let $\sigma_i(\mathbf{x})$ be the column vector $(x_{\sigma_i(0)}, x_{\sigma_i(1)}, \ldots, x_{\sigma_i(m-1)})$ and let $\mathbf{u}_i$ be the column vector $(a_{0,\sigma_i(0)}, a_{1,\sigma_i(1)}, \ldots, a_{m-1,\sigma_i(m-1)})$. Then a quick computation shows $\mathbf{A} \cdot \mathbf{x} = \sum_{i=0}^{m-1} \mathbf{u}_i \odot \sigma_i(\mathbf{x})$.

On the other hand, a real linear map from $\mathbb{C}^m$ to $\mathbb{R}^m$ is always of the form $\mathbf{A} \cdot \mathbf{x} + \overline{\mathbf{A}} \cdot \overline{\mathbf{x}}$, where $\mathbf{A}$ is a complex matrix and $\mathbf{x} \in \mathbb{C}^m$ [9]. To be relevant to this paper, we will look at the case where there is further permutation of the coordinates of $\overline{\mathbf{x}}$ represented by a matrix $\mathbf{S}$, and the linear transformation becomes $\mathbf{A} \cdot \mathbf{x} + \overline{\mathbf{A}} \cdot \mathbf{S}^{-1} \cdot (\mathbf{S}\overline{\mathbf{x}})$, which can always be written into

$$\mathbf{A} \cdot \mathbf{x} + \overline{\mathbf{A}} \cdot \overline{\mathbf{x}} = \sum_{i=0}^{m-1} \mathbf{u}_i \odot \sigma_i(\mathbf{x}) + \sum_{i=0}^{m-1} \mathbf{u}_i' \odot \sigma_i(\mathbf{S}\overline{\mathbf{x}}).$$

Here, the new vector $\mathbf{u}_i'$ depends on $\overline{\mathbf{A}}$, $\mathbf{S}$ and σ_i.

Now we are ready to construct the **CoeffToSlot** for the ring $\mathcal{R}$. Notice that the encoding map is a real linear map $\sigma^{-1} : \mathbb{C}^{n/4 \times n/2} \to \mathbb{R}^{n/2 \times n/2}$, i.e., it linearly maps a complex matrix $(z_{i,j}) \in \mathbb{C}^{n/4 \times n/2}$ to a real coefficient matrix $\mathbf{M} \in \mathbb{R}^{n/2 \times n/2}$. Let us now define two projection maps $\mathbf{P}_0, \mathbf{P}_1$ from $\mathbb{R}^{n/2 \times n/2}$ to $\mathbb{R}^{n/4 \times n/2}$ such that $\mathbf{P}_0$ (resp. $\mathbf{P}_1$) projects $\mathbf{M}$ to its up half (resp. down half). Then both $\mathbf{P}_0 \circ \sigma^{-1}$ and $\mathbf{P}_1 \circ \sigma^{-1}$ are real linear maps from $\mathbb{C}^{n/4 \times n/2}$ to $\mathbb{R}^{n/4 \times n/2}$.

Suppose $\mathbf{M}$ decodes to $\mathbf{z} = (z_{i,j}) \in \mathbb{C}^{n/4 \times n/2}$, then we define $\sigma_{a,b}(\mathbf{z})$ to be the decoding of $\sigma_{a,b}(\mathbf{M})$. From the properties of $\mathrm{Gal}(F/\mathbb{Q})$, we learn that $\{\sigma_{a,b} : (a, b) \in N\}$ defines permutations of $\mathbf{z} = (z_{i,j})$ that satisfy the condition in the previous paragraphs. (Of course, we first need to flatten $(z_{i,j})$ to a column vector.) We immediately deduce that for every pair $(a, b) \in N$, there exist

$n/4 \times n/2$ complex matrices $\mathbf{U}_{a,b}^0$, $\mathbf{U}_{a,b}'^0$, $\mathbf{U}_{a,b}^1$, $\mathbf{U}_{a,b}'^1$ such that

$$\mathbf{M}_0 = \mathbf{P}_0 \circ \sigma^{-1}(\mathbf{z}) = \sum_{(a,b) \in N} \left(\mathbf{U}_{a,b}^0 \odot \sigma_{a,b}(\mathbf{z}) + \mathbf{U}_{a,b}'^0 \odot \sigma_{a,b}(\sigma_{n-1,0}(\mathbf{z})) \right),$$

$$\mathbf{M}_1 = \mathbf{P}_1 \circ \sigma^{-1}(\mathbf{z}) = \sum_{(a,b) \in N} \left(\mathbf{U}_{a,b}^1 \odot \sigma_{a,b}(\mathbf{z}) + \mathbf{U}_{a,b}'^1 \odot \sigma_{a,b}(\sigma_{n-1,0}(\mathbf{z})) \right).$$

These matrices can be computed by brute force. The encoding of these two equations give us

$$\sigma^{-1}(\mathbf{M}_0) = \sum_{(a,b) \in N} \left(\sigma^{-1}(\mathbf{U}_{a,b}^0) \cdot \sigma_{a,b}(\mathbf{M}) + \sigma^{-1}(\mathbf{U}_{a,b}'^0) \cdot \sigma_{a,b} \circ \sigma_{n-1,0}(\mathbf{M}) \right),$$

$$\sigma^{-1}(\mathbf{M}_1) = \sum_{(a,b) \in N} \left(\sigma^{-1}(\mathbf{U}_{a,b}^1) \cdot \sigma_{a,b}(\mathbf{M}) + \sigma^{-1}(\mathbf{U}_{a,b}'^1) \cdot \sigma_{a,b} \circ \sigma_{n-1,0}(\mathbf{M}) \right),$$

which gives us the Coefficients-to-Slots map for the new ring $\mathcal{R}$. In conclusion, the new variant is bootstrappable.

A The Order-LWE Distribution is Pseudorandom

In this section, we give a reduction from the Search Order-LWE to the Average-Case Decision Order-LWE by adapting the methods in [19] to the new ring $\mathcal{R}$, thereby showing that the Order-LWE distribution over $\mathcal{R}$ is pseudorandom.

A.1 The Factorization of Primes in $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$

Suppose we have a prime number q such that $X^{n/2}+1 \equiv 0 \bmod q$ has a solution $\alpha \in \mathbb{F}_q$ and $Y^n - 2 \equiv 0 \bmod q$ has a solution $\beta \in \mathbb{F}_q$. Through replacing β by $\alpha\beta$ if necessary, we can always assume $\beta^{n/2} = \alpha^{n/8} - \alpha^{3n/8} \bmod q$. Such a prime number q splits completely in $\mathbb{Z}[\zeta_n]$ into a product of prime ideals $\prod_{i=0}^{n/2-1} \mathfrak{p}_{2i+1}$, where $\mathfrak{p}_{2i+1} \subset \mathbb{Z}[\zeta_n]$ is [19, 22]

$$\mathfrak{p}_{2i+1} = \langle q, \zeta_n - \alpha^{2i+1} \rangle = q\mathbb{Z}[\zeta_n] + (\zeta_n - \alpha^{2i+1})\mathbb{Z}[\zeta_n]. \tag{A.1}$$

The automorphism $\varsigma_j : \zeta_n \to \zeta_n^j$, $j \in (\mathbb{Z}/n\mathbb{Z})^\times$, of $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ acts transitively on the $n/2$ prime ideals via $\varsigma_j : \mathfrak{p}_{2i+1} \to \mathfrak{p}_{(2i+1)/j}$, where the quotient $(2i+1)/j$ is computed in $(\mathbb{Z}/n\mathbb{Z})^\times$ [19]. The residue class degree of $\mathfrak{p}_{2i+1}$ is 1 for every i, i.e., $f(\mathfrak{p}_{2i+1}/q) = 1$, hence we have an isomorphism $\mathbb{Z}/q\mathbb{Z} \cong \mathbb{Z}[\zeta_n]/\mathfrak{p}_{2i+1}$ [22, 34]. From definition, the norm $\mathrm{Nm}(\mathfrak{p}_{2i+1})$ of $\mathfrak{p}_{2i+1}$ is q [22].

Let us now look at the factorization of the prime ideal $\mathfrak{p}_{2i+1}$ in $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$. If $i \equiv 0, 3 \bmod 4$, we have a factorization

$$Y^{n/2} - \left(\zeta_n^{n/8} - \zeta_n^{3n/8} \right) = \prod_{j=0}^{n/2-1} (Y - \alpha^{2j}\beta) \bmod \mathfrak{p}_{2i+1}.$$

In this case, let the prime ideal $\mathfrak{P}_{2i+1,2j}$ of $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$ be

$$\mathfrak{P}_{2i+1,2j} = q\mathbb{Z}[\zeta_n, \sqrt[n]{2}] + (\zeta_n - \alpha^{2i+1})\mathbb{Z}[\zeta_n, \sqrt[n]{2}] + (\sqrt[n]{2} - \alpha^{2j}\beta)\mathbb{Z}[\zeta_n, \sqrt[n]{2}].$$

Because $\mathfrak{P}_{2i+1,2j} \cap \mathbb{Z}[\zeta_n] = \mathfrak{p}_{2i+1}$ for every j, so $\mathfrak{p}_{2i+1}$ splits completely into $\prod_{j=0}^{n/2-1} \mathfrak{P}_{2i+1,2j}$ in $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$. If $i \equiv 1, 2 \bmod 4$, we have a factorization of the form

$$Y^{n/2} - \left(\zeta_n^{n/8} - \zeta_n^{3n/8}\right) = \prod_{j=0}^{n/2-1} (Y - \alpha^{2j+1}\beta) \bmod \mathfrak{p}_{2i+1}.$$

In this case, let the prime ideal $\mathfrak{P}_{2i+1,2j+1}$ of $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$ be

$$\mathfrak{P}_{2i+1,2j+1} = q\mathbb{Z}[\zeta_n, \sqrt[n]{2}] + (\zeta_n - \alpha^{2i+1})\mathbb{Z}[\zeta_n, \sqrt[n]{2}] + (\sqrt[n]{2} - \alpha^{2j+1}\beta)\mathbb{Z}[\zeta_n, \sqrt[n]{2}].$$

Similarly, $\mathfrak{p}_{2i+1}$ splits completely into $\prod_{j=0}^{n/2-1} \mathfrak{P}_{2i+1,2j+1}$ in $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$. In conclusion, the prime number q splits completely in $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$ into

$$q\mathbb{Z}[\zeta_n, \sqrt[n]{2}] = \prod_{(a,b)\in G} \mathfrak{P}_{a,b}. \tag{A.2}$$

The action of the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ on $\mathfrak{P}_{a,b}$ can be obtained straightforwardly: $\sigma_{a,b}(\mathfrak{P}_{c,d}) = \mathfrak{P}_{c/a,d-b}$. Moreover, since q splits completely in $\mathbb{Z}[\zeta_n, \sqrt[n]{2}]$, we have an isomorphism $\mathbb{Z}/q\mathbb{Z} \cong \mathbb{Z}[\zeta_n, \sqrt[n]{2}]/\mathfrak{P}_{a,b}$ for every $(a,b) \in G$, from which we deduce that the norm $\mathrm{Nm}(\mathfrak{P}_{a,b})$ of $\mathfrak{P}_{a,b}$ is q [22].

