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Development of Hybrid Block Integrator for the Numerical solution of Stiff and Oscillatory Differential Equations for first order Ordinary Differential Equations

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ABSTRACT

This paper discussed the development of a new numerical hybrid block integrator for the numerical solution of stiff and oscillatory differential equations for first order ordinary differential equations. The method of interpolation and collocation at some selected grid point to generate the continuous scheme was adopted. The research also investigates the consistency, convergence, zero-stability and region of absolute stability of the integrator using matlab and the integrator was tested on some numerical experiments for comparism. The analysis of the method showed that the method is Zero-stable, consistent, convergent and computationally stable. The method handles stiff and Oscillatory differential equations effectively.

AMS Subject Classification: 65L05, 65L06, 65D30

KEYWORDS

Consistency, Convergence, Zero-Stability and Region of Absolute Stability.

1. Introduction

The field of sciences, engineering, mathematics, fracture mechanics and management involves rate of change of one or more quantities in relation to one another which may result to general first order ordinary differential equations (ODEs). Engineering problem is formed as an accurate physical model and this physical model is then transformed to mathematical model which is typically described in ODEs.

In this research, a new numerical hybrid block integrator for the solution of stiff and oscillatory higher order initial value problems of ODEs of the form

$$y' = f(x,y)\,, y(x_0) = y_0, \qquad x\varepsilon[a,b] \quad (1)$$

Method of collocation of the differential system and interpolation of the approximate solution which is a combination of power series and exponential function at some selected grid and off-grid points to generate a linear multistep method which will be implemented in block method and can be of the form.

The solution of (1) has been discussed by various researchers among them are: Onumanyi et al., (1994), James and Adesanya (2011). Adoption of collocation and interpolation of power series approximate solution to developed block method for solution of initial value problems have been studied by many scholars, among them are James et al.,(2013), Fasasi et al.,(2014), Areo and Adeniyi (2013).These authors independently implemented their methods such that the solutions are simultaneously generated at different grid points within the interval of integration and they reported that block method is more efficient than the existing method in terms of time of development and execution. Moreover, block method gives better approximation than the predictor - corrector method and enables the nature of the problem to be understood at the selected grid points. The introduction of hybrid method to avoid the Dalhquist stability barrier has been studied by some scholars which include Anake et al., (2012) and Adamu et al.,(2013). This scholars reported that though hybrid method are difficult to develop but enables the reduction in the step lenght.These scholars equally reported that lower k step method gives better result than the higher k step method.

Fatunla, (2011) classified numerical integration schemes into three:

(i) Schemes in which the approximate solutions are given by a linear combination of independent functions. Examples are Taylor series, Lie approach and Picard integration scheme. These methods generate the series of solutions whose convergence cannot be guaranteed, and when the series do converge, the rate of convergence may be rather too slow.

(ii) Methods whereby the theoretical solution is approximated by the first few terms of an expansion in orthogonal functions

$$y_M = \sum_{i=1}^{M} N_i V_i(x) \quad (2)$$

The expansion coefficient N_i can depend on M and possibly on $V_i(x)$. The function $V_i(x)$ may be defined over the integration interval (a, b) as in the case of chebyshev Polynomial.

(iii) Methods that consists of the discrete variable methods where by numerical approximations are obtained at some specified points in the integration interval. This approach is universally applicable, since it is more amenable to computer implementation.

The two major discrete variable methods we have are (i) one-step (single step) methods and (ii) the multistep methods especially the linear multistep methods (LMMs). The development of numerical integration formulas for (1) has attracted considerable attention in the past, most conventional numerical integration methods cannot efficiently cope with (1) because they lack adequate stability characteristics, (Fatunla,2011).

