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Research article

On the study of the recurrence relations and characterizations based on progressive first-failure censoring

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Abstract: In this research, the progressive first-failure censored data (PFFC) from the Kumaraswamy modified inverse-Weibull distribution (KMIWD) were used to obtain the recurrence relations and characterizations for single and product moments. The recurrence relationships allow for a rapid and efficient assessment of the means, variances and covariances for any sample size. Additionally, the paper outcomes can be boiled down to the traditional progressive type-II censoring. Also, some special cases are limited to some lifetime distributions as the exponentiated modified inverse Weibull and Kumaraswamy inverse exponential.

Keywords: KMIWD; progressive first-failure censoring; product moments; recurrence relations and characterizations; single moments
Mathematics Subject Classification: 62E10, 62N99

1. Introduction

In industrial and reliability experiments, it is important to save cost and money when observing the product's failure time. Censoring is the most suitable technique for achieving this aim through the lifetime experiments, where we observe some lifetimes or failure times and not all the lifetimes of the units under the test. There are different methods of censoring. One of the most popular types of censorship is type-II censoring where n items are placed on the lifetime test and the test is continued until the occurrence of the m^{th} failure time, where $1 \leq m \leq n$. Progressive censoring of type-II (PTIIC) is the modified version of type-II censoring, where researchers can exclude some of the survived units during the experiment running. PTIIC can be explained as follows: Assume that n units are subjected to a lifespan test, and that m failures will be detected by the test's completion. When the initial failure occurs, R_1 of the survived units are chosen at random and excluded from the test, when the second failure happens, R_2 of the survived units are chosen at random and excluded from the test. This process will continue until the m^{th} failure is obtained at that time the remaining survived units $(n - m - R_1 - R_2 - \cdots - R_m)$ are removed from the test. For extensive reading about PTIIC see [1,2], where they presented a variety of progressive censorship features.

When the lifespan of an item is relatively ongoing and its testing establishments are few, but the testing units are inexpensive, one can test $n \times k$ units by storing them in sets of k, with each group tested as one unit. The lifespan test is then run by testing each of these unit sets separately until the point at which each set has its first failure. First-failure censoring is the term for this type of censorship, which was first developed by Balasooriya [3]. Different authors have conducted the study of the first-failure censoring, such as Wu et al. [4] and Wu and Yu [5]. Blending the first-failure censoring and PTIIC will result in the PFFC scheme, in which we test groups of units with the privilege of removing some survived groups of units during the test operation; this was contributed by Wu and Kus [6]. Different works have discussed the concept of PFFC; see [7–13].

PFFC can be described as follows: Suppose a lifetime test is administered to n separate groups with k items in each group. Upon the occurrence of the first failure $X_{1:m:n,k}^{(R_1,R_2,\dots,R_m)}$, R_1 groups and the group exhibiting the first failure are arbitrarily eliminated from the test. At the occurrence of the second failure $X_{2:m:n,k}^{(R_1,R_2,\dots,R_m)}$, R_2 groups and the group exhibiting the second failure are arbitrarily eliminated from the test and so on until the m^{th} failure $X_{m:m:n,k}^{(R_1,R_2,\dots,R_m)}$ is occurred. The unobserved groups

$$R_m = n - m - R_1 - R_2 - \cdots - R_{m-1}$$

are eliminated from the test. Then

$$X_{1:m:n,k}^{(R_1,R_2,\dots,R_m)} < \cdots < X_{m:m:n,k}^{(R_1,R_2,\dots,R_m)}$$

are called PFFC sample with the censoring scheme $(R_1, R_2, \dots, R_m)$, where $n = m + \sum_{i=1}^{m} R_i$. Suppose that the failuretimes of the $n \times k$ units under the test follow a continuous distribution with CDF $F(x)$ and PDF $f(x)$, then the joint pdf for $(X_{1:m:n,k}^{(R_1,R_2,\dots,R_m)}, \dots, X_{m:m:n,k}^{(R_1,R_2,\dots,R_m)})$ is defined as follows:

$$f_{X_{1:m:n},\dots,X_{m:m:n}}(x_1, x_2, \dots, x_m) = I_{(n,m-1)} k^m \prod_{i=1}^{m} f(x_i)[\bar{F}(x_i)]^{N_i}, 0 < x_1 < \cdots < x_m < \infty, \quad (1.1)$$

where,

$$I_{(n,m-1)} = n(n - R_1 - 1) \dots (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1),$$

$$N_i = kR_i + k - 1.$$

In mathematical statistics, recurrence relations are of great use in variety of domains as they reduce the number of direct computations quite considerably. They are also useful in obtaining the moment generating functions, moments and in characterizing distributions. Different authors have discussed the recurrence relations with characterizations: Aggarwala and Balakrishnan [14] obtained the RR for both product and single moments of PTIIRC from exponential distribution; El-Din et al. [15,16] derived RR of moments of the Gompertz and generalized Pareto distributions based on general PTIIRC with characterizations; Sadek et al. [17] discussed the characterization and the RR based on general PTIIRC; and El-Din and Sharawy [18] derived RR for the generalized exponential distribution based on general PTIIRC. However, no studies about the RR under the PFFC exist in the literature. In this paper, we derive the RR and characterizations for the KMIWD based on PFFC.