A.2 The Main Reduction Theorems

We first prove the main reduction theorems using the lemmas proved later in Section A.3 and Section A.4.

Theorem A.1. *Suppose δ and q satisfy $\delta q \geq \eta_\epsilon(\mathcal{R}^\vee)$ for some negligible function $\delta = \delta(n^2/4)$. Then there exists a randomized polynomial-time reduction from $\mathsf{OLWE}_{q,\Psi_{\leq\delta}}$ to $\mathsf{DOLWE}_{q,\Upsilon_\delta}$.*

Proof. The proof comes from a sequence of reductions given in Section A.3 and Section A.4. From Lemma A.5, we have a deterministic polynomial-time reduction from $\mathsf{OLWE}_{q,\Psi_{\leq\delta}}$ to $\mathfrak{P}_{a,b}\text{-}\mathsf{OLWE}_{q,\Psi_{\leq\delta}}$. From Lemma A.8, there exists a probabilistic polynomial-time reduction from $\mathfrak{P}_{a,b}\text{-}\mathsf{OLWE}_{q,\Psi_{\leq\delta}}$ to $\mathsf{WDOLWE}_{q,\Psi_{\leq\delta}}^{(a,b)}$. From Lemma A.10, there exists a randomized polynomial-time reduction from $\mathsf{WDOLWE}_{q,\Psi_{\leq\delta}}^{(a,b)}$ to $\mathsf{DOLWE}_{q,\Upsilon_\delta}^{(a,b)}$. Then using Lemma A.12, we obtain a reduction to $\mathsf{DOLWE}_{q,\Upsilon_\delta}$. $\square$

From Lemma 5.2, we have $\eta_{2^{-n^2/2}}(\mathcal{R}^\vee) \leq n/(2\lambda_1(\mathcal{R})) = 1$, where we have used $\lambda_1(\mathcal{R}) = n/2$. So in Theorem A.1, it suffices to require $\delta q \geq 1$.

Theorem A.2. *Suppose δ and q are as in Theorem A.1, and let $\ell \geq 1$. Then there exists a randomized polynomial time reduction from solving $\mathsf{OLWE}_{q,\Psi_{\leq\delta}}$ to solving DOLWE_{q,D_ξ} given only ℓ samples, where $\xi = \delta \cdot \left(n^2\ell/(4\log(n^2\ell/4))\right)^{1/4}$.*

Proof. The proof also comes from a similar sequence of reductions as in the proof of Theorem A.1. From Lemma A.5, we have a deterministic polynomial-time reduction from $\mathsf{OLWE}_{q,\Psi_{\leq\delta}}$ to $\mathfrak{P}_{a,b}\text{-}\mathsf{OLWE}_{q,\Psi_{\leq\delta}}$. From Lemma A.8, there exists a probabilistic polynomial-time reduction from $\mathfrak{P}_{a,b}\text{-}\mathsf{OLWE}_{q,\Psi_{\leq\delta}}$ to $\mathsf{WDOLWE}_{q,\Psi_{\leq\delta}}^{(a,b)}$. From Lemma A.13, there exists a randomized polynomial-time reduction from $\mathsf{WDOLWE}_{q,\Psi_{\leq\delta}}^{(a,b)}$ to $\mathsf{DOLWE}_{q,D_\xi}^{(a,b)}$ given only ℓ samples. Then using Lemma A.12, we obtain a reduction to DOLWE_{q,D_ξ}. $\square$

Theorem A.3. *Suppose δ and q are as in Theorem A.1, then there exists a randomized polynomial time reduction from solving* $\mathsf{OLWE}_{q,D_\delta}$ *to solving* $\mathsf{DOLWE}_{q,D_\delta}$.

Proof. Use the same sequence of reductions as in the proof of Theorem A.1, except that Lemma A.10 must be modified so that the error distribution is not randomized but only the secret **s** is randomized. $\square$

A.3 The Reduction of Search to Worst-Case Decision

Let q be a prime number that satisfies the conditions in Section A.1, i.e., q splits completely into the product of $n^2/4$ prime ideals in $\mathcal{R}$. From the Chinese Remainder Theorem, there is an efficiently computable $\mathcal{R}$-module isomorphism [3, 19]

$$\mathcal{R}_q^\vee \cong \prod_{(a,b)\in G} \left(\mathcal{R}^\vee/\mathfrak{P}_{a,b}\mathcal{R}^\vee\right). \tag{A.3}$$

Definition A.4 (OLWE over $\mathfrak{P}_{a,b}$)**.** *The* $\mathfrak{P}_{a,b}$-$\mathsf{OLWE}_{q,\Psi}$ *problem is to find* **s** *mod* $\mathfrak{P}_{a,b}\mathcal{R}^\vee$ *given access to samples from* $A_{\mathbf{s},\psi}$ *for arbitrary* $\mathbf{s}\in\mathcal{R}_q^\vee$ *and* $\psi\in\Psi$.

Lemma A.5 (OLWE to $\mathfrak{P}_{a,b}$-OLWE)**.** *Suppose the family of distributions* Ψ *over* $F_\mathbb{R}$ *is closed under the actions of* $\mathrm{Gal}(F/\mathbb{Q})$: $\sigma_{a,b}(\psi)\in\Psi$ *for every* $\psi\in\Psi$ *and* $\sigma_{a,b}\in\mathrm{Gal}(F/\mathbb{Q})$. *Then for every* $(a,b)\in G$, *there exists a deterministic polynomial-time reduction from* $\mathsf{OLWE}_{q,\Psi}$ *to* $\mathfrak{P}_{a,b}$-$\mathsf{OLWE}_{q,\Psi}$.

Proof. Suppose we are given an oracle for $\mathfrak{P}_{a,b}$-$\mathsf{OLWE}_{q,\Psi}$ for an arbitrary $(a,b)\in G$, then using it we can recover the value **s** mod $\mathfrak{P}_{a,b}\mathcal{R}^\vee$ given access to samples from $A_{\mathbf{s},\psi}$ for any $\psi\in\Psi$. If we can recover **s** mod $\mathfrak{P}_{c,d}\mathcal{R}^\vee$ for every $(c,d)\in G$, then we can efficiently reconstruct $\mathbf{s}\in\mathcal{R}^\vee$ using the Chinese Remainder Theorem, i.e., Eq. (A.3), thus completes the proof.

We now show how to recover **s** mod $\mathfrak{P}_{c,d}\mathcal{R}^\vee$ from **s** mod $\mathfrak{P}_{a,b}\mathcal{R}^\vee$ using Galois action. Let $\sigma_{a',b'}$ be an element of $\mathrm{Gal}(F/\mathbb{Q})$ such that $\sigma_{a',b'}(\mathfrak{P}_{c,d})=\mathfrak{P}_{a,b}$. Notice that such an element always exists [22]. Suppose we are given an arbitrary sample $(\mathbf{a},\mathbf{b})\leftarrow A_{\mathbf{s},\psi}$, where $\mathbf{b}=(\mathbf{a}\cdot\mathbf{s})/q+\mathbf{e}$ mod $\mathcal{R}^\vee$. We transform this sample to $(\sigma_{a',b'}(\mathbf{a}),\sigma_{a',b'}(\mathbf{b}))$, which satisfies

$$\sigma_{a',b'}(\mathbf{b}) = \left(\sigma_{a',b'}(\mathbf{a})\cdot\sigma_{a',b'}(\mathbf{s})\right)/q+\sigma_{a',b'}(\mathbf{e}) \text{ mod } \mathcal{R}^\vee.$$

Notice that $\sigma_{a',b'}(\mathbf{a})$ is a uniformly random element of $\mathcal{R}_q$ since $\sigma_{a',b'}$ fixes $\mathcal{R}_q$. Thus $(\sigma_{a',b'}(\mathbf{a}),\sigma_{a',b'}(\mathbf{b}))$ is distributed according to $A_{\sigma_{a',b'}(\mathbf{s}),\psi'}$ where ψ' is $\sigma_{a',b'}(\psi)\in\Psi$. If we feed the transformed samples to the oracle, it returns an answer $\mathbf{t}\in\mathcal{R}^\vee/\mathfrak{P}_{a,b}\mathcal{R}^\vee$. Then we efficiently obtain $\mathbf{s}\equiv\sigma_{a',b'}^{-1}(\mathbf{t})$ mod $\mathfrak{P}_{c,d}\mathcal{R}^\vee$.
$\square$

Let us define a lexicographic order on G. Recall from Section 2.3 that an element of G is chosen to be of the form (a,b) with $a,b\in\{0,1,\cdots,n-1\}$. Define $(a,b)<(c,d)$ if $a<c$ or $a=c$ and $b<d$. The smallest element is $(1,0)$ and the largest element is $(n-1,n-2)$. Given $(a,b)\in G$, let $(a,b)-$ be the largest element in G that is less than (a,b), while let $(1,0)-$ be $(0,0)$ [19].

Definition A.6 (**Hybrid** RLWE **Distribution**)**.** *Given* $(a,b)\in G$, $\mathbf{s}\in\mathcal{R}_q^\vee$, *and a distribution* ψ *over* $F_\mathbb{R}$, *the distribution* $A_{\mathbf{s},\psi}^{(a,b)}$ *over* $\mathcal{R}_q\times\mathbb{T}$ *is defined as follows. If* $(a,b)=(0,0)$, *define* $A_{\mathbf{s},\psi}^{(0,0)}$ *to be* $A_{\mathbf{s},\psi}$. *Otherwise, sample* $(\mathbf{a},\mathbf{b})\leftarrow A_{\mathbf{s},\psi}$

and output $(\mathbf{a}, \mathbf{b} + \mathbf{h}/q)$, *where* $\mathbf{h} \in \mathcal{R}_q^\vee$ *is uniformly random and independent* $\mathrm{mod}\, \mathfrak{P}_{(c,d)}\mathcal{R}^\vee$ *for all* $(c,d) \leq (a,b)$, *and is* 0 *mod all the remaining* $\mathfrak{P}_{(c,d)}\mathcal{R}^\vee$.