The conventional method of solving (1) is by reduction to system of first order ordinary differential equation and appropriate numerical method could be employed to solve the resultant systems. Awoyemi, et.al,. (2010), reported that in spite of the success of this approach, the setback of the method is in writing computer program which is often complicated especially when subroutine are incorporated to supply the starting values required for the method. The consequence is in longer computer time and human effort. Furthermore,
Adesanya et. al (2012a) stated that this method does not utilize additional information associated with specific ordinary differential equation such as the oscillatory nature of the solution. In addition, Olaide et.al, (2012) reported that more serious disadvantage of the method of reduction is the fact that the given system of equation to be solved cannot be solved explicitly with respect to the derivatives of the highest order. For these reasons, this method is inefficient and not suitable for general purpose. Scholars later developed method to solve (1) directly without reducing it to systems of first order ordinary differential equations and concluded that direct method is better than method of reduction, among these authors are Sagir (2014), Awoyemi et.al, (2010), Adesanya et.al (2012b). In order to cater for the above mentioned setback, researchers came up with block methods which simultaneously generate approximation at different grid points within the interval of integration. Block method is less expensive in terms of the number of function evaluations compared to the linear multistep method or Runge-Kutta method, above all, it does not require predictors or starting values. Fotta, et al,. (2015), proposed continuous hybrid block method for the solution of general second order initial value problem of ordinary differentialequations which when evaluated at selected grid points gives discrete block. The Continuous blockmethod possesses the same properties as the continuous hybrid linear multistep.

2. Methodology

We considered an approximate solution that combines power series and exponential function of the form

$$y(x)=\sum_{i=0}^{1}a_i e^{i}+\sum_{i=2}^{4}a_i e^{-ix} \qquad (3)$$

where a_i's are unknown parameters to be determined. The first derivative of (3) gives

$$y'(x)=\sum_{i=0}^{1}ia_i e^{i-1}-\sum_{i=2}^{4}ia_i e^{-ix} \qquad (4)$$

Substituting (4) into (3) gives

$$f(x,y)=\sum_{i=0}^{1}ia_i e^{i-1}-\sum_{i=2}^{4}ia_i e^{-ix} \qquad (5)$$

Interpolating (3) at $x = x_n$ and collocating (5) at $x = x_{n+s}, s = 0, \frac{1}{3}, \frac{2}{3}, 1$ gives a system of non-linear equation in the form

$$XA = U \quad (6)$$

where

$$X = \begin{bmatrix} 1 & x_n & e^{-2x_n} & e^{-3x_n} & e^{-4x_n} \\ 0 & 1 & -2e^{-2x_n} & -3e^{-3x_n} & -4e^{-4x_n} \\ 0 & 1 & -2e^{-2x_{n+\frac{1}{3}}} & -3e^{-3x_{n+\frac{1}{3}}} & -4e^{-4x_{n+\frac{1}{3}}} \\ 0 & 1 & -2e^{-2x_{n+\frac{2}{3}}} & -3e^{-3x_{n+\frac{2}{3}}} & -4e^{-4x_{n+\frac{2}{3}}} \\ 0 & 1 & -2e^{-2x_{n+1}} & -3e^{-3x_{n+1}} & -4e^{-4x_{n+1}} \end{bmatrix}$$

and $A = [a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4]^T, U = [y_n \quad f_n \quad f_{n+\frac{1}{3}} \quad f_{n+\frac{2}{3}} \quad f_{n+1}]^T$

Solving (6) for the a_i's and substituting back into (5) gives a linear multistep method of the form

$$y(t) = \alpha_0(t) y_n + h(\beta_0 f_n + \beta_{\frac{1}{3}} f_{n+\frac{1}{3}} + \beta_{\frac{2}{3}} f_{n+\frac{2}{3}} + \beta_1 f_{n+1}) \quad (7)$$

Where

$$t = \frac{x - x_n}{h}, \; f_{n+j} = f(x_n + jh, y(x_n + jh))$$

$$\begin{aligned} \alpha_0 &= 1 \\ \beta_0 &= -\frac{1}{8} h(9t^4 - 24t^3 + 22t^2 - 8t), \\ \beta_{\frac{1}{3}} &= \frac{3}{8} h(9t^4 - 20t^3 + 12t^2), \\ \beta_{\frac{2}{3}} &= -\frac{3}{8} h(9t^4 - 16t^3 + 6t^2), \\ \beta_1 &= \frac{1}{8} h(9t^4 - 12t^3 + 4t^2). \end{aligned} \quad (8)$$

evaluating (8) at $t = 0, \frac{1}{3}, \frac{2}{3}, 1$ and writing in block form gives a discrete block formula of the form