A new contribution for enhancing the existing distributions has been added to the literature (see for example [19–22]). In 2015, Aryal and Elbatal [23] proposed a new modified distribution called the KMIWD. This is an extremely flexible model that approaches different distributions with different parameters. It has many applications in engineering, computer sciences and hydrology. The PDF of the KMIWD is given by

$$f(x,a,b,\alpha,\beta,\theta)=ab\left(\frac{\beta}{x^2}+\frac{\theta\alpha}{x^{\alpha+1}}\right)exp\left[-a\left(\frac{\beta}{x}+\frac{\theta}{x^{\alpha}}\right)\right]\left\{1-exp\left[-a\left(\frac{\beta}{x}+\frac{\theta}{x^{\alpha}}\right)\right]\right\}^{b-1}, \quad (1.2)$$

where,

$$a,b,\alpha,\beta,\theta>0, x>0.$$

The corresponding CDF of KMIWD is given by

$$F(x,a,b,\alpha,\beta,\theta)=1-\left\{1-exp\left[-a\left(\frac{\beta}{x}+\frac{\theta}{x^{\alpha}}\right)\right]\right\}^{b}. \quad (1.3)$$

The relation between (1.2) and (1.3) is given by

$$\left\{exp\left[a\left(\frac{\beta}{x}+\frac{\theta}{x^{\alpha}}\right)\right]-1\right\}f(x)=ab\left(\frac{\beta}{x^2}+\frac{\theta\alpha}{x^{\alpha+1}}\right)[1-F(x)]. \quad (1.4)$$

Many existence distributions can be obtained from the KMIWD by changing its parameters as follows in Table 1.

Table 1. Subdistributions that can be obtained from KMIWD.

Case	Values of parameters	Distribution
1	b=1	exponentiated modified inverse Weibull
2	α=2	Kumaraswamy modified inverse Rayleigh
3	α=1	Kumaraswamy inverse exponential
4	β=0	Kumaraswamy inverse Weibull
5	a=1 and b=1	the modified inverse Weibull
6	a=1, b=1 and β=0	the inverse Weibull

The i^{th} single moment for $X_{q:m:n,k}$ based on the PFFC is defined as follows

$$\mu_{q:m:n,k}^{(N_1,\dots,N_m)(i)}=E\left[X_{q:m:n,k}^{(N_1,\dots,N_m)}\right]^i=I_{(n,m-1)}\iint\dots\int_{0<x_1<\dots<x_m<\infty}x_q^i k^m f(x_1)[\bar{F}(x_1)]^{N_1}$$

$$\times f(x_2)[\bar{F}(x_2)]^{N_2}\dots f(x_m)[\bar{F}(x_m)]^{N_m}dx_1\dots dx_m, \quad (1.5)$$

while the i^{th} and r^{th} product moment for $X_{q:m:n,k}$ and $X_{s:m:n,k}$ ($X_{q:m:n,k}<X_{s:m:n,k}$) based on the PFFC has the following definition:

$$\mu_{q,s:m:n,k}^{(N_1,\dots,N_m)^{(i,r)}} = E\left[X_{q:m:n,k}^{(N_1,\dots,N_m)^{i}} X_{s:m:n,k}^{(N_1,\dots,N_m)^{r}}\right] = I_{(n,m-1)} \iint \cdots \int_{0<x_1<\cdots<x_m<\infty} x_q^i x_s^j k^m$$

$$\times f(x_1)[\bar{F}(x_1)]^{N_1} f(x_2)[\bar{F}(x_2)]^{N_2} \dots f(x_m)[\bar{F}(x_m)]^{N_m} dx_1 \dots dx_m. \tag{1.6}$$

Our paper is motivated by the unfortunate lack of literature on recurrence relations and characterization based on the PFFC, particularly under a significant and general distribution like the KMIWD. This is how the rest of the article is organized: In Section 2, both the single and product RR are obtained based on the PFFC. The characterizations are analyzed in Section 3. Finally, Section 4 concludes the proposed work in this article.

2. Recurrence relations of progressive first failure censoring

In this section, we propose the single and product RR of KMIWD based on PFFC. In Theorem 2.1, we propose the recurrence relation associated with the single moment of PFFC.

Theorem 2.1. For $2 \le r \le m-1, m \le n$ and $i \ge 0$, then

$$\sum_{h=0}^{\infty}\sum_{l=0}^{\infty} \frac{(i-1)(a\beta)^h(a\theta)^l}{abh!\,l!} \mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i-h-\alpha l)}} = \frac{(i-1)}{ab} \mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i)}}$$

$$+\beta(N_r+1)\mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i-1)}} + \theta\alpha(i-1)\left(\frac{N_r+1}{i-\alpha-2}\right)\mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i-\alpha-2)}} - (n-R_1-\cdots-R_{r-1}-r+1)$$

$$\left[\beta\mu_{r-1:m-1:n,k}^{(N_1,\dots,N_{r-2},(N_{r-1}+N_r+1),N_{r+1},\dots,N_m)^{(i-1)}} + \frac{(i-1)\theta\alpha}{i-\alpha-2}\mu_{r-1:m-1:n,k}^{(N_1,\dots,N_{r-2},(N_{r-1}+N_r+1),N_{r+1},\dots,N_m)^{(i-\alpha-2)}}\right]$$

$$+(n-R_1-\cdots-R_r-r)$$

$$\left[\beta\mu_{r:m-1:n,k}^{(N_1,\dots,N_{r-1},(N_r+N_{r+1}+1),N_{r+2},\dots,N_m)^{(i-1)}} + \frac{(i-1)\theta\alpha}{i-\alpha-2}\mu_{r:m-1:n,k}^{(N_1,\dots,N_{r-1},(N_r+N_{r+1}+1),N_{r+2},\dots,N_m)^{(i-\alpha-2)}}\right]. \tag{2.1}$$