Definition A.7 (**Worst-Case Decision** Order-LWE **relative to** $\mathfrak{P}_{a,b}$)**.** *Given* $(a,b) \in G$ *and a family of distributions* Ψ, *the Worst-Case Decision* Order-LWE *relative to* $\mathfrak{P}_{a,b}$, *denoted by* $\mathsf{WDOLWE}_{q,\Psi}^{(a,b)}$, *is defined as follows: given access to samples from* $A_{\mathbf{s},\psi}^{(c,d)}$ *for arbitrary* $\mathbf{s} \in \mathcal{R}_q^\vee$, $\psi \in \Psi$ *and* $(c,d) \in \{(a,b)-, (a,b)\}$, *find* (c,d).

Lemma A.8 (**Search to Decision**)**.** *For any* $(a,b) \in G$, *there exists a probabilistic polynomial-time reduction from* $\mathfrak{P}_{a,b}\text{-}\mathsf{OLWE}_{q,\Psi}$ *to* $\mathsf{WDOLWE}_{q,\Psi}^{(a,b)}$.

Proof. Let $\mathbf{v} \in \mathcal{R}_q$ be uniformly random mod $\mathfrak{P}_{a,b}$, but equals to 0 mod all the other $\mathfrak{P}_{c,d}$. Let $\mathbf{h} \in \mathcal{R}_q^\vee$ be uniformly random and independent mod $\mathfrak{P}_{c,d}\mathcal{R}^\vee$ for all $(c,d) < (a,b)$, but is 0 mod all the remaining $\mathfrak{P}_{c,d}\mathcal{R}^\vee$. Suppose we have an element $\mathbf{g} \in \mathcal{R}_q^\vee$ and we are given a sample $(\mathbf{a}, \mathbf{b}) \leftarrow A_{\mathbf{s},\psi}$, we transform it to

$$(\mathbf{a}', \mathbf{b}') = (\mathbf{a} + \mathbf{v}, \mathbf{b} + (\mathbf{h} + \mathbf{v}\mathbf{g})/q) \in \mathcal{R}_q \times \mathbb{T}.$$

Since $\mathbf{a}$ is uniformly random in $\mathcal{R}_q$, whence $\mathbf{a}'$ is also uniformly random in $\mathcal{R}_q$. In terms of $\mathbf{a}'$, $\mathbf{b}'$ can also be written as

$$\mathbf{b}' = \left(\mathbf{a}' \cdot \mathbf{s} + \mathbf{h} + \mathbf{v}(\mathbf{g} - \mathbf{s})\right)/q + \mathbf{e},$$

where $\mathbf{e}$ is sampled according to ψ. Depending on the value of $\mathbf{g}$, there are two different cases:

1. $\mathbf{g} \equiv \mathbf{s} \bmod \mathfrak{P}_{a,b}\mathcal{R}^\vee$. If so, the Chinese Remainder Theorem, i.e., Eq. (A.3) tells us that $\mathbf{v}(\mathbf{g} - \mathbf{s}) = 0$. Hence the distribution of $(\mathbf{a}', \mathbf{b}')$ is $A_{\mathbf{s},\psi}^{(a,b)-}$.

2. $\mathbf{g} \not\equiv \mathbf{s} \bmod \mathfrak{P}_{a,b}\mathcal{R}^\vee$. As $\mathcal{R}/\mathfrak{P}_{a,b}$ is a field, $\mathbf{v}(\mathbf{g} - \mathbf{s}) \in \mathcal{R}_q^\vee$ is distributed uniformly mod $\mathfrak{P}_{a,b}\mathcal{R}^\vee$ and is 0 mod all other $\mathfrak{P}_{c,d}\mathcal{R}^\vee$. Therefore, $\mathbf{v}(\mathbf{g} - \mathbf{s}) + \mathbf{h}$ is uniformly random and independent mod $\mathfrak{P}_{c,d}\mathcal{R}^\vee$ for all $(c,d) \leq (a,b)$, and is 0 mod all the remaining $\mathfrak{P}_{c,d}\mathcal{R}^\vee$. Hence the distribution of $(\mathbf{a}', \mathbf{b}')$ is $A_{\mathbf{s},\psi}^{(a,b)}$.

If we are given an oracle for $\mathsf{WDOLWE}_{q,\Psi}^{(a,b)}$, then we can distinguish the above two cases for $\mathbf{g}$. Now, we randomly take some $\mathbf{g} \in \mathcal{R}^\vee$ and use the oracle to determine whether $\mathbf{g} \equiv \mathbf{s} \bmod \mathfrak{P}_{a,b}\mathcal{R}^\vee$ or not. If not, we change to a different $\mathbf{g}$ and repeat the process. Since there are only $\mathrm{Nm}(\mathfrak{P}_{a,b}) = q = \mathrm{poly}(n)$ possible values for $\mathbf{s} \bmod \mathfrak{P}_{a,b}\mathcal{R}^\vee$, we certainly can enumerate over all the values and efficiently discover the correct one.

$\square$

A.4 Worst-Case Decision to Average-Case Decision

We now give a reduction from the worst-case problem $\mathsf{WDOLWE}_{q,\Psi}^{(a,b)}$ to an average-case problem, which distinguishes $A_{\mathbf{s},\psi}$ from the uniform distribution for a random choice of $\mathbf{s}$ and ψ. But the error distribution ψ must be drawn randomly from a certain distribution Υ and is kept secret.

Definition A.9 (Average-Case Decision Order-LWE relative to $\mathfrak{P}_{a,b}$). *For $(a,b) \in G$ and a distribution Υ defined over the error distributions Ψ, an algorithm is said to solve the problem $\mathsf{DOLWE}_{q,\Upsilon}^{(a,b)}$ with a non-negligible probability over the choice of a random $(\mathbf{s},\psi) \leftarrow \mathcal{U}(\mathcal{R}_q^\vee) \times \Upsilon$ if there is a non-negligible difference in acceptance probability between inputs from $A_{\mathbf{s},\psi}^{(a,b)}$ and inputs from $A_{\mathbf{s},\psi}^{(a,b)-}$.*

Recall that Claim 5.11 of the paper [19] tells us that for $P = \Gamma(2,1)^{n^2/4}$ and

$$Q(z_1,\ldots,z_{n^2/4}) = (\Gamma(2,1) - z_1) \times \cdots \times (\Gamma(2,1) - z_{n^2/4}),$$

where $0 \leq z_1,\ldots,z_{n^2/4} \leq 2/n$, any set $S \subset \mathbb{R}^n$ with non-negligible measure under P also has non-negligible measure under Q.

Lemma A.10 (Worst-case to average-case). *For any $\delta > 0$ and $(a,b) \in G$, there exists a randomized polynomial-time reduction from $\mathsf{WDOLWE}_{q,\Psi_{\leq\delta}}^{(a,b)}$ to $\mathsf{DOLWE}_{q,\Upsilon_\delta}^{(a,b)}$.*

Proof. Suppose we are given $\mathbf{s}' \in \mathcal{R}_q^\vee$, $\mathbf{r}' \in (\mathbb{R}^+)^{n^2/4}$ and $(c,d) \in G$. Let $\mathbf{h} \in \mathcal{R}_q^\vee$ be uniformly random and independent mod $\mathfrak{P}_{e,f}\mathcal{R}^\vee$ for all $(e,f) \leq (c,d)$, and is 0 mod all the remaining $\mathfrak{P}_{e,f}\mathcal{R}^\vee$. Now consider the transformation

$$(\mathbf{a},\mathbf{b}) \to (\mathbf{a}, \mathbf{b} + (\mathbf{a} \cdot \mathbf{s}' + \mathbf{h})/q + \mathbf{e}'),$$

where $\mathbf{e}'$ is sampled from $D_{\mathbf{r}'}$. For all $\mathbf{s} \in \mathcal{R}_q^\vee$ and $(a,b) \in G$, this transformation maps $A_{\mathbf{s},\psi}^{(a,b)}$ to $A_{\mathbf{s}+\mathbf{s}',\psi+D_{\mathbf{r}'}}^{\max\{(a,b),(c,d)\}}$.

The reduction repeats the following procedure a polynomial number of times. Choose a uniform $\mathbf{s}' \in \mathcal{R}_q^\vee$ and a real positive $x_{e,f}$, $(e,f) \in N$, independently from $\Gamma(2,1)$. Let $\mathbf{r}' \in (\mathbb{R}^+)^{n^2/4}$ be

$$r_{e,f}'^2 = r_{(n-1)e,(n-1)f}'^2 = n\delta^2 x_{e,f}/2, \ \forall (e,f) \in N.$$

Then estimate the acceptance probability of the oracle for $\mathsf{DOLWE}_{q,\Upsilon_\epsilon}^{(a,b)}$ on the following two input distributions:

1. apply the transformation with $\mathbf{s}'$, $\mathbf{r}'$ and $(a,b)-$ to input samples.

2. apply the transformation with $\mathbf{s}'$, $\mathbf{r}'$ and (a,b) to input samples.

If in any of these polynomial numbers of attempts a non-negligible difference is observed between the two acceptance probabilities, output $(a,b)-$. Otherwise, output (a,b). If our input samples are from $A_{\mathbf{s},\psi}^{(a,b)}$, then both transformations map $A_{\mathbf{s},\psi}^{(a,b)}$ to $A_{\mathbf{s}+\mathbf{s}',\psi+D_{\mathbf{r}'}}^{(a,b)}$. Then the oracle's acceptance probability will be exactly the same, thus we output (a,b) with overwhelming probability.