$$A^{(0)} Y_m = e y_n + h d f(y_n) + h b F(Y_m) \quad (9)$$

where

$$Y_m = [y_{n+\frac{1}{3}} \quad y_{n+\frac{2}{3}} \quad y_{n+1}]^T,$$

$$y_n = [y_{n-\frac{1}{3}} \quad y_{n-\frac{2}{3}} \quad y_n]^T,$$

$$F(Y_m) = [f_{n+\frac{1}{3}} \quad f_{n+\frac{2}{3}} \quad f_{n+1}]^T,$$

$$f(y_n) = [f_{n-\frac{1}{3}} \quad f_{n-\frac{2}{3}} \quad f_{n-1}]^T$$

$$A^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, e = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, d = \begin{bmatrix} 0 & 0 & \frac{1}{9} \\ 0 & 0 & \frac{1}{9} \\ 0 & 0 & \frac{1}{9} \end{bmatrix}, b = \begin{bmatrix} \frac{17}{75} & -\frac{5}{75} & \frac{1}{75} \\ \frac{4}{9} & \frac{1}{9} & 0 \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix}$$

3.0 Analysis of Basic Properties

3.1 Order of the block method

Let the linear operator $L\{y(x):h\}$ associated with the block integrator (9) be defined as

$$L\{y(x):h\} = A^{(0)}Y_m - ey_n + hdf(y_n) + hbF(Y_m) \quad (10)$$

Expanding using Taylor series and comparing the coefficients of h gives

$$L\{y(x):h\} = C_0 y(x) + C_1 hy'(x) + \ldots + C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) + \ldots \quad (11)$$

3.2 Order of Block Method developed

The linear operator L and associated block method are said to be of order p if $c_0 = c_1 = c_2 = \ldots = c_p = 0, c_{p+1} \neq 0. c_{p+1}$ is called the error constant and implies that the truncation error is given by

$$t_{n+k} = C_{p+1} h^{p+1} y^{p+1}(x) + O(h^{p+2}) \quad (12)$$

For our method

$$L\{y(x):h\} = \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-\frac{1}{3}} \\ y_{n-\frac{2}{3}} \\ y_n \end{bmatrix} - h \begin{bmatrix} \frac{1}{8} & \frac{19}{72} & -\frac{5}{72} & \frac{1}{72} \\ \frac{1}{9} & \frac{4}{9} & \frac{1}{9} & 0 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \end{bmatrix} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(13)

Expanding (13) in Taylor series, we find that

$$\begin{bmatrix} \sum_{j=0}^{\infty}\frac{(\frac{1}{3}h)^j}{j!}y'_n - y_n - \frac{1h}{8}y'_n - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{19}{72}(\frac{1}{3})^j - \frac{5}{72}(\frac{2}{3})^j + \frac{1}{72}(1)^j\right\} \\ \sum_{j=0}^{\infty}\frac{(\frac{2h}{3})^j}{j!}y'_n - y_n - \frac{1h}{9}y'_n - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{4}{9}(\frac{1}{3})^j - \frac{1}{9}(\frac{2}{3})^j + 0(1)^j\right\} \\ \sum_{j=0}^{\infty}\frac{(h)^j}{j!}y'_n - y_n - \frac{1h}{8}y'_n - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{3}{8}(\frac{1}{3})^j + \frac{3}{8}(\frac{2}{3})^j + \frac{1}{8}(1)^j\right\} \end{bmatrix} \quad (14)$$

Hence, $c_0 = c_1 = c_2 = c_3 = c_4 = 0, c_5 \neq 0$, LTE $= [-1.0860(-04), -4.5725(-05), -1.5432(-4)]^T$.

The hybrid block integrator is of order 4.

3.3 Zero – Stability of the method developed

The block integrator (9) is said to be zero stable, if the roots z_s, s = 1, 2,..., k of the first characteristics polynomial $\rho(z)$ defined by $\rho(z) = \det(zA^{(0)} - e)$ satisfies $|z_s| \leq 1$ and if every root with modulus , $|z_s| = 1$ have multiplicity not exceeding the order of the differential equation. Moreover as $h \to 0, \rho(z) = z^{r-\mu}(z-1)^{\mu}$, where μ is the order of the differential equation, r is the order of the matrices $A^{(0)}$ and e.

$$\rho(z) = \left| z\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$\rho(z) = z^3 - z^2 = z^2(z-1) = 0 \Rightarrow z_1 = z_2 = 0. z_3 = 1.$ Hence our integrator is zero - stable.