Proof. From (1.4) and (1.5), we get

$$\sum_{h=0}^{\infty}\sum_{l=0}^{\infty} \frac{(a\beta)^h(a\theta)^l}{abh!\,l!} \mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i-h-\alpha l)}} - \frac{1}{ab}\mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i)}}$$

$$= I_{(n,m-1)} \iint \dots \int_{0<x_1<\cdots<x_{r-1}<x_{r+1}<\cdots<x_m<\infty} k^m W_1(x_{r-1},x_{r+1}) \dots f(x_1)[\bar{F}(x_1)]^{N_1} \dots f(x_{r-1})$$

$$\times [\bar{F}(x_{r-1})]^{N_{r-1}} f(x_{r+1})[\bar{F}(x_{r+1})]^{N_{r+1}} \dots f(x_m)[\bar{F}(x_m)]^{N_m} x_1 \dots dx_{r-1} dx_{r+1} \dots dx_m, \tag{2.2}$$

where,

$$W_1(x_{r-1},x_{r+1}) = \int_{x_{r-1}}^{x_{r+1}} x_r^i \left(\frac{\beta}{x_r^2} + \frac{\theta\alpha}{x_r^{\alpha+1}}\right)[\bar{F}(x_r)]^{N_r+1} dx_r. \tag{2.3}$$

Using integrating by parts, we get

$$W_1(x_{r-1},x_{r+1}) = \frac{\beta x_{r+1}^{i-1}[\bar{F}(x_{r+1})]^{N_r+1} - \beta x_{r-1}^{i-1}[\bar{F}(x_{r-1})]^{N_r+1}}{i-1}$$
$$+\frac{\theta\alpha x_{r+1}^{i-\alpha-2}[\bar{F}(x_{r+1})]^{N_r+1} - \theta\alpha x_{r-1}^{i-\alpha-2}[\bar{F}(x_{r-1})]^{N_r+1}}{i-\alpha-2}$$
$$+\beta\left(\frac{N_r+1}{i-1}\right)\int_{x_{r-1}}^{x_{r+1}} x_r^{i-1} f(x_r)[\bar{F}(x_r)]^{N_r}dx_r + \frac{\theta\alpha(N_r+1)}{i-\alpha-2}\int_{x_{r-1}}^{x_{r+1}} x_r^{i-\alpha-2} f(x_r)[\bar{F}(x_r)]^{N_r}dx_r. \quad (2.4)$$

By substituting the obtained expression of $W_1(x_{r-1},x_{r+1})$ from (2.4) in (2.2) and simplifying, yields (2.1). This brings the proof to a close.

In the coming theorems, we discuss the product moments of PFFC.

Theorem 2.2. For $1 \le r < s \le m-1, m \le n$ and $i,j \ge 0$,

$$\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\frac{(i-1)(a\beta)^h(a\theta)^l}{h!\,l!}\mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i-h-\alpha l,j)}} = \frac{(i-1)}{ab}\mu_{r,s:m:n,k}^{(N_1,\dots,N_m)^{(i,j)}} + \beta(N_r+1)\mu_{r,s:m:n,k}^{(N_1,\dots,N_m)^{(i-1,j)}}$$
$$+\theta\alpha(i-1)\left(\frac{N_r+1}{i-\alpha-2}\right)\mu_{r,s:m:n,k}^{(N_1,\dots,N_m)^{(i-\alpha,j)}} - (n-R_1-\dots-R_{r-1}-r+1)$$
$$\times\left[\beta\mu_{r-1,s-1:m-1:n,k}^{(N_1,\dots,N_{r-2},(N_{r-1}+N_r+1),N_{r+1},\dots N_m)^{(i-1,j)}} + \frac{(i-1)\theta\alpha}{i-\alpha-2}\mu_{r-1,s-1:m-1:n,k}^{(N_1,\dots,N_{r-2},(N_{r-1}+N_r+1),N_{r+1},\dots N_m)^{(i-\alpha-2,j)}}\right]$$
$$+(n-R_1-R_2-\dots-R_r-r)$$
$$\left[\beta\mu_{r,s-1:m-1:n,k}^{(N_1,\dots,N_{r-1},(N_r+N_{r+1}+1),N_{r+2},\dots,N_m)^{(i-1,j)}} + \frac{(i-1)\theta\alpha}{i-\alpha-2}\mu_{r,s-1:m-1:n,k}^{(N_1,\dots,N_{r-1},(N_r+N_{r+1}+1),N_{r+2},\dots,N_m)^{(i-\alpha-2,j)}}\right]. \quad (2.5)$$

From (1.6), we get

$$\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\frac{(a\beta)^h(a\theta)^l}{h!\,l!}\mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i-h-\alpha l,j)}} - \frac{1}{ab}\mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i,j)}}$$
$$= [\bar{F}(x_{r-1})]^{N_{r-1}} f(x_{r+1})[\bar{F}(x_{r+1})]^{N_{r+1}} \dots f(x_m)[\bar{F}(x_m)]^{N_m}dx_1 \dots dx_{r-1}dx_{r+1}\dots dx_m. \quad (2.6)$$

Substituting by the obtained expression of $W_1(x_{r-1},x_{r+1})$ from (2.4) in (2.6) and simplifying, yields (2.5). This brings the proof for a close.

Theorem 2.3. For $1 \le r < s \le m-1, m \le n$ and $i,j \ge 0$, then

$$\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\frac{(j-1)(a\beta)^h(a\theta)^l}{h!\,l!}\mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i,j-h-\alpha l)}} = \frac{(j-1)}{ab}\mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i,j)}} + \beta(N_s+1)\mu_{r,s:m:n,k}^{(N_1,\dots,N_m)^{(i,j-1)}}$$
$$+\theta\alpha(j-1)\left(\frac{N_s+1}{j-\alpha-2}\right)\mu_{r,s:m:n,k}^{(N_1,\dots,N_m)^{(i,j-\alpha-2)}} - (n-R_1-\dots-R_{s-1}-s+1)$$
$$\times\left[\beta\mu_{r,s-1:m-1:n,k}^{(N_1,\dots,N_{s-2},(N_{s-1}+N_s+1),N_{s+1},\dots N_m)^{(i,j-1)}} + \frac{(j-1)\theta\alpha}{j-\alpha-2}\mu_{r,s-1:m-1:n,k}^{(N_1,\dots,N_{s-2},(N_{s-1}+N_s+1),N_{s+1},\dots N_m)^{(i,j-\alpha-2)}}\right]$$
$$+(n-R_{p+1}-R_{p+2}-\dots-R_s-s)$$
$$\left[\beta\mu_{r,s:m-1:n,k}^{(N_1,\dots,N_{s-1},(N_s+N_{s+1}+1),N_{s+2},\dots,N_m)^{(i,j-1)}} + \frac{(j-1)\theta\alpha}{j-\alpha-2}\mu_{r,s:m-1:n,k}^{(N_1,\dots,N_{s-1},(N_s+N_{s+1}+1),N_{s+2},\dots,N_m)^{(i,j-\alpha-2)}}\right].$$

Proof. The proof can easily derived similarly as in Theorem 2.2.