If our input samples are from $A_{\mathbf{s},D_{\mathbf{r}}}^{(a,b)-}$ for some $\mathbf{r}$ such that all $r_{e,f}$ are in $[0,\delta]$. Then the transformation with parameters $\mathbf{s}'$, $\mathbf{r}'$ and $(a,b)-$ transforms $A_{\mathbf{s},D_{\mathbf{r}}}^{(a,b)-}$ to $A_{\mathbf{s}+\mathbf{s}',D_{\mathbf{r}}+D_{\mathbf{r}'}}^{(a,b)-}$. The transformation with parameters $\mathbf{s}'$, $\mathbf{r}'$ and (a,b) transforms $A_{\mathbf{s},D_{\mathbf{r}}}^{(a,b)-}$ to $A_{\mathbf{s}+\mathbf{s}',D_{\mathbf{r}}+D_{\mathbf{r}'}}^{(a,b)}$. Moreover, we have

$$D_{\mathbf{r}} + D_{\mathbf{r}'} = D_{\mathbf{r}''} \text{ with } r_{e,f}''^2 = r_{e,f}^2 + r_{e,f}'^2, \ \forall (e,f) \in N.$$

Let S be the set of all pairs $(\mathbf{s}, \psi)$ such that the oracle has a non-negligible difference in acceptance probability on $A_{\mathbf{s},\psi}^{(a,b)-}$ and $A_{\mathbf{s},\psi}^{(a,b)}$. Here "non-negligible" means the measure of S under the distribution $\mathcal{U}(\mathcal{R}_q^\vee) \times \Upsilon_\delta$ is non-negligible. By Claim 5.11 of [19], the probability of $(\mathbf{s}+\mathbf{s}', D_{\mathbf{r}}+D_{\mathbf{r}'}) \in S$ is also non-negligible, hence we will output $(a,b)-$. Thus proves the lemma.

$\square$

Lemma A.11. *Suppose $\delta \geq \eta_\epsilon(\mathcal{R}^\vee)/q$ for some $\epsilon > 0$, then for any ψ in the support of Υ_δ and $\mathbf{s} \in \mathcal{R}_q^\vee$, the distribution $A_{\mathbf{s},\psi}^{(n-1,n-2)}$ is within statistical distance $\epsilon/2$ from the uniform distribution over $\mathcal{R}_q \times \mathbb{T}$.*

Proof. From its definition, a sample from the distribution $A_{\mathbf{s},\psi}^{(n-1,n-2)}$ is of the form $(\mathbf{a}, (\mathbf{a} \cdot \mathbf{s} + \mathbf{h})/q + \mathbf{e})$, where $\mathbf{a} \leftarrow \mathcal{U}(\mathcal{R}_q)$, $\mathbf{h} \leftarrow \mathcal{U}(\mathcal{R}_q^\vee)$ and $\mathbf{e} \leftarrow \psi$. Thus it suffices to prove that conditioned on an arbitrary value of $\mathbf{a}$, the second component of the pair is within statistical distance ϵ of the uniform distribution over $\mathbb{T}$. Now, take an $\mathbf{a}$ and keep its valued fixed, then $(\mathbf{a} \cdot \mathbf{s} + \mathbf{h})/q$ is distributed like a uniformly random element of $(q^{-1}\mathcal{R}^\vee)/\mathcal{R}^\vee$. Moreover, any noise distribution ψ in the support of Υ_δ can be written as the sum of two independent Gaussian distributions $D_{\mathbf{r}}+D_{\mathbf{r}'}$, where the first is with parameters $r_{e,f} = \delta$ and the second is with parameters $r_{e,f}'^2 = x_{e,f} \geq 0$. From Lemma 2.3 of [19] and our assumption on δ, the sum of a uniform element of $(q^{-1}\mathcal{R}^\vee)/\mathcal{R}^\vee$ and a noise sampled from $D_{\mathbf{r}}$ is within statistical distance $\epsilon/2$ from the uniform distribution on $\mathbb{T}$, while this remains the case even after adding the independent noise $D_{\mathbf{r}'}$.

$\square$

Lemma A.12 (Hybrid). *Suppose Υ is a distribution over error distributions such that for any ψ in the support of Υ and any $\mathbf{s} \in \mathcal{R}_q^\vee$, the distribution $A_{\mathbf{s},\psi}^{(n-1,n-2)}$ is within negligible statistical distance from uniform. Then for any oracle that solve $\mathsf{DOLWE}_{q,\Upsilon}$, there exists an $(a,b) \in G$ and an efficient algorithm that solves $\mathsf{DOLWE}_{q,\Upsilon}^{(a,b)}$ using the oracle.*

Proof. Let $(\mathbf{s}, \psi)$ be any pair for which the oracle distinguishes between $A_{\mathbf{s},\psi}$ and the uniform inputs with a non-negligible advantage. From Markov's inequality, the probability measure of such pairs is non-negligible. Since $A_{\mathbf{s},\psi}^{(0,0)} = A_{\mathbf{s},\psi}$ and $A_{\mathbf{s},\psi}^{(n-1,n-2)}$ is far from the uniform distribution, there exists an element $(a,b) \in G$ such that the oracle distinguishes between $A_{\mathbf{s},\psi}^{(a,b)}$ and $A_{\mathbf{s},\psi}^{(a,b)-}$ with a non-negligible advantage. The lemma follows immediately by choosing the (a,b) associated with the set of pairs $(\mathbf{s}, \psi)$ with highest probability. $\square$

Suppose $r_1, \ldots, r_{n^2/4} \in \mathbb{R}^+$ and $s_1, \ldots, s_{n^2/4} \in \mathbb{R}^+$ satisfy $|s_i/r_i - 1| < 2\sqrt{\log(n^2/4)}/n$ for all i. Then Claim 5.15 of [19] says that any set S with non-negligible measure under the Gaussian distribution $D_{r_1} \times \cdots D_{r_{n^2/4}}$ also has non-negligible measure under $D_{s_1} \times \cdots \times D_{s_{n^2/4}}$.

Lemma A.13 (Worst-case to average-case with spherical noise). *For any $\delta > 0$, $\ell \geq 1$, and every $(a,b) \in G$, there exists a randomized polynomial-time reduction from solving $\mathsf{WDOLWE}_{q,\Psi_{\leq\delta}}^{(a,b)}$ to solving $\mathsf{DOLWE}_{q,D_\xi}^{(a,b)}$ given only ℓ samples, where $\xi = \delta \cdot \left(n^2\ell/(4\log(n^2\ell/4))\right)^{1/4}$.*

Proof. For some $\mathbf{s}' \in \mathcal{R}_q^\vee$, $(c,d) \in G$ and $\mathbf{e}_1, \ldots, \mathbf{e}_\ell \in \mathbb{T}$, consider the transformation

$$(\mathbf{a}_i, \mathbf{b}_i) \to (\mathbf{a}_i, \mathbf{b}_i + (\mathbf{a}_i \cdot \mathbf{s}' + \mathbf{h}_i)/q + \mathbf{e}_i), \ 1 \le i \le \ell, \qquad (\text{A.4})$$

where for every i, $\mathbf{h}_i \in \mathcal{R}_q^\vee$ is chosen independently to be uniformly random $\mod \mathfrak{P}_{e,f} \mathcal{R}^\vee$ for all $(e,f) \le (c,d)$ and be 0 mod all the remaining $\mathfrak{P}_{e,f} \mathcal{R}^\vee$. Then for any $\mathbf{s} \in \mathcal{R}_q^\vee$, ψ, $\mathbf{r}'$, and $(a,b) \in G$, if we sample from $(A_{\mathbf{s},\psi}^{(a,b)})^\ell$, i.e., ℓ independent samples from $A_{\mathbf{s},\psi}^{(a,b)}$, and apply this transformation with $\mathbf{e}_1, \ldots, \mathbf{e}_\ell$ sampled independently from $D_{\mathbf{r}'}$, then averaged over the choice of $\mathbf{e}_i$ the output is distributed according to $\left(A_{\mathbf{s}+\mathbf{s}',\psi+D_{\mathbf{r}'}}^{\max\{(a,b),(c,d)\}}\right)^\ell$.

The reduction repeats the following process a polynomial number of times. Choose $\mathbf{s}' \in \mathcal{R}_q^\vee$ uniformly randomly, and sample $\mathbf{e}_1, \ldots, \mathbf{e}_\ell$ independently from D_ξ. Now let us estimate the acceptance probability of the oracle on the following two different input distributions:

1. apply the transformation with parameters $\mathbf{s}'$, $\mathbf{e}_1, \ldots, \mathbf{e}_\ell$ and $(a,b)-$.

2. apply the transformation with parameters $\mathbf{s}'$, $\mathbf{e}_1, \ldots, \mathbf{e}_\ell$ and (a,b)

If during any of the polynomial number of attempts a non-negligible difference is observed between the two acceptance probabilities, output $(a,b)-$; otherwise output (a,b).

If the input distribution is $A_{\mathbf{s},\psi}^{(a,b)}$, then in each of the two attempts, the two distributions on which we estimate the oracle's acceptance probability are the same, hence we output (a,b) with overwhelming probability. If the input distribution is $A_{\mathbf{s},D_\mathbf{r}}^{(a,b)-}$ for a vector $\mathbf{r}$ with all $r_{c,d}$ in $[0, \epsilon]$. Let the distributions $B^{(a,b)-}(\mathbf{s}', \mathbf{e}_1, \ldots, \mathbf{e}_\ell)$ and $B^{(a,b)}(\mathbf{s}', \mathbf{e}_1, \ldots, \mathbf{e}_\ell)$ be on ℓ pairs which our reduction uses as input to the oracle. Let the vector $\mathbf{r}'$ be given by $r_{c,d}'^2 = \xi - r_{c,d}^2$ so that $D_\mathbf{r} + D_{\mathbf{r}'} = D_\xi$. From our analysis above, the average of $B^{(a,b)-}(\mathbf{s}', \mathbf{e}_1, \ldots, \mathbf{e}_\ell)$ over $\mathbf{e}_1, \ldots, \mathbf{e}_\ell$ sampled independently from $D_{\mathbf{r}'}$ is $(A_{\mathbf{s}+\mathbf{s}',D_\xi}^{(a,b)-})^\ell$, and similarly with $B^{(a,b)}$ and $A^{(a,b)}$. Let S be the set of all $(\mathbf{s}, \mathbf{e}_1, \ldots, \mathbf{e}_\ell)$ for which the oracle has a non-negligible difference in acceptance probability on $B^{(a,b)-}(\mathbf{s}', \mathbf{e}_1, \ldots, \mathbf{e}_\ell)$ and $B^{(a,b)}(\mathbf{s}', \mathbf{e}_1, \ldots, \mathbf{e}_\ell)$. From our assumption and a Markov argument, the measure of S under $\mathcal{U}(\mathcal{R}_q^\vee) \times (D_{\mathbf{r}'})^\ell$ is non-negligible [19]. From the inequalities

$$1 \le \frac{\xi}{\sqrt{\xi^2 - r_{c,d}^2}} \le \frac{\xi}{\sqrt{\xi^2 - \epsilon^2}} \le 1 + \sqrt{\frac{\log(n^2\ell/4)}{n^2\ell/4}},$$

we deduce that the measure of S under $\mathcal{U}(\mathcal{R}_q^\vee) \times (D_\xi)^\ell$ is also non-negligible, thus proves the lemma. $\qquad\square$

B   The Twin: the Splitting Field of $Y^n + 2$

In this section, we study the splitting field $\widehat{F}$ of $Y^n + 2$, where n (≥ 8) is a power-of-two integer. We show that all the results we have obtained for $F = \mathbb{Q}(\zeta_n, \sqrt[n]{2})$ can be directly generalized to $\widehat{F} = \mathbb{Q}(\zeta_n, \sqrt[n]{-2})$.