3.4 Consistency of the method developed

The block integrator (9) is consistent since it has order greater than one. From the above analysis, it is obvious that our integrator is consistent.

3.5 Convergence of the method developed

The block integrator is convergent by the consequence of Dahlquist theorem below.

Theorem 1:

The necessary and sufficient conditions for a linear multistep method to be convergent are that it be consistent and zero stable.

3.6 Region of Absolute Stability of the method developed

Region of absolute stability is a region in the complex z plane, where $z = \lambda h$. It is defined as those values of z such that the numerical solution of $y' = -\lambda y$ satisfies $j_j \to 0$ as $j \to \infty$ for any initial condition.

To determine the absolute stability region of the new block Integrator, we adopt the boundary locus method. This is achieved by substituting the test equation

$$y' = -\lambda y \quad (15)$$

into the block formula (9).This gives

$$A^{(0)} Y_m(w) = e y_n(w) - h\lambda D y_n(w) - h\lambda B Y_m(w) \quad (16)$$

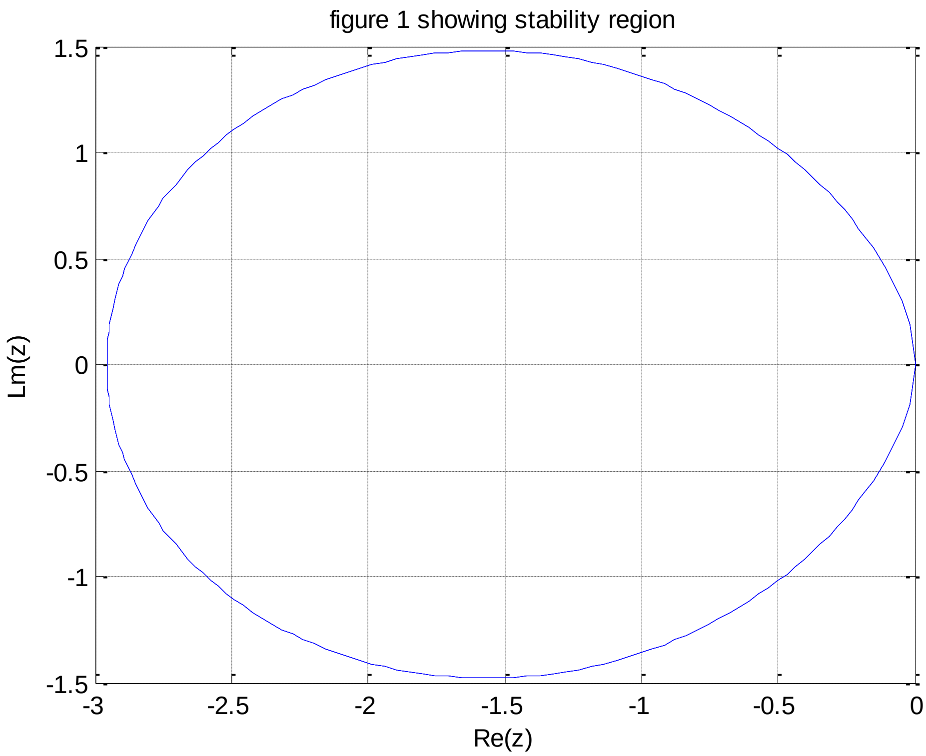
Thus,

$$\bar{h}(w) = -\left(\frac{A^{(0)} Y_m(w) - e y_n(w)}{D y_n(w) + B Y_m(w)} \right) \quad (17)$$

Since $\bar{h}$ is given by $\bar{h} = \lambda h$ and $w = e^{i\theta}$. Equation (17) is our characteristics or stability polynomial for our integrator, equation (16) is given by

$$\bar{h}(w) = \left(-\frac{1}{108} w^3 - \frac{1}{108} w^2 \right) h^3 + \left(\frac{11}{108} w^3 - \frac{11}{108} w^2 \right) h^2 + \left(-\frac{1}{2} w^3 - \frac{1}{2} w^2 \right) h + \left(w^3 - w^2 \right)$$

The Region of Absolute Stability for the method is shown in figure 1 below.