3. Characterizations

During this section, we proposed the characterization of the KMIWD depending RR for PFFC.

3.1. Characterizations via differential equation for KMIWD

In Theorem 3.1, we discuss the characterization of the KMIWD.

Theorem 3.1. Let X be a continuous variable with $[\bar{F}(\cdot) = 1 - F(.)]$. Then X has KMIWD iff

$$\left\{exp\left[a\left(\frac{\beta}{x}+\frac{\theta}{x^{\alpha}}\right)\right]-1\right\}f(x) = ab\left(\frac{\beta}{x^2}+\frac{\theta\alpha}{x^{\alpha+1}}\right)[\bar{F}(x)], x \geq 0. \quad (3.1)$$

Proof. **Necessary direction:** From (1.2) and (1.3), we can easily obtain (3.1).

Sufficiency direction: Suppose that (3.1) is true, then we get

$$\frac{-d[\bar{F}(x)]}{\bar{F}(x)} = \frac{ab\left(\frac{\beta}{x^2}+\frac{\theta\alpha}{x^{\alpha+1}}\right)}{\left\{exp\left[a\left(\frac{\beta}{x}+\frac{\theta}{x^{\alpha}}\right)\right]-1\right\}}dx = \frac{ab\left(\frac{\beta}{x^2}+\frac{\theta\alpha}{x^{\alpha+1}}\right)exp\left[-a\left(\frac{\beta}{x}+\frac{\theta}{x^{\alpha}}\right)\right]}{1-exp\left[-a\left(\frac{\beta}{x}+\frac{\theta}{x^{\alpha}}\right)\right]}dx.$$

By integrating, we get

$$-\ln|\bar{F}(x)| = -b\ln\left|1-exp\left[-a\left(\frac{\beta}{x}+\frac{\theta}{x^{\alpha}}\right)\right]\right| + C,$$

where C is an arbitrary constant.

Now, we get $C = 0$, when $x = 0$.

Therefore,

$$\ln|\bar{F}(x)| = \ln\left\{1-exp\left[-a\left(\frac{\beta}{x}+\frac{\theta}{x^{\alpha}}\right)\right]\right\}^{b}.$$

Hence,

$$F(x) = 1-\left\{1-exp\left[-a\left(\frac{\beta}{x}+\frac{\theta}{x^{\alpha}}\right)\right]\right\}^{b}.$$

Which is the CDF of KMIWD. This brings the proof to a close.

3.2. Characterization via single moment of KMIWD

In Theorem 3.2, we discuss the characterization of the KMIWD depending on the single moment of PFFC.

Theorem 3.2. With a survival function $[\bar{F}(\cdot)]$, let X be a continuous random variable where $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be a random ordered sample with size n. Then X has KMIWD iff for $2 \leq r \leq m-1, m \leq n$ and $i \geq 0$,

$$\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\frac{(i-1)(a\beta)^h(a\theta)^l}{h!\,l!}\mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i-h-\alpha l)}} = \frac{i-1}{ab}\mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i)}} + \beta(N_r+1)\mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i-1)}}$$

$$+(i-1)\theta\alpha\left(\frac{N_r+1}{i-\alpha-2}\right)\mu_{r:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i-\alpha-2)}} - (n-R_1-\cdots-R_{r-1}-r+1)$$

$$\times\left[\beta\mu_{r-1:m-1:n,k}^{(N_1,\dots,N_{r-2},(N_{r-1}+N_r+1),N_{r+1},\dots N_m)^{(i-1)}} + \frac{(i-1)\theta\alpha}{i-\alpha-2}\mu_{r-1:m-1:n,k}^{(N_1,\dots,N_{r-2},(N_{r-1}+N_r+1),N_{r+1},\dots N_m)^{(i-\alpha-2)}}\right]$$

$$+(n-R_1-\cdots-R_r-r)$$

$$\left[\beta\mu_{r:m-1:n,k}^{(N_1,\dots,N_{r-1},(N_r+N_{r+1}+1),N_{r+2},\dots,N_m)^{(i-1)}} + \frac{(i-1)\theta\alpha}{i-\alpha-2}\mu_{r:m-1:n,k}^{(N_1,\dots,N_{r-1},(N_r+N_{r+1}+1),N_{r+2},\dots,N_m)^{(i-\alpha-2)}}\right]. \quad (3.2)$$

Proof. **Necessary direction:** Theorem 2.1 provides the proof for the necessary side for this theorem.

Sufficiency direction: Assume that X be a random variable has a continous PDF $f(\cdot)$ and CDF $F(\cdot)$.