B.1 The Splitting Field of $Y^n + 2$

The splitting field $\widehat{F}$ of $Y^n + 2$ is generated by ζ_n and $\sqrt[n]{-2}$ over $\mathbb{Q}$: $\widehat{F} = \mathbb{Q}(\zeta_n, \sqrt[n]{-2})$. The two numbers satisfy the algebraic relations

$$\zeta_n^{n/2} = -1 \text{ and } \sqrt{-2} = \zeta_n^{n/8} + \zeta_n^{3n/8}. \tag{B.1}$$

The proof in the webpage [35] can be immediately adapted to prove that the degree of $\widehat{F}$ over $\mathbb{Q}$ is also $n^2/4$. An element of the Galois group $\mathrm{Gal}(\widehat{F}/\mathbb{Q})$ is determined by the actions

$$\widehat{\sigma}_{a,b}(\zeta_n) = \zeta_n^a \text{ and } \widehat{\sigma}_{a,b}(\sqrt[n]{-2}) = \zeta_n^b \sqrt[n]{-2}.$$

To preserve the relations in Eq. (B.1), a and b must satisfy

$$a \equiv \begin{cases} 1, 3 \bmod 8, & \text{if } b \text{ is even,} \\ 5, 7 \bmod 8, & \text{if } b \text{ is odd.} \end{cases} \tag{B.2}$$

For later convenience, we define the finite set $\widehat{G}$ to be

$$\widehat{G} = \left\{ (a,b) \in (\mathbb{Z}/n\mathbb{Z})^2 : (a,b) \text{ satisfies the condition in Eq. (B.2)} \right\},$$

then $\mathrm{Gal}(\widehat{F}/\mathbb{Q})$ is given by $\{\widehat{\sigma}_{a,b} : (a,b) \in \widehat{G}\}$. The identity of $\mathrm{Gal}(\widehat{F}/\mathbb{Q})$ is $\widehat{\sigma}_{1,0}$ and the complex conjugation is $\widehat{\sigma}_{n-1,n-1}$. Let $\widehat{N}$ be the subset

$$\widehat{N} = \left\{ (a,b) \in \widehat{G} : a \equiv 1 \text{ or } 5 \bmod 8 \right\}.$$

Then the Galois group $\mathrm{Gal}(\widehat{F}/\mathbb{Q}(\sqrt{-1}))$ is the subgroup

$$\mathrm{Gal}\left(\widehat{F}/\mathbb{Q}(\sqrt{-1}) \right) = \left\{ \widehat{\sigma}_{a,b} \in \mathrm{Gal}(\widehat{F}/\mathbb{Q}) : (a,b) \in \widehat{N} \right\}.$$

Moreover, the Galois group $\mathrm{Gal}(\widehat{F}/\mathbb{Q}(\zeta_n))$ is the subgroup generated by $\widehat{\sigma}_{1,2}$, whose order is $n/2$. Through studying the subfields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\zeta_n)$ using the fundamental theorem of Galois theory, we can obtain an explicit algebraic structure of $\mathrm{Gal}(\widehat{F}/\mathbb{Q})$ in the same way as in Section 2, which is omitted here.

B.2 The Canonical Embedding

The canonical embedding of $\widehat{F}$ into $\mathbb{C}^{n^2/4}$ is given by

$$\widehat{\sigma} : x \in \widehat{F} \mapsto (\widehat{\sigma}_{a,b}(x))_{(a,b)\in\widehat{G}} \in \mathbb{C}^{n^2/4},$$

which induces an injective real linear map from $\widehat{F}_{\mathbb{R}} = \widehat{F} \otimes_{\mathbb{Q}} \mathbb{R}$ to $\mathbb{C}^{n^2/4}$ via

$$\widehat{\sigma} : x \otimes r \mapsto (\widehat{\sigma}_{a,b}(x) \cdot r)_{(a,b)\in\widehat{G}}, \text{ where } x \in \widehat{F} \text{ and } r \in \mathbb{R}. \tag{B.3}$$

The complex conjugation of $\widehat{\sigma}_{a,b}(x)$, denoted by $\overline{\widehat{\sigma}_{a,b}(x)}$, satisfies

$$\overline{\widehat{\sigma}_{a,b}(x)} = \widehat{\sigma}_{(n-1)a,(n-1)b+(n-1)}(x) \tag{B.4}$$

for every $x \in \widehat{F}$ and $(a, b) \in \widehat{G}$. Therefore, the image of $\widehat{F}_{\mathbb{R}}$ under $\widehat{\sigma}$ is the real subspace

$$\widehat{H} = \left\{ (z_{a,b})_{(a,b) \in \widehat{G}} \in \mathbb{C}^{n^2/4} : z_{(n-1)a,(n-1)b+(n-1)} = \overline{z_{a,b}} \right\},$$

which is isomorphic to $\widehat{F}_{\mathbb{R}}$. As a result, we often implicitly identify $\widehat{H}$ with $\widehat{F}_{\mathbb{R}}$. The homomorphism in Eq. (B.3) can also be written as

$$\widehat{\sigma} : x \otimes r \mapsto (\widehat{\sigma}_{a,b}(x) \cdot r, \widehat{\sigma}_{(n-1)a,(n-1)b+(n-1)}(x) \cdot r)_{(a,b) \in \widehat{N}}.$$

An automorphism $\widehat{\sigma}_{c,d} \in \mathrm{Gal}(\widehat{F}/\mathbb{Q})$ of $\widehat{F}$ induces an automorphism of $\widehat{H}$ by making the following diagram commute

$$
\begin{array}{ccc}
x \otimes r & \xrightarrow{\ \widehat{\sigma}\ } & (\widehat{\sigma}_{a,b}(x) \cdot r, \widehat{\sigma}_{(n-1)a,(n-1)b+(n-1)}(x) \cdot r)_{(a,b) \in \widehat{N}} \\
\Big\downarrow \widehat{\sigma}_{c,d} & & \Big\downarrow \widehat{\sigma}_{c,d} \\
\widehat{\sigma}_{c,d}(x) \otimes r & \xrightarrow{\ \widehat{\sigma}\ } & (\widehat{\sigma}_{a,b}(\widehat{\sigma}_{c,d}(x)) \cdot r, \widehat{\sigma}_{(n-1)a,(n-1)b+(n-1)}(\widehat{\sigma}_{c,d}(x)) \cdot r)_{(a,b) \in \widehat{N}}
\end{array}
.
$$

Therefore, the effect of the automorphism $\widehat{\sigma}_{c,d}$ is to permute the components of vectors in $\widehat{H}$.

Let $\widehat{\mathbf{e}}_{a,b} \in \mathbb{C}^{n^2/4}$ be the vector with 1 in its (a, b)-th coordinate and 0 elsewhere, which forms a natural basis of $\mathbb{C}^{n^2/4}$. For every $(a, b) \in \widehat{N}$, we define

$$\widehat{\mathbf{h}}_{(a,b)} = \frac{1}{\sqrt{2}} \left(\widehat{\mathbf{e}}_{a,b} + \widehat{\mathbf{e}}_{(n-1)a,(n-1)b+(n-1)} \right),$$

$$\widehat{\mathbf{h}}_{((n-1)a,(n-1)b+(n-1))} = \frac{\sqrt{-1}}{\sqrt{2}} \left(\widehat{\mathbf{e}}_{a,b} - \widehat{\mathbf{e}}_{(n-1)a,(n-1)b+(n-1)} \right),$$

which forms a real basis for $\widehat{H}$. The restriction of the inner product $\langle \cdot, \cdot \rangle$ of $\mathbb{C}^{n^2/4}$ on $\widehat{H}$ is a real positive definite inner product, with respect to which $\{\widehat{\mathbf{h}}_{a,b} : (a, b) \in \widehat{G}\}$ forms an orthonormal basis.

Proposition B.1. *Under the canonical embedding, the basis $\zeta_n^{k_0} \sqrt[n]{-2}^{k_1}$, $0 \leq k_0, k_1 < n/2$, of $\widehat{F}$ is sent to an orthogonal basis of $\mathbb{C}^{n^2/4}$:*

$$\langle \widehat{\sigma}(\zeta_n^{k_0} \sqrt[n]{-2}^{k_1}), \widehat{\sigma}(\zeta_n^{l_0} \sqrt[n]{-2}^{l_1}) \rangle = \begin{cases} \sqrt[n]{2}^{2k_1} \cdot n^2/4, & \text{if } k_0 = l_0 \text{ and } k_1 = l_1; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Same as the proof of Proposition 3.1. $\qquad\square$

Theorem B.2. $Tr(\zeta_n^{k_0+l_0} \sqrt[n]{-2}^{k_1+l_1})$ *is always 0 unless $k_1 = l_1 = 0$ or $k_1 + l_1 = n/2$. When $k_1 = l_1 = 0$, we have*

$$Tr(\zeta_n^{k_0+l_0}) = \begin{cases} n^2/4, & \text{if } k_0 = l_0 = 0; \\ -n^2/4, & \text{if } k_0 + l_0 = n/2. \end{cases}$$

when $k_1 + l_1 = n/2$, we have

$$Tr(\zeta_n^{k_0+l_0} \sqrt[n]{-2}^{n/2}) = \begin{cases} -n^2/4, & \text{if } k_0 + l_0 = n/8 \text{ or } 3n/8; \\ n^2/4, & \text{if } k_0 + l_0 = 5n/8 \text{ or } 7n/8. \end{cases}$$

Proof. Same as the proof of Theorem 4.1. □

From this theorem, we can compute the absolute discriminant of $\mathbb{Z}[\zeta_n, \sqrt[n]{-2}]$ using the method given in Section 4.2, which is also equal to $2^{n(n-2)/8}(n^2/4)^{n^2/4}$, a power-of-two integer. The dual of the natural integral basis $\zeta_n^{k_0} \sqrt[n]{-2}^{k_1}$ of $\mathbb{Z}[\zeta_n, \sqrt[n]{-2}]$ is a set of elements $\{\widehat{e}_{l_0,l_1} \in \widehat{F} : 0 \leq l_0, l_1 < n/2\}$ such that $\mathrm{Tr}(\widehat{e}_{l_0,l_1} \cdot \zeta_n^{k_0} \sqrt[n]{-2}^{k_1})$ is equal to 1 if $l_0 = k_0$ and $l_1 = k_1$, and 0 otherwise.