4. Numerical Examples

We shall evaluate the performance of the block integrator on some challenging stiff and oscillatory problems which have appeared in literature and compare the results with solutions from some methods of similar derivation. The numerical results are obtained using MATLAB.

The following notations shall be used in the tables below.

ERR - |Exact solution - Computed solution| (Error in new method)

ERS - Error in Sunday, et al., (2014).

ERO - Error in Okunuga, et al., (2022).

ERA - Error in Adebayo and Umar, et al., (2013).

4.1 Problem 1

Consider the highly stiff ordinary differential equation:

$$y' = -10(y-1)^2, y(0) = 2$$

Exact solution: $y(x) = 1 + \dfrac{1}{1+10x}$

This problem was earlier discussed by Lambert (2020), the newly derived hybrid block integrator is used for the integration of this problem within the interval $0 \le x \le 0.1$. Okunuga and Sofoluwe et al., (2022) solved this stiff problem by adopting a new 2 - point hybrid block method with step size ratio at r = 1. Sunday et al., (2014) also solved problem 1 by applying a self - starting block integrator.

4.2 Problem 2

Consider the Prothero - Robinson Oscillatory ODE.

$$y' = L(y - \sin x) + \cos x, L = -1, y(0) = 0.$$

Exact solution: $y(x) = \sin x$

Adebayo and Umar (2013) solved this problem by adopting a generalized rational approximation method via Pade approximants with step number k = 6, r = 1. In a similar manner Sunday et al., (2014) solved problem 2 by applying a self - starting block integrator.

Table 1: Showing the results for stiff Problem 1

X	ERR	ERS	ERO
0.0100	2.879519e-011	3.414671e-006	2.07e-03
0.0200	4.097789e-011	2.749635e-006	3.30e-03
0.0300	4.514411e-011	1.342943e-005	3.21e-03
0.0400	4.539880e-011	9.090648e-005	6.30e-03
0.0500	4.378098e-011	7.969685e-005	8.53e-03
0.0600	4.132628e-011	6.994886e-005	9.00e-03
0.0700	3.856981e-011	6.270048e-005	9.98e-03
0.0800	3.578560e-011	6.017101e-005	2.06e-02

0.0900	3.311063e-011	5.411308e-005	2.10e-02
0.1000	3.060774e-011	4.880978e-005	2.12e-02

Table 2: Showing the results for Prothero - Robinson Oscillatory Problem 2

X	ERR	ERS	ERA
0.1000	1.473405e-013	1.452952e-011	3.0e-11
0.2000	2.791656e-013	1.621117e-011	4.0e-11
0.3000	3.954059e-013	2.131013e-011	2.0e-10
0.4000	4.962142e-013	1.379910e-011	3.0e-10
0.5000	5.817569e-013	2.744084e-011	2.0e-10
0.6000	6.520340e-013	1.111424e-011	3.0e-10
0.7000	7.072121e-013	2.865663e-011	2.0e-10
0.8000	7.477352e-013	1.921784e-010	3.0e-10
0.9000	7.736034e-013	1.239202e-010	4.0e-10
1.0000	7.857048e-013	1.471102e-010	4.0e-10

4.3 Discussion of Results

We have considered two numerical examples to test the efficiency of our method. The first problem (which is stiff) was solved by Okunuga and Sofoluwe (2022), where they both applied 2-point block method with step-size ratio at r = 1, while the second problem (which is oscillatory) was solved by Adebayo and Umar (2013) where they adopted generalized rational approximation method via Pade approximants with step number k = 6. We solved the two problems using the new hybrid block integrator developed. Tables 1 and 2 above showed that the hybrid block integrator gives better results than the existing methods. It should be noted that the method performs better when the step-size is chosen within the stability interval. The problems are first order ODEs; hence, the method performs better when the step size is within the stability interval.

4.4 Conclusion

A first order stiff and oscillatory initial value problems have been developed. The method are found to be convergent and A-stable. A MATLAB code using Newton Raphson's method for solution of equations was written to implement the block method. Results of the numerical examples show that the methods are good for stiff and oscillatory problems.

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