Let (3.2) is satisfied, then we have:

$$\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\frac{(a\beta)^h(a\theta)^l}{h!\,l!}\mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i-h-\alpha l)}} = \frac{1}{ab}\mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i)}} + \beta\left(\frac{N_r+1}{i-1}\right)\mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i-1)}}$$

$$+\theta\alpha\left(\frac{N_r+1}{i-\alpha-2}\right)\mu_{r:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(i-\alpha-2)}} - (n-R_1-\cdots-R_{r-1}-r+1)$$

$$\times\left[\frac{\beta}{i-1}\mu_{r-1:m-1:n,k}^{(N_1,\dots,N_{r-2},(N_{r-1}+N_r+1),N_{r+1},\dots N_m)^{(i-1)}} + \frac{\theta\alpha}{i-\alpha-2}\mu_{r-1:m-1:n,k}^{(N_1,\dots,N_{r-2},(N_{r-1}+N_r+1),N_{r+1},\dots N_m)^{(i-\alpha-2)}}\right]$$

$$+(n-R_1-\cdots-R_r-r)$$

$$\left[\frac{\beta}{i-1}\mu_{r:m-1:n,k}^{(N_1,\dots,N_{r-1},(N_r+N_{r+1}+1),N_{r+2},\dots,N_m)^{(i-1)}} + \frac{\theta\alpha}{i-\alpha-2}\mu_{r:m-1:n,k}^{(N_1,\dots,N_{r-1},(N_r+N_{r+1}+1),N_{r+2},\dots,N_m)^{(i-\alpha-2)}}\right], \quad (3.3)$$

where,

$$\mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i-1)}} = I_{(n,m-1)}\iint\cdots\int_{0<x_1<\cdots<x_{r-1}<x_{r+1}<\cdots<x_m<\infty} k^m W_2(x_{r-1},x_{r+1}) f(x_1)[\bar{F}(x_1)]^{N_1}\dots$$

$$\times f(x_{r-1})[\bar{F}(x_{r-1})]^{N_{r-1}} f(x_{r+1})[\bar{F}(x_{r+1})]^{N_{r+1}}\dots f(x_m)[\bar{F}(x_m)]^{N_m} dx_1\dots dx_{r-1}dx_{r+1}\dots dx_m, \quad (3.4)$$

where,

$$W_2(x_{r-1},x_{r+1}) = \int_{x_{r-1}}^{x_{r+1}} x_r^{i-1} f(x_r)[\bar{F}(x_r)]^{N_r} dx_r. \quad (3.5)$$

By integrating (3.5) by parts, we obtain

$$W_2(x_{r-1},x_{r+1}) = \frac{-1}{N_r+1}x_{r+1}^{i-1}[\bar{F}(x_{r+1})]^{N_r+1} + \frac{1}{R_r+1}x_{r-1}^{i-1}[\bar{F}(x_{r-1})]^{N_r+1} + \frac{i-1}{N_r+1}\int_{x_{r-1}}^{x_{r+1}} x_r^{i-2}[\bar{F}(x_r)]^{N_r+1}dx_r.$$

Now by substituting in Eq (3.4), we get

$$\mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i-1)}} = \frac{i-1}{N_r+1} I_{(n,m-1)} \iint \cdots \int_{0<x_1<\cdots<x_{r-1}<x_{r+1}<\cdots<x_m<\infty} k^m f(x_1)$$

$$\times [\bar{F}(x_1)]^{N_1} \dots \int_{x_{r-1}}^{x_{r+1}} x_r^{i-2} [\bar{F}(x_r)]^{N_r+1} dx_r\, f(x_{r-1}) [\bar{F}(x_{r-1})]^{N_{r-1}}$$

$$f(x_{r+1}) [\bar{F}(x_{r+1})]^{N_{r+1}} \dots f(x_m) [\bar{F}(x_m)]^{N_m} dx_1 \dots dx_{r-1} dx_{r+1} \dots dx_m$$

$$+ \frac{I_{(n,m-1)}}{N_r+1} \iint \cdots \int_{0<x_1<\cdots<x_{r-1}<x_{r+1}<\cdots<x_m<\infty} x_{r-1}^{i-1} k^m f(x_1)$$

$$\times [\bar{F}(x_1)]^{N_1} \dots f(x_{r-1}) [\bar{F}(x_{r-1})]^{N_{r-1}+N_r+1} f(x_{r+1})$$

$$\times [\bar{F}(x_{r+1})]^{N_{r+1}} \dots f(x_m) [\bar{F}(x_m)]^{N_m} dx_1 \dots dx_{r-1} dx_{r+1} \dots dx_m$$

$$- \frac{I_{(n,m-1)}}{N_r+1} \iint \cdots \int_{0<x_1<\cdots<x_{r-1}<x_{r+1}<\cdots<x_m<\infty} x_{r+1}^{i-1} k^m f(x_1)$$

$$\times [\bar{F}(x_1)]^{N_1} \dots f(x_{r-1}) [\bar{F}(x_{r-1})]^{N_{r-1}} f(x_{r+1})$$

$$\times [\bar{F}(x_{r+1})]^{N_r+N_{r+1}+1} \dots f(x_m) [\bar{F}(x_m)]^{N_m} dx_1 \dots dx_{r-1} dx_{r+1} \dots dx_m$$

$$= I_{(n,m-1)} \frac{i-1}{N_r+1} \iint \cdots \int_{0<x_1<\cdots<x_{r-1}<x_{r+1}<\cdots<x_m<\infty} k^m f(x_1)$$

$$\times [\bar{F}(x_1)]^{N_1} \dots \int_{x_{r-1}}^{x_{r+1}} x_r^{i-2} [\bar{F}(x_r)]^{N_r+1} dx_r\, f(x_{r-1}) [\bar{F}(x_{r-1})]^{N_{r-1}}$$

$$f(x_{r+1}) [\bar{F}(x_{r+1})]^{N_{r+1}} \dots f(x_m) [\bar{F}(x_m)]^{N_m} dx_1 \dots dx_{r-1} dx_{r+1} \dots dx_m$$

$$+ \frac{(n-R_1-\cdots-R_r-r)}{N_r+1} \mu_{r:m-1:n,k}^{(N_1,\dots,N_{r-1},(N_r+N_{r+1}+1),N_{r+2},\dots N_m)^{(i-1)}}$$