Proposition B.3. *When $l_1 = 0$, $\widehat{e}_{l_0,0}$ is given by*

$$\widehat{e}_{0,0} = \frac{4}{n^2} \ and \ \widehat{e}_{l_0,0} = -\frac{4}{n^2} \cdot \zeta_n^{n/2-l_0} \ if \ l_0 > 0.$$

When $0 < l_1 < n/2$, $\widehat{e}_{l_0,l_1}$ is given by

$$\widehat{e}_{l_0,l_1} = \begin{cases} -\frac{2}{n^2}\zeta_n^{n/8-l_0} \sqrt[n]{-2}^{n/2-l_1} - \frac{2}{n^2}\zeta_n^{3n/8-l_0} \sqrt[n]{-2}^{n/2-l_1}, & if \ 0 \leq l_0 \leq \frac{n}{8}, \\ -\frac{2}{n^2}\zeta_n^{3n/8-l_0} \sqrt[n]{-2}^{n/2-l_1} + \frac{2}{n^2}\zeta_n^{5n/8-l_0} \sqrt[n]{-2}^{n/2-l_1}, & if \ \frac{n}{8} < l_0 \leq \frac{3n}{8}, \\ \frac{2}{n^2}\zeta_n^{5n/8-l_0} \sqrt[n]{-2}^{n/2-l_1} + \frac{2}{n^2}\zeta_n^{7n/8-l_0} \sqrt[n]{-2}^{n/2-l_1}, & if \ \frac{3n}{8} < l_0 < \frac{n}{2}. \end{cases}$$

B.3   The Error Distributions on the Ideal Lattices

Let $\widehat{\mathbf{r}} = (\widehat{r}_{a,b})_{(a,b)\in\widehat{G}} \in (\mathbb{R}^+)^{n^2/4}$ be a vector of positive real numbers such that $\widehat{r}_{a,b} = \widehat{r}_{((n-1)a,(n-1)b+(n-1))}$ for every pair $(a,b) \in \widehat{N}$. Then a sample from the elliptical Gaussian distribution $D_{\widehat{\mathbf{r}}}$ is a vector $\sum_{(a,b)\in\widehat{G}} \widehat{x}_{a,b}\widehat{\mathbf{h}}_{a,b}$, where each $\widehat{x}_{a,b}$ is sampled independently from the one-dimensional Gaussian distribution $D_{\widehat{r}_{a,b}}$.

Definition B.4. *Given a positive real number $\widehat{\varrho}$, let $\widehat{\Psi}_{\leq\widehat{\varrho}}$ be the set of all elliptic Gaussian distribution $D_{\widehat{\mathbf{r}}}$ over $\widehat{H}$ ($\cong \widehat{F}_{\mathbb{R}}$) where for every $(a,b) \in \widehat{N}$ we have $\widehat{r}_{a,b} = \widehat{r}_{((n-1)a,(n-1)b+(n-1))}$ and $\widehat{r}_{a,b} \leq \widehat{\varrho}$.*

Lemma B.5. *For any $\widehat{\varrho}$, the family of distributions $\widehat{\Psi}_{\leq\widehat{\varrho}}$ is closed under the action of any element of $\mathrm{Gal}(\widehat{F}/\mathbb{Q})$.*

Proof. Same as the proof of Lemma 5.4. □

The $\widehat{\mathcal{R}}\text{-}\mathsf{DGS}_\gamma$ problem is: given a fractional ideal $\widehat{\mathcal{I}}$ associated with $\widehat{\mathcal{R}}$ and a number $\widehat{s} \geq \widehat{\gamma} = \widehat{\gamma}(\widehat{\mathcal{I}})$, output a sample from the distribution $D_{\widehat{\mathcal{I}},\widehat{s}}$.

Definition B.6. *Given a positive real number $\widehat{\delta}$, $\widehat{\Upsilon}_{\widehat{\delta}}$ is a distribution over distributions: a sample from $\widehat{\Upsilon}_{\widehat{\delta}}$ is an elliptical Gaussian distribution $D_{\widehat{\mathbf{r}}}$ on $\widehat{H}$ with parameters*

$$\widehat{r}_{a,b}^2 = \widehat{r}_{(n-1)a,(n-1)b+(n-1)}^2 = \widehat{\delta}^2 \left(1 + n\widehat{x}_{a,b}/2\right), \ (a,b) \in \widehat{N},$$

where $\widehat{x}_{a,b}$ is sampled independently from $\Gamma(2,1)$.

B.4 Reduction Theorems for RLWE

Recall that via sending ζ_n to X and $\sqrt[n]{-2}$ to Y, the integral ring $\mathbb{Z}[\zeta_n, \sqrt[n]{-2}]$ is isomorphic to a quotient polynomial ring

$$\widehat{\mathcal{R}} = \mathbb{Z}[X,Y]/\langle X^{n/2} + 1, Y^{n/2} - (X^{n/8} + X^{3n/8})\rangle.$$

The results in Section 4.3 can be extended to $\widehat{F}$. In particular, we have

$$\lambda_1(\widehat{\mathcal{R}}) = n/2, \qquad \lambda_{n^2/4}(\widehat{\mathcal{R}}) \leq 2^{-\frac{1}{2} - \frac{1}{n}} n,$$
$$\lambda_1(\widehat{\mathcal{R}}^\vee) = 2^{\frac{1}{2} + \frac{1}{n}}/n, \quad \lambda_{n^2/4}(\widehat{\mathcal{R}}^\vee) \leq 2/n.$$

The worst-case hardness of the Search Order-LWE in $\widehat{\mathcal{R}}$ comes from Theorem 4.1 of [19].

Proposition B.7. *Let $\widehat{\varrho} = \widehat{\varrho}(n^2/4)$ (> 0) and let $\widehat{q} = \widehat{q}(n^2/4)$ be such that $\widehat{\varrho}\widehat{q} \geq 2 \cdot \widehat{\omega}\left(\sqrt{\log(n^2/4)}\right)$. For some negligible function $\widehat{\epsilon} = \widehat{\epsilon}(n^2/4)$, there exists a probabilistic polynomial-time reduction from $\widehat{\mathcal{R}}$-DGS$_{\widehat{\gamma}}$ to OLWE$_{\widehat{q}, \widehat{\Psi}_{\leq \widehat{\varrho}}}$, where $\widehat{\gamma}$ is*

$$\widehat{\gamma} = \max\left\{\eta_{\widehat{\epsilon}}\left(\widehat{\mathcal{I}}\right) \cdot \left(\sqrt{2}/\widehat{\varrho}\right) \cdot \widehat{\omega}\left(\sqrt{\log(n^2/4)}\right), \sqrt{n^2/2}/\lambda_1\left(\widehat{\mathcal{I}}^\vee\right)\right\}.$$

Here $\widehat{\omega}(\sqrt{\log(n^2/4)})$ denotes a fixed but arbitrary function that grows asymptotically faster than $\sqrt{\log(n^2/4)}$. The proof of the following theorem is almost exactly the same as that of Theorem 5.10. In particular, we need a lexicographic order on $\widehat{G}$: $(a,b) < (c,d)$ if $a < c$ or $a = c$ and $b < d$.

Theorem B.8. *Let $\widehat{\delta} < 2\sqrt{\log(n^2/4)}/n$ and let $\widehat{q} = \widehat{q}(n^2) \geq 3$ be a poly(n)-bounded prime number such that both $X^{n/2} \equiv -1 \bmod \widehat{q}$ and $Y^n \equiv -2 \bmod \widehat{q}$ have solutions. Then there exists a polynomial-time reduction from $\widetilde{O}(n/(2\widehat{\delta}))$-approximate SIVP (or SVP) in the ideal lattices of $\widehat{F}$ to DOLWE$_{\widehat{q}, \Upsilon_{\widehat{\delta}}}$. Moreover, for any $\widehat{\ell} \geq 1$, we can replace the target problem in the reduction by the problem of solving DOLWE$_{\widehat{q}, D_{\widehat{\xi}}}$ given only $\widehat{\ell}$ samples, where $\widehat{\xi} = \widehat{\delta} \cdot \left(n^2\widehat{\ell}/(4\log(n^2\widehat{\ell}/4))\right)^{1/4}$.*

B.5 The 2NTT

Via sending ζ_n to X and $\sqrt[n]{-2}$ to Y, the integral ring $\mathbb{Z}[\zeta_n, \sqrt[n]{-2}]$ is isomorphic to a quotient polynomial ring

$$\widehat{\mathcal{R}} = \mathbb{Z}[X,Y]/\langle X^{n/2} + 1, Y^{n/2} - (X^{n/8} + X^{3n/8})\rangle.$$

We choose a prime number $\widehat{p}$ such that $X^{n/2} + 1 \equiv 0 \bmod \widehat{p}$ has a solution $\widehat{\alpha}$ and $Y^n + 2 \equiv 0 \bmod \widehat{p}$ has a solution $\widehat{\beta}$. By replacing $\widehat{\beta}$ by $\widehat{\alpha}\widehat{\beta}$ if necessary, we can always assume $\widehat{\beta}^{n/2} = \widehat{\alpha}^{n/8} + \widehat{\alpha}^{3n/8} \bmod \widehat{p}$. For such a prime number $\widehat{p}$, the polynomial equations

$$X^{n/2} \equiv -1 \bmod \widehat{p} \text{ and } Y^{n/2} \equiv X^{n/8} + X^{3n/8} \bmod \widehat{p} \tag{B.5}$$

have $n^2/4$ solutions in $\mathbb{F}_{\widehat{p}}$. The first equation has $n/2$ solutions $\{\widehat{\alpha}^{2i+1} : 0 \leq i < n/2\}$. For every solution $X = \widehat{\alpha}^{2i+1}$, the second equation also has $n/2$ solutions for Y:

$$Y = \begin{cases} \widehat{\alpha}^{2j}\widehat{\beta}, \ 0 \leq j < n/2, \text{ if } i \equiv 0, 1 \bmod 4; \\ \widehat{\alpha}^{2j+1}\widehat{\beta}, \ 0 \leq j < n/2, \text{ if } i \equiv 2, 3 \bmod 4. \end{cases}$$

An element of $\widehat{\mathcal{R}}_{\widehat{p}} = \widehat{\mathcal{R}}/\widehat{p}\widehat{\mathcal{R}}$ is of the form $\widehat{\mathbf{F}}(X,Y) = \sum_{k,l=0}^{n/2-1} \widehat{f}_{k,l} X^k Y^l$ with $\widehat{f}_{k,l} \in \mathbb{F}_{\widehat{p}}$.