$$- \frac{(n-R_1-\cdots-R_{r-1}-r+1)}{N_r+1} \mu_{r-1:m-1:n,k}^{(N_1,\dots,N_{r-2},(N_{r-1}+N_r+1),N_{r+1},\dots,N_m)^{(i-1)}} \qquad (3.7)$$

and

$$\mu_{r:m:n,k}^{(N_1,\dots,kR_m+k-1)^{(i-\alpha-2)}} = I_{(n,m-1)} \frac{i-\alpha-1}{kR_r+k} \iint \cdots \int_{0<x_1<\cdots<x_{r-1}<x_{r+1}<\cdots<x_m<\infty} k^m$$

$$\times f(x_1) [1-F(x_1)]^{N_1} \dots \int_{x_{r-1}}^{x_{r+1}} x_r^{i-\alpha-1} [\bar{F}(x_r)]^{N_r+1} dx_r\, f(x_{r-1})$$

$$\times [\bar{F}(x_{r-1})]^{N_{r-1}} f(x_{r+1}) [\bar{F}(x_{r+1})]^{N_{r+1}} \dots f(x_m) [\bar{F}(x_m)]^{N_m} dx_1 \dots dx_{r-1} dx_{r+1} \dots dx_m$$

$$+ \frac{(n-R_1-\cdots-R_r-r)}{N_r+1} \mu_{r:m-1:n,k}^{(N_1,\dots,N_{r-1},(N_r+N_{r+1}+1),N_{r+2},\dots,N_m)^{(i-\alpha-2)}}$$

$$- \frac{(n-R_1-\cdots-R_{r-1}-r+1)}{N_r+1} \mu_{r-1:m-1:n,k}^{(N_1,\dots,N_{r-2},(N_{r11}+N_r+1),N_{r+1},\dots,N_m)^{(i-\alpha-2)}}. \qquad (3.8)$$

Now by substituting for $\mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i-1)}}$ and $\mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(i-\alpha-2)}}$ from (3.7) and (3.8) in (3.3), we obtain

$$I_{(n,m-1)} \iint \cdots \int_{0<x_1<\cdots<x_m<\infty} x_r^i \left[\sum_{h=0}^{\infty} \sum_{l=0}^{\infty} \frac{(a\beta)^h (a\theta)^l}{h!\, l!} \mu_{r:m:n,k}^{(N_1,\dots,N_m)^{(-h-\alpha l)}} - 1 \right] k^m$$

$$\times f(x_1)[\bar{F}(x_1)]^{N_1} \dots f(x_{r-1})[\bar{F}(x_{r-1})]^{N_{r-1}} f(x_r)[\bar{F}(x_r)]^{N_r+1} f(x_{r+1})[\bar{F}(x_{r+1})]^{N_{r+1}} \dots$$

$$\times f(x_m)[\bar{F}(x_m)]^{N_m} dx_1 \dots dx_m$$

$$= I_{(n,m-1)} \alpha\theta^2 \iint \cdots \int_{0<x_1<\cdots<x_m<\infty} x_r^i \left(\frac{\beta}{x_r^2} + \frac{\theta\alpha}{x_r^{\alpha+1}} \right) k^m f(x_1)[\bar{F}(x_1)]^{N_1} \dots$$

$$\times f(x_{r-1})[\bar{F}(x_{r-1})]^{N_{r-1}} [\bar{F}(x_r)]^{N_r+1} f(x_{r+1})[\bar{F}(x_{r+1})]^{N_{r+1}} \dots f(x_m)[\bar{F}(x_m)]^{N_m} dx_1 \dots dx_m.$$

We get

$$I_{(n,m-1)} \iint \cdots \int_{0<x_1<\cdots<x_m<\infty} x_r^i f(x_r)[\bar{F}(x_r)]^{N_r+1} k^m$$

$$\times \left\{ \left\{ exp\left[a\left(\frac{\beta}{x} + \frac{\theta}{x^\alpha} \right) \right] - 1 \right\} f(x_r) - \left(\frac{\beta}{x_r^2} + \frac{\theta\alpha}{x_r^{\alpha-1}} \right) [\bar{F}(x_r)] \right\} f(x_1)[\bar{F}(x_1)]^{N_1} \dots$$

$$\times f(x_{r-1})[\bar{F}(x_{r-1})]^{N_{r-1}} f(x_{r+1})[\bar{F}(x_{r+1})]^{N_{r+1}} \dots f(x_m)[\bar{F}(x_m)]^{N_m} dx_1 \dots dx_m$$

$$= 0.$$

Using Muntz-Szasz theorem in [24], we get

$$\left[\left\{ exp\left[a\left(\frac{\beta}{x} + \frac{\theta}{x^\alpha} \right) \right] - 1 \right\} \right] f(x_r) = \left(\frac{\beta}{x_r^2} + \frac{\theta\alpha}{x_r^{\alpha+1}} \right) [\bar{F}(x_r)].$$

By Theorem 3.1, we obtain

$$F(x) = 1 - \left\{ 1 - exp\left[-a\left(\frac{\beta}{x} + \frac{\theta}{x^\alpha} \right) \right] \right\}^b.$$

Which is the CDF of KMIWD. This brings the proof to a close.

Special cases:

(1) This theorem is going to hold for the PTIIRC when $k = 1$,

$$\sum_{h=0}^{\infty} \sum_{l=0}^{\infty} \frac{(a\beta)^h (a\theta)^l}{abh!\, l!} \mu_{r:m:n}^{(R_1,\dots,R_m)^{(i-h-\alpha l)}} = \frac{1}{ab} \mu_{r:m:n}^{(R_1,\dots,R_m)^{(i)}} + \beta \left(\frac{R_r + 1}{i-1} \right) \mu_{r:m:n}^{(R_1,\dots,R_m)^{(i-1)}}$$

$$+\theta\alpha \left(\frac{R_r + 1}{i - \alpha - 2} \right) \mu_{r:m:n}^{(R_1,\dots,R_m)^{(i-\alpha-2)}} - (n - R_1 - \cdots - R_{r-1} - r + 1)$$

$$\left[\frac{\beta}{i-1} \mu_{r-1:m-1:n}^{(R_1,\dots,R_{r-1}+R_r+1,\dots,R_m)^{(i-1)}} + \frac{\theta\alpha}{i-\alpha-2} \mu_{r-1:m-1:n}^{(R_1,\dots,R_{r-1}+R_r+1,\dots,R_m)^{(i-\alpha-2)}} \right]$$

$$+(n - R_1 - R_2 - \cdots - R_r - r)$$

$$\left[\frac{\beta}{i-1} \mu_{r:m-1:n}^{(R_1,\dots,(R_r+R_{r+1}+1,\dots,R_m)^{(i-1)}} + \frac{\theta\alpha}{i-\alpha-2} \mu_{r:m-1:n}^{(R_1,\dots,R_r+R_{r+1}+1,\dots,R_m)^{(i-\alpha-2)}} \right].$$

(2) For $k = 1$ and $r = m$,

$$\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\frac{(a\beta)^h(a\theta)^l}{abh!\,l!}\mu_{m:m:n}^{(R_1,\dots,R_m)^{(i-h-\alpha l)}} = \frac{1}{ab}\mu_{m:m:n}^{(R_1,\dots,R_m)^{(i)}} + \beta\left(\frac{R_m+1}{i-1}\right)\mu_{m:m:n}^{(R_1,\dots,R_m)^{(i-1)}}$$

$$+\theta\alpha\left(\frac{R_m+1}{i-\alpha-2}\right)\mu_{m:m:n}^{(R_1,\dots,R_m)^{(i-\alpha-2)}} - (n - R_1 - \cdots - R_{m-1} - m + 1)$$

$$\left[\frac{\beta}{i-1}\mu_{m-1:m-1:n}^{(R_1,\dots,R_{m-1}+R_m+1)^{(i-1)}} + \frac{\theta\alpha}{i-\alpha-2}\mu_{m-1:m-1:n}^{(R_1,\dots,R_{m-1}+R_m+1)^{(i-\alpha-2)}}\right].$$

(3) For $k = 1$ and $2 \le m \le n$,

$$\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\frac{(a\beta)^h(a\theta)^l}{abh!\,l!}\mu_{1:m:n}^{(R_1,\dots,R_m)^{(i-h-\alpha l)}} = \frac{1}{ab}\mu_{1:m:n}^{(R_1,\dots,R_m)^{(i)}} + \beta\left(\frac{R_1+1}{i-1}\right)\mu_{1:m:n}^{(R_1,\dots,R_m)^{(i-1)}}$$

$$+\theta\alpha\left(\frac{R_1+1}{i-\alpha-2}\right)\mu_{1:m:n}^{(R_1,\dots,R_m)^{(i-\alpha-2)}} - (n - R_1 - 1)$$

$$\left[\frac{\beta}{i-1}\mu_{1:m-1:n}^{((R_1+R_2+1),R_3,\dots,R_m)^{(i-1)}} + \frac{\theta\alpha}{i-\alpha-2}\mu_{1:m-1:n}^{((R_1+R_2+1),R_3,\dots,R_m)^{(i-\alpha-2)}}\right].$$

(4) For $k = 1,\ m = 1$ and $n = 1,2,\dots$,

$$\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\frac{(a\beta)^h(a\theta)^l}{abh!\,l!}\mu_{1:1:n}^{(n-1)^{(i-h-\alpha l)}} = \frac{1}{ab}\mu_{1:1:n}^{(n-1)^{(i)}} + \left(\frac{\beta}{i-1}\right)\mu_{1:1:n}^{(n-1)^{(i-1)}} + \left(\frac{\theta\alpha}{i-\alpha-2}\right)\mu_{1:1:n}^{(n-1)^{(i-\alpha-2)}}.$$

(5) For $k = 1,\ m = 1, n = 1$ and $R_1 = \cdots = R_m = 0$,

$$\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\frac{(a\beta)^h(a\theta)^l}{abh!\,l!}\mu^{(i-h-\alpha l)} = \frac{\mu^{(i)}}{ab} + \beta\frac{\mu^{(i-1)}}{i-1} + \theta\alpha\frac{\mu^{(i-\alpha-2)}}{i-\alpha-2},$$

using Theorem 3.1 we get

$$E(X^i) = ab!\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{h+l}[a\theta(h+1)]^l}{h!\,l!\,(b-h-1)!}\left[\frac{\beta(\alpha l - i)!}{[a\beta(h+1)]^{\alpha l+1-i}} + \frac{\theta\alpha(\alpha l + \alpha - i - 1)!}{[a\beta(h+1)]^{\alpha l+\alpha-i}}\right].$$

The mathematical expectation, variance, skewness and kurtosis of the KMIWD:

$$Mean(X) = E(x),$$

$$Variance(X) = E(x^2) - E^2(x),$$

$$Skewness(X) = \frac{E(x^3) - 3E(x)E(x^2) + 2E^3(x)}{Var^{\frac{3}{2}}(x)},$$

$$Kurtosis(X) = \frac{E(x^4) - 4E(x)E(x^3) + 6E(x^2)E^2(x) - 3E^4(x)}{Var^2(x)}.$$

3.3. Characterizations via product moments of KMIWD

In this subsection, we characterize the KMIWD using product moments of PFFC.