Definition B.9. *The 2NTT of a polynomial $\widehat{\mathbf{F}} \in \widehat{\mathcal{R}}_{\widehat{p}}$ is the evaluation of $\widehat{\mathbf{F}}$ at the $n^2/4$ solutions of the two polynomial equations in Eq.* (B.5).

The vector butterflies for the 2NTT of $\widehat{\mathbf{F}}$ follows immediately from Section 6:

1. Transverse vector butterfly. We use generalized one-variable NTT to evaluate $\widehat{\mathbf{F}}$ at the $n/2$ roots $X = \widehat{\alpha}^{2i+1}$ with $0 \leq i < n/2$, the output of which are $n/2$ vectors $\widehat{\mathbf{F}}(\widehat{\alpha}^{2i+1}, Y)$. There are $\log_2(n/2)$ stages in this phase, and each stage consumes $O(n)$ vector operations.

2. Transpose and longitudinal vector butterfly. The output of the transverse vector butterfly naturally falls into two groups: Group 1 with $i \equiv 0,1 \bmod 4$ and Group 2 with $i \equiv 2,3 \bmod 4$. Each group forms an $n/4 \times n/2$ matrix. Now we evaluate the polynomial (with column vector coefficients) constructed from the column vectors of Group 1 at the roots $\{\widehat{\alpha}^{2j}\widehat{\beta} : 0 \leq j < n/2\}$ using a vector butterfly with a new set of twiddle factors. Then we evaluate the polynomial constructed from the column vectors of Group 2 at the roots $\{\widehat{\alpha}^{2j+1}\widehat{\beta} : 0 \leq j < n/2\}$ using a vector butterfly with another new set of twiddle factors. Each evaluation consists of $\log_2(n/2)$ stages and each stage consumes $O(n)$ vector operations with the length of the vectors now being $n/4$.

B.6 The Factorization of Primes in $\mathbb{Z}[\zeta_n, \sqrt[n]{-2}]$

Suppose now we have a prime number $\widehat{q}$ such that $X^{n/2} + 1 \equiv 0 \bmod \widehat{q}$ has a solution $\widehat{\alpha} \in \mathbb{F}_{\widehat{q}}$ and $Y^n + 2 \equiv 0 \bmod \widehat{q}$ has a solution $\widehat{\beta} \in \mathbb{F}_{\widehat{q}}$. Through replacing $\widehat{\beta}$ by $\widehat{\alpha}\widehat{\beta}$ if necessary, we can always assume $\widehat{\beta}^{n/2} = \widehat{\alpha}^{n/8} + \widehat{\alpha}^{3n/8} \bmod \widehat{q}$. Such a prime number $\widehat{q}$ splits completely in $\mathbb{Z}[\zeta_n]$ into a product of prime ideals $\prod_{i=0}^{n/2-1} \widehat{\mathfrak{p}}_{2i+1}$, where $\widehat{\mathfrak{p}}_{2i+1} \subset \mathbb{Z}[\zeta_n]$ is given by [19, 22]

$$\widehat{\mathfrak{p}}_{2i+1} = \langle \widehat{q}, \zeta_n - \widehat{\alpha}^{2i+1} \rangle = \widehat{q}\mathbb{Z}[\zeta_n] + (\zeta_n - \widehat{\alpha}^{2i+1})\mathbb{Z}[\zeta_n]. \tag{B.6}$$

The automorphism ς_j acts transitively on the $n/2$ prime ideals in the factorization via $\varsigma_j : \widehat{\mathfrak{p}}_{2i+1} \to \widehat{\mathfrak{p}}_{(2i+1)/j}$, where the quotient $(2i + 1)/j$ is computed in $(\mathbb{Z}/n\mathbb{Z})^\times$ [19]. The residue class degree of $\widehat{\mathfrak{p}}_{2i+1}$ is 1, i.e., $f(\widehat{\mathfrak{p}}_{2i+1}/\widehat{q}) = 1$, for every i, hence we have a field isomorphism $\mathbb{Z}/\widehat{q}\mathbb{Z} \cong \mathbb{Z}[\zeta_n]/\widehat{\mathfrak{p}}_{2i+1}$ [22, 34]. By definition, the norm $\mathrm{Nm}(\widehat{\mathfrak{p}}_{2i+1})$ of $\widehat{\mathfrak{p}}_{2i+1}$ is $\widehat{q}$ [22].

The prime ideal $\widehat{\mathfrak{p}}_{2i+1}$ splits completely in $\mathbb{Z}[\zeta_n, \sqrt[n]{-2}]$. If $i \equiv 0,1 \bmod 4$, we have a factorization

$$Y^{n/2} - \left(\zeta_n^{n/8} + \zeta_n^{3n/8}\right) = \prod_{j=0}^{n/2-1} (Y - \widehat{\alpha}^{2j}\widehat{\beta}) \bmod \widehat{\mathfrak{p}}_{2i+1}.$$

In this case, let the prime ideal $\widehat{\mathfrak{P}}_{2i+1,2j}$ of $\mathbb{Z}[\zeta_n, \sqrt[n]{-2}]$ be

$$\widehat{\mathfrak{P}}_{2i+1,2j} = \langle \widehat{q}, \zeta_n - \widehat{\alpha}^{2i+1}, \sqrt[n]{-2} - \widehat{\alpha}^{2j}\widehat{\beta} \rangle.$$

Because $\widehat{\mathfrak{P}}_{2i+1,2j} \cap \mathbb{Z}[\zeta_n] = \widehat{\mathfrak{p}}_{2i+1}$ for every j, so $\widehat{\mathfrak{p}}_{2i+1}$ splits completely into $\prod_{j=0}^{n/2-1} \widehat{\mathfrak{P}}_{2i+1,2j}$ in $\mathbb{Z}[\zeta_n, \sqrt[n]{-2}]$. If $i \equiv 2, 3 \bmod 4$, we have a factorization

$$Y^{n/2} - \left(X^{n/8} + X^{3n/8}\right) = \prod_{j=0}^{n/2-1} \left(Y - \widehat{\alpha}^{2j+1}\widehat{\beta}\right) \bmod \widehat{\mathfrak{p}}_{2i+1}.$$

In this case, let the prime ideal $\widehat{\mathfrak{P}}_{2i+1,2j+1}$ of $\mathbb{Z}[\zeta_n, \sqrt[n]{-2}]$ be

$$\widehat{\mathfrak{P}}_{2i+1,2j+1} = \langle \widehat{q}, \zeta_n - \widehat{\alpha}^{2i+1}, \sqrt[n]{-2} - \widehat{\alpha}^{2j+1}\widehat{\beta} \rangle,$$

and similarly $\widehat{\mathfrak{p}}_{2i+1}$ splits completely into $\prod_{j=0}^{n/2-1} \widehat{\mathfrak{P}}_{2i+1,2j+1}$ in $\mathbb{Z}[\zeta_n, \sqrt[n]{-2}]$. In conclusion, the prime number $\widehat{q}$ splits completely in $\mathbb{Z}[\zeta_n, \sqrt[n]{-2}]$ into

$$\widehat{q}\,\mathbb{Z}[\zeta_n, \sqrt[n]{-2}] = \prod_{(a,b)\in\widehat{G}} \widehat{\mathfrak{P}}_{a,b}. \tag{B.7}$$

The action of the Galois group $\mathrm{Gal}(\widehat{F}/\mathbb{Q})$ on $\widehat{\mathfrak{P}}_{a,b}$, $(a,b) \in \widehat{G}$, can be obtained straightforwardly: $\widehat{\sigma}_{a,b}(\widehat{\mathfrak{P}}_{c,d}) = \widehat{\mathfrak{P}}_{c/a,d-b}$. Moreover, since $\widehat{q}$ splits completely in $\mathbb{Z}[\zeta_n, \sqrt[n]{-2}]$, we have an $\mathbb{Z}/\widehat{q}\mathbb{Z} \cong \mathbb{Z}[\zeta_n, \sqrt[n]{-2}]/\widehat{\mathfrak{P}}_{a,b}$ for all $(a,b) \in \widehat{G}$, from which we deduce that $\mathrm{Nm}(\widehat{\mathfrak{P}}_{a,b}) = \widehat{q}$ [22].

C   The Splitting Field of $Y^n - r$

In this section, we look at the splitting field F_r of $Y^n - r$, where $n \ (\geq 8)$ is a power-of-two integer and $|r| \ (\geq 3)$ is a prime number. We compute the Galois group $\mathrm{Gal}(F_r/\mathbb{Q})$ and introduce a 2NTT for $\mathbb{Z}[\zeta_n, \sqrt[n]{r}]$.

C.1   The Computation of the Galois Group

The splitting field F_r of $Y^n - r$ is generated by ζ_n and $\sqrt[n]{r}$: $F_r = \mathbb{Q}(\zeta_n, \sqrt[n]{r})$. In order to compute the degree of F_r over $\mathbb{Q}$, we first need to find all the quadratic subfields of $\mathbb{Q}(\zeta_n)$.