Theorem 3.3. Let X is a continuous random variable has a survival function $[\bar{F}(\cdot)]$. Let $X_{1:n}\leq\ldots\leq X_{n:n}$ be a random ordered sample of size n. Then X has KMIWD iff, for $1\leq r<s\leq m-1,\ m\leq n$ and $i,j\geq 0$,

$$\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\frac{(i-1)(a\beta)^h(a\theta)^l}{h!\,l!}\mu_{r:m:n,k}^{(N_1,\ldots,N_m)^{(i-h-\alpha l,j)}}=\frac{(i-1)}{ab}\mu_{r,s:m:n,k}^{(N_1,\ldots,N_m)^{(i,j)}}$$

$$+\beta(N_r+1)\mu_{r,s:m:n,k}^{(N_1,\ldots,N_m)^{(i-1,j)}}+(i-1)\theta\alpha\left(\frac{N_r+1}{i-\alpha-2}\right)\mu_{r,s:m:n,k}^{(N_1,\ldots,N_m)^{(i-\alpha-2,j)}}$$

$$-(n-R_1-\cdots-R_{r-1}-r+1)$$

$$\times\left[\beta\mu_{r-1,s-1:m-1:n,k}^{(N_1,\ldots,N_{r-2},(N_{r-1}+N_r+1),N_{r+1},\ldots N_m)^{(i-1,j)}}+\frac{(i-1)\theta\alpha}{i-\alpha-2}\mu_{r-1,s-1:m-1:n,k}^{(N_1,\ldots,N_{r-2},(N_{r-1}+N_r+1),N_{r+1},\ldots,N_m)^{(i-\alpha-2,j)}}\right]$$

$$+(n-R_1-\cdots-R_r-r)$$

$$\left[\beta\mu_{r,s-1:m-1:n,k}^{(N_1,\ldots,N_{r-1},(N_r+N_{r+1}+1),N_{r+2},\ldots,N_m)^{(i-1,j)}}+\frac{(i-1)\theta\alpha}{i-\alpha-2}\mu_{r,s-1:m-1:n,k}^{(N_1,\ldots,N_{r-1},(N_r+N_{r+1}+1),N_{r+2},\ldots,N_m)^{(i-\alpha-2,j)}}\right]. \quad (3.10)$$

Proof. **Necessary direction:** Theorem2.2 leads to prove the necessary side for this theorem.

Sufficiency direction: The proof is easily obtained as in Theorem 3.2, we derive the CDF of KMIWD as follows

$$F(x)=1-\left\{1-exp\left[-a\left(\frac{\beta}{x}+\frac{\theta}{x^{\alpha}}\right)\right]\right\}^{b}.$$

That is the CDF of KMIWD and the proof is now complete.

Theorem 3.4. Let X be a random continuous variable having a survival function $[\bar{F}(\cdot)]$. Let $X_{1:n}\leq X_{2:n}\leq\ldots\leq X_{n:n}$ be a random ordered sample with size n having KMIWD iff, for $1\leq r<s\leq m-1,\ m\leq n$ and $i,j\geq 0$,

$$\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\frac{(j-1)(a\beta)^h(a\theta)^l}{h!\,l!}\mu_{r:m:n,k}^{(N_1,\ldots,N_m)^{(i,j-h-\alpha l)}}=\frac{(j-1)}{ab}\mu_{r:m:n,k}^{(N_1,\ldots,N_m)^{(i,j)}}+\beta(N_s+1)\mu_{r,s:m:n,k}^{(N_1,\ldots,N_m)^{(i,j-1)}}$$

$$+(j-1)\theta\alpha\left(\frac{N_s+1}{j-\alpha-2}\right)\mu_{r,s:m:n,k}^{(N_1,\ldots,N_m)^{(i,j-\alpha-2)}}-(n-R_1-\cdots-R_{s-1}-s+1)$$

$$\times\left[\beta\mu_{r,s-1:m-1:n,k}^{(N_1,\ldots,N_{s-2},(N_{s-1}+N_s+1),N_{s+1},\ldots N_m)^{(i,j-1)}}+\frac{(j-1)\theta\alpha}{j-\alpha-2}\mu_{r,s-1:m-1:n,k}^{(N_1,\ldots,N_{s-2},(N_{s-1}+N_s+1),N_{s+1},\ldots N_m)^{(i,j-\alpha-2)}}\right]$$

$$+(n-R_1-R_2-\cdots-R_s-s)$$

$$\left[\beta\mu_{r,s:m-1:n,k}^{(N_1,\ldots,N_{s-1},(N_s+N_{s+1}+1),N_{s+2},\ldots,N_m)^{(i,j-1)}}+\frac{(j-1)\theta\alpha}{j-\alpha-2}\mu_{r,s:m-1:n,k}^{(N_1,\ldots,N_{s-1},(N_s+N_{s+1}+1),N_{s+2},\ldots,N_m)^{(i,j-\alpha-2)}}\right]. \quad (3.11)$$

Proof. **Necessary direction:** Theorem 2.3 leads to prove the necessary side for this theorem.

Sufficiency direction: The proof is easily obtained as well as in Theorem 3.2 we get the CDF of KMIWD as follows

$$F(x)=1-\left\{1-exp\left[-a\left(\frac{\beta}{x}+\frac{\theta}{x^{\alpha}}\right)\right]\right\}^{b}.$$

That is the CDF of KMIWD and the proof is now complete.

4. Conclusions

In this research, some Recurrence relationships for single moments and for product moments of the PFFC data from the KMIWD have been established. Further, the characterization of the KMIWD have been studied. The results showed that for all censoring techniques and sample sizes, we can easily and recursively acquire both the single and product moments of any PFFC with direct computations which saves time, money and effort. Recurrence relationships for the product and single moments for different special cases have been obtained as the case of the progressive type-II censoring. Also, this work can be reduced to a special distribution, as shown in Table 1.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There are no conflicts of interest declared by the authors.

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