Lemma C.1. *Given a power-of-two integer $n \ (\geq 8)$, the only quadratic subfields of $\mathbb{Q}(\zeta_n)$ are $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{-2})$.*

Proof. From the fundamental theorem of Galois theory [24], the quadratic subfields of $\mathbb{Q}(\zeta_n)$ correspond to the subgroups of $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ with index being 2. From Section 2.3.2, $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ is $\langle \varsigma_5 \rangle \times \langle \varsigma_{n-1} \rangle$, where the order of ς_5 is $n/4$ and the order of ς_{n-1} is 2. Define pr to be the natural projection map

$$\mathrm{pr} : \langle \varsigma_5 \rangle \times \langle \varsigma_{n-1} \rangle \to \langle \varsigma_5 \rangle,$$

the kernel of which is the subgroup $\langle \varsigma_{n-1} \rangle$ of order 2.

Suppose H is a subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ of index 2. If $\langle \varsigma_{n-1} \rangle$ is a subgroup of H, then $\mathrm{pr}(H)$ is a subgroup of $\langle \varsigma_5 \rangle$ of index 2, namely the order of $\mathrm{pr}(H)$ is $n/8$. Since $\langle \varsigma_5 \rangle$ is a cyclic group of order $n/4$, we immediately deduce that $\mathrm{pr}(H)$ is $\langle \varsigma_5^2 \rangle$, hence H is $\langle \varsigma_5^2 \rangle \times \langle \varsigma_{n-1} \rangle$. The subfield of $\mathbb{Q}(\zeta_n)$ fixed by H is $\mathbb{Q}(\zeta_n^{n/8} + \zeta_n^{7n/8})$, i.e., $\mathbb{Q}(\sqrt{2})$.

If $\langle \varsigma_{n-1} \rangle$ is not a subgroup of H, then pr must be surjective since the order of H is $n/4$, which is equal to the order of $\langle \varsigma_5 \rangle$. Therefore, we deduce $H = \langle \varsigma_5 \rangle$ or $H = \langle \varsigma_5 \circ \varsigma_{n-1} \rangle$. If $H = \langle \varsigma_5 \rangle$, then the subfield of $\mathbb{Q}(\zeta_n)$ fixed by it is $\mathbb{Q}(\sqrt{-1})$. If $H = \langle \varsigma_5 \circ \varsigma_{n-1} \rangle$, then the subfield of $\mathbb{Q}(\zeta_n)$ fixed by it is $\mathbb{Q}(\sqrt{-2})$. $\qquad\square$

We now adapt the proof in [35] to prove the following proposition.

Proposition C.2. *The degree of F_r over $\mathbb{Q}$ is $n^2/2$.*

Proof. As the degree of $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$ is $n/2$ and the degree of $\mathbb{Q}(\sqrt[n]{r})$ over $\mathbb{Q}$ is n, so the degree of F_r over $\mathbb{Q}$ satisfies [24]

$$[F_r : \mathbb{Q}] = \frac{[\mathbb{Q}(\zeta_n) : \mathbb{Q}] \cdot [\mathbb{Q}(\sqrt[n]{r}) : \mathbb{Q}]}{[\mathbb{Q}(\zeta_n) \cap \mathbb{Q}(\sqrt[n]{r}) : \mathbb{Q}]} = \frac{n^2/2}{[\mathbb{Q}(\zeta_n) \cap \mathbb{Q}(\sqrt[n]{r}) : \mathbb{Q}]}.$$

Let $[\mathbb{Q}(\zeta_n) \cap \mathbb{Q}(\sqrt[n]{r}) : \mathbb{Q}]$ be m, which is also a power-of-two integer and $m \leq n/2$. Let $\mathbb{Q}(\zeta_n) \cap \mathbb{Q}(\sqrt[n]{r})$ be denoted by K, we now show $K = \mathbb{Q}(\sqrt[m]{r})$. The norm of $\sqrt[n]{r}$ over K, denoted by $\mathrm{Nm}_{\mathbb{Q}(\sqrt[n]{r})/K}(\sqrt[n]{r})$, is given by the product of n/m conjugates of $\sqrt[n]{r}$ over K. But each of these conjugates is of the form $\zeta_n^i \cdot \sqrt[n]{r}$, hence this norm of $\sqrt[n]{r}$ must be of the form

$$\mathrm{Nm}_{\mathbb{Q}(\sqrt[n]{r})/K}(\sqrt[n]{r}) = \zeta_n^j \cdot \sqrt[n]{r}^{n/m} = \zeta_n^j \cdot \sqrt[m]{r} \in K.$$

But K also lies in $\mathbb{Q}(\zeta_n)$, thus $\zeta_n^j \cdot \sqrt[m]{r}$ is in $\mathbb{Q}(\zeta_n)$, hence $\sqrt[m]{r}$ is also in $\mathbb{Q}(\zeta_n)$. The upshot is that $\sqrt[m]{r} \in K$. But $[\mathbb{Q}(\sqrt[m]{r}) : \mathbb{Q}]$ is also m, so $K = \mathbb{Q}(\sqrt[m]{r})$.

The Galois group of $\mathbb{Q}(\zeta_n)$ is abelian, therefore every subfield is a Galois extension of $\mathbb{Q}$, which tells us K is a Galois extension of $\mathbb{Q}$ [24]. As a result, K contains all the conjugates of $\sqrt[m]{r}$. In particular, K contains $\zeta_m \cdot \sqrt[m]{r}$, which implies $\zeta_m \in K$. But as a subfield of $\mathbb{Q}(\sqrt[n]{r})$, K is a real field, whence $m \leq 2$. From Lemma C.1, $\mathbb{Q}(\sqrt[2]{r})$ cannot be in $\mathbb{Q}(\zeta_n)$ if $|r| \geq 3$, hence $m = 1$ and $K = \mathbb{Q}$. Thus completes the proof. $\qquad\square$

The Galois group of F_r over $\mathbb{Q}$, $\mathrm{Gal}(F_r/\mathbb{Q})$, can be computed similarly. An element of $\mathrm{Gal}(F_r/\mathbb{Q})$ is determined by its actions on ζ_n and $\sqrt[n]{r}$:

$$\sigma_{a,b}(\zeta_n) = \zeta_n^a \text{ and } \sigma_{a,b}(\sqrt[n]{r}) = \zeta_n^b \cdot \sqrt[n]{r},$$

where $a \in (\mathbb{Z}/n\mathbb{Z})^\times$ and $b \in \mathbb{Z}/n\mathbb{Z}$. From Proposition C.2, there is no further restriction on the values of a and b. Let G_r be the finite set

$$G_r = \left\{ (a,b) : a \in (\mathbb{Z}/n\mathbb{Z})^\times \text{ and } b \in \mathbb{Z}/n\mathbb{Z} \right\},$$

then $\mathrm{Gal}(F_r/\mathbb{Q})$ is given by $\{\sigma_{a,b} : (a,b) \in G_r\}$. The canonical embedding of F_r into $\mathbb{C}^{n^2/2}$ can be constructed similarly:

$$\sigma : x \in F_r \mapsto (\sigma_{a,b}(x))_{(a,b) \in G_r} \in \mathbb{C}^{n^2/2}. \tag{C.1}$$

Proposition C.3. *Under the canonical embedding of F_r into $\mathbb{C}^{n^2/2}$, we have:*

$$\langle \sigma(\zeta_n^{k_0} \sqrt[n]{r}^{k_1}), \sigma(\zeta_n^{l_0} \sqrt[n]{r}^{l_1}) \rangle = \begin{cases} \sqrt[n]{|r|}^{2k_1} \cdot n^2/2, & \text{if } k_0 = l_0 \text{ and } k_1 = l_1; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The proof uses similar method as in the proof of Proposition 3.1. $\qquad\square$

However, we only know $\mathbb{Z}[\zeta_n, \sqrt[n]{r}]$ is an order of F_r [4, 22]. The hardness of Order-LWE in $\mathbb{Z}[\zeta_n, \sqrt[n]{r}]$ can be analysed as in the paper [4]. A mild issue is that under the canonical embedding, the length of the shortest vector $\sigma(\zeta_n^{k_0})$ is $n/\sqrt{2}$, while the length of the longest vector $\sigma(\zeta_n^{k_0} \sqrt[n]{r}^{n-1})$ is $|r|^{1-1/n} n/\sqrt{2}$, which is almost $|r|$ times larger than the shortest ones. This could potentially lead to some technical issues when choosing the error distributions for Order-LWE in $\mathbb{Z}[\zeta_n, \sqrt[n]{r}]$, which deserves further studies [4, 19].

C.2   The 2NTT

Via sending ζ_n to X and $\sqrt[n]{r}$ to Y, $\mathbb{Z}[\zeta_n, \sqrt[n]{r}]$ is isomorphic to the ring

$$\mathcal{R}' = \mathbb{Z}[X, Y]/\langle X^{n/2} + 1, Y^n - r \rangle.$$

The 2NTT for the ring $\mathcal{R}'$ can be defined similarly as in Section 6. Suppose p is a prime number such that there exist $\alpha, \beta \in \mathbb{F}_p$ that satisfy $\alpha^{n/2} + 1 \equiv 0 \bmod p$ and $\beta^n - r \equiv 0 \bmod p$. Then the roots of $X^{n/2} + 1 \equiv 0 \bmod p$ are $\{\alpha^{2i+1} : 0 \leq i < n/2\}$ and the roots of $Y^n - r \equiv 0 \bmod p$ are $\{\alpha^j \beta : 0 \leq j < n\}$. The 2NTT of a polynomial $\mathbf{F}(X, Y) \in \mathcal{R}'_p$ is to evaluate it at the $n^2/2$ root-pairs

$$\left\{ (\alpha^{2i+1}, \alpha^j \beta) : 0 \leq i < n/2, \ 0 \leq j < n \right\}.$$

The vector butterfly for the 2NTT follows immediately from Section 6:

1. Transverse vector butterfly, which is the evaluation of $\mathbf{F}(X, Y)$ at the roots $X = \alpha^{2i+1}$, $0 \leq i < n/2$. It consists of $\log_2(n/2)$ stages, and each stage costs $O(n)$ vector operations. The output is an $n/2 \times n$ matrix, with each row the coefficient vector of the one-variable polynomial $\mathbf{F}(\alpha^{2i+1}, Y)$ for some $0 \leq i < n/2$.

2. Transpose and longitudinal vector butterfly, which is the evaluation of a vector-valued polynomial constructed from the column vectors of the output of the first phase at the roots $Y = \alpha^j \beta$, $0 \leq j < n$. It consists of $\log_2 n$ stages and each stage costs $O(n)$ vector operations. The output is an $n/2 \times n$ matrix.

Hence, the total complexity of 2NTT is $O(n \log_2 n)$ vector operations.